Chapter 2
Linear Programming

Since G.B. Dantzig first proposed the simplex method around 1947, linear programming, as an optimization method of maximizing or minimizing a linear objective function subject to linear constraints, has been extensively studied and, with the significant advances in computer technology, widely used in the fields of operations research, industrial engineering, systems science, management science, and computer science.

In this chapter, after an overview of the basic concepts of linear programming via a simple numerical example, the standard form of linear programming and fundamental concepts and definitions are introduced. The simplex method and the two-phase method are presented with the details of the computational procedures. By reviewing the procedure of the simplex method, the revised simplex method, which provides a computationally efficient implementation, is also discussed. Associated with linear programming problems, dual problems are formulated, and duality theory is discussed which also leads to the dual simplex method.

2.1 Algebraic Approach to Two-Dimensional Linear Programming

In Sect. 1.1, we have presented a graphical method for solving the two-dimensional production planning problem of Example 1.1.

Minimize the opposite of the linear total profit

$$z = -3x_1 - 8x_2$$

subject to the linear inequality constraints

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$$
\begin{aligned}
2x_1 + 6x_2 &\leq 27 \\
3x_1 + 2x_2 &\leq 16 \\
4x_1 + x_2 &\leq 18
\end{aligned}
$$

and nonnegativity conditions for all decision variables

$$x_1 \geq 0,\ x_2 \geq 0.$$

Since in multiple dimensions more than two the graphical method used in Sect. 1.1 cannot be applied, it becomes necessary to develop an algebraic method. In this section, as a prelude to the development of the general theory, consider an algebraic approach to two-dimensional linear programming problems for understanding the basic ideas of linear programming. To do so, by introducing the amounts, x_3 (≥ 0), x_4 (≥ 0), and x_5 (≥ 0), of unused (idle) materials for M_1, M_2, and M_3, respectively, and converting the inequalities into the equalities, the problem with the equation $-3x_1 - 8x_2 - z = 0$ for the objective function can then be stated as follows:

Find values of $x_j \geq 0$, $j = 1, 2, 3, 4, 5$ so as to minimize z, satisfying the augmented system of linear equations

$$
\left.
\begin{array}{rrrrrrl}
2x_1 & +\ 6x_2 & +\ x_3 & & & & = 27 \\
3x_1 & +\ 2x_2 & & +\ x_4 & & & = 16 \\
4x_1 & +\ x_2 & & & +\ x_5 & & = 18 \\
-3x_1 & -\ 8x_2 & & & & -z & = \ 0.
\end{array}
\right\}
\tag{2.1}
$$

In (2.1), setting $x_1 = x_2 = 0$ yields $x_3 = 27$, $x_4 = 16$, $x_5 = 18$, and $z = 0$, which corresponds to the extreme point A in Fig. 1.1. Now, from the fourth equation of (2.1) for the objective function, we see that any increase in the values of x_1 and x_2 from 0 to positive would decrease the value of the objective function z. Considering that the profit of P_2 is larger than that of P_1 (in the above formulation, the opposite of the profit is smaller), choose to increase x_2 from 0 to a positive value, while keeping $x_1 = 0$. In Fig. 1.1, this corresponds to the movement from the extreme point A to E along the edge AE. From (2.1), if x_2 can be made positive, the values of x_3, x_4, and x_5 decrease. However, since x_3, x_4, and x_5 cannot become negative, the increase amount of x_2 is restricted by the first three equations of (2.1). In the first three equations of (2.1), remaining $x_1 = 0$, the values of x_2 to be increased are restricted to at most $27/6 = 4.5$, $16/2 = 8$, and $18/1 = 18$, respectively. Hence, the largest permissible value of x_2 not yielding the negative values of x_3, x_4, and x_5 is the smallest of 4.5, 8, and 18, that is, 4.5. Increasing the values of x_2 from 0 to 4.5 yields $x_3 = 0$, which implies that the available amount of material M_1 is used up.

Dividing the first equation of (2.1) by the coefficient 6 of x_2 and eliminating x_2 from the second, third, and fourth equations yields

$$\left.\begin{array}{rrrrrr} \frac{1}{3}x_1 & + \, x_2 & + \, \frac{1}{6}x_3 & & & = 4.5 \\ \frac{7}{3}x_1 & & - \, \frac{1}{3}x_3 & + \, x_4 & & = 7 \\ \frac{11}{3}x_1 & & - \, \frac{1}{6}x_3 & & + \, x_5 & = 13.5 \\ -\frac{1}{3}x_1 & & + \, \frac{4}{3}x_3 & & & -z = 36. \end{array}\right\} \tag{2.2}$$

In (2.2), setting $x_1 = x_3 = 0$ yields $x_2 = 4.5$, $x_4 = 7$, $x_5 = 13.5$, and $z = -36$. This implies the resulting point $(x_1, x_2) = (0, 4.5)$ corresponds to the extreme point E and the value of the objective function z is decreased from 0 to -36.

Next, from the fourth equation of (2.2), keeping $x_3 = 0$, by increasing the value of x_1 from 0 to positive, the value of z can be decreased. This corresponds to the movement from the extreme point E to D along the edge ED in Fig. 1.1. From the first three equations of (2.2), to keep the values of x_2, x_4, and x_5 nonnegative, the values of x_1 to be increased are restricted to at most $4.5/(1/3) = 13.5$, $7/(7/3) = 3$, and $13.5/(11/3) \simeq 3.682$, respectively. Hence, increasing the values of x_2 from 0 to 3, the smallest among them, yields $x_4 = 0$, which implies that the available amount of material M_2 is used up.

Dividing the second equation of (2.2) by the coefficient 7/3 of x_1 and eliminating x_1 from the first, third, and fourth equations yields

$$\left.\begin{array}{rrrrrr} & x_2 & + \, \frac{3}{14}x_3 & - \, \frac{1}{7}x_4 & & = 3.5 \\ x_1 & & - \, \frac{1}{7}x_3 & + \, \frac{3}{7}x_4 & & = 3 \\ & & \frac{5}{14}x_3 & - \, \frac{11}{7}x_4 & + \, x_5 & = 2.5 \\ & & \frac{9}{7}x_3 & + \, \frac{1}{7}x_4 & & -z = 37. \end{array}\right\} \tag{2.3}$$

In (2.3), setting $x_3 = x_4 = 0$ yields $x_1 = 3$, $x_2 = 3.5$, $x_5 = 2.5$, and $z = -37$, which corresponds to the extreme point D in Fig. 1.1, and the value of z is decreased from -36 to -37.

From the fourth equation of (2.3), both coefficients of x_3 and x_4 are positive. This means that increasing the value of x_3 or x_4 increases the value of z. Therefore, the minimum of z is -37, that is, the maximum of the total profit is 37 million yen, and the production numbers of products $P_1$and P_2 are 3 and 3.5 tons, respectively.

2.2 Typical Examples of Linear Programming Problems

Thus far, we have outlined linear programming through the two-dimensional production planning problem, which can be generalized as the following production planning problem with n decision variables.

Example 2.1 (Production planning problem). A manufacturing company has fixed amounts of m different resources at its disposal. These resources are used to produce n different commodities. The company knows that to produce one unit of commodity j, a_{ij} units of resource i are required. The total number of units of resource i available is b_i. It also knows that a profit per unit of commodity j is c_j. It desires to produce a combination of commodities which will maximize the total profit.

Let x_j denote a decision variable for the production amount of commodity j. Since the amount of resource i that is used must be less than or equal to the available number b_i of units of resource i, we have, for each $i = 1, 2, \ldots, m$, a linear inequality

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i.$$

As a negative x_j has no appropriate interpretation, it is required that $x_j \geq 0$, $j = 1, 2, \ldots, n$. The profit arising from producing x_j units of commodity j is calculated as $c_j x_j$. Our formulation is represented as a linear programming problem where the linear profit function

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n \tag{2.4}$$

is maximized subject to the linear inequality constraints

$$\left.\begin{array}{rcccccccl} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & \leq & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & \leq & b_2 \\ & & \cdots & \cdots & \cdots & \cdots & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & \leq & b_m \end{array}\right\} \tag{2.5}$$

and nonnegativity conditions for all decision variables

$$x_j \geq 0, \quad j = 1, 2, \ldots, n. \tag{2.6}$$

$\Diamond$

Compared with such a production planning problem maximizing the linear objective function of the total profit subject to the linear inequality constraints in a direction of the less than or equal to symbol $\leq$, the following diet problem minimizing the linear objective function of the total cost subject to the linear inequality constraints in a direction of the greater than or equal to symbol $\geq$ is well known as a nearly symmetric one. It should be noted here that both of the problems have the nonnegativity conditions for all decision variables $x_j \geq 0$, $j = 1, \ldots, n$.

Example 2.2 (Diet problem). How can we determine the most economical diet that satisfies the basic minimum nutritional requirements for good health? Assume n different foods are available at the market and the selling price for food j is c_j per unit. Moreover, there are m basic nutritional ingredients for the human body, and at least b_i units of nutrient i are required everyday to achieve a balanced diet for good health. In addition, assume that each unit of food j contains a_{ij} units of nutrient i. The problem is to determine the most economical diet that satisfies the basic minimum nutritional requirements.

For this problem, let x_j, $j = 1, \ldots, n$ denote a decision variable for the number of units of food j in the diet, and then it is required that $x_j \geq 0$, $j = 1, \ldots, n$. The total amount of nutrient i

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$$

contained in the purchased foods must be greater than or equal to the daily requirement b_i of nutrient i. Thus, the economic diet can be represented as a linear programming problem where the linear cost function

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n \tag{2.7}$$

is minimized subject to the linear constraints

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \geq b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq b_m \end{array}\right\} \tag{2.8}$$

and nonnegativity conditions for all decision variables

$$x_j \geq 0, \quad j = 1, 2, \ldots, n. \tag{2.9}$$

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To develop a better understanding, as a simple numerical example of the diet problem, we present the following diet problem with two decision variables and three constraints.

Example 2.3 (Diet problem with 2 decision variables and 3 constraints). A housewife is planning a menu by utilizing two foods F_1 and F_2 containing three nutrients N_1, N_2, and N_3 in order to meet the nutritional requirements at a minimum cost.

Each 1 g (gram) of the food F_1 contains 1 mg (milligram) of N_1, 1 mg of N_2, and 2 mg of N_3; and each 1 g of the food F_2 contains 3 mg of N_1, 2 mg of N_2, and 1 mg of N_3. The recommended amounts of the nutrients N_1, N_2, and N_3 are known

Table 2.1 Data for two foods diet problem

	Food F_1 (g)	Food F_2 (g)	Minimum requirement
Nutrient N_1 (mg)	1	3	12
Nutrient N_2 (mg)	1	2	10
Nutrient N_3 (mg)	2	1	15
Price (thousand yen)	4	3	

to be at least 12 mg, 10 mg, and 15 mg, respectively. Also, it is known that the costs per gram of the foods F_1 and F_2 are, respectively, 4 and 3 thousand yen. These data concerning the nutrients and foods are summarized in Table 2.1.

The housewife's problem is to determine the purchase volumes of foods F_1 and F_2 which minimize the total cost satisfying the nutritional requirements for the nutrients N_1, N_2, and N_3.

Let x_j denote a decision variable for the number of units of food F_j to be purchased, and then we can formulate the corresponding linear programming problem minimizing the linear cost function

$$4x_1 + 3x_2 \tag{2.10}$$

subject to the linear constraints

$$\left.\begin{aligned} x_1 + 3x_2 &\geq 12 \\ x_1 + 2x_2 &\geq 10 \\ 2x_1 + x_2 &\geq 15 \end{aligned}\right\} \tag{2.11}$$

and nonnegativity conditions for all variables

$$x_1 \geq 0, \ x_2 \geq 0. \tag{2.12}$$

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2.3 Standard Form of Linear Programming

In order to deal with such nearly symmetrical production planning problems and diet problems in a unified way, the standard form of linear programming is defined as follows:

The standard form of linear programming is to minimize the linear objective function

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \tag{2.13}$$

subject to the linear equality constraints

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\} \tag{2.14}$$

and nonnegativity conditions for all decision variables

$$x_j \geq 0, \quad j = 1, 2, \ldots, n, \tag{2.15}$$

where the a_{ij}, b_i, and c_j are fixed real constants. In particular, b_i is called a right-hand side constant, and c_j is sometimes called a cost coefficient in a minimization problem, while called a profit coefficient in a maximization one.

In this book, the standard form of linear programming is written in the following form:

$$\left.\begin{array}{ll} \text{minimize} & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\ & x_j \geq 0, \quad j = 1, 2, \ldots, n, \end{array}\right\} \tag{2.16}$$

or using summation notation, it is compactly rewritten as

$$\left.\begin{array}{ll} \text{minimize } z = & \displaystyle\sum_{j=1}^{n} c_j x_j \\ \text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, \ldots, m \\ & x_j \geq 0, \quad j = 1, \ldots, n. \end{array}\right\} \tag{2.17}$$

By introducing an n dimensional row vector $\boldsymbol{c}$, an $m \times n$ matrix A, an n dimensional column vector $\boldsymbol{x}$, and an m dimensional column vector $\boldsymbol{b}$, the standard form of linear programming can be then written in a more compact vector–matrix form as follows:

$$\left.\begin{array}{ll} \text{minimize } z = & \boldsymbol{c}\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}, \end{array}\right\} \tag{2.18}$$

where

$$\boldsymbol{c} = (c_1, c_2, \ldots, c_n), \tag{2.19}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}, \tag{2.20}$$

and $\mathbf{0}$ is an n dimensional column vector with zero components.

Moreover, by denoting the jth column of an $m \times n$ matrix A by

$$\boldsymbol{p}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}, \ j = 1, 2, \ldots, n \tag{2.21}$$

and writing $A = [\boldsymbol{p}_1 \ \boldsymbol{p}_2 \ \cdots \ \boldsymbol{p}_n]$, the standard form linear programming (2.16) can also be represented in column form:

$$\left.\begin{array}{ll} \text{minimize } z = & c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \\ \text{subject to} & \boldsymbol{p}_1 x_1 + \boldsymbol{p}_2 x_2 + \cdots + \boldsymbol{p}_n x_n = \boldsymbol{b} \\ & x_j \geq 0, \quad j = 1, 2, \ldots, n. \end{array}\right\} \tag{2.22}$$

In the standard form of linear programming (2.16), the objective function

$$z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

can be treated as just another equation, i.e.,

$$-z + c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = 0, \tag{2.23}$$

and by including it in an augmented system of equations, the problem can then be stated as follows:

Find values of the nonnegative decision variables $x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$ so as to minimize z, satisfying the augmented system of linear equations

$$\left.\begin{array}{rcl} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n & = & b_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n & = & b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots & & \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n & = & b_m \\ -z + c_1 x_1 + c_2 x_2 + \cdots + c_n x_n & = & 0. \end{array}\right\} \tag{2.24}$$

It should be noted here that the standard form of linear programming deals with a linear minimization problem with nonnegative decision variables and linear equality constraints. We introduce a mechanism to convert any general linear programming problem into the standard form. A linear inequality can be easily converted into an equality. When the ith constraint is represented as

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, \ldots, m, \tag{2.25}$$

by adding a nonnegative slack variable $x_{n+i} \geq 0$ such that

$$\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i, \quad i = 1, 2, \ldots, m, \tag{2.26}$$

the inequality (2.25) becomes the equality (2.26).

Similarly, if the ith constraint is

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad i = 1, 2, \ldots, m, \tag{2.27}$$

by subtracting a nonnegative surplus variable $x_{n+i} \geq 0$ such that

$$\sum_{j=1}^{n} a_{ij} x_j - x_{n+i} = b_i, \quad i = 1, 2, \ldots, m, \tag{2.28}$$

we can also transform the inequality (2.27) into the equality (2.28). It should be noted here that both the slack variables and the surplus variables must be nonnegative in order that the inequalities (2.25) and (2.27) are satisfied for all $i = 1, 2, \ldots, m$.

If, in the original formulation of the problem, some decision variable x_k is not restricted to be nonnegative, it can be replaced with the difference of two nonnegative variables, i.e.,

$$x_k = x_k^+ - x_k^-, \; x_k^+ \geq 0, \; x_k^- \geq 0. \tag{2.29}$$

If an objective function is to be maximized, we simply multiply the objective function by -1 to convert a maximization problem into a minimization problem.

Recall that, in the algebraic method for the production planning problem of Example 1.1, multiplying the objective function by -1 and introducing the three nonnegative slack variables x_3, x_4, and x_5 yields the following standard form of linear programming:

$$\left.\begin{array}{lllllll}
\text{minimize } z = & -3x_1 & - \; 8x_2 & & & & \\
\text{subject to} & 2x_1 & + \; 6x_2 & + \; x_3 & & & = 27 \\
 & 3x_1 & + \; 2x_2 & & + \; x_4 & & = 16 \\
 & 4x_1 & + \; \;\; x_2 & & & + \; x_5 & = 18 \\
 & \multicolumn{6}{l}{x_j \geq 0, \quad j = 1, 2, 3, 4, 5.}
\end{array}\right\} \tag{2.30}$$

For the general production planning problem with n decision variables, by introducing the m nonnegative slack variables x_{n+i} (≥ 0), $i = 1, \ldots, m$, it can be converted into the following standard form of linear programming:

$$\left.\begin{array}{ll}\text{minimize} & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} \qquad\qquad = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \qquad + x_{n+2} \qquad = b_2 \\ & \cdots\cdots\cdots\cdots\cdots\cdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \qquad\qquad + x_{n+m} = b_m \\ & x_j \geq 0, \quad j = 1, 2, \ldots, n, n+1, \ldots, n+m. \end{array}\right\} \tag{2.31}$$

Similarly, for the diet problem with n decision variables, introducing the m nonnegative surplus variables x_{n+i} (≥ 0), $i = 1, \ldots, m$ yields the following standard form of linear programming:

$$\left.\begin{array}{ll}\text{minimize} & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n - x_{n+1} \qquad\qquad = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \qquad - x_{n+2} \qquad = b_2 \\ & \cdots\cdots\cdots\cdots\cdots\cdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \qquad\qquad - x_{n+m} = b_m \\ & x_j \geq 0, \quad j = 1, 2, \ldots, n, n+1, \ldots, n+m. \end{array}\right\} \tag{2.32}$$

The basic ideas of linear programming are to first detect whether solutions satisfying equality constraints and nonnegativity conditions exist and, if so, to find a solution yielding the minimum value of z.

However, in the standard form of linear programming (2.16) or (2.18), if there is no solution satisfying the equality constraint, or if there exists only one, we do not need optimization. Also, if any of the equality constraints is redundant, i.e., a linear combination of the others, it could be deleted without changing any solutions of the system. Therefore, we are mostly interested in the case where the system of linear equations (2.16) is nonredundant and has an infinite number of solutions.

For that purpose, assume that the number of variables exceeds the number of equality constraints, i.e.,

$$n > m \tag{2.33}$$

and the system of linear equations is linearly independent, i.e.,

$$\text{rank}(A) = m. \tag{2.34}$$

Under these assumptions, we introduce a number of definitions for the standard form of linear programming (2.16) or (2.18).[1]

[1]These assumptions, introduced to establish the principle theoretical results, will be relaxed in Sect. 2.5 and are no longer necessary when solving general linear programming problems.

Definition 2.1 (Feasible solution). A feasible solution to the linear programming problem (2.16) is a vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ which satisfies the linear equalities and the nonnegativity conditions of (2.16).[2]

Definition 2.2 (Basis matrix). A basis matrix is an $m \times m$ nonsingular submatrix formed by choosing some m columns of the rectangular matrix A. Observe that A contains at least one basis matrix due to $\operatorname{rank}(A) = m$.

Definition 2.3 (Basic solution). A basic solution to the linear programming problem (2.16) is a solution obtained by setting $n - m$ variables (called nonbasic variables) equal to zeros and solving for the remaining m variables (called basic variables). A basic solution is also a unique vector determined by choosing a basis matrix from the $m \times n$ matrix A and solving the resulting square, nonsingular system of equations for the m variables. The set of all basic variables is called the basis.

Definition 2.4 (Basic feasible solution). A basic feasible solution to the linear programming problem (2.16) is a basic solution which satisfies not only the linear equations but also the nonnegativity conditions of (2.16), that is, all basic variables are nonnegative. Observe that at most m variables can be positive by Definition 2.3.

Definition 2.5 (Nondegenerate basic feasible solution). A nondegenerate basic feasible solution to the linear programming problem (2.16) is a basic solution with exactly m positive x_j, that is, all basic variables are positive.

Definition 2.6 (Optimal solution). An optimal solution to the linear programming problem (2.16) is a feasible solution which also minimizes z in (2.16). The corresponding value of z is called the optimal value.

The number of basic solutions is the number of ways that m variables are selected from a group of n variables, i.e.,

$$ {}_nC_m = \frac{n!}{(n-m)!m!}. $$

Example 2.4 (Basic solutions). Consider the basic solutions of the standard form of the linear programming (2.30) discussed in Example 1.1.

Choosing x_3, x_4, and x_5 as basic variables, we have the corresponding basic solution $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 27, 16, 18)$ which is a nondegenerate basic feasible solution and corresponds to the extreme point A in Fig. 1.1. After making another choice of x_1, x_2, and x_4 as basic variables, solving

$$
\begin{aligned}
2x_1 + 6x_2 \phantom{{}+x_4} &= 27 \\
3x_1 + 2x_2 + x_4 &= 16 \\
4x_1 + x_2 \phantom{{}+x_4} &= 18
\end{aligned}
$$

[2] In this book, the superscript T denotes the transpose operation for a vector or a matrix.

yields $x_1 = 81/22$, $x_2 = 36/11$, and $x_4 = -35/22$. The resulting basic solution $(x_1, x_2, x_3, x_4, x_5) = (81/22, 36/11, 0, -35/22, 0)$ is not feasible.

