

# Chapter 1

## Propositional Logic

*Either this man is dead or my watch has stopped.*

- Groucho Marx.

Like her three older brothers before her, little Amanda always wants to know “Why”: “Why do I have to go to school?” “Why does it only snow in Winter?” As young as she is, she can understand that – logically – the responses she gets satisfy each and every one of her queries: “You go