Choosing x_1, x_2, and x_5 as basic variables, we solve

$$\begin{array}{rcrcrcr} 2x_1 & + & 6x_2 & & & = & 27 \\ 3x_1 & + & 2x_2 & & & = & 16 \\ 4x_1 & + & x_2 & + & x_5 & = & 18, \end{array}$$

and then we have a basic feasible solution $(x_1, x_2, x_3, x_4, x_5) = (3, 3.5, 0, 0, 2.5)$. It corresponds to the extreme point D in Fig. 1.1 which is an optimal solution. $\diamondsuit$

2.4 Simplex Method

For generalizing the basic ideas of linear programming grasped in the algebraic approach to the two-dimensional production planning problem of Example 1.1, consider the following linear programming problem with basic variables $x_1, x_2, \ldots, x_m$:

Find values of $x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$ so as to minimize z, satisfying the augmented system of linear equations

$$\left.\begin{array}{lllllllll} x_1 & & & + \bar{a}_{1,m+1}x_{m+1} & + \bar{a}_{1,m+2}x_{m+2} & + \cdots + & \bar{a}_{1n}x_n & = & \bar{b}_1 \\ & x_2 & & + \bar{a}_{2,m+1}x_{m+1} & + \bar{a}_{2,m+2}x_{m+2} & + \cdots + & \bar{a}_{2n}x_n & = & \bar{b}_2 \\ & & \cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots & \cdots\cdots & & \\ & & x_m & + \bar{a}_{m,m+1}x_{m+1} & + \bar{a}_{m,m+2}x_{m+2} & + \cdots + & \bar{a}_{mn}x_n & = & \bar{b}_m \\ & & -z + & \bar{c}_{m+1}x_{m+1} & + \bar{c}_{m+2}x_{m+2} & + \cdots + & \bar{c}_n x_n & = & -\bar{z}. \end{array}\right\} \tag{2.35}$$

As in the previous section, here it is assumed that $n > m$ and the system of m equality constrains is nonredundant. As with the augmented system of equations (2.35), a system of linear equations in which each of the variables $x_1, x_2, \ldots, x_m$ has a coefficient of unity in one equation and zeros elsewhere is called a canonical form or a basic form. In a canonical form, the variables $x_1, x_2, \ldots, x_m$ and $(-z)$ are called basic variables, and the remaining variables $x_{m+1}, x_{m+2}, \ldots, x_n$ are called nonbasic variables. In such a canonical form, observing that $(-z)$ always is a basic variable, with no further notice, only $x_1, x_2, \ldots, x_m$ are called basic variables.

It is useful to set up such a canonical form (2.35) in tableau form as shown in Table 2.2. This table is called a simplex tableau, in which only the coefficients of the algebraic representation in (2.35) are given.

From the canonical form (2.35) or the simplex tableau given in Table 2.2, it follows directly that a basic solution with basic variables $x_1, x_2, \ldots, x_m$ becomes

$$x_1 = \bar{b}_1,\ x_2 = \bar{b}_2, \ldots,\ x_m = \bar{b}_m,\ x_{m+1} = x_{m+2} = \cdots = x_n = 0 \tag{2.36}$$

Table 2.2 Simplex tableau

Basis	x_1	x_2	$\cdots$	x_m	x_{m+1}	x_{m+2}	$\cdots$	x_n	Constants
x_1	1				$\bar{a}_{1,m+1}$	$\bar{a}_{1,m+2}$	$\cdots$	$\bar{a}_{1n}$	$\bar{b}_1$
x_2		1			$\bar{a}_{2,m+1}$	$\bar{a}_{2,m+2}$	$\cdots$	$\bar{a}_{2n}$	$\bar{b}_2$
$\vdots$			$\ddots$		$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
x_m				1	$\bar{a}_{m,m+1}$	$\bar{a}_{m,m+2}$	$\cdots$	$\bar{a}_{mn}$	$\bar{b}_m$
$-z$					$\bar{c}_{m+1}$	$\bar{c}_{m+2}$	$\cdots$	$\bar{c}_n$	$-\bar{z}$

and the value of the objective function is

$$z = \bar{z}. \tag{2.37}$$

If

$$\bar{b}_1 \geq 0,\ \bar{b}_2 \geq 0, \ldots, \bar{b}_m \geq 0, \tag{2.38}$$

then the solution $(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n) = (\bar{b}_1, \ldots, \bar{b}_m, 0, \ldots, 0)$ is a basic feasible solution. In this case, the corresponding canonical form (tableau) is called a feasible canonical form (tableau). If, for one or more i, $\bar{b}_i = 0$ holds, then it is said that the basic feasible solution is degenerate.

As an example that we can directly formulate a feasible canonical form, consider the production planning problem of Example 2.1. For this problem, by introducing m slack variables $x_{n+i} \geq 0$, $i = 1, 2, \ldots, m$ and multiplying the objective function by -1 to convert the maximization problem into a minimization problem, the following canonical form is obtained:

$$\left.\begin{array}{rcrcccrcccccl}
a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & + & x_{n+1} & & & & & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & & & + & x_{n+2} & & & = & b_2 \\
\multicolumn{5}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots} & & & & & & & & \\
a_{m1}x_1 & + & a_{m,2}x_2 & + \cdots + & a_{mn}x_n & & & & & + & x_{n+m} & = & b_m \\
c_1x_1 & + & c_2x_2 & + \cdots + & c_nx_n & & & & & & -z & = & 0.
\end{array}\right\} \tag{2.39}$$

In this formulation, by using the m slack variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ as basic variables, it is evident that (2.39) is a canonical form, and then the corresponding basic solution is

$$x_1 = x_2 = \cdots = x_n = 0, x_{n+1} = b_1, \ldots, x_{n+m} = b_m. \tag{2.40}$$

From the fact that the right-hand side constant b_i means the available amount of resource i, it should be nonnegative, i.e., $b_i \geq 0$, $i = 1, 2, \ldots, m$, and therefore this canonical form is feasible.

In contrast, for the diet problem of Example 2.2, introducing m surplus variables $x_{n+i} \geq 0$, $i = 1, 2, \ldots, m$ and then multiplying both sides of the resulting constraints by -1 yields a basic solution

$$x_1 = x_2 = \cdots = x_n = 0, x_{n+1} = -b_1, \ldots, x_{n+m} = -b_m. \tag{2.41}$$

Unfortunately, however, since $b_i \geq 0$, $i = 1, 2, \ldots, m$, this operation cannot lead a feasible canonical form.

In the following discussions of this section, assume that the canonical form (2.35) is feasible. That is, starting with the canonical form (2.35) with the basic solution

$$x_1 = \bar{b}_1,\ x_2 = \bar{b}_2, \ldots,\ x_m = \bar{b}_m,\ x_{m+1} = x_{m+2} = \cdots = x_n = 0,$$

we assume that this basic solution is feasible, i.e.,

$$\bar{b}_1 \geq 0,\ \bar{b}_2 \geq 0, \ldots, \bar{b}_m \geq 0.$$

From the last equation in (2.35), we have

$$z = \bar{z} + \bar{c}_{m+1}x_{m+1} + \bar{c}_{m+2}x_{m+2} + \cdots + \bar{c}_n x_n.$$

Since $x_{m+1} = x_{m+2} = \cdots = x_n = 0$, one finds $z = \bar{z}$. This equation provides even more valuable information than this. By merely glancing at the numbers $\bar{c}_j$, $j = m + 1, m + 2, \ldots, n$, one can tell if this basic feasible solution is optimal or not. Furthermore, one can find a better basic feasible solution if it is not optimal. Consider first the optimality of the canonical form, given by the following theorem.

Theorem 2.1 (Optimality test). *In the feasible canonical form (2.35), if all coefficients $\bar{c}_{m+1}$, $\bar{c}_{m+2}$, ..., $\bar{c}_n$ of the last equation are nonnegative, i.e.,*

$$\bar{c}_j \geq 0, \quad j = m + 1, m + 2, \ldots, n, \tag{2.42}$$

then the basic feasible solution is optimal.

Proof. The last equation of (2.35) can be rewritten as

$$z = \bar{z} + \bar{c}_{m+1}x_{m+1} + \bar{c}_{m+2}x_{m+2} + \cdots + \bar{c}_n x_n.$$

The nonbasic variables $x_{m+1}, x_{m+2}, \ldots, x_n$ are presently zeros, and they are restricted to be nonnegative. If $\bar{c}_j \geq 0$ for $j = m + 1, m + 2, \ldots, n$, then from $\bar{c}_j x_j \geq 0$, $j = m+1, m+2, \ldots, n$, increasing any x_j cannot decrease the objective function z. Thus, since any change in the nonbasic variables cannot decrease z, the present solution must be optimal. □

The coefficient $\bar{c}_j$ of x_j in (2.35) represents the rate of change of z with respect to the nonbasic variable x_j. From this observation, the coefficient $\bar{c}_j$ is called the relative cost coefficient or, alternatively, the reduced cost coefficient.

The optimality condition (2.42) is sometimes referred to as the optimality criterion or the simplex criterion. The feasible canonical form satisfying the optimality criterion is called the optimal canonical form or the optimal basic form, and the simplex tableau satisfying the optimality criterion is also called the optimal tableau.

Note that since $\bar{c}_j = 0$ for all basic variables, the optimality criterion (2.42) could also be stated simply as $\bar{c}_j \geq 0$ for all $j = 1, 2, \ldots, n$ in place of $\bar{c}_j \geq 0$ for all $j = m + 1, \ldots, n$.

In addition to the optimality, the relative cost coefficients can also tell if there are multiple optima. Assume that for all nonbasic variables x_j, $\bar{c}_j \geq 0$, and for some nonbasic variable x_k, $\bar{c}_k = 0$. In that case, if the increase of x_k does not violate the constraints, there are multiple optima because no change in z results. Hence, the following theorem can be derived.

Theorem 2.2 (Unique optimal solution). *In the feasible canonical form (2.35), if $\bar{c}_j > 0$ for all nonbasic variables, then the basic feasible solution is the unique optimal solution.*

Of course, if, for some nonbasic variable x_j, $\bar{c}_j < 0$, then z can be decreased by increasing x_j. Consider a method for finding better solutions than the current nonoptimal solution.

If there is at least one negative coefficient, say $\bar{c}_j < 0$, then, under the assumption of nondegeneracy, i.e., $\bar{b}_i > 0$ for all i, it is always possible to generate another basic feasible solution with an improved value of the objective function. If there are two or more negative coefficients, we choose a variable x_s with the smallest relative cost coefficient

$$\bar{c}_s = \min_{\bar{c}_j < 0} \bar{c}_j \tag{2.43}$$

and increase the value of x_s.

Although this choice may not lead to the greatest possible decrease in z (since only a limited extent of increase of x_s may be allowed), it is at least intuitively a good rule for choosing a variable to be made a basic one. It is the one used in practice today because (i) it is simple and (ii) it generally leads to an optimal solution in fewer iterations than just choosing any $\bar{c}_s < 0$.

After a nonbasic variable x_s is selected to be a basic one, we increase the value of x_s from zero, holding the other nonbasic variables zeros. Observe the effect of this operation on the current basic variables. From (2.35), each of the current basic variables can be represented as a function of x_s:

$$\left.\begin{aligned} x_1 &= \bar{b}_1 - \bar{a}_{1s} x_s \\ x_2 &= \bar{b}_2 - \bar{a}_{2s} x_s \\ &\cdots\cdots\cdots\cdots \\ x_m &= \bar{b}_m - \bar{a}_{ms} x_s \\ z &= \bar{z} + \bar{c}_s x_s. \end{aligned}\right\} \tag{2.44}$$

Since the coefficient $\bar{c}_s$ of the last equation in (2.44) is negative, i.e., $\bar{c}_s < 0$, increasing the value of x_s decreases the value of z. The only factor limiting the increase of x_s is that all of the variables $x_1, x_2, \ldots, x_m$ must be nonnegative. In other words, keeping the feasibility of the solution requires

$$x_i = \bar{b}_i - \bar{a}_{is} x_s \geq 0, \quad i = 1, 2, \ldots, m. \tag{2.45}$$

However, if all the coefficients $\bar{a}_{is}$, $i = 1, 2, \ldots, m$ are nonpositive, i.e.,

$$\bar{a}_{is} \leq 0, \quad i = 1, 2, \ldots, m, \tag{2.46}$$

then x_s can increase infinitely. Hence since $\bar{c}_s < 0$, from the last equation of (2.44), it follows that

$$z = \bar{z} + \bar{c}_s x_s \to -\infty.$$

Thus, we have the following theorem.

Theorem 2.3 (Unboundedness). *If in the feasible canonical form (2.35), for some nonbasic variable x_s, the coefficients $\bar{a}_{is}$, $i = 1, 2, \ldots, m$ are nonpositive and the coefficient $\bar{c}_s$ is negative, i.e.,*

$$\bar{a}_{is} \leq 0, \ i = 1, 2, \ldots, m, \quad \text{and} \quad \bar{c}_s < 0, \tag{2.47}$$

then the optimal value is unbounded.

If, however, at least one $\bar{a}_{is}$ is positive, then x_s cannot be increased indefinitely since eventually some basic variable, say x_i, will decrease beyond zero and become negative. From (2.44), x_i becomes zero when the coefficient $\bar{a}_{is}$ is positive and x_s raises to $\bar{b}_i / \bar{a}_{is}$, i.e.,

$$x_s = \frac{\bar{b}_i}{\bar{a}_{is}}, \quad \bar{a}_{is} > 0. \tag{2.48}$$

The value of x_s is maximized under the condition of the nonnegativity of the basic variables x_i, $i = 1, 2, \ldots, m$, and it is given by

$$\min_{\bar{a}_{is} > 0} \frac{\bar{b}_i}{\bar{a}_{is}} = \frac{\bar{b}_r}{\bar{a}_{rs}} = \theta. \tag{2.49}$$

The basic variable x_r determined by (2.49) then becomes nonbasic, and instead, the nonbasic variable x_s becomes basic. That is, x_r becomes zero while x_s increases from zero to $\bar{b}_r / \bar{a}_{rs} = \theta \ (\geq 0)$. Also, from the last equation of (2.44), the value of objective function z decreases by $|\bar{c}_s x_s| = |\bar{c}_s \theta|$.

A new canonical form in which x_s is selected as a basic variable in place of x_r can be easily obtained by pivoting on $\bar{a}_{rs}$, which is called the pivot element determined by (2.43) and (2.49). That is, finding $\bar{c}_s = \min_{\bar{c}_j < 0} \bar{c}_j$ tells us that the pivot term is in column s, and finding the minimum $\bar{b}_r / \bar{a}_{rs}$ of all the ratios $\bar{b}_i / \bar{a}_{is}$ such that $\bar{a}_{is} > 0$ tells us that it is in row r.

Fundamental to linear programming is a pivot operation defined as follows.

Definition 2.7 (Pivot operation). A pivot operation consists of m elementary steps for replacing a linear system with an equivalent system in which a specified variable has a coefficient of unity in one equation and zeros elsewhere. The detailed steps are as follows:

(i) Select the nonzero element a_{rs} in row (equation) r and column s, which is called the pivot element.
(ii) Replace the rth equation with the rth equation multiplied by $1/a_{rs}$.
(iii) For each $i = 1, 2, \ldots, m$ except $i = r$, replace the ith equation with the sum of the ith equation and the replaced rth equation multiplied by $-a_{is}$.

In linear programming, pivot operations are sometimes counted by the term "cycle." Now, a pivot operation on $\bar{a}_{rs} (\neq 0)$ is performed to the feasible canonical form

$$\left.\begin{array}{llllllll}
x_1 & & & + \bar{a}_{1,m+1}x_{m+1} + \cdots + & \bar{a}_{1s}x_s & + \cdots + & \bar{a}_{1n}x_n = & \bar{b}_1 \\
& x_2 & & + \bar{a}_{2,m+1}x_{m+1} + \cdots + & \bar{a}_{2s}x_s & + \cdots + & \bar{a}_{2n}x_n = & \bar{b}_2 \\
& & & \cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & & \\
& & x_r & + \bar{a}_{r,m+1}x_{m+1} + \cdots + & \bar{a}_{rs}x_s & + \cdots + & \bar{a}_{rn}x_n = & \bar{b}_r \\
& & & \cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & & \\
& & x_m & + \bar{a}_{m,m+1}x_{m+1} + \cdots + & \bar{a}_{ms}x_s & + \cdots + & \bar{a}_{mn}x_n = & \bar{b}_m \\
& & -z + & \bar{c}_{m+1}x_{m+1} + \cdots + & \bar{c}_s x_s & + \cdots & + \bar{c}_n x_n = & -\bar{z},
\end{array}\right\} \tag{2.50}$$

where $\bar{b}_i \geq 0$, $i = 1, 2, \ldots .m$, and then we have the new canonical form

$$\left.\begin{array}{llllllll}
x_1 & + \bar{a}^*_{1r}x_r & & + \bar{a}^*_{1,m+1}x_{m+1} + \cdots + & 0 & + \cdots + & \bar{a}^*_{1n}x_n = & \bar{b}^*_1 \\
x_2 & + \bar{a}^*_{2r}x_r & & + \bar{a}^*_{2,m+1}x_{m+1} + \cdots + & 0 & + \cdots + & \bar{a}^*_{2n}x_n = & \bar{b}^*_2 \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & & & & \\
& \bar{a}^*_{rr}x_r & & + \bar{a}^*_{r,m+1}x_{m+1} + \cdots + & x_s & + \cdots + & \bar{a}^*_{rn}x_n = & \bar{b}^*_r \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & & & & \\
& \bar{a}^*_{mr}x_r & + x_m & + \bar{a}^*_{m,m+1}x_{m+1} + \cdots + & 0 & + \cdots + & \bar{a}^*_{mn}x_n = & \bar{b}^*_m \\
& \bar{c}^*_r x_r & -z + & \bar{c}^*_{m+1}x_{m+1} + \cdots + & 0 & + \cdots + & \bar{c}^*_n x_n = & -\bar{z}^*,
\end{array}\right\} \tag{2.51}$$

where the superscript $*$ is added to a revised coefficient, and the revised coefficients for $j = r, m+1, m+2, \ldots, n$ are calculated as follows:

$$\bar{a}^*_{rj} = \frac{\bar{a}_{rj}}{\bar{a}_{rs}}, \quad \bar{b}^*_r = \frac{\bar{b}_r}{\bar{a}_{rs}}, \tag{2.52}$$

$$\bar{a}^*_{ij} = \bar{a}_{ij} - \bar{a}_{is}\frac{\bar{a}_{rj}}{\bar{a}_{rs}}, \quad \bar{b}^*_i = \bar{b}_i - \bar{a}_{is}\frac{\bar{b}_r}{\bar{a}_{rs}}, \; i = 1, 2, \ldots, m; \; i \neq r, \tag{2.53}$$

$$\bar{c}^*_j = \bar{c}_j - \bar{c}_s\frac{\bar{a}_{rj}}{\bar{a}_{rs}}, \quad -\bar{z}^* = -\bar{z} - \bar{c}_s\frac{\bar{b}_r}{\bar{a}_{rs}}. \tag{2.54}$$

Table 2.3 Pivot operation on $\bar{a}_{rs}$

Cycle	Basis	x_1	$\cdots$	x_r	$\cdots$	x_m	x_{m+1}	$\cdots$	x_s	$\cdots$	x_n	Constants
ℓ	x_1	1					$\bar{a}_{1,m+1}$	$\cdots$	$\bar{a}_{1s}$	$\cdots$	$\bar{a}_{1n}$	$\bar{b}_1$
	$\vdots$		$\ddots$				$\vdots$		$\vdots$		$\vdots$	$\vdots$
	x_r			1			$\bar{a}_{r,m+1}$	$\cdots$	$[\bar{a}_{rs}]$	$\cdots$	$\bar{a}_{rn}$	$\bar{b}_r$
	$\vdots$				$\ddots$		$\vdots$		$\vdots$		$\vdots$	$\vdots$
	x_m					1	$\bar{a}_{m,m+1}$	$\cdots$	$\bar{a}_{ms}$	$\cdots$	$\bar{a}_{mn}$	$\bar{b}_m$
	$-z$						$\bar{c}_{m+1}$	$\cdots$	$\bar{c}_s$	$\cdots$	$\bar{c}_n$	$-\bar{z}$
$\ell+1$	x_1	1		$\bar{a}^*_{1r}$			$\bar{a}^*_{1,m+1}$	$\cdots$	0	$\cdots$	$\bar{a}^*_{1n}$	$\bar{b}^*_1$
	$\vdots$		$\ddots$				$\vdots$		$\vdots$		$\vdots$	$\vdots$
	x_s			$\bar{a}^*_{rr}$			$\bar{a}^*_{r,m+1}$	$\cdots$	1	$\cdots$	$\bar{a}^*_{rn}$	$\bar{b}^*_r$
	$\vdots$				$\ddots$		$\vdots$		$\vdots$		$\vdots$	$\vdots$
	x_m			$\bar{a}^*_{mr}$		1	$\bar{a}^*_{m,m+1}$	$\cdots$	0	$\cdots$	$\bar{a}^*_{mn}$	$\bar{b}^*_m$
	$-z$			$\bar{c}^*_r$	$\cdots$		$\bar{c}^*_{m+1}$	$\cdots$	0	$\cdots$	$\bar{c}^*_n$	$-\bar{z}^*$

$\bar{a}^*_{rj} = \frac{\bar{a}_{rj}}{\bar{a}_{rs}}, \quad \bar{b}^*_r = \frac{\bar{b}_r}{\bar{a}_{rs}}$

$\bar{a}^*_{ij} = \bar{a}_{ij} - \bar{a}_{is}\frac{\bar{a}_{rj}}{\bar{a}_{rs}} = \bar{a}_{ij} - \bar{a}_{is}\bar{a}^*_{rj}, \quad \bar{b}^*_i = \bar{b}_i - \bar{a}_{is}\frac{\bar{b}_r}{\bar{a}_{rs}} = \bar{b}_i - \bar{a}_{is}\bar{b}^*_r \ (i \neq r)$

$\bar{c}^*_j = \bar{c}_j - \bar{c}_s\frac{\bar{a}_{rj}}{\bar{a}_{rs}} = \bar{c}_j - \bar{c}_s\bar{a}^*_{rj}, \quad -\bar{z}^* = -\bar{z} - \bar{c}_s\frac{\bar{b}_r}{\bar{a}_{rs}} = -\bar{z} - \bar{c}_s\bar{b}^*_r$

Since the pivot element $\bar{a}_{rs}$ is determined by (2.43) and (2.49), it is expected that the new canonical form (2.51) with basic variables $x_1, x_2, \ldots, x_{r-1}, x_s, x_{r+1}, \ldots, x_m$ also becomes feasible. This fact can be formally verified as follows.

It is obvious that $\bar{b}^*_r = \bar{b}_r/\bar{a}_{rs} \geq 0$. For i $(i \neq r)$ such that $\bar{a}_{is} > 0$, from (2.49), it follows that

$$\bar{b}^*_i = \bar{b}_i - \frac{\bar{a}_{is}}{\bar{a}_{rs}}\bar{b}_r = \bar{a}_{is}\left(\frac{\bar{b}_i}{\bar{a}_{is}} - \frac{\bar{b}_r}{\bar{a}_{rs}}\right) \geq 0,$$

and for i $(i \neq r)$ such that $\bar{a}_{is} \leq 0$, one finds that

$$\bar{b}^*_i = \bar{b}_i - \frac{\bar{a}_{is}}{\bar{a}_{rs}}\bar{b}_r \geq 0.$$

Hence, it holds that $\bar{b}^*_i \geq 0$ for all i, and then (2.51) is a feasible canonical form.

The pivot operation on $\bar{a}_{rs}$ replacing x_r with x_s as a new basic variable can be summarized in Table 2.3.

As described so far, starting with a feasible canonical form and updating it through a series of pivot operations, the simplex method seeks for an optimal solution satisfying the optimality criterion or the unboundedness information. The procedure of the simplex method, starting with a feasible canonical form, can be summarized as follows.

Procedure of the Simplex Method

Start with a feasible canonical form (simplex tableau).

Step 1 If all of the relative cost coefficients are nonnegative, i.e., $\bar{c}_j \geq 0$ for all indices j of the nonbasic variables, then the current solution is optimal, and stop. Otherwise, by using the relative cost coefficients $\bar{c}_j$, find the index s such that

$$\min_{\bar{c}_j < 0} \bar{c}_j = \bar{c}_s.$$

Step 2 If all of the coefficients in column s are nonpositive, i.e., $\bar{a}_{is} \leq 0$ for all indices i of the basic variables, then the optimal value is unbounded, and stop.

Step 3 If some of $\bar{a}_{is}$ are positive, find the index r such that

$$\min_{\bar{a}_{is} > 0} \frac{\bar{b}_i}{\bar{a}_{is}} = \frac{\bar{b}_r}{\bar{a}_{rs}} = \theta.$$

Step 4 Perform the pivot operation on $\bar{a}_{rs}$ for obtaining a new feasible canonical form (simplex tableau) with x_s replacing x_r as a new basic variable. The coefficients of the new feasible canonical form after pivoting on $\bar{a}_{rs} \neq 0$ are calculated as follows:

(i) Replace row r (the rth equation) with row r multiplied by $1/\bar{a}_{rs}$ (divide row r by $\bar{a}_{rs}$), i.e.,

$$\bar{a}^*_{rj} = \frac{\bar{a}_{rj}}{\bar{a}_{rs}}, \quad \bar{b}^*_r = \frac{\bar{b}_r}{\bar{a}_{rs}}.$$

(ii) For each $i = 1, 2, \ldots, m$ except $i = r$, replace row i (the ith equation) with the sum of row i and the revised row r multiplied by $-\bar{a}_{is}$, i.e.,

$$\bar{a}^*_{ij} = \bar{a}_{ij} - \bar{a}_{is}\bar{a}^*_{rj}, \quad \bar{b}^*_i = \bar{b}_i - \bar{a}_{is}\bar{b}^*_r.$$

(iii) Replace row $m+1$ (the $(m+1)$th equation for the objective function) with the sum of row $m+1$ and the revised row r multiplied by $-\bar{c}_s$, i.e.,

$$\bar{c}^*_j = \bar{c}_j - \bar{c}_s\bar{a}^*_{rj}, \quad -\bar{z}^* = -\bar{z} - \bar{c}_s\bar{b}^*_r.$$

Return to step 1.

It should be noted here that when multiple candidates exist for the index s of the variable entering the basis in step 1 or the index r of the variable leaving the basis in step 3, for the sake of convenience, we choose the smallest index.

Table 2.4 Simplex tableau for Example 1.1

Cycle	Basis	x_1	x_2	x_3	x_4	x_5	Constants
0	x_3	2	[6]	1			27
	x_4	3	2		1		16
	x_5	4	1			1	18
	$-z$	-3	-8				0
1	x_2	1/3	1	1/6			4.5
	x_4	[7/3]		$-1/3$	1		7
	x_5	11/3		$-1/6$		1	13.5
	$-z$	$-1/3$		4/3			36
2	x_2		1	3/14	$-1/7$		3.5
	x_1	1		$-1/7$	3/7		3
	x_5			5/14	$-11/7$	1	2.5
	$-z$			9/7	1/7		37

Example 2.5 (Simplex method for the production planning problem of Example 1.1). Using the simplex method, solve the production planning problem in the standard form given in Example 1.1:

$$\begin{array}{llllllll} \text{minimize } z = & -3x_1 & - 8x_2 & & & & & \\ \text{subject to} & 2x_1 & + 6x_2 & + x_3 & & & = & 27 \\ & 3x_1 & + 2x_2 & & + x_4 & & = & 16 \\ & 4x_1 & + x_2 & & & + x_5 & = & 18 \\ & x_j \geq 0, & j = 1, 2, 3, 4, 5. & & & & & \end{array}$$

Introducing the slack variables x_3, x_4, and x_5 and using them as the basic variables, we have the initial basic feasible solution

$$x_1 = x_2 = 0, \quad x_3 = 27, \quad x_4 = 16, \quad x_5 = 18,$$

which is shown at cycle 0 of the simplex tableau given in Table 2.4.

At cycle 0, since the minimum of $\bar{c}_1$ and $\bar{c}_2$ is

$$\min(-3, -8) = -8 < 0,$$

x_2 becomes a new basic variable. The minimum ratio, $\min_{\bar{a}_{i2} > 0} \bar{b}_i / \bar{a}_{i2}$, is calculated as

$$\min\left(\frac{27}{6}, \frac{16}{2}, \frac{18}{1}\right) = \frac{27}{6} = 4.5,$$

and then x_3 becomes a nonbasic variable. From $s = 2$ and $r = 1$, the pivot element is 6 bracketed by [ ] in Table 2.4. After the pivot operation on 6, the result at cycle 1 is obtained.

Table 2.5 Simplex tableau with multiple optima

Cycle	Basis	x_1	x_2	x_3	x_4	x_5	Constants
0	x_3	2	[6]	1			27
	x_4	3	2		1		16
	x_5	4	1			1	18
	$-z$	-1	-3				
1	x_2	1/3	1	1/6			4.5
	x_4	[7/3]		$-1/3$	1		7
	x_5	11/3		$-1/6$		1	13.5
	$-z$	0		1/2			13.5
2	x_2		1	3/14	$-1/7$		3.5
	x_1	1		$-1/7$	3/7		3
	x_5			5/14	$-11/7$	1	2.5
	$-z$			1/2	0		13.5

At cycle 1, since the negative relative cost coefficient is only $-1/3$, x_1 becomes a basic variable. Since

$$\min\left(\frac{4.5}{1/3}, \frac{7}{7/3}, \frac{13.5}{11/3}\right) = \frac{7}{7/3} = 3,$$

7/3 bracketed by [ ] becomes the pivot element. After the pivot operation on 7/3, the result at cycle 2 is obtained. At cycle 2, all of the relative cost coefficients become positive, and then the following optimal solution is obtained:

$$x_1 = 3,\ x_2 = 3.5\ (x_3 = x_4 = 0, x_5 = 2.5), \quad z = -37$$

The above optimal solution corresponds to the extreme point D in Fig. 1.1. ◊

Example 2.6 (Example with multiple optima). To show a simple linear programming problem having multiple optima, consider the following modified production planning problem in which the coefficients of x_1 and x_2 in the original objective function given in Example 1.1 are changed to 1 and 3, respectively:

$$\begin{array}{llrl} \text{minimize} & z = -x_1 - 3x_2 & & \\ \text{subject to} & 2x_1 + 6x_2 + x_3 & & = 27 \\ & 3x_1 + 2x_2 \quad\quad + x_4 & & = 16\,. \\ & 4x_1 + \ \ x_2 \quad\quad\quad\quad + x_5 & & = 18 \\ & x_j \geq 0, \quad j = 1, 2, 3, 4, 5 & & \end{array}$$

By using the simplex method, at cycle 1 in Table 2.5, an optimal solution

$$x_1 = 0,\ x_2 = 4.5\ (x_3 = 0, x_4 = 7, x_5 = 13.5), \quad z = -13.5$$

is obtained, observing the relative cost coefficient of x_1 is zero, which means that the value of the objective function is unchanged even if x_1 becomes positive, provided that it is not violating the constraints. Replacing x_1 with x_4 as a basic variable yields an alternative optimal solution

$$x_1 = 3, x_2 = 3.5\ (x_3 = x_4 = 0, x_5 = 2.5), \quad z = -13.5$$

giving the same value of the objective function. It should be noted here that the optimal solutions obtained in cycles 1 and 2, respectively, correspond to the extreme points E and D in Fig. 1.1, and all of the points on the line segment ED are also optimal. ♢

2.5 Two-Phase Method

The simplex method requires a basic feasible solution as a starting point. Such a starting point is not always easy to find, and in fact none will exist if the constraints are inconsistent. Phase I of the simplex method finds an initial basic feasible solution or derives the information that no feasible solution exists. Phase II then proceeds from this starting point to an optimal solution or derives the information that the optimal value is unbounded. Both phases use the procedure of the simplex method given in the previous section.

Phase I starts with a linear programming problem in the standard form (2.24), where all the constants b_i are nonnegative. For this purpose, if some b_i is negative, multiply the corresponding equation by -1. In order to set up an initial feasible solution for phase I, the linear programming problem in the standard form is augmented with a set of nonnegative variables $x_{n+1} \geq 0$, $x_{n+2} \geq 0$, ..., $x_{n+m} \geq 0$, so that the problem becomes as follows:

Find values of $x_j \geq 0$, $j = 1, 2, \ldots, n, n+1, \ldots, n+m$ so as to minimize z, satisfying the augmented system of linear equations

$$\left.\begin{array}{lllllll}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & +x_{n+1} & & & & = b_1\ (\geq 0) \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & & +x_{n+2} & & & = b_2\ (\geq 0) \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & & & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & & & +x_{n+m} & & = b_m\ (\geq 0) \\
c_1x_1 + c_2x_2 + \cdots + c_nx_n & & & & -z & = 0.
\end{array}\right\} \tag{2.55}$$

The newly introduced nonnegative variables $x_{n+1} \geq 0$, $x_{n+2} \geq 0$, ..., $x_{n+m} \geq 0$ are called artificial variables.

In the canonical form (2.55), using the artificial variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ as basic variables, the following initial basic feasible solution is directly obtained:

$$x_1 = x_2 = \cdots = x_n = 0, \quad x_{n+1} = b_1 \geq 0, \ldots, x_{n+m} = b_m \geq 0. \tag{2.56}$$

Although a basic feasible solution to (2.55) such as (2.56) is not always feasible to the original system, basic feasible solutions to (2.55) such that all the artificial variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ are equal to zeros are also feasible to the original system. Thus, one way to find a basic feasible solution to the original system is to start from the initial basic solution (2.56) and use the simplex method to drive a basic feasible solution such that all the artificial variables are equal to zeros. This can be done by minimizing a function of the artificial variables

$$w = x_{n+1} + x_{n+2} + \cdots + x_{n+m} \tag{2.57}$$

subject to the equality constraints (2.55) and the nonnegativity conditions for all variables. By its very nature, the function (2.57) is sometimes called the infeasibility form.

That is, the phase I problem is to find values of $x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$, $x_{n+1} \geq 0, \ldots, and x_{n+m} \geq 0$ so as to minimize w, satisfying the augmented system of linear equations

$$\left.\begin{array}{llllll}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & +x_{n+1} & & & & = b_1 \ (\geq 0) \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & & +x_{n+2} & & & = b_2 \ (\geq 0) \\
\cdots\cdots\cdots\cdots\cdots\cdots & & & & & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & & & +x_{n+m} & & = b_m \ (\geq 0) \\
c_1x_1 + \ c_2x_2 + \cdots + \ c_nx_n & & & & -z & = 0 \\
 & x_{n+1} & +x_{n+2} + \cdots & +x_{n+m} & -w & = 0.
\end{array}\right\} \tag{2.58}$$

Since the artificial variables are nonnegative, the function w which is the sum of the artificial variables is obviously larger than or equal to zero. In particular, if the optimal value of w is zero, i.e., $w = 0$, then all the artificial variables are zeros, i.e., $x_{n+i} = 0$ for all $i = 1, 2, \ldots, m$. In contrast, if it is positive, i.e., $w > 0$, then no feasible solution to the original system exists because some artificial variables are not zeros and then the corresponding original constraints are not satisfied. Given an initial basic feasible solution, the simplex method generates other basic feasible solutions in turn, and then the end product of phase I must be a basic feasible solution to the original system if such a solution exists.

It should be mentioned that a full set of m artificial variables may not be necessary. If the original system has some variables that can be used as initial basic variables, then they should be chosen in preference to artificial variables. The result is less work in phase I.

For obtaining an initial basic feasible solution through the minimization of w with the simplex method, it is necessary to convert the augmented system (2.58) into the canonical form with the row of $-w$, where w must be expressed by the current nonbasic variables $x_1, x_2, \ldots, x_n$. Since from (2.58) the artificial variables are represented by using the nonbasic variables, i.e.,

$$x_{n+i} = b_i - a_{i1}x_1 - a_{i2}x_2 - \cdots - a_{in}x_n, \quad i = 1, 2, \ldots, m,$$

(2.57) can be rewritten as

$$w = \sum_{i=1}^{m} x_{n+i} = \sum_{i=1}^{m} \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right) = \sum_{i=1}^{m} b_i - \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} \right) x_j, \tag{2.59}$$

which is now expressed by the nonbasic variables $x_1, x_2, \ldots, x_n$. Defining

$$w_0 = \sum_{i=1}^{m} b_i \ (\geq 0), \quad d_j = -\sum_{i=1}^{m} a_{ij}, \quad j = 1, 2, \ldots, n, \tag{2.60}$$

the row of $-w$ is compactly expressed as

$$-w + d_1 x_1 + d_2 x_2 + \cdots + d_n x_n = -w_0. \tag{2.61}$$

In this way, the augmented system (2.58) is converted into the following initial feasible canonical form for phase I with the row of $-w$ in which the artificial variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ are selected as basic variables:

$$\left.\begin{array}{lllllll}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & +x_{n+1} & & & & & = b_1 \ (\geq 0) \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & & +x_{n+2} & & & & = b_2 \ (\geq 0) \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & & & & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & & & +x_{n+m} & & & = b_m \ (\geq 0) \\
c_1x_1 + c_2x_2 + \cdots + c_nx_n & & & & -z & & = 0 \\
d_1x_1 + d_2x_2 + \cdots + d_nx_n & & & & & -w & = -w_0.
\end{array}\right\} \tag{2.62}$$

Now it becomes possible to solve the phase I problem as given by (2.62) using the simplex method. Finding the pivot element $\bar{a}_{rs}$ by using the rule

$$\bar{d}_s = \min_{\bar{d}_j < 0} \ \bar{d}_j \tag{2.63}$$

and

$$\frac{\bar{b}_r}{\bar{a}_{rs}} = \min_{\bar{a}_{is} > 0} \ \frac{\bar{b}_i}{\bar{a}_{is}} \tag{2.64}$$

and performing the pivot operation on it, we minimize the objective function w in phase I. When

$$\bar{d}_j \geq 0, \quad j = 1, \ldots, n, n+1, \ldots, n+m; \quad w = 0, \tag{2.65}$$

all the artificial variables become zeros. In this case, if all the artificial variables become nonbasic ones, an initial basic feasible solution to the original problem is obtained. Hence, after eliminating all the artificial variables together with the row of $-w$, initiate phase II of the simplex method for minimizing the original objective function z.

In the two-phase method, phase I finds an initial basic feasible solution or derives the information that no feasible solution exists, and phase II then proceeds from this starting point to an optimal solution or derives the information that the optimal value is unbounded.

Whenever the original system contains redundancies and often when degenerate solutions occur, artificial variables will remain in the basis at the end of phase I. Thus, it is necessary to prevent their values from becoming positive in phase II. One possible way is to drop all nonartificial variables whose relative cost coefficients for w are positive and all nonbasic artificial variables before starting phase II. To see this, we note that the equation for w at the end of phase I satisfies

$$\sum_{j=1}^{n+m} \bar{d}_j x_j = w - w_0, \tag{2.66}$$

where $\bar{d}_j \geq 0$ and $w_0 = 0$ since feasible solutions to the original problem exist. For feasibility, w must remain zero in phase II, which means that every x_j corresponding to $\bar{d}_j > 0$ must be zero; hence, all such x_j can be set equal to zero and eliminated from further consideration in phase II. We can also drop any nonbasic artificial variables because we no longer need to consider them. That is, eliminate the columns of the artificial variables leaving from the basis and those of nonbasic variable x_j with $d_j > 0$ in the optimal simplex tableau of phase I. Due to this operation, the objective function w of phase I will not become positive again, and also the values of the artificial variables remaining in the basis will not become positive in phase II. This means that basic solutions generated in phase II are always feasible.

$$d_j = -\sum_{i=1}^{m} a_{ij}, \quad -w_0 = -\sum_{i=1}^{m} b_i$$

Before summarizing the procedure of the two-phase method, the following useful remarks are given. In the simplex tableau, it is customary to omit the artificial variable columns because these, once dropped from the basis, can be eliminated from further consideration. Moreover, if the pivot operations for minimizing w in phase I are also simultaneously performed on the row of $-z$, the original objective function z will be expressed in terms of nonbasic variables at each cycle. Thus, if an initial basic feasible solution is found for the original problem, the simplex method can be initiated immediately on z. Therefore, the row of $-z$ is incorporated into the pivot operations in phase I.

Following the above discussions, the procedure of the two-phase method can be summarized as follows.

Table 2.6 Initial tableau of two-phase method

Basis	x_1	x_2	$\cdots$	x_j	$\cdots$	x_n	Constants
x_{n+1}	a_{11}	a_{12}	$\cdots$	a_{1j}	$\cdots$	a_{1n}	b_1
x_{n+2}	a_{21}	a_{22}	$\cdots$	a_{2j}	$\cdots$	a_{2n}	b_2
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_{n+i}	a_{i1}	a_{i2}	$\cdots$	a_{ij}	$\cdots$	a_{in}	b_i
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_{n+m}	a_{m1}	a_{m2}	$\cdots$	a_{mj}	$\cdots$	a_{mn}	b_m
$-z$	c_1	c_2	$\cdots$	c_j	$\cdots$	c_n	0
$-w$	d_1	d_2	$\cdots$	d_j	$\cdots$	d_n	$-w_0$

Procedure of Two-Phase Method

Phase I Starting with the simplex tableau in Table 2.6, perform the simplex method with the row of $-w$ as an objective function in phase I, where the pivot element is not selected from the row of $-z$, but the pivot operation is performed to the row of $-z$. When an optimal tableau is obtained, if $w > 0$, no feasible solution exists to the original problem. Otherwise, i.e., if $w = 0$, proceed to phase II.

Phase II After dropping all columns of x_j such that $\bar{d}_j > 0$ and the row of $-w$, perform the simplex method with the row of $-z$ as the objective function in phase II.

Example 2.7 (Two-phase method for diet problem with two decision variables). Using the two-phase method, solve the diet problem in the standard form given in Example 2.3.

$$
\begin{array}{lrcrcrcrcrcr}
\text{minimize } z = & 4x_1 & + & 3x_2 & & & & & & & & \\
\text{subject to} & x_1 & + & 3x_2 & - & x_3 & & & & & = & 12 \\
& x_1 & + & 2x_2 & & & - & x_4 & & & = & 10 \\
& 2x_1 & + & x_2 & & & & & - & x_5 & = & 15 \\
& \multicolumn{11}{l}{x_j \geq 0, \quad j = 1, 2, 3, 4, 5.}
\end{array}
$$

After introducing artificial variables x_6, x_7, and x_8 as basic variables in phase I, the two-phase method starts from cycle 0 as shown in Table 2.7, and then the value of w becomes zero, i.e., $w = 0$ at cycle 3. In this example, when phase I has finished at cycle 3, since all of the relative cost coefficients of the row of $-z$ are positive, phase II also finishes. Thus, an optimal solution

$$
x_1 = 6.6, \; x_2 = 1.8 \; (x_3 = 0, x_4 = 0.2, x_5 = 0) \quad z = 31.8
$$

is obtained. ◊

Table 2.7 Simplex tableau of two-phase method for Example 2.3

Cycle	Basis	x_1	x_2	x_3	x_4	x_5	Constants
0	x_6	1	[3]	-1			12
	x_7	1	2		-1		10
	x_8	2	1			-1	15
	$-z$	4	3				0
	$-w$	-4	-6	1	1	1	-37
1	x_2	1/3	1	$-1/3$			4
	x_7	[1/3]		2/3	-1		2
	x_8	5/3		1/3		-1	11
	$-z$	3		1			-12
	$-w$	-2		-1	1	1	-13
2	x_2		1	-1	1		2
	x_1	1		2	-3		6
	x_8			-3	[5]	-1	1
	$-z$			-5	9		-30
	$-w$			3	-5	1	-1
3	x_2		1	-0.4		0.2	1.8
	x_1	1		0.2		-0.6	6.6
	x_4			-0.6	1	-0.2	0.2
	$-z$			0.4		1.8	-31.8
	$-w$			0		0	0

Example 2.8 (Example of infeasible problem with two decision variables and four constraints). As an example of simple infeasible problem, consider a linear programming problem in the standard form for the diet problem of Example 2.7 including the additional inequality constraint

$$4x_1 + 5x_2 \le 8.$$

Introducing a slack variable x_6, the problem is converted into the following standard form of linear programming:

$$\begin{array}{lrcrcrcrcrcrcl}
\text{minimize } z = & 4x_1 & + & 3x_2 & & & & & & & & & & \\
\text{subject to} & x_1 & + & 3x_2 & - & x_3 & & & & & & & = & 12 \\
& x_1 & + & 2x_2 & & & - & x_4 & & & & & = & 10 \\
& 2x_1 & + & x_2 & & & & & - & x_5 & & & = & 15 \\
& 4x_1 & + & 5x_2 & & & & & & & + & x_6 & = & 8 \\
& \multicolumn{13}{l}{x_j \ge 0, \quad j = 1, 2, 3, 4, 5, 6.}
\end{array}$$

Using the slack variable x_6 and the artificial variables x_7, x_8, and x_9 as initial basic variables, phase I of the simplex method is performed. As shown in Table 2.8, phase I is terminated at cycle 1 because of $d_1 > 0$, $d_3 > 0$, $d_4 > 0$, $d_5 > 0$, and $d_6 > 0$. However, from $w = 27.4 > 0$, no feasible solution exists to this problem.

Table 2.8 Infeasible simplex tableau

Cycle	Basis	x_1	x_2	x_3	x_4	x_5	x_6	Constants
0	x_7	1	3	-1				12
	x_8	1	2		-1			10
	x_9	2	1			-1		15
	x_6	4	[5]				1	8
	$-z$	4	3					0
	$-w$	-4	-6	1	1	1		-37
1	x_7	-1.4		-1			-0.6	7.2
	x_8	-0.6			-1		-0.4	6.8
	x_9	1.2				-1	-0.2	13.4
	x_2	0.8	1				0.2	1.6
	$-z$	1.6					-0.6	-4.8
	$-w$	0.8		1	1	1	1.2	-27.4

It should be noted here that since the slack variable x_6 is used as a basic variable, the row of $-w$ is calculated only from the rows of x_7, x_8, and x_9 in cycle 0. For example, $d_1 = -(1 + 1 + 2) = -4$. ◊

Example 2.9 (Example of artificial variables left in the basis). As an example where artificial variables remain as a part of basic variables, consider the following problem:

$$\begin{array}{lrcrcrcrcl}
\text{minimize } z = & 3x_1 & + & x_2 & + & 2x_3 & & & & \\
\text{subject to} & x_1 & + & x_2 & + & x_3 & & & = & 10 \\
& 3x_1 & + & x_2 & + & 4x_3 & - & x_4 & = & 30 \\
& 4x_1 & + & 3x_2 & + & 3x_3 & + & x_4 & = & 40 \\
& & & & & \multicolumn{5}{l}{x_j \geq 0, \quad j = 1, 2, 3, 4.}
\end{array}$$

Using the artificial variables x_5, x_6, and x_7 as basic variables, phase I of the simplex method is performed. As shown in Table 2.9, phase I is terminated with $w = 0$ at cycle 1. However, the artificial variables x_6 and x_7 still remain in the basis as a part of basic variables. Since $\bar{d}_2 = 3 > 0$, after dropping the columns of x_2 and the row of $-w$, phase II of the simplex method is performed. At cycle 3, an optimal solution

$$x_1 = 0, \ x_2 = 0, \ x_3 = 10, \ x_4 = 10, \ (x_5 = 0, x_6 = 0, x_7 = 0) \quad z = 20$$

is obtained. ◊

The procedure of the simplex method considered thus far provides a means of going from one basic feasible solution to another one such that the objective function z is lower than the previous value of z if there is no degeneracy or at least equal to it

Table 2.9 Example of artificial variables left in the basis

Cycle	Basis	x_1	x_2	x_3	x_4	Constants
0	x_5	[1]	1	1		10
	x_6	3	1	4	−1	30
	x_7	4	3	3	1	40
	$-z$	3	1	2	0	0
	$-w$	−8	−5	−8	0	−80
1	x_1	1	1	1		10
	x_6		−2	[1]	−1	0
	x_7		−1	−1	1	0
	$-z$	0	−2	−1	0	−30
	$-w$	0	3	0	0	0
2	x_1	1			[1]	10
	x_3			1	−1	0
	x_7					0
	$-z$	0		0	−1	−30
3	x_4	1			1	10
	x_3	1		1		10
	x_7					0
	$-z$	1		0	0	−20

(as can occur in the degenerate case). It continues until (i) the condition of optimality test (2.42) is satisfied, or (ii) the information of unboundedness on the optimal value is provided. Therefore, in case of no degeneracy, the following convergence theorem can be easily understood.

Theorem 2.4 (Finite convergence of simplex method (nondegenerate case)). *Assuming nondegeneracy at each iteration, the simplex method will terminate in a finite number of iterations.*

Proof. Since the number of basic feasible solutions is at most ${}_nC_m$ and it is finite, the algorithm of the simplex method fails to finitely terminate only if the same basic feasible solution repeatedly appears. Such repetition implies that the value of the objective function z is the same. Under nondegeneracy, however, since each value of z is lower than the previous, no repetition can occur and therefore the algorithm finitely terminates. □

Recall that there is at least one basic variable whose value is zero in a degenerate basic feasible solution. Such degeneracy may occur in an initial feasible canonical form, and it is also possible that after some pivot operations in the procedure of the simplex method, degenerate basic feasible solution may occur.

For example, in step 3 of the procedure of the simplex method, if the minimum of $\{\bar{b}_i/\bar{a}_{is}$ for all $i \mid \bar{a}_{is} > 0\}$ is attained by two or more basic variables, i.e.,

$$\min_{\bar{a}_{is}>0} \frac{\bar{b}_i}{\bar{a}_{is}} = \theta = \frac{\bar{b}_{r_1}}{\bar{a}_{r_1 s}} = \frac{\bar{b}_{r_2}}{\bar{a}_{r_2 s}}, \tag{2.67}$$

either x_{r_1} or x_{r_2} can be removed from the basis and the other remains in the basis. In either case, both x_{r_1} and x_{r_2} become zeros, i.e.,

$$\left.\begin{aligned} x_{r_1} &= \bar{b}_{r_1} - \bar{a}_{r_1 s}\theta = 0 \\ x_{r_2} &= \bar{b}_{r_2} - \bar{a}_{r_2 s}\theta = 0. \end{aligned}\right\} \tag{2.68}$$

Thus, since there is at least one basic variable whose value is zero, the new basic feasible solution is degenerate.

This in itself does not undermine the feasibility of the solution. However, if at some iteration a basic feasible solution is degenerate, the value of objective function z could remain the same for some number of subsequent iterations. Moreover, there is a possibility that after a series of pivot operations without decrease of z, the same basis appears, and then the simplex method may be trapped into an endless loop without termination. This phenomenon is called cycling or circling.[3]

The following example given by H.W. Kuhn shows that the simplex method could be trapped into the cycling problem if the smallest index is used as tie breaker.

Example 2.10 (Kuhn's example of cycling). As an example of cycling, consider the following problem given by H.W. Kuhn:

$$\begin{array}{llrrrrrrl} \text{minimize } z = & & & - 2x_4 & - 3x_5 & + x_6 & + 12x_7 & \\ \text{subject to} & x_1 & & - 2x_4 & - 9x_5 & + x_6 & + 9x_7 & = 0 \\ & x_2 & & + \frac{1}{3}x_4 & + x_5 & - \frac{1}{3}x_6 & - 2x_7 & = 0\,. \\ & & x_3 & + 2x_4 & + 3x_5 & - x_6 & - 12x_7 & = 2 \\ & x_j \geq 0, & j = 1, 2, \ldots, 7 \end{array}$$

Using x_1, x_2, and x_3 as the initial basic variables and performing the simplex method, we have the result shown in Table 2.10. Observing that the tableau of cycle 6 is completely identical to that of cycle 0 in Table 2.10, one finds that cycling occurs. ◊

To avoid the trap of cycling, some means to prevent the procedure from cycling is required. Observe that in the absence of degeneracy the objective function values in a series of iterations of the simplex method form a strictly decreasing monotone sequence that guarantees the same basis does not repeatedly appear. With a degenerate basic solution, the sequence is no longer strictly decreasing. To prevent the procedure from revisiting the same basis, we need to incorporate another rule to keep a strictly monotone decreasing sequence.

[3]In his famous 1963 book, G.B. Dantzig adopted the term "circling" for avoiding possible confusion with the term "cycle," which was used synonymously with iteration.

Table 2.10 Simplex tableau for Kuhn's example of cycling

Cycle	Basis	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Constants
0	x_1	1			-2	-9	1	9	0
	x_2		1		1/3	[1]	$-1/3$	-2	0
	x_3			1	2	3	-1	-12	2
	$-z$				-2	-3	1	12	0
1	x_1	1	9		[1]		-2	-9	0
	x_5		1		1/3	1	$-1/3$	-2	0
	x_3		-3	1	1		0	-6	2
	$-z$		3		-1		0	6	0
2	x_4	1	9		1		-2	-9	0
	x_5	$-1/3$	-2			1	1/3	[1]	0
	x_3	-1	-12	1			2	3	2
	$-z$	1	12				-2	-3	0
3	x_4	-2	-9		1	9	[1]		0
	x_7	$-1/3$	-2			1	1/3	1	0
	x_3	0	-6	1		-3	1		2
	$-z$	0	6			3	-1		0
4	x_6	-2	-9		1	9	1		0
	x_7	1/3	[1]		$-1/3$	-2		1	0
	x_3	2	3	1	-1	-12			2
	$-z$	-2	-3		1	12			0
5	x_6	[1]			-2	-9	1	9	0
	x_2	1/3	1		$-1/3$	-2		1	0
	x_3	1		1	0	-6		-3	2
	$-z$	-1			0	6		3	0
6	x_1	1			-2	-9	1	9	0
	x_2		1		1/3	[1]	$-1/3$	-2	0
	x_3			1	2	3	-1	-12	2
	$-z$				-2	-3	1	12	0

Several methods besides the random choice rule exist for avoiding cycling in the simplex method. Among them, a very simple and elegant (but not necessarily efficient) rule due to Bland (1977) is theoretically interesting. Bland's rule is summarized as follows:

(i) Among all candidates to enter the basis, choose the one with the smallest index.
(ii) Among all candidates to leave the basis, choose the one with the smallest index.

The procedure of the simplex method incorporating Bland's anticycling rule, just specifying the choice of both the entering and leaving variables, can now be given in the following.

Procedure of Simplex Method Incorporating Bland's Rule

Step 1B If all of the relative cost coefficients are nonnegative, i.e., $\bar{c}_j \geq 0$ for all indices j of the nonbasic variables, then the current solution is optimal, and stop. Otherwise, by using the relative cost coefficients $\bar{c}_j$, find the index s such that

$$\min\left\{j \mid \bar{c}_j < 0\right\} = s. \quad (j : \text{nonbasic})$$

That is, if there are two or more indices j such that $\bar{c}_j < 0$ for all indices of the nonbasic variables, choose the smallest index s as the index of a nonbasic variable newly entering the basis.

Step 2 If all of the coefficients in column s are nonpositive, i.e., $\bar{a}_{is} \leq 0$ for all indices i of the basic variables, then the optimal value is unbounded, and stop.

Step 3B If some of $\bar{a}_{is}$ are positive, find the index r such that

$$\min_{\bar{a}_{is}>0} \frac{\bar{b}_i}{\bar{a}_{is}} = \frac{\bar{b}_r}{\bar{a}_{rs}} = \theta.$$

If there is a tie in the minimum ratio test, choose the smallest index r as the index of a basic variable leaving the basis.

Step 4 Perform the pivot operation on $\bar{a}_{rs}$ for obtaining a new feasible canonical form with x_s replacing x_r as a basic variable. Return to step 1B.

It is interesting to note here that the use of Bland's rule for cycling prevention can be proven by contradiction on the basis of the following observation.

In a degenerate pivot operation, if some variable x_q enters the basis, then x_q cannot leave the basis until some other variable with a higher index than q, which was nonbasic when x_q entered, also enters the basis. If this holds, then cycling cannot occur because in a cycle any variable that enters must also leave the basis, which means that there exists some highest indexed variable that enters and leaves the basis. This contradicts the foregoing monotone feature.[4]

In practice, however, such a procedure is found to be unnecessary because the simplex procedure generally does not enter a cycle even if degenerate solutions are encountered. However, an anticycling procedure is simple, and therefore many codes incorporate such a procedure for the sake of safety.

Example 2.11 (Simplex method incorporating Bland's rule for Kuhn's example). Apply the simplex method incorporating Bland's rule to the example given by Kuhn. After only two pivot operations, the algorithm stops, and the result is shown in Table 2.11. At cycle 2 in Table 2.11, degeneracy is ended, and an optimal solution

$$x_1 = 2, x_2 = 0, x_3 = 0, x_4 = 2, x_5 = 0, x_6 = 2, x_7 = 0, \quad z = -2$$

is obtained. ◊

[4]The interested reader should refer to the solution of Problem 2.8 for a full discussion of the proof.

Table 2.11 Simplex tableau incorporating Bland's rule

Cycle	Basis	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Constants
0	x_1	1			−2	−9	1	9	0
	x_2		1		[1/3]	1	−1/3	−2	0
	x_3			1	2	3	−1	−12	2
	$-z$				−2	−3	1	12	0
1	x_1	1	6			−3	−1	−3	0
	x_4		3		1	3	−1	−6	0
	x_3		−6	1		−3	[1]	0	2
	$-z$		6			3	−1	0	0
2	x_1	1	0	1		−6		−3	2
	x_4		−3	1	1	0		−6	2
	x_6		−6	1		−3	1	0	2
	$-z$		0	1		0		0	2

2.6 Revised Simplex Method

In performing the simplex method, all the information contained in the tableau is not necessarily used. Only the following items are needed:

Information Needed for Updating the Simplex Tableau

(i) Using the relative cost coefficients $\bar{c}_j$, find the index s such that

$$\min_{\bar{c}_j<0} \bar{c}_j = \bar{c}_s.$$

(ii) Assuming $\bar{c}_s < 0$, we require the elements of the sth column (pivot column)

$$\bar{\boldsymbol{p}}_s = (\bar{a}_{1s}, \bar{a}_{2s}, \ldots, \bar{a}_{ms})^T$$

and the values of the basic variables

$$\bar{\boldsymbol{b}} = (\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_m)^T.$$

By using these values, the quotients

$$\frac{\bar{b}_r}{\bar{a}_{rs}} = \min_{\bar{a}_{is}>0} \frac{\bar{b}_i}{\bar{a}_{is}}$$

are calculated for finding the index r. Then, a pivot operation is performed on $\bar{a}_{rs}$ for updating the tableau.

From the above discussion, note that only one nonbasic column $\bar{\boldsymbol{p}}_s$ in the current tableau is required. Since there are many more columns than rows in a linear programming problem, dealing with all the columns $\bar{\boldsymbol{p}}_j$ wastes much computation time and computer storage. A more efficient procedure is to calculate first the relative cost coefficients $\bar{c}_j$ and then the pivot column $\bar{\boldsymbol{p}}_s$ from the data of the original problem. The revised simplex method does precisely this, and the inverse of the current basis matrix is what is needed to calculate them.

We assume again that the $m \times n$ rectangular matrix $A = [\boldsymbol{p}_1\ \boldsymbol{p}_2\ \cdots\ \boldsymbol{p}_n]$ for the constraints has the rank of m and $n > m$. Moreover, we assume that a linear programming problem in the standard form is feasible. A basis matrix B is defined as an $m \times m$ nonsingular submatrix formed by selecting some m linearly independent columns from the n columns of matrix A. Note that matrix A contains at least one basis matrix B due to $\mathrm{rank}(A) = m$ and $n > m$.

For notational simplicity, without loss of generality, assume that the basis matrix B is formed by selecting the first m columns of matrix A, i.e.,

$$B = [\boldsymbol{p}_1\ \boldsymbol{p}_2\ \cdots\ \boldsymbol{p}_m]. \tag{2.69}$$

Let

$$\boldsymbol{x}_B = (x_1, x_2, \ldots, x_m)^T \text{ and } \boldsymbol{c}_B = (c_1, c_2, \ldots, c_m) \tag{2.70}$$

be the corresponding vectors of basic variables and coefficients of the objective function, respectively. Note that $\boldsymbol{c}_B$ is a row vector. The vector $\boldsymbol{x}_B$ satisfies

$$B\boldsymbol{x}_B = \boldsymbol{b}, \tag{2.71}$$

and one finds that

$$\boldsymbol{x}_B = B^{-1}\boldsymbol{b} = \bar{\boldsymbol{b}}. \tag{2.72}$$

Assume that the basis matrix B is feasible, i.e.,

$$\boldsymbol{x}_B \geq \boldsymbol{0}. \tag{2.73}$$

As shown earlier, it is convenient to deal with the objective function z as the $(m+1)$th equation and keep the variable $-z$ in the basis. This augmented system can be written in column form as follows:

$$\sum_{j=1}^{n} \begin{pmatrix} \boldsymbol{p}_j \\ \hdashline c_j \end{pmatrix} x_j + \begin{pmatrix} \boldsymbol{0} \\ \hdashline 1 \end{pmatrix} (-z) = \begin{pmatrix} \boldsymbol{b} \\ \hdashline 0 \end{pmatrix}. \tag{2.74}$$

By using the corresponding basis $\boldsymbol{x}_B = (x_1, x_2, \ldots, x_m)^T$, $(-z)$ and the nonbasis $\boldsymbol{x}_N = (x_{m+1}, \ldots, x_n)^T$, (2.74) is also rewritten as

$$\left[\begin{array}{cccc|c} \boldsymbol{p}_1 & \boldsymbol{p}_2 & \cdots & \boldsymbol{p}_m & \boldsymbol{0} \\ \hline c_1 & c_2 & \cdots & c_m & 1 \end{array}\right] \left(\begin{array}{c} \boldsymbol{x}_B \\ \hline -z \end{array}\right) + \left[\begin{array}{ccc} \boldsymbol{p}_{m+1} & \cdots & \boldsymbol{p}_n \\ \hline c_{m+1} & \cdots & c_n \end{array}\right] \boldsymbol{x}_N = \left(\begin{array}{c} \boldsymbol{b} \\ \hline 0 \end{array}\right). \tag{2.75}$$

Since the basis matrix B is feasible, the $(m + 1) \times (m + 1)$ matrix

$$\hat{B} = \left[\begin{array}{cccc|c} \boldsymbol{p}_1 & \boldsymbol{p}_2 & \cdots & \boldsymbol{p}_m & \boldsymbol{0} \\ \hline c_1 & c_2 & \cdots & c_m & 1 \end{array}\right] = \left[\begin{array}{c|c} B & \boldsymbol{0} \\ \hline \boldsymbol{c}_B & 1 \end{array}\right] \tag{2.76}$$

is also a feasible basis matrix for the enlarged system (2.74). It is easily verified by direct matrix multiplication that the inverse of $\hat{B}$ is

$$\hat{B}^{-1} = \left[\begin{array}{c|c} B^{-1} & \boldsymbol{0} \\ \hline -\boldsymbol{c}_B B^{-1} & 1 \end{array}\right]. \tag{2.77}$$

Such an $(m+1)\times(m+1)$ matrix $\hat{B}$ is called an enlarged basis matrix and its inverse $\hat{B}^{-1}$ is called an enlarged basis inverse matrix.

Introducing a simplex multiplier vector

$$\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_m) = \boldsymbol{c}_B B^{-1} \tag{2.78}$$

associated with the basis matrix B, the enlarged basis inverse matrix $\hat{B}^{-1}$ is written more compactly as

$$\hat{B}^{-1} = \left[\begin{array}{c|c} B^{-1} & \boldsymbol{0} \\ \hline -\boldsymbol{\pi} & 1 \end{array}\right]. \tag{2.79}$$

Premultiplying the enlarged system (2.75) by $\hat{B}^{-1}$, (2.75) becomes as

$$\left[\begin{array}{c|c} I & \boldsymbol{0} \\ \hline \boldsymbol{0}^T & 1 \end{array}\right] \left(\begin{array}{c} \boldsymbol{x}_B \\ \hline -z \end{array}\right) + \hat{B}^{-1} \left[\begin{array}{ccc} \boldsymbol{p}_{m+1} & \cdots & \boldsymbol{p}_n \\ \hline c_{m+1} & \cdots & c_n \end{array}\right] \boldsymbol{x}_N = \hat{B}^{-1} \left(\begin{array}{c} \boldsymbol{b} \\ \hline 0 \end{array}\right) \tag{2.80}$$

which results in the following canonical form:

$$\left(\begin{array}{c} \boldsymbol{x}_B \\ \hline -z \end{array}\right) + \sum_{j=m+1}^{n} \left(\begin{array}{c} \bar{\boldsymbol{p}}_j \\ \hline \bar{c}_j \end{array}\right) x_j = \left(\begin{array}{c} \bar{\boldsymbol{b}} \\ \hline -\bar{z} \end{array}\right), \tag{2.81}$$

or equivalently

$$\left.\begin{array}{r}\begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_m \end{bmatrix} + \displaystyle\sum_{j=m+1}^{n} \bar{\boldsymbol{p}}_j x_j = \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_m \end{pmatrix} \\ -z + \displaystyle\sum_{j=m+1}^{n} \bar{c}_j x_j = -\bar{z}, \end{array}\right\} \tag{2.82}$$

where

$$\begin{pmatrix} \bar{\boldsymbol{p}}_j \\ \hdashline \bar{c}_j \end{pmatrix} = \hat{B}^{-1} \begin{pmatrix} \boldsymbol{p}_j \\ \hdashline c_j \end{pmatrix} = \left[\begin{array}{c:c} B^{-1} & \boldsymbol{0} \\ \hdashline -\boldsymbol{\pi} & 1 \end{array}\right] \begin{pmatrix} \boldsymbol{p}_j \\ \hdashline c_j \end{pmatrix} = \begin{pmatrix} B^{-1}\boldsymbol{p}_j \\ \hdashline c_j - \boldsymbol{\pi}\boldsymbol{p}_j \end{pmatrix}, \tag{2.83}$$

$$\begin{pmatrix} \bar{\boldsymbol{b}} \\ \hdashline -\bar{z} \end{pmatrix} = \hat{B}^{-1} \begin{pmatrix} \boldsymbol{b} \\ \hdashline 0 \end{pmatrix} = \left[\begin{array}{c:c} B^{-1} & \boldsymbol{0} \\ \hdashline -\boldsymbol{\pi} & 1 \end{array}\right] \begin{pmatrix} \boldsymbol{b} \\ \hdashline 0 \end{pmatrix} = \begin{pmatrix} B^{-1}\boldsymbol{b} \\ \hdashline -\boldsymbol{\pi}\boldsymbol{b} \end{pmatrix}. \tag{2.84}$$

In particular, from (2.83), the updated column vector of the coefficients is represented as

$$\bar{\boldsymbol{p}}_j = B^{-1}\boldsymbol{p}_j, \tag{2.85}$$

and the relative cost coefficient is

$$\bar{c}_j = c_j - \boldsymbol{\pi}\boldsymbol{p}_j. \tag{2.86}$$

These two formulas are fundamental to calculation in the revised simplex method, and as seen in (2.85) and (2.86), $\bar{c}_j$ and $\bar{\boldsymbol{p}}_j$, which are used in each iteration of the simplex method, can be calculated by using the original coefficients c_j and $\boldsymbol{p}_j$ given in the initial simplex tableau, provided that the enlarged basis inverse matrix $\hat{B}^{-1}$, or equivalently the basis inverse matrix B^{-1} and the simplex multipliers $\boldsymbol{\pi}$ are given.

Now, assume that the smallest $\bar{c}_s$ is found among the relative cost coefficients $\bar{c}_j$ calculated by (2.86) for all nonbasic variables and the corresponding column vector $\bar{\boldsymbol{p}}_s = (\bar{a}_{1s}, \ldots, \bar{a}_{ms})^T$ is obtained by (2.85). If the vector $\bar{\boldsymbol{b}}$ of the values of the basic variables is known at the beginning of each cycle, the pivot element $\bar{a}_{rs}$ is immediately determined. Furthermore, however, we need the inverse matrix $\hat{B}^{*-1}$ of the revised enlarged basis matrix $\hat{B}^*$ corresponding to a new basis made by replacing x_r in the current basis with the new basic variable x_s. From the current $(m+1) \times (m+1)$ enlarged basis matrix

$$\hat{B} = \left[\begin{array}{cccccc:c} \boldsymbol{p}_1 & \cdots & \boldsymbol{p}_{r-1} & \boldsymbol{p}_r & \boldsymbol{p}_{r+1} & \cdots & \boldsymbol{p}_m & \boldsymbol{0} \\ \hdashline c_1 & \cdots & c_{r-1} & c_r & c_{r+1} & \cdots & c_m & 1 \end{array}\right], \tag{2.87}$$

by removing $\boldsymbol{p}_r$ and c_r and entering $\boldsymbol{p}_s$ and c_s instead, the new enlarged basis matrix

$$\hat{B}^* = \left[\begin{array}{ccccccc|c} \boldsymbol{p}_1 & \cdots & \boldsymbol{p}_{r-1} & \boldsymbol{p}_s & \boldsymbol{p}_{r+1} & \cdots & \boldsymbol{p}_m & \boldsymbol{0} \\ \hline c_1 & \cdots & c_{r-1} & c_s & c_{r+1} & \cdots & c_m & 1 \end{array}\right] \tag{2.88}$$

is obtained.

It can be shown that, without the direct calculations of inverse matrices,[5] the new enlarged basis inverse matrix $\hat{B}^{*-1}$ can be obtained from $\hat{B}^{-1}$ by performing the pivot operation on $\bar{a}_{rs}$ in $\hat{B}^{-1}$. Since $\hat{B}^{*-1}$ is the same as $\hat{B}^{-1}$ except the rth column, the product of $\hat{B}^{-1}$ and $\hat{B}^*$ can be represented as

$$\hat{B}^{-1}\hat{B}^* = \left[\begin{array}{cccc|c} 1 & \bar{a}_{1s} & & & \\ \ddots & \vdots & & & \\ & \bar{a}_{rs} & & & \\ & \vdots & \ddots & & \\ & \bar{a}_{ms} & & 1 & \\ \hline & \bar{c}_s & & & 1 \end{array}\right], \tag{2.89}$$

where the rth column $(\bar{a}_{1s}, \ldots, \bar{a}_{rs}, \ldots, \bar{a}_{ms}, \bar{c}_s)^T = \begin{pmatrix} \bar{\boldsymbol{p}}_s \\ \hline \bar{c}_s \end{pmatrix}$ is $\hat{B}^{-1}\begin{pmatrix} \boldsymbol{p}_s \\ \hline c_s \end{pmatrix}$, and the i column ($i \neq r$) is the $(m+1)$ dimensional unit vectors such that the ith element is one.

After introducing the $(m+1) \times (m+1)$ nonsingular square matrix

$$\hat{E} = \left[\begin{array}{cccc|c} 1 & -\bar{a}_{1s}/\bar{a}_{rs} & & & \\ \ddots & \vdots & & & \\ & 1/\bar{a}_{rs} & & & \\ & \vdots & \ddots & & \\ & -\bar{a}_{ms}/\bar{a}_{rs} & & 1 & \\ \hline & -\bar{c}_s/\bar{a}_{rs} & & & 1 \end{array}\right] \tag{2.90}$$

that differs from the $(m+1) \times (m+1)$ unit matrix $\hat{I}$ in only the rth column, premultiplying (2.89) by $\hat{E}$ yields

$$\hat{E}\hat{B}^{-1}\hat{B}^* = \hat{I}. \tag{2.91}$$

Hence, by postmultiplying both sides of (2.91) by $\hat{B}^{*-1}$, we have the new enlarged basis inverse matrix

[5] Annoying calculations of the inverse matrix make no sense of the revised method.

$$\hat{B}^{*-1} = \hat{E}\hat{B}^{-1}. \tag{2.92}$$

Setting

$$\hat{B}^{*-1} = \left[\begin{array}{c|c} \beta_{ij}^{*} & \mathbf{0} \\ \hline -\pi_j^{*} & 1 \end{array}\right], \quad \hat{B}^{-1} = \left[\begin{array}{c|c} \beta_{ij} & \mathbf{0} \\ \hline -\pi_j & 1 \end{array}\right] \tag{2.93}$$

and calculating the right-hand side of (2.92), we have

$$\left.\begin{aligned} \beta_{rj}^{*} &= \frac{1}{\bar{a}_{rs}}\beta_{rj}, \quad j = 1, 2, \ldots, m, \\ \beta_{ij}^{*} &= \beta_{ij} - \frac{\bar{a}_{is}}{\bar{a}_{rs}}\beta_{rj}, \quad i = 1, 2, \ldots, m,\ i \neq r,\ j = 1, 2, \ldots, m, \\ -\pi_j^{*} &= -\pi_j - \frac{\bar{c}_s}{\bar{a}_{rs}}\beta_{rj}, \quad j = 1, 2, \ldots, m. \end{aligned}\right\} \tag{2.94}$$

This means that performing the pivot operation on $\bar{a}_{rs}$ to the current enlarged basis inverse matrix $\hat{B}^{-1}$ gives the new enlarged basis inverse matrix $\hat{B}^{*-1}$.

By adding the superscript $*$ to the values of $\bar{\boldsymbol{b}}$ and $\bar{z}$ for the new enlarged basis matrix $\hat{B}^{*}$ and premultiplying the enlarged system (2.75) corresponding to $\hat{B}^{*}$ by $\hat{B}^{*-1}$, the right-hand side becomes as

$$\left(\begin{array}{c} \bar{\boldsymbol{b}}^{*} \\ \hline -\bar{z}^{*} \end{array}\right) = \hat{B}^{*-1}\left(\begin{array}{c} \boldsymbol{b} \\ \hline 0 \end{array}\right) = \hat{E}\hat{B}^{-1}\left(\begin{array}{c} \boldsymbol{b} \\ \hline 0 \end{array}\right) = \hat{E}\left(\begin{array}{c} \bar{\boldsymbol{b}} \\ \hline -\bar{z} \end{array}\right). \tag{2.95}$$

Each element of (2.95) is represented by

$$\left.\begin{aligned} \bar{b}_r^{*} &= \frac{\bar{b}_r}{\bar{a}_{rs}}, \\ \bar{b}_r^{*} &= \bar{b}_i - \frac{\bar{a}_{is}}{\bar{a}_{rs}}\bar{b}_r, \quad (i \neq r) \\ -\bar{z}^{*} &= -\bar{z} - \frac{\bar{c}_s}{\bar{a}_{rs}}\bar{b}_r. \end{aligned}\right\} \tag{2.96}$$

This also means that performing the pivot operation on $\bar{a}_{rs}$ to the current $\bar{\boldsymbol{b}}$ and $-\bar{z}$ gives the new constants $\bar{\boldsymbol{b}}^{*}$ and $-\bar{z}^{*}$. As just described, since premultiplying the current enlarged basis inverse matrix $\hat{B}^{-1}$ and the right-hand side of (2.81) by $\hat{E}$ corresponds to a pivot operation, $\hat{E}$ is called a pivot matrix or an elementary matrix.

Now, the procedure of the revised simplex method, starting with an initial basic feasible solution, can be summarized as follows.

Procedure of the Revised Simplex Method

Assume that the coefficients A, $\boldsymbol{b}$, and $\boldsymbol{c}$ of the initial feasible canonical form and the inverse matrix B^{-1} of the initial feasible basis are available.

Step 0 By using B^{-1}, calculate

$$\boldsymbol{\pi} = \boldsymbol{c}_B B^{-1},\ \boldsymbol{x}_B = \bar{\boldsymbol{b}} = B^{-1}\boldsymbol{b},\ \bar{z} = \boldsymbol{\pi}\boldsymbol{b},$$

and put them in the revised simplex tableau shown in Table 2.12.

Step 1 Calculate the relative cost coefficients $\bar{c}_j$ for all indices j of the nonbasic variables by

$$\bar{c}_j = c_j - \boldsymbol{\pi}\boldsymbol{p}_j.$$

If all of the relative cost coefficients are nonnegative, i.e., $\bar{c}_j \geq 0$, then the current solution is optimal, and stop. Otherwise, find the index s such that

$$\min_{\bar{c}_j < 0} \bar{c}_j = \bar{c}_s.$$

Step 2 Calculate

$$\bar{\boldsymbol{p}}_s = B^{-1}\boldsymbol{p}_s.$$

If all of the elements of $\bar{\boldsymbol{p}}_s = (\bar{a}_{1s}, \bar{a}_{2s}, \ldots, \bar{a}_{ms})^T$ are nonpositive, i.e., $\bar{a}_{is} \leq 0$ for all indices i of the basic variables, then the optimal value is unbounded, and stop.

Step 3 If some of $\bar{a}_{is}$ are positive, put the values of $\hat{\bar{\boldsymbol{p}}}_s = (\bar{\boldsymbol{p}}_s, \bar{c}_s)^T$ in the revised simplex tableau of Table 2.12, and find the index r such that

$$\frac{\bar{b}_r}{\bar{a}_{rs}} = \min_{\bar{a}_{is} > 0} \frac{\bar{b}_i}{\bar{a}_{is}}.$$

Step 4 Perform the pivot operation on $\bar{a}_{rs}$ to B^{-1}, $-\boldsymbol{\pi}$, $\bar{\boldsymbol{b}}$, and $-\bar{z}$ of Table 2.12, and replace x_r with x_s as a new basic variable. The pivot operation for B^{-1} and $-\boldsymbol{\pi}$ is given by (2.94), and that for $\bar{\boldsymbol{b}}$ and $-\bar{z}$ is also given by (2.96). After updating the revised simplex tableau in Table 2.12, return to step 1.

It should be noted here that in the procedure of the revised simplex method, since the $(m+1)$th column of the enlarged basis inverse matrix $\hat{B}^{-1}$ is always $\begin{pmatrix} \boldsymbol{0} \\ \hdashline 1 \end{pmatrix}$, it is recommended to neglect it and to use the revised simplex tableau without the $(m+1)$th column of $\hat{B}^{-1}$ as given in Table 2.12.

Table 2.12 Revised simplex tableau

Basis	Basis inverse matrix	Constants	$\hat{\bar{\boldsymbol{p}}}_s$
x_1			
$\vdots$			
x_r	B^{-1}	$\bar{\boldsymbol{b}}$	$\bar{\boldsymbol{p}}_s$
$\vdots$			
x_m			
$-z$	$-\boldsymbol{\pi}$	$-\bar{z}$	$\bar{c}_s$

When starting the revised simplex method with phase I of the two-phase method, the enlarged basis inverse matrix $\hat{B}^{-1}$ can be considered as

$$\hat{B}^{-1} = \left[\begin{array}{c|c|c} B^{-1} & \mathbf{0} & \mathbf{0} \\ \hline -\boldsymbol{\pi} & 1 & 0 \\ \hline -\boldsymbol{\sigma} & 0 & 1 \end{array}\right], \tag{2.97}$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_m)$ is a vector of the simplex multipliers for the objective function w of phase I, and the initial enlarged basis inverse matrix $\hat{B}^{-1}$ is an $(m + 2) \times (m + 2)$ unit matrix.

In phase I, the relative cost coefficients $\bar{d}_j$ are calculated by

$$\bar{d}_j = d_j - \boldsymbol{\sigma} \boldsymbol{p}_j, \quad (j : \text{nonbasic}) \tag{2.98}$$

and the pivot column is determined by $\min_{\bar{d}_j < 0} \bar{d}_j$. Including the row of $-w$, the pivot operations are performed according to step 4 in the procedure of the revised simplex method. After eliminating the row of $-w$, i.e., the $(m + 2)$th row at the beginning of phase II, the above-mentioned revised simplex method is continued.

Example 2.12 (Revised simplex method for production planning problem of Example 1.1). Using the revised simplex method, solve the standard form of the production planning problem of Example 1.1.

$$\begin{array}{lrcrcrcrcrcl} \text{minimize} & z = -3x_1 & - & 8x_2 & & & & & & & & \\ \text{subject to} & 2x_1 & + & 6x_2 & + & x_3 & & & & & = & 27 \\ & 3x_1 & + & 2x_2 & & & + & x_4 & & & = & 16 \\ & 4x_1 & + & x_2 & & & & & + & x_5 & = & 18 \\ & \multicolumn{11}{l}{x_j \geq 0, \quad j = 1, 2, 3, 4, 5.} \end{array}$$

Employing the slack variables x_3, x_4, and x_5 as basic variables, one finds that the basis matrix B is a 3×3 unit matrix and its inverse B^{-1} is also the same unit matrix. Thus, from (2.78) and (2.84), it follows that

$$\boldsymbol{\pi} = \boldsymbol{c}_B B^{-1} = (0, 0, 0), \quad \bar{\boldsymbol{b}} = B^{-1}\boldsymbol{b} = \boldsymbol{b} = (27, 16, 18)^T, \quad \bar{z} = \boldsymbol{\pi}\boldsymbol{b} = 0.$$

Table 2.13 Revised simplex tableau of Example 1.1

Cycle	Basis	Basis inverse matrix			Constants	$\hat{\bar{\boldsymbol{p}}}_s$
0	x_3	1			27	[6]
	x_4		1		16	2
	x_5			1	18	1
	$-z$				0	-8
1	x_2	1/6			4.5	1/3
	x_4	$-1/3$	1		7	[7/3]
	x_5	$-1/6$		1	13.5	11/3
	$-z$	4/3			36	$-1/3$
2	x_2	3/14	$-1/7$		3.5	
	x_1	$-1/7$	3/7		3	
	x_5	5/14	$-11/7$	1	2.5	
	$-z$	9/7	1/7		37	

Putting these values in the revised simplex tableau at cycle 0 of Table 2.13.

After calculating $\bar{c}_j$ for nonbasic variables, the minimum of them is calculated in order to select a new basic variable as follows:

$$\bar{c}_1 = c_1 - \boldsymbol{\pi}\boldsymbol{p}_1 = -3 - (0,0,0)\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = -3,$$

$$\bar{c}_2 = c_2 - \boldsymbol{\pi}\boldsymbol{p}_2 = -8 - (0,0,0)\begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix} = -8,$$

$$\min_{\bar{c}_j < 0} \bar{c}_j = (-3, -8) = \bar{c}_2 = -8 < 0.$$

Since $\bar{c}_2$ is the minimum, x_2 becomes a new basic variable. The corresponding coefficient column vector $\bar{\boldsymbol{p}}_2$ is calculated as

$$\bar{\boldsymbol{p}}_2 = B^{-1}\boldsymbol{p}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix},$$

and then it is filled in on the rightmost column of the revised simplex tableau. Since

$$\min\left(\frac{\bar{b}_3}{\bar{a}_{32}}, \frac{\bar{b}_4}{\bar{a}_{42}}, \frac{\bar{b}_5}{\bar{a}_{52}}\right) = \min\left(\frac{27}{6}, \frac{16}{2}, \frac{18}{1}\right) = \frac{27}{6} = 4.5,$$

x_3 becomes a nonbasic variable, and it follows that the pivot element is $\bar{a}_{32} = 6$, which is bracketed by [ ] in Table 2.13. After replacing x_3 with x_2 as a new basic variable, the pivot operation on $\bar{a}_{32} = 6$ is performed to the basis inverse matrix and the constants at cycle 0 of Table 2.13, and the result of cycle 1 is obtained. These values become B^{-1}, $-\boldsymbol{\pi}$, $\boldsymbol{b}$, and $-\bar{z}$ for the new basis matrix $B = [\boldsymbol{p}_2\ \boldsymbol{p}_4\ \boldsymbol{p}_5]$ and the procedure returns to step 1.

From

$$\bar{c}_1 = c_1 - \boldsymbol{\pi}\boldsymbol{p}_1 = -3 - \left(-\frac{4}{3}, 0, 0\right)\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = -\frac{1}{3}$$

$$\bar{c}_3 = c_3 - \boldsymbol{\pi}\boldsymbol{p}_3 = 0 - \left(-\frac{4}{3}, 0, 0\right)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{4}{3}$$

$$\min_{\bar{c}_j<0} \bar{c}_j = \bar{c}_1 = -\frac{1}{3} < 0,$$

x_1 becomes a new basic variable. By using B^{-1} at cycle 1, $\bar{\boldsymbol{p}}_1$ is calculated as

$$\bar{\boldsymbol{p}}_1 = B^{-1}\boldsymbol{p}_1 = \begin{bmatrix} 1/6 & 0 & 0 \\ -1/3 & 1 & 0 \\ -1/6 & 0 & 1 \end{bmatrix}\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 7/3 \\ 11/3 \end{pmatrix},$$

and it is filled in on the rightmost column of the revised simplex tableau. Since

$$\min\left(\frac{\bar{b}_2}{\bar{a}_{21}}, \frac{\bar{b}_4}{\bar{a}_{41}}, \frac{\bar{b}_5}{\bar{a}_{51}}\right) = \min\left(\frac{4.5}{1/3}, \frac{7}{7/3}, \frac{13.5}{11/3}\right) = \frac{7}{7/3} = 3,$$

x_4 becomes a nonbasic variable. After replacing x_4 with x_1, the pivot operation on $\bar{a}_{41} = 7/3$ bracketed by [ ] is performed to the basis inverse matrix and the constants at cycle 1. This yields the result at cycle 2 in Table 2.13. The next basis matrix becomes as $B = [\boldsymbol{p}_2\ \boldsymbol{p}_1\ \boldsymbol{p}_5]$, and since

$$\bar{c}_3 = c_3 - \boldsymbol{\pi}\boldsymbol{p}_3 = 0 - \left(-\frac{9}{7}, -\frac{1}{7}, 0\right)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{9}{7} > 0$$

$$\bar{c}_4 = c_4 - \boldsymbol{\pi}\boldsymbol{p}_4 = 0 - \left(-\frac{9}{7}, -\frac{1}{7}, 0\right)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{7} > 0,$$

an optimal solution

$$x_1 = 3, \quad x_2 = 3.5 \quad (x_3 = x_4 = 0, x_5 = 2.5), \quad z = -37$$

is obtained. ♢

2.7 Duality

The notion of duality is one of the most important concepts in linear programming. Basically, associated with a linear programming problem (we may call it the primal problem), defined by the constraint matrix A, the right-hand side constant vector $\boldsymbol{b}$, and the cost coefficient vector $\boldsymbol{c}$, there is the corresponding linear programming problem (called the dual problem) which is specified by the same set of coefficients A, $\boldsymbol{b}$, and $\boldsymbol{c}$. These two problems bear interesting and useful relationships to one another.

Consider the standard form of linear programming

$$\left.\begin{array}{ll} \text{minimize } z = & \boldsymbol{cx} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}, \end{array}\right\} \tag{2.99}$$

where $\boldsymbol{c}$ is an n dimensional row vector, $\boldsymbol{x}$ is an n dimensional column vector, A is an $m \times n$ matrix A, and $\boldsymbol{b}$ is an n dimensional column vector. Introducing an m dimensional row vector $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_m)$, we define an associated linear programming problem:

$$\left.\begin{array}{ll} \text{maximize } v = & \boldsymbol{\pi b} \\ \text{subject to} & \boldsymbol{\pi} A \leq \boldsymbol{c}. \end{array}\right\} \tag{2.100}$$

It should be noted here that problem (2.100) is a maximization problem with m unrestricted variables and n inequality constraints. The roles of the variables and constraints are somewhat reversed in problems (2.99) and (2.100). Usually, the original problem (2.99) is called the primal problem and the related problem (2.100) is called the dual problem. The two problems make a primal–dual pair. Similarly, an element of the vector $\boldsymbol{x}$ is called a primal variable, and that of the vector $\boldsymbol{\pi}$ is called a dual variable.

The constraint of the dual problem (2.100), which is a product of the m dimensional row vector $\boldsymbol{\pi}$ and the $m \times n$ constraint matrix A, is alternatively expressed as

$$\left.\begin{array}{c} a_{11}\pi_1 + a_{21}\pi_2 + \cdots + a_{m1}\pi_m \leq c_1 \\ a_{12}\pi_1 + a_{22}\pi_2 + \cdots + a_{m2}\pi_m \leq c_2 \\ \cdots\cdots\cdots \\ a_{1n}\pi_1 + a_{2n}\pi_2 + \cdots + a_{mn}\pi_m \leq c_n, \end{array}\right\} \tag{2.101}$$

which implies that the coefficients of the system of inequalities (2.101) are given by the transposed matrix A^T of A.

Any primal problem can be changed into a linear programming problem in a different format by using the following devices: (i) replace an unconstrained variable with the difference of two nonnegative variables; (ii) replace an equality constraint

Table 2.14 Primal–dual relationships

Minimization problem		Maximization problem
Constraints		Variables
$\geq$	$\Leftrightarrow$	≥ 0
$\leq$	$\Leftrightarrow$	≤ 0
$=$	$\Leftrightarrow$	Unrestricted
Variables		Constraints
≥ 0	$\Leftrightarrow$	$\leq$
≤ 0	$\Leftrightarrow$	$\geq$
Unrestricted	$\Leftrightarrow$	$=$

with two opposing inequalities; and (iii) replace an inequality constraint with an equality by adding a slack or surplus variable.

For example, consider the following linear programming problem involving not only equality constraints but also inequality constraints and free variables:

$$\left.\begin{array}{ll}\text{minimize } z = & \boldsymbol{c}^1\boldsymbol{x}^1 + \boldsymbol{c}^2\boldsymbol{x}^2 + \boldsymbol{c}^3\boldsymbol{x}^3 \\ \text{subject to} & A_{11}\boldsymbol{x}^1 + A_{12}\boldsymbol{x}^2 + A_{13}\boldsymbol{x}^3 \geq \boldsymbol{b}^1 \\ & A_{21}\boldsymbol{x}^1 + A_{22}\boldsymbol{x}^2 + A_{23}\boldsymbol{x}^3 \leq \boldsymbol{b}^2 \\ & A_{31}\boldsymbol{x}^1 + A_{32}\boldsymbol{x}^2 + A_{33}\boldsymbol{x}^3 = \boldsymbol{b}^3 \\ & \boldsymbol{x}^1 \geq \boldsymbol{0},\ \boldsymbol{x}^2 \leq \boldsymbol{0}. \end{array}\right\} \tag{2.102}$$

By converting this problem to its standard form by introducing slack and surplus variables, and substituting $\boldsymbol{x}^2 = -\boldsymbol{x}^{2+}$ ($\boldsymbol{x}^{2+} \geq \boldsymbol{0}$) and $\boldsymbol{x}^3 = \boldsymbol{x}^{3+} - \boldsymbol{x}^{3-}$ ($\boldsymbol{x}^{3+} \geq \boldsymbol{0}$, $\boldsymbol{x}^{3-} \geq \boldsymbol{0}$), it can be easily understood that its dual becomes

$$\left.\begin{array}{ll}\text{maximize } v = & \boldsymbol{\pi}^1\boldsymbol{b}^1 + \boldsymbol{\pi}^2\boldsymbol{b}^2 + \boldsymbol{\pi}^3\boldsymbol{b}^3 \\ \text{subject to} & \boldsymbol{\pi}^1 A_{11} + \boldsymbol{\pi}^2 A_{21} + \boldsymbol{\pi}^3 A_{31} \leq \boldsymbol{c}^1 \\ & \boldsymbol{\pi}^1 A_{12} + \boldsymbol{\pi}^2 A_{22} + \boldsymbol{\pi}^3 A_{32} \geq \boldsymbol{c}^2 \\ & \boldsymbol{\pi}^1 A_{13} + \boldsymbol{\pi}^2 A_{23} + \boldsymbol{\pi}^3 A_{33} = \boldsymbol{c}^3 \\ & \boldsymbol{\pi}^1 \geq \boldsymbol{0},\ \boldsymbol{\pi}^2 \leq \boldsymbol{0}. \end{array}\right\} \tag{2.103}$$

Carefully comparing this dual problem (2.103) with the primal problem (2.102) gives the relationships between the primal and dual pair summarized in Table 2.14. For example, an unrestricted variable corresponds to an equality constraint.

By utilizing the relationships in Table 2.14, it is possible to write the dual problem for a given linear programming problem without going through the intermediate step of converting the problem to the standard form. From Table 2.14, the symmetric primal-dual pair given in Table 2.15 is immediately obtained. In a symmetric form, it is especially easy to see that the dual of the dual is the primal.

The relationship between the primal and dual problems is called duality. The following theorem, sometimes called the weak duality theorem, is easily proven and gives us an important relationship between the two problems. In the following, it is convenient to deal with a primal problem in the standard form.

Table 2.15 Symmetric primal–dual pair

Primal		Dual	
Minimize	$z = \boldsymbol{cx}$	Maximize	$v = \boldsymbol{\pi b}$
Subject to	$A\boldsymbol{x} \geq \boldsymbol{b}$	Subject to	$\boldsymbol{\pi} A \leq \boldsymbol{c}$
	$\boldsymbol{x} \geq \boldsymbol{0}$		$\boldsymbol{\pi} \geq \boldsymbol{0}$

Theorem 2.5 (Weak duality theorem). *If $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{\pi}}$ are feasible primal and dual solutions, then*

$$\bar{z} = \boldsymbol{c}\bar{\boldsymbol{x}} \geq \bar{\boldsymbol{\pi}}\boldsymbol{b} = \bar{v}. \tag{2.104}$$

Proof. From the dual feasibility of $\bar{\boldsymbol{\pi}}$ and the primal feasibility of $\bar{\boldsymbol{x}}$, we have

$$\boldsymbol{c} \geq \bar{\boldsymbol{\pi}} A, \text{ and } A\bar{\boldsymbol{x}} = \boldsymbol{b},\ \bar{\boldsymbol{x}} \geq \boldsymbol{0},$$

which implies

$$\boldsymbol{c}\bar{\boldsymbol{x}} \geq \bar{\boldsymbol{\pi}} A\bar{\boldsymbol{x}} = \bar{\boldsymbol{\pi}}\boldsymbol{b}.$$

□

This theorem shows that the primal (minimization) problem is always bounded below by the dual (maximization) problem and the dual (maximization) problem is always bounded above by the primal (minimization) problem if they are feasible.

From the weak duality theorem, several corollaries can be immediately obtained.

Corollary 2.1. *If $\bar{\boldsymbol{x}}^o$ and $\bar{\boldsymbol{\pi}}^o$ are feasible primal and dual solutions and $\boldsymbol{cx}^o = \boldsymbol{\pi}^o\boldsymbol{b}$ holds, then $\bar{\boldsymbol{x}}^o$ and $\bar{\boldsymbol{\pi}}^o$ are optimal solutions to their respective problems.*

This corollary implies that if a pair of feasible solutions can be found to the primal and dual problems with the same objective value, then they are both optimal.

Corollary 2.2. *If the primal problem is unbounded below, then the dual problem is infeasible.*

Corollary 2.3. *If the dual problem is unbounded above, then the primal problem is infeasible.*

With these results, the following duality theorem, sometimes called the strong duality theorem, can be established as a stronger result.

Theorem 2.6 (Strong duality theorem).

(i) If either the primal or the dual problem has a finite optimal solution, then so does the other, and the corresponding values of the objective functions are the same.
(ii) If one problem has an unbounded objective value, then the other problem has no feasible solution.

Proof. (i) It is sufficient, in proving the first statement, to assume that the primal has a finite optimal solution, and then we show that the dual has a solution with the same value of the objective function.

To show that the optimal values are the same, let $\boldsymbol{x}^o$ solve the primal. Since the primal must have a basic optimal solution, we may as well assume $\boldsymbol{x}^o$ as the basic, with the optimal basis matrix B^o, and the vector of basic variables $\boldsymbol{x}^o_{B^o}$. Thus

$$B^o \boldsymbol{x}^o_{B^o} = \boldsymbol{b}, \quad \boldsymbol{x}^o_{B^o} \geq \boldsymbol{0}.$$

The simplex multiplier vector associated with B^o is

$$\boldsymbol{\pi}^o = \boldsymbol{c}_{B^o}(B^o)^{-1},$$

where $\boldsymbol{c}_{B^o}$ is the vector of cost coefficients of basic variables. Since $\boldsymbol{x}^o$ is optimal, the relative cost coefficients $\bar{c}_j$ given by (2.86) are nonnegative:

$$\bar{c}_j = c_j - \boldsymbol{\pi}^o \boldsymbol{p}_j \geq 0, \quad j = 1, \ldots, n,$$

or, in matrix form,

$$\boldsymbol{\pi}^o A \leq \boldsymbol{c}.$$

Thus, $\boldsymbol{\pi}^o$ satisfies the dual constraints, and the corresponding objective value is

$$v^o = \boldsymbol{\pi}^o \boldsymbol{b} = \boldsymbol{c}_{B^o}(B^o)^{-1}\boldsymbol{b} = \boldsymbol{c}_{B^o}\boldsymbol{x}^o_{B^o} = z^o.$$

Hence, from Corollary 2.1, it directly follows that $\boldsymbol{\pi}^o$ is an optimal solution to the dual problem.

(ii) The second statement is an immediate consequence of Corollaries 2.2 and 2.3. □

The preceding proof illustrates some important points.

(i) The constraints of the dual problem exactly represent the optimality conditions of the primal problem, and the relative cost coefficients $\bar{c}_j$ can be interpreted as slack variables in them.
(ii) The simplex multiplier vector $\boldsymbol{\pi}^o$ associated with a primal optimal basis solves the corresponding dual problem. Since, as shown in the previous section, the vector $-\boldsymbol{\pi}$ is contained in the bottom row of the revised simplex tableau for the primal problem, the optimal revised simplex tableau inherently provides a dual optimal solution.

For interpreting the relationships between the primal and dual problems, recall the two-variable diet problem of Example 2.3. Associated with this problem, we examine the following problem, though it is somewhat intentional.

Example 2.13 (Dual problem for the diet problem of Example 2.3). A drug company wants to maximize the total profit by producing three pure tablets V_1, V_2, and V_3 which contain exactly one mg (milligram) of the nutrients N_1, N_2, and N_3, respectively. To do so, the company attempts to determine the prices of three tablets which compare favorably with those of the two foods F_1 and F_2. Let π_1, π_2, and π_3 denote the prices in yens of one tablet of V_1, V_2, and V_3, respectively.

One gram of the food F_1 provides 1, 1, and 2 mg of N_1, N_2, and N_3 and costs 4 yen. If the housewife replaces one gram of this food F_1 with tablets of V_1, V_2, and V_3, one tablet of V_1, one tablet of V_2, and two tablets of V_3 are needed. This would cost $\pi_1+\pi_2+2\pi_3$, which should be less than or equal to the price of one gram of the food F_1, i.e., $\pi_1 + \pi_2 + 2\pi_3 \leq 4$. Similarly, one gram of the food F_2 provides 1, 1, and 2 mg of N_1, N_2, and N_3 and costs 3 yen. Thus, the inequality $3\pi_1 + 2\pi_2 + \pi_3 \leq 3$ is imposed. Since the housewife understands that the daily requirements of the nutrients N_1, N_2, and N_3 are 12, 10, and 15 mg, respectively, the cost of meeting these requirements by using the tablets would be $v = 12\pi_1+10\pi_2+15\pi_3$. Thus, the company should determine the prices of the tablets V_1, V_2, and V_3 so as to maximize this function subject to the above two inequalities. That is, the company determines the prices of the three tablets which maximize the profit function

$$v = 12\pi_1 + 10\pi_2 + 15\pi_3$$

subject to the constraints

$$\begin{array}{l} \pi_1 + \pi_2 + 2\pi_3 \leq 4 \\ 3\pi_1 + 2\pi_2 + \pi_3 \leq 3 \\ \pi_1 \geq 0,\ \pi_2 \geq 0,\ \pi_3 \geq 0. \end{array}$$

It should be noted here that this linear programming problem is precisely the dual of the original diet problem of Example 2.3. ◇

As thus far discussed, the dual variables, corresponding to the constraints of the primal problem, coincide with the simplex multipliers for the optimal basic solution of the primal problem. Consider the economic interpretation of the simplex multiplier. Let

$$\boldsymbol{x}^o = (x_1^o, x_2^o, \ldots, x_n^o)^T \text{ and } \boldsymbol{\pi}^o = (\pi_1^o, \pi_2^o, \ldots, \pi_m^o)$$

be the optimal solutions of the primal and dual problems, respectively. From the strong duality theorem, it follows that

$$z^o = c_1x_1^o + c_2x_2^o + \cdots + c_nx_n^o = \pi_1^ob_1 + \pi_2^ob_2 + \cdots + \pi_m^ob_m = v^o.$$

In this relation, one finds that

$$z^o = \pi_1^ob_1 + \pi_2^ob_2 + \cdots + \pi_m^ob_m, \tag{2.105}$$

and then it can be intuitively understood that when one unit of the right-hand side constant b_i of the ith constraint $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$ of the primal problem is changed from b_i to $b_i + 1$, the value of the objective function will increase by π_i^o as long as the basis does not change.

To be more precise, from (2.105), the amount of change in the objective function z for a small change in b_i is obtained by partially differentiating z with respect to the right-hand side b_i, i.e.,

$$\pi_i^o = \frac{\partial z^o}{\partial b_i}, \quad i = 1, \ldots, m. \tag{2.106}$$

Thus, the simplex multiplier π_i indicates how much the value of the objective function varies for a small change in the right-hand side of the constraint, and therefore it is referred to as the shadow price or the marginal price.

Using the duality theorem, the following result, known as Farkas's theorem concerning systems of linear equalities and inequalities, can be easily proven.

Theorem 2.7 (Farkas's theorem). *One and only one of the following two alternatives holds.*

(i) There exists a solution $\boldsymbol{x} \geq \boldsymbol{0}$ *such that* $A\boldsymbol{x} = \boldsymbol{b}$.
(ii) There exists a solution $\boldsymbol{\pi}$ *such that* $\boldsymbol{\pi} A \leq \boldsymbol{0}^T$ *and* $\boldsymbol{\pi}\boldsymbol{b} > 0$.

Proof. Consider the (primal) linear problem

$$\left.\begin{array}{ll} \text{minimize} & z = \boldsymbol{0}^T\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}, \end{array}\right\} \tag{2.107}$$

and its dual

$$\left.\begin{array}{ll} \text{maximize} & v = \boldsymbol{\pi}\boldsymbol{b} \\ \text{subject to} & \boldsymbol{\pi} A \leq \boldsymbol{0}^T. \end{array}\right\} \tag{2.108}$$

If the statement (i) holds, the primal problem is feasible. Since the value of the objective function z is always zero, any feasible solution is optimal. From the strong duality theorem, the value of the objective function v of the dual is zero. Thus, the statement (ii) does not hold.

Conversely, if the statement (ii) holds, the dual problem has a feasible solution such that the objective function v is positive. From the weak duality theorem, this implies that the objective function z of the primal is positive, and therefore the primal problem has no feasible solution. □

Associated with Farkas's theorem, Gordon's theorem also plays an important role for deriving the optimality conditions of nonlinear programming.

Theorem 2.8 (Gordon's theorem). *One and only one of the following two alternatives holds.*

(i) There exists a solution $\boldsymbol{x} \geq \boldsymbol{0}$, $\boldsymbol{x} \neq \boldsymbol{0}$ *such that* $A\boldsymbol{x} = \boldsymbol{0}$.
(ii) There exists a solution $\boldsymbol{\pi}$ *such that* $\boldsymbol{\pi} A < \boldsymbol{0}^T$.

The following theorem, relating the primal and dual problems, is often useful.

Theorem 2.9 (Complementary slackness theorem). *Let* $\boldsymbol{x}$ *be a feasible solution to the primal problem (2.99) and* $\boldsymbol{\pi}$ *be a feasible solution to the dual problem (2.100). Then they are respectively optimal if and only if the complementary slackness condition*

$$(\boldsymbol{c} - \boldsymbol{\pi}^o A)\boldsymbol{x}^o = 0 \tag{2.109}$$

is satisfied.

2.8 Dual Simplex Method

There are a number of algorithms for linear programming which start with an infeasible solution to the primal and iteratively force a sequence of solutions to become feasible as well as optimal. The most prominent among such methods is the dual simplex method (Lemke 1954). Operationally, its procedure still involves a sequence of pivot operations, but with different rules for choosing the pivot element.

Consider a primal problem in the standard form

$$\left.\begin{array}{ll} \text{minimize } z = & \boldsymbol{cx} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}, \end{array}\right\}$$

and its dual

$$\left.\begin{array}{ll} \text{maximize } v = & \boldsymbol{\pi b} \\ \text{subject to} & \boldsymbol{\pi} A \leq \boldsymbol{c}. \end{array}\right\}$$

Consider the canonical form of the primal problem starting with the basis $(x_1, x_2, \ldots, x_m)$ expressed as

$$\left.\begin{array}{r} \begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_m \end{bmatrix} + \displaystyle\sum_{j=m+1}^{n} \bar{\boldsymbol{p}}_j x_j = \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_m \end{pmatrix} \\ -z + \displaystyle\sum_{j=m+1}^{n} \bar{c}_j x_j = -\bar{z}, \end{array}\right\} \tag{2.110}$$

where not all right-hand side constants $\bar{b}_i$ may be nonnegative, i.e., for some i, $\bar{b}_i \geq 0$ may not hold.

In this canonical form, if $\bar{c}_j = c_j - \boldsymbol{\pi} \boldsymbol{p}_j \geq 0$ for all $j = m+1, \ldots, n$, which can be alternatively expressed as $\boldsymbol{\pi} A \leq \boldsymbol{c}$ in a vector-matrix form, $\boldsymbol{\pi}$ is a feasible solution to the dual problem. Thus, the canonical form of the primal problem (2.110) satisfying $\bar{c}_j \geq 0$, $j = m+1, \ldots, n$ is called the dual feasible canonical one. Obviously, if the dual feasible canonical form is also feasible to the primal problem, i.e., for all i, $b_i \geq 0$ hold, then it is an optimal canonical form.

Now, in a quite similar way to the selection rule of $\bar{c}_s$ in the simplex method, find the pivot row by

$$\min_{\bar{b}_i < 0} \bar{b}_i = \bar{b}_r.$$

It should be noted that if $\bar{b}_r \geq 0$ for all r, it follows that an optimal solution is obtained.

If $\bar{a}_{rj} \geq 0$ for all j, from $\bar{b}_r < 0$, in the rth equation

$$x_r = \bar{b}_r - \sum_{j=m+1}^{n} \bar{a}_{rj} x_j,$$

the right-hand side is negative for $x_j \geq 0$, $j = m+1, \ldots, n$, which implies that the value of the basic variable x_r is negative, i.e., $x_r < 0$ for all the nonnegative nonbasic variables x_j. This means that the primal problem is infeasible, and then the following theorem is obtained.

Theorem 2.10 (Infeasibility of primal problem). *In the rth row of the canonical form (2.110), if*

$$\bar{b}_r < 0, \quad \bar{a}_{rj} \geq 0, \ j = m+1, m+2, \ldots, n, \tag{2.111}$$

then the primal problem is infeasible.

Now, in the dual feasible canonical form (2.110), let the rth row be the pivot one. Moreover, assume that $\bar{b}_r$ is negative and for some j at least one $\bar{a}_{rj}$ is negative. Then, if the pivot column is found by

$$\min_{\bar{a}_{rj} < 0} \frac{\bar{c}_j}{-\bar{a}_{rj}} = \frac{\bar{c}_s}{-\bar{a}_{rs}} = \Delta, \tag{2.112}$$

$\bar{a}_{rs}$ is chosen as the pivot element.

By performing the pivot operation on $\bar{a}_{rs}$ which means that x_r is replaced with x_s as a new basic variable, as shown in Table 2.3, the resulting new relative cost coefficients $\bar{c}_j^*$ for nonbasic variables, which are discriminated by adding the superscript $*$, are represented as

$$\bar{c}_j^* = \bar{c}_j - \bar{c}_s \bar{a}_{rj}^* = \bar{c}_j - \bar{c}_s \frac{\bar{a}_{rj}}{\bar{a}_{rs}}.$$

Obviously, for any column index j of a nonbasic variable such that $\bar{a}_{rj} \geq 0$ holds, from $\bar{c}_s > 0$ and $\bar{a}_{rs} < 0$, it directly follows that its relative cost coefficient is nonnegative, i.e.,

$$\bar{c}_j^* = \bar{c}_j - \bar{c}_s \frac{\bar{a}_{rj}}{\bar{a}_{rs}} \geq \bar{c}_j \geq 0.$$

For any column index j of a nonbasic variable such that $\bar{a}_{rj} < 0$ holds, from (2.112), it follows that its relative cost coefficient is also nonnegative, i.e.,

$$\bar{c}_j^* = \bar{a}_{rj} \left(\frac{\bar{c}_s}{-\bar{a}_{rs}} - \frac{\bar{c}_j}{-\bar{a}_{rj}} \right) \geq 0.$$

Hence, it holds that $\bar{c}_j^* \geq 0$ for all j, the resulting new canonical form (tableau) is a dual feasible canonical form.

Moreover, by the pivot operation on $\bar{a}_{rs}$, we also have the updated value of the objective function

$$\bar{z}^* = \bar{z} + \bar{c}_s \frac{\bar{b}_r}{\bar{a}_{rs}} = \bar{z} - \bar{b}_r \Delta,$$

and from $\bar{b}_r < 0$ and $\Delta \geq 0$, the value is increased by $|\bar{b}_r \Delta|$ compared to the previous value of $\bar{z}$.[6]

After starting with the dual feasible canonical form, the dual simplex method improves feasible solutions of the dual problems through a series of pivot operations in order to seek for an optimal solution. Although the dual simplex method uses the pivot operations in a similar way to the simplex method, it employs a different rule for choosing the pivot element and the value of the objective function increases with the number of iterations. The procedure of the dual simplex method, starting with the dual feasible canonical form, can be summarized as follows.

Procedure of the Dual Simplex Method

Start with the dual feasible canonical form. That is, assume that $\bar{c}_j \geq 0$ for all j.

Step 1 If $\bar{b}_i \geq 0$ for all indices i of the basic variables, then the current solution is optimal, and stop. Otherwise, choose the index r for the pivot row such that

[6]If $\bar{c}_s = 0$ and dual degeneracy occurs, it is possible to avoid cycling by utilizing the similar anticycling rule in the simplex method.

$$\min_{\bar{b}_i < 0} \bar{b}_i = \bar{b}_r.$$

Step 2 If $\bar{a}_{rj} \geq 0$ for all indices j of the nonbasic variables, then the primal problem is infeasible, and stop.

Step 3 If some of $\bar{a}_{rj}$ are negative, find the index s for the pivot column such that

$$\min_{\bar{a}_{rj} < 0} \frac{\bar{c}_j}{-\bar{a}_{rj}} = \frac{\bar{c}_s}{-\bar{a}_{rs}} = \Delta.$$

Step 4 Perform the pivot operation on $\bar{a}_{rs}$ for obtaining a new dual feasible canonical form with x_s replacing x_r as a basic variable. Return to step 1.

Example 2.14 (Dual simplex method for the diet problem of Example 2.3). Using the dual simplex method, solve the diet problem in the standard form given in Example 2.3:

$$\begin{array}{llrcrcrcrcrcl}
\text{minimize} & z = & 4x_1 & + & 3x_2 & & & & & & & & \\
\text{subject to} & & x_1 & + & 3x_2 & - & x_3 & & & & & = & 12 \\
& & x_1 & + & 2x_2 & & & - & x_4 & & & = & 10 \\
& & 2x_1 & + & x_2 & & & & & - & x_5 & = & 15 \\
& & \multicolumn{11}{l}{x_j \geq 0, \quad j = 1,2,3,4,5.}
\end{array}$$

Multiplying both sides of the three equations of the constraints by -1 yields the dual feasible canonical form

$$\begin{array}{rcrcrcrcrcrcl}
-x_1 & - & 3x_2 & + & x_3 & & & & & & & = & -12 \\
-x_1 & - & 2x_2 & & & + & x_4 & & & & & = & -10 \\
-2x_1 & - & x_2 & & & & & + & x_5 & & & = & -15\,. \\
4x_1 & + & 3x_2 & & & & & & & - & z & = & 0 \\
\multicolumn{13}{l}{x_j \geq 0, \quad j = 1,2,3,4,5}
\end{array}$$

Since $\bar{c}_1 = 4 > 0$ and $\bar{c}_2 = 3 > 0$, this canonical form with basic variables x_3, x_4, and x_5 is dual feasible. However, it is not primal feasible because $\bar{b}_1 = -12 < 0$, $\bar{b}_2 = -10 < 0$ and $\bar{b}_3 = -9 < 0$.

At cycle 0 in Table 2.16, from

$$\min(\bar{b}_3, \bar{b}_4, \bar{b}_5) = \min(-12, -10, -15) = -15 < 0,$$

x_5 becomes a nonbasic variable in the next cycle. From

$$\min\left(\frac{\bar{c}_1}{-\bar{a}_{51}}, \frac{\bar{c}_2}{-\bar{a}_{52}}\right) = \min\left(\frac{4}{2}, \frac{3}{1}\right) = \frac{4}{2},$$

Table 2.16 Simplex tableau of Example 2.3 (dual simplex method)

Cycle	Basis	x_1	x_2	x_3	x_4	x_5	Constants
0	x_3	-1	-3	1			-12
	x_4	-1	-2		1		-10
	x_5	[-2]	-1			1	-15
	$-z$	4	3				0
1	x_3		[-2.5]	1		-0.5	-4.5
	x_4		-1.5		1	-0.5	-2.5
	x_1	1	0.5			-0.5	7.5
	$-z$		1			2	-30
2	x_2		1	-0.4		0.2	1.8
	x_4			-0.6	1	-0.2	0.2
	x_1	1		0.2		-0.6	6.6
	$-z$			0.4		1.8	-31.8

x_1 becomes a basic variable in the next cycle, and the pivot element is determined at $\bar{a}_{51} = -2$ bracketed by [ ] in Table 2.16. After performing the pivot operation on $\bar{a}_{51} = -2$, the tableau at cycle 1 is obtained. At cycle 1, from $\bar{b}_1 > 0$ and

$$\min(\bar{b}_3, \bar{b}_4) = \min(-4.5, -2.5) = -4.5 < 0$$

x_3 becomes a nonbasic variable in the next cycle. From

$$\min\left(\frac{\bar{c}_2}{-\bar{a}_{32}}, \frac{\bar{c}_5}{-\bar{a}_{35}}\right) = \min\left(\frac{1}{2.5}, \frac{2}{0.5}\right) = \frac{1}{2.5}$$

x_2 becomes a basic variable in the next cycle, and the pivot element is determined at $\bar{a}_{32} = -2.5$ bracketed by [ ]. After performing the pivot operation on $\bar{a}_{32} = -2.5$, the tableau at cycle 2 is obtained. At cycle 2, all of the constants $\bar{b}_i$ become positive, and an optimal solution

$$x_1 = 6.6,\ x_2 = 1.8\ (x_3 = 0, x_4 = 0.2, x_5 = 0), \quad z = 31.8$$

is obtained. Observe that the tableau of cycle 2 in Table 2.16 coincides with that of cycle 3 in Table 2.7 when the row of $-w$ is dropped. ◇

It should be noted here that the idea of the revised simplex method can be employed in the discussion of the dual simplex method. In the dual simplex method, in addition to the data of the initial feasible canonical form A, $\boldsymbol{b}$, and $\boldsymbol{c}$, the coefficients $\bar{a}_{rj}$ for all indices j of the nonbasic variables with respect to x_r left from the basis and the relative cost coefficients $\bar{c}_j$ for all indices j of the nonbasic variables are required, where $\bar{c}_j$ can be computed by the formula $\bar{c}_j = c_j - \boldsymbol{\pi}\boldsymbol{p}_j$ of the revised simplex method. Hence, if the formula for calculating $\bar{a}_{rj}$ for all indices j of the nonbasic variables through the basis inverse matrix B^{-1} is given, the dual simplex method can be expressed in a style followed by the revised simplex method. Since the coefficient $\bar{a}_{rj}$ is the rth element of $\bar{\boldsymbol{p}}_j$, by using the rth row vector of B^{-1}, denoted by $[B^{-1}]_{r\cdot}$, it can be calculated just as

$$\bar{a}_{rj} = [B^{-1}]_{r\cdot}\,\boldsymbol{p}_j, \quad j : \text{nonbasic.} \tag{2.113}$$

With the above discussion, the procedure of the revised dual simplex method can be summarized as follows.

Procedure of the Revised Dual Simplex Method

Assume that the coefficients A, $\boldsymbol{b}$, and $\boldsymbol{c}$ of the initial dual feasible canonical form and the inverse matrix B^{-1} of the initial dual feasible basis are available.

Step 0 Using B^{-1}, calculate

$$\boldsymbol{\pi} = \boldsymbol{c}_B B^{-1},\ \boldsymbol{x}_B = \bar{\boldsymbol{b}} = B^{-1}\boldsymbol{b},\ \bar{z} = \boldsymbol{\pi}\boldsymbol{b}$$

and put them in the revised simplex tableau shown in Table 2.12.

Step 1 If $\bar{b}_i \geq 0$ for all indices i of the basic variables, then the current solution is optimal, and stop. Otherwise, choose the index r for the pivot row such that

$$\min_{\bar{b}_i < 0} \bar{b}_i = \bar{b}_r.$$

Step 2 For all indices j of the nonbasic variables, calculate

$$\bar{a}_{rj} = [B^{-1}]_{r\cdot}\,\boldsymbol{p}_j.$$

If $\bar{a}_{rj} \geq 0$ for all indices j of the nonbasic variables, then the primal problem is infeasible, and stop.

Step 3 If some of $\bar{a}_{rj}$ are negative, calculate

$$\bar{c}_j = c_j - \boldsymbol{\pi}\boldsymbol{p}_j$$

and find the index s for the pivot column such that

$$\min_{\bar{a}_{rj} < 0} \frac{\bar{c}_j}{-\bar{a}_{rj}} = \frac{\bar{c}_s}{-\bar{a}_{rs}} = \Delta.$$

In Table 2.12, replace x_r with x_s as a basic variable.

Step 4 Calculate

$$\bar{\boldsymbol{p}}_s = B^{-1}\boldsymbol{p}_s$$

Table 2.17 Revised dual simplex tableau of Example 2.3

Cycle	Basis	Basis inverse matrix			Constants	$\hat{\bar{\boldsymbol{p}}}_s$
0	x_3	1			-12	-1
	x_4		1		-10	-1
	x_5			1	-15	$[-2]$
	$-z$					4
1	x_3	1		$-1/2$	$-9/2$	$[-5/2]$
	x_4		1	$-1/2$	$-5/2$	$-3/2$
	x_1			$-1/2$	$15/2$	$1/2$
	$-z$			2	-30	1
2	x_2	$-2/5$		$1/5$	$9/5$	
	x_4	$-3/5$	1	$-1/5$	$1/5$	
	x_1	$1/5$		$-3/5$	$33/5$	
	$-z$	$2/5$		$9/5$	$-159/5$	

and put the values of $\hat{\bar{\boldsymbol{p}}}_s = (\bar{\boldsymbol{p}}_s, \bar{c}_s)^T$ in the column $\hat{\bar{\boldsymbol{p}}}_s$ of Table 2.12. Perform the pivot operation on $\bar{a}_{rs}$ to B^{-1}, $-\boldsymbol{\pi}$, $\bar{\boldsymbol{b}}$, $-\bar{z}$ of Table 2.12, and return to step 1.

Example 2.15 (Revised dual simplex method for the diet problem of Example 2.3). The canonical form

$$
\begin{array}{rcl}
-x_1 - 3x_2 + x_3 & & = -12 \\
-x_1 - 2x_2 \qquad + x_4 & & = -10 \\
-2x_1 - \ x_2 \qquad\qquad + x_5 & & = -15 \\
4x_1 + 3x_2 \qquad\qquad\qquad & - z = & \ \ 0 \\
x_j \geq 0, \quad j = 1,2,3,4,5, & &
\end{array}
$$

for the diet problem discussed in Examples 2.3 and 2.14, where x_3, x_4, and x_5 are basic variables, is dual feasible because $\bar{c}_1 = 4 > 0$ and $\bar{c}_2 = 3 > 0$. However, since $\bar{b}_1 = -12 < 0$, $\bar{b}_2 = -10 < 0$, and $\bar{b}_3 = -9 < 0$, the primal problem is not feasible. The initial basis matrix B is the 3×3 unit matrix and its inverse B^{-1} is also the same unit matrix. Hence, from (2.78) and (2.84), it follows that

$$
\boldsymbol{\pi} = \boldsymbol{c}_B B^{-1} = (0,0,0), \quad \bar{\boldsymbol{b}} = B^{-1}\boldsymbol{b} = \boldsymbol{b} = (-12,-10,-15)^T, \quad \bar{z} = \boldsymbol{\pi}\boldsymbol{b} = 0.
$$

Putting these values in the revised dual simplex tableau at cycle 0 of Table 2.17.

At cycle 0 in Table 2.17, since

$$
\min(\bar{b}_3, \bar{b}_4, \bar{b}_5) = \min(-12,-10,-15) = -15 < 0,
$$

x_5 becomes a nonbasic variable in the next cycle and the index r of the variable leaving the basis is determined as $r = 3$.

According to (2.113), we calculate the coefficients $\bar{a}_{rj}$, $r = 3$, $j = 1,2$ for the nonbasic variables. That is, using the third row $[B^{-1}]_{3\cdot}$ of B^{-1}, $\boldsymbol{p}_1$, and $\boldsymbol{p}_2$, we have

$$\bar{a}_{31} = [B^{-1}]_{3\cdot}\,\boldsymbol{p}_1 = (0, 0, 1)(-1, -1, -2)^T = -2$$
$$\bar{a}_{32} = [B^{-1}]_{3\cdot}\,\boldsymbol{p}_2 = (0, 0, 1)(-3, -2, -1)^T = -1.$$

Also the relative cost coefficients $\bar{c}_j$, $j = 1, 2$ are calculated as

$$\bar{c}_1 = c_1 - \boldsymbol{\pi}\boldsymbol{p}_1 = 4 + (0, 0, 0)(-1, -1, -2)^T = 4$$
$$\bar{c}_2 = c_2 - \boldsymbol{\pi}\boldsymbol{p}_2 = 3 + (0, 0, 0)(-3, -2, -1)^T = 3.$$

From

$$\min_{\bar{a}_{rj}<0} \frac{\bar{c}_j}{-\bar{a}_{rj}} = \min\left(\frac{\bar{c}_1}{-\bar{a}_{31}}, \frac{\bar{c}_2}{-\bar{a}_{32}}\right) = \min\left(\frac{4}{2}, \frac{3}{1}\right) = \frac{4}{2},$$

x_1 becomes a basic variable in the next cycle. We calculate $\bar{\boldsymbol{p}}_1$ as

$$\bar{\boldsymbol{p}}_1 = B^{-1}\boldsymbol{p}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix}$$

and put $\bar{\boldsymbol{p}}_1$ and $\bar{c}_1 = 4$ in the column of $\hat{\bar{\boldsymbol{p}}}_s$ at cycle 0 in Table 2.17. Since $r = 3$, the pivot element is -2 bracketed by [ ]. By performing the pivot operation on -2 at cycle 0, the tableau at cycle 1 is obtained.

At cycle 1, the variables x_3, x_4, and x_1 are basic variables, and in Table 2.17, since $\bar{b}_1 > 0$ and

$$\min(\bar{b}_3, \bar{b}_4) = \min(-9/2, -5/2) = -9/2 < 0,$$

x_3 becomes a nonbasic variable in the next cycle, and the index r of the variable leaving the basis is determined as $r = 1$.

From (2.113), we calculate the coefficients $\bar{a}_{rj}$, $r = 1$, $j = 2, 5$ for nonbasic variables. Using the first row $[B^{-1}]_{1\cdot}$ of B^{-1}, $\boldsymbol{p}_2$, and $\boldsymbol{p}_5$, we have

$$\bar{a}_{12} = [B^{-1}]_{1\cdot}\,\boldsymbol{p}_2 = (1, 0, -1/2)(-3, -2, -1)^T = -5/2$$
$$\bar{a}_{15} = [B^{-1}]_{1\cdot}\,\boldsymbol{p}_5 = (1, 0, -1/2)(0, 0, 1)^T = -1/2.$$

Also the relative cost coefficients $\bar{c}_j$, $j = 2, 5$ are calculated as

$$\bar{c}_2 = c_2 - \boldsymbol{\pi}\boldsymbol{p}_2 = 3 + (0, 0, 2)(-3, -2, -1)^T = 1$$
$$\bar{c}_5 = c_5 - \boldsymbol{\pi}\boldsymbol{p}_5 = 0 + (0, 0, 2)(0, 0, 1)^T = 2.$$

From

$$\min_{\bar{a}_{rj}<0} \frac{\bar{c}_j}{-\bar{a}_{rj}} = \min\left(\frac{\bar{c}_2}{-\bar{a}_{12}}, \frac{\bar{c}_5}{-\bar{a}_{15}}\right) = \min\left(\frac{1}{5/2}, \frac{2}{1/2}\right) = \frac{1}{5/2},$$

x_2 becomes a basic variable in the next cycle. We calculate $\bar{\boldsymbol{p}}_2$ as

$$\bar{\boldsymbol{p}}_2 = B^{-1}\boldsymbol{p}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} -5/2 \\ -3/2 \\ 1/2 \end{pmatrix}$$

and put $\bar{\boldsymbol{p}}_2$ and $\bar{c}_2 = 1$ in the column of $\hat{\bar{\boldsymbol{p}}}_s$ at cycle 1 in Table 2.17. Since $r = 1$, the pivot element is $-5/2$ bracketed by [ ]. By performing the pivot operation on $-5/2$ at cycle 1, the tableau at cycle 2 is obtained.

At cycle 2, the variables x_2, x_4 and x_1 are basic variables. Since all of the constants $\bar{b}_i$ are positive, an optimal solution

$$x_1 = \frac{33}{5},\ x_2 = \frac{9}{5}\ \left(x_3 = 0, x_4 = \frac{1}{5}, x_5 = 0\right), \quad z = \frac{159}{5}$$

is obtained. $\diamond$

Finally, consider the sensitivity analysis, which examines the effects of small changes in the parameters of a linear programming problem on its optimal solution. In particular, we deal with a case where the right-hand side vector is changed, which is closely related to the dual simplex method.

Assume that in the standard form of linear programming

$$\left.\begin{array}{ll} \text{minimize } z = & \boldsymbol{c}\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}, \end{array}\right\} \tag{2.114}$$

an optimal basis B is known, and then the corresponding optimal basic solution $\boldsymbol{x}_B$ is

$$\boldsymbol{x}_B = \bar{\boldsymbol{b}} = B^{-1}\boldsymbol{b}. \tag{2.115}$$

Moreover, the corresponding simplex multiplier vector $\boldsymbol{\pi}$ is

$$\boldsymbol{\pi} = \boldsymbol{c}_B B^{-1}, \tag{2.116}$$

and the value of the objective function $\bar{z}$ is also calculated as

$$\bar{z} = \boldsymbol{c}_B\boldsymbol{x}_B = \boldsymbol{c}_B\bar{\boldsymbol{b}} = \boldsymbol{\pi}\boldsymbol{b}. \tag{2.117}$$

Obviously, the optimality criterion

$$\bar{c}_j = c_j - \boldsymbol{\pi} \boldsymbol{p}_j \geq 0 \quad \text{for all } j \text{ of the nonbasic variables} \tag{2.118}$$

is satisfied.

In discussing changes in the right-hand side vector, assume that $\boldsymbol{b}$ is changed to $\boldsymbol{b} + \Delta\boldsymbol{b}$. Consider the following linear programming problem:

$$\left.\begin{array}{rr} \text{minimize } z = & \boldsymbol{cx} \\ \text{subject to} \quad & A\boldsymbol{x} = \boldsymbol{b} + \Delta\boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{array}\right\} \tag{2.119}$$

Since the simplex multiplier vector $\boldsymbol{\pi}$ and the relative cost coefficients $\bar{c}_j$ for all indices j of the nonbasic variables do not depend on $\boldsymbol{b}$ as shown in (2.116) and (2.118), they remain the same even if $\boldsymbol{b}$ is changed to $\boldsymbol{b} + \Delta\boldsymbol{b}$. However, the basic solution $\boldsymbol{x}_B$ itself may no longer be feasible.

The new basic solution and the value of the objective function are calculated as

$$\boldsymbol{x}_B^* = B^{-1}(\boldsymbol{b} + \Delta\boldsymbol{b}) = \boldsymbol{x}_B + B^{-1}\Delta\boldsymbol{b} \tag{2.120}$$

and

$$\bar{z}^* = \boldsymbol{\pi}(\boldsymbol{b} + \Delta\boldsymbol{b}) = \bar{z} + \boldsymbol{\pi}\Delta\boldsymbol{b}, \tag{2.121}$$

respectively.

Therefore, the following statements hold:

(i) If $\boldsymbol{x}_B^* \geq \boldsymbol{0}$ holds, then $\boldsymbol{x}_B^*$ is an optimal solution, and the variation in the objective function is $\boldsymbol{\pi}\Delta\boldsymbol{b}$.
(ii) If $\boldsymbol{x}_B^* \geq \boldsymbol{0}$ does not hold, since the optimality condition $\bar{c}_j \geq 0$ for all indices j of the nonbasic variables is satisfied, the dual simplex method can be used to find a new optimal solution.

Example 2.16 (Sensitivity analysis for the production planning problem of Example 1.1). In the production planning problem of Example 1.1, we calculate optimal solutions when the total amounts of available materials are changed as follows:

(i) The available amounts of material M_1 is changed from 27 tons to 32 tons.
(ii) The available amounts of material M_2 is changed from 16 tons to 23 tons.

Although the optimal solution to the original problem is given at cycle 2 in the revised simplex method of Table 2.13, for the sake of convenience, we rewrite the initial tableau (cycle 0) and the optimal tableau (cycle 2) in Table 2.18.

From the optimal tableau, one finds that the basic variables are $\boldsymbol{x}_B = (x_2, x_1, x_5)^T$, the basis inverse matrix is

Table 2.18 Initial and optimal tableaux of Example 1.1

Cycle	Basis	Basis inverse matrix			Constants	$\hat{\bar{p}}_s$
Cycle 0 (initial)	x_3	1			27	[6]
	x_4		1		16	2
	x_5			1	18	1
	$-z$				0	-8
Cycle 2 (optimal)	x_2	3/14	$-1/7$		3.5	
	x_1	$-1/7$	3/7		3	
	x_5	5/14	$-11/7$	1	2.5	
	$-z$	9/7	1/7		37	

$$B^{-1} = \begin{bmatrix} 3/14 & -1/7 & 0 \\ -1/7 & 3/7 & 0 \\ 5/14 & -11/7 & 1 \end{bmatrix},$$

and the simplex multiplier vector is

$$\boldsymbol{\pi} = (-9/7, -1/7, 0).$$

(i) Let the amounts of changes be $\Delta \boldsymbol{b} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}$, and from $\boldsymbol{b} = \begin{pmatrix} 27 \\ 16 \\ 18 \end{pmatrix}$, it follows that

$$\begin{aligned} \boldsymbol{x}_B^* &= B^{-1}(\boldsymbol{b} + \Delta \boldsymbol{b}) \\ &= \begin{bmatrix} 3/14 & -1/7 & 0 \\ -1/7 & 3/7 & 0 \\ 5/14 & -11/7 & 1 \end{bmatrix} \begin{pmatrix} 32 \\ 16 \\ 18 \end{pmatrix} = \begin{pmatrix} 32/7 \\ 16/7 \\ 30/7 \end{pmatrix} \end{aligned}$$

$$\bar{z}^* = \boldsymbol{\pi}(\boldsymbol{b} + \Delta \boldsymbol{b}) = (-9/7, -1/7, 0) \begin{pmatrix} 32 \\ 16 \\ 18 \end{pmatrix} = -304/7.$$

Since $\boldsymbol{x}_B^* \geq \boldsymbol{0}$ holds, $\boldsymbol{x}_B^*$ is an optimal basic solution, and then an optimal solution

$$x_2 = 32/7,\ x_1 = 16/7,\ x_5 = 30/7,\ (x_3 = x_4 = 0) \quad z = -304/7$$

is obtained.

Table 2.19 Revised simplex tableau after change of $b_2 = 23$

Cycle	Basis	Basis inverse matrix			Constants	$\hat{\bar{\boldsymbol{p}}}_s$
1	x_2	3/14	$-1/7$		5/2	$-1/7$
	x_1	$-1/7$	3/7		6	3/7
	x_5	5/14	$-11/7$	1	$-17/2$	$[-11/7]$
	$-z$	9/7	1/7		38	1/7
2	x_2	2/11		1/11	36/11	
	x_1	$-1/22$		3/11	81/22	
	x_4	$-5/22$	1	$-7/11$	119/22	
	$-z$	29/22		1/11	819/22	

(ii) Let the amounts of changes be $\Delta\boldsymbol{b} = \begin{pmatrix} 0 \\ 7 \\ 0 \end{pmatrix}$, and from $\boldsymbol{b} = \begin{pmatrix} 27 \\ 16 \\ 18 \end{pmatrix}$, it follows that

$$\boldsymbol{x}_B^* = B^{-1}(\boldsymbol{b} + \Delta\boldsymbol{b}) = B^{-1}\begin{pmatrix} 27 \\ 23 \\ 18 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 6 \\ -17/2 \end{pmatrix}$$

$$\bar{z}^* = \boldsymbol{\pi}(\boldsymbol{b} + \Delta\boldsymbol{b}) = (-9/7, -1/7, 0)\begin{pmatrix} 27 \\ 23 \\ 18 \end{pmatrix} = -38.$$

Since the negative component $-17/2$ appears in $\boldsymbol{x}_B^*$, using the revised dual simplex method, we can obtain an optimal tableau shown in Table 2.19.

That is, using the third row $[B^{-1}]_{3\cdot}$ of B^{-1}, $\boldsymbol{p}_3$, and $\boldsymbol{p}_4$, we have

$$\bar{a}_{33} = [B^{-1}]_{3\cdot}\,\boldsymbol{p}_3 = (5/14, -11/7, 1)(1, 0, 0)^T = 5/14,$$

$$\bar{a}_{34} = [B^{-1}]_{3\cdot}\,\boldsymbol{p}_4 = (5/14, -11/7, 1)(0, 1, 0)^T = -11/7.$$

Thus, x_4 becomes a basic variable in the next cycle. The relative cost coefficients $\bar{c}_4$ is calculated as

$$\bar{c}_4 = c_4 - \boldsymbol{\pi}\boldsymbol{p}_4 = 0 - (-9/7, -1/7, 0)(0, 1, 0)^T = 1/7.$$

We calculate $\bar{\boldsymbol{p}}_4$ as

$$\begin{bmatrix} 3/14 & -1/7 & 0 \\ -1/7 & 3/7 & 0 \\ 5/14 & -11/7 & 1 \end{bmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/7 \\ 3/7 \\ -11/7 \end{pmatrix}.$$

These values are put in the column of $\hat{\bar{\boldsymbol{p}}}_s$ in the tableau. By performing the pivot operation on $[-11/7]$, a new tableau is obtained.

In this example, after the only one pivot operation, an optimal solution

$$x_2 = 36/11,\ x_1 = 81/22 \quad (x_4 = 119/22,\ x_3 = x_5 = 0), \quad z = -819/22$$

is obtained. ◊

When the coefficients of the objective function are changed, since only the changes in the cost coefficients $\boldsymbol{c}$ affect the optimality criterion and the value of the objective function, the (revised) simplex method is used for finding the new optimal solution only when some relative cost coefficients become negative, i.e., $\bar{c}_j < 0$ for some j.

Problems

2.1 Convert the following problems to the standard form of linear programming:

(i) (Absolute value problem)

$$\begin{aligned} \text{minimize} \quad & z = \sum_{j=1}^{n} c_j |x_j| \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, 2, \ldots, m, \end{aligned}$$

where $c_j > 0$, $j = 1, 2, \ldots, n$, and x_j, $j = 1, 2, \ldots, n$ are free variables.

(ii) (Fractional programming problem)

$$\begin{aligned} \text{minimize} \quad & z = \frac{\displaystyle\sum_{j=1}^{n} c_j x_j + c_0}{\displaystyle\sum_{j=1}^{n} d_j x_j + d_0} \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, 2, \ldots, m \\ & x_j \geq 0, \quad j = 1, 2, \ldots, n, \end{aligned}$$

where $\sum_{j=1}^{n} d_j x_j + d_0 > 0$ holds for all feasible solutions.

(iii) (Minimax problem)

$$\begin{array}{ll} \text{minimize} & z = \max\left(\sum_{j=1}^{n} c_j^1 x_j, \sum_{j=1}^{n} c_j^2 x_j, \ldots, \sum_{j=1}^{n} c_j^L x_j\right) \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, 2, \ldots, m \\ & x_j \geq 0, \quad j = 1, 2, \ldots, n. \end{array}$$

2.2 Formulate the following problems as linear programming ones.

(i) A manufacturing company produces two products A and B. There are 40 h of labor available each day, and 1 kg (kilogram) of product A requires 2 h of labor, whereas 1 kg of product B requires 5 h. There are up to 30 machine-hours available per day, and machine processing time for 1 kg of product A is 3 h and for 1 kg of product B is 1 h. There are 39 kg of raw material available each day, and 1 kg of product A requires 3 kg of the material, whereas 1 kg of product B requires 4 kg. The daily profit for product A is 30 thousand yen per 1 kg, while B is 20 thousand yen per 1 kg, and the manager wishes to maximize the daily profit.

(ii) A firm manufactures cattle feed by mixing two ingredients A and B. Each ingredient contains three nutrients C, D, and E. Each 1 g (gram) of the ingredient A contains 9 mg (milligram) of C, 1 mg of D, and 1 mg of E. Each 1 g of the ingredient B contains 2 mg of C, 5 mg of D, and 1 mg of E. Each 1 g of the feed must contain at least 54 g, 25 g, and 13 g of C, D, and E, respectively. The costs per gram of the ingredients A and B are 9 yen and 15 yen, respectively, and the manager wishes to find the best feed mix that has the minimum cost per gram.

2.3 Assume that all $\boldsymbol{x}^l = (x_1^l, x_2^l, \ldots, x_n^l)^T$, $l = 1, 2, \ldots, L$ are optimal solutions to a certain linear programming problem. Show that $\boldsymbol{x}^* = \sum_{l=1}^{L} \lambda_l \boldsymbol{x}^l$ is also an optimal solution to the problem, where λ_l, $l = 1, \ldots, L$ are nonnegative constants satisfying $\sum_{l=1}^{L} \lambda_l = 1$.

2.4 For a linear programming problem involving a free variable x_k, assume that we substitute the difference of two nonnegative variables $x_k^+ - x_k^-$, $x_k^+ \geq 0$, $x_k^- \geq 0$ for x_k. Explain why both x_k^+ and x_k^- cannot be in the same basis simultaneously.

2.5 Consider the two linear programming problems

$$\begin{array}{lll} \text{minimize } z = \boldsymbol{c}\boldsymbol{x} & & \text{minimize } z = (\mu\boldsymbol{c})\boldsymbol{x} \\ \text{subject to} \quad A\boldsymbol{x} = \boldsymbol{b} & \text{and} & \text{subject to} \quad A\boldsymbol{x} = (\lambda\boldsymbol{b}) \\ \qquad\qquad\quad \boldsymbol{x} \geq \boldsymbol{0}, & & \qquad\qquad\quad \boldsymbol{x} \geq \boldsymbol{0}, \end{array}$$

where λ and μ are positive real numbers. Explain the relationships between these two problems. What happens if either λ or μ is negative?

2.6 Solve the following problems using the simplex method:

(i) Minimize $-2x_1 - 5x_2$

$$\begin{aligned} \text{subject to } \quad & 2x_1 + 6x_2 \le 27 \\ & 8x_1 + 6x_2 \le 45 \\ & 3x_1 + x_2 \le 15 \\ & x_j \ge 0,\ j = 1,2 \end{aligned}$$

(ii) Minimize $-3x_1 - 2x_2$

$$\begin{aligned} \text{subject to } \quad & 2x_1 + 5x_2 \le 130 \\ & 6x_1 + 3x_2 \le 110 \\ & x_j \ge 0,\ j = 1,2 \end{aligned}$$

(iii) Minimize $-3x_1 - 4x_2$

$$\begin{aligned} \text{subject to } \quad & 3x_1 + 12x_2 \le 400 \\ & 6x_1 + 3x_2 \le 600 \\ & 8x_1 + 7x_2 \le 800 \\ & x_j \ge 0,\ j = 1,2 \end{aligned}$$

(iv) Minimize $-2.5x_1 - 5x_2 - 3.4x_3$

$$\begin{aligned} \text{subject to } \quad & -5x_1 + 10x_2 + 6x_3 \le 425 \\ & 2x_1 - 5x_2 + 4x_3 \le 400 \\ & 3x_1 - 10x_2 + 8x_3 \le 600 \\ & x_j \ge 0,\ j = 1,2,3 \end{aligned}$$

(v) minimize $-12x_1 - 18x_2 - 8x_3 - 40x_4$

$$\begin{aligned} \text{subject to } \quad & 2x_1 + 5.5x_2 + 6x_3 + 10x_4 \le 80 \\ & 4x_1 + x_2 + 4x_3 + 20x_4 \le 50 \\ & x_j \ge 0,\ j = 2,3,4;\ x_1\text{: a free variable} \end{aligned}$$

(vi) Minimize $2x_1 - 3x_2 - x_3 + 2x_4$

$$\begin{aligned} \text{subject to } \quad & -3x_1 + 2x_2 - x_3 + 3x_4 = 2 \\ & -x_1 + 2x_2 + x_3 + 2x_4 = 3 \\ & x_j \ge 0,\ j = 1,2,3,4 \end{aligned}$$

2.7 Solve the following problems using the simplex method:

(i) Minimize $|x_1| + 4|x_2| + 2|x_3|$

$$\begin{aligned} \text{subject to } \quad & 2x_1 + x_2 \le 3 \\ & x_1 + 2x_2 + x_3 = 5 \end{aligned}$$

(ii) Minimize $\dfrac{-x_1 + 4x_2 + x_3 + 1}{x_1 + 2x_2 + x_3 + 1}$

$$\begin{aligned} \text{subject to } \quad & 2x_1 - 2x_2 + x_3 \le 1 \\ & x_1 + 2x_2 - x_3 \ge 1.5 \\ & x_j \ge 0,\ j = 1,2,3 \end{aligned}$$

(iii) Minimize $\max(-x_1 + 2x_2 - x_3, -2x_1 + 3x_2 - 2x_3, x_1 - x_2 - 2x_3)$

$$\begin{array}{rl} \text{subject to} & 2x_1 + \;\; x_2 + \;\; x_3 \le 5 \\ & 2x_1 + 2x_2 + 5x_3 \le 10 \\ & x_j \ge 0, \;\; j = 1,2,3 \end{array}$$

2.8 Prove by contradiction that the use of Bland's rule prevents cycling in the following way.

(i) Let T be the index set of all variables that enter the basis during cycling, and let q be the largest index in T, i.e., $q = \max\{j \mid j \in T\}$. The variable x_q enters the basis during cycling, and then x_q must also leave the basis. Let I be the index set of basic variables before x_q enters the basis, and let $J = \{1, 2, \cdots, n\} - I$ be the index set of nonbasic variables. The corresponding canonical form is represented by

$$x_i + \sum_{j \in J} \bar{a}_{ij} x_j = \bar{b}_i, \;\; i \in I, \qquad -z + \sum_{j \in J} \bar{c}_j x_j = -z.$$

Furthermore, let I' be the index set of basic variables when x_q leaves the basis, and let $J' = \{1, 2, \cdots, n\} - I'$ be the index set of nonbasic variables. The corresponding canonical form is represented by

$$x_i + \sum_{j \in J'} \bar{a}'_{ij} x_j = \bar{b}_i, \;\; i \in I', \qquad -z + \sum_{j \in J'} \bar{c}'_j x_j = -z.$$

Let $t \in J'$ be the index of the basic variable that enters I' instead of x_q. By the definitions of q and t, it follows that $\bar{c}_q < 0$, $\bar{c}'_t < 0$, $\bar{a}'_{qt} > 0$, $t \in T$, $t < q$. In the canonical form for I' and J', assume that $x_t = -1$ for $t \in J'$ and $x_j = 0$ for all $j \in J' - \{t\}$. Explain that the relation $-\bar{c}'_t = \sum_{j \in J} \bar{c}_j x_j$ holds.

(ii) From $\bar{c}'_t < 0$, there must be a positive term in $\sum_{j \in J} \bar{c}_j x_j$ of the above relation. Let the term be $\bar{c}_r x_r > 0$, $r \in J$. Show that $r < q$.

(iii) Show $x_r = \bar{a}'_{rt} > 0$ and derive the contradiction.

2.9 Apply the standard simplex method to the following linear programming problem due to E.M.L. Beale, starting with x_5, x_6, and x_7 as the initial basic variables, and verify that the procedure of the simplex method cycles:

$$\begin{array}{lrrrrrrrl} \text{minimize} & (-3/4)x_1 & +150x_2 & -(1/50)x_3 & +6x_4 & & & & \\ \text{subject to} & (1/4)x_1 & -60x_2 & -(1/25)x_3 & +9x_4 & +x_5 & & & = 0 \\ & (1/2)x_1 & -90x_2 & -(1/50)x_3 & +3x_4 & & +x_6 & & = 0 \\ & & & x_3 & & & & +x_7 & = 1 \\ & \multicolumn{8}{l}{x_j \ge 0, \quad j = 1, 2, \ldots, 7.} \end{array}$$

Solve the problem using the simplex method incorporating Bland's rule.

2.10 A vector $\boldsymbol{\pi}$ of the simplex multipliers can also be defined as follows: Multiply $\boldsymbol{\pi}$ by a vector $\boldsymbol{b}$ of the right-hand side constants of the original equation system, and subtract it from the objective function z. Then, π_i is determined such that the coefficient of a basic variable x_i is zero. Explain that the above definition and the original definition of the simplex multiplier are equivalent.

2.11 Solve problem 2.6 by the revised simplex method.

2.12 Show that the dual to the linear programming problem

$$\begin{array}{rrrrl} \text{minimize} & x_1 & +x_2 & +x_3 & \\ \text{subject to} & & -x_2 & +x_3 & \geq -1 \\ & x_1 & & -x_3 & \geq -1 \\ & -x_1 & +x_2 & & \geq -1 \\ & & x_j \geq 0, & j = 1, 2, 3 & \end{array}$$

is equivalent to the primal problem. Such a pair of linear programming is known as self-dual. Assuming A is a square matrix, derive the conditions for $\boldsymbol{c}$, A, and $\boldsymbol{b}$ for which the linear programming problem

$$\begin{array}{rl} \text{minimize} & \boldsymbol{cx} \\ \text{subject to} & A\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

is self-dual.

2.13 Prove the complementary slackness theorem.

2.14 Prove Gordon's theorem.

2.15 Solve the following problems using the dual simplex method:

(i) Minimize $4x_1 + 3x_2$

$$\begin{array}{rl} \text{subject to} & x_1 + 3x_2 \geq 12 \\ & x_1 + 2x_2 \geq 10 \\ & 2x_1 + x_2 \geq 9 \\ & x_j \geq 0, \ j = 1, 2 \end{array}$$

(ii) Minimize $3x_1 + 5x_2$

$$\begin{array}{rl} \text{subject to} & 2x_1 + 3x_2 \geq 20 \\ & 2x_1 + 5x_2 \geq 22 \\ & 5x_1 + 3x_2 \geq 25 \\ & x_j \geq 0, \ j = 1, 2 \end{array}$$

(iii) Minimize $4x_1 + 2x_2 + 3x_3$

$$\begin{array}{rl} \text{subject to} & 5x_1 + 3x_2 - 2x_3 \geq 10 \\ & 3x_1 + 2x_2 + 4x_3 \geq 8 \\ & x_j \geq 0, \ j = 1, 2, 3 \end{array}$$

(iv) $$\begin{aligned} \text{Minimize } & 4x_1 + 8x_2 + 3x_3 \\ \text{subject to } & 2x_1 + 5x_2 + 3x_3 \geq 185 \\ & 3x_1 + 2.5x_2 + 8x_3 \geq 155 \\ & 8x_1 + 10x_2 + 4x_3 \geq 600 \\ & x_j \geq 0, \ j = 1, 2, 3 \end{aligned}$$

2.16 In the production planning problem of Example 1.1, assume that the total amounts of available materials are changed as follows:

(i) The total amount of M_1 is changed from 27 tons to 33 tons.
(ii) The total amount of M_2 is changed from 16 tons to 21 tons.

In each case, find a new optimal solution starting from the last optimal tableau.

2.17 In the linear programming problem solved in problem 2.6 (i), assume that the right-hand side constants are changed as follows:

(i) The right-hand side constant 27 is changed to 30.
(ii) The right-hand side constant 45 is changed to 51.

In each case, find a new optimal solution starting from the last optimal tableau.

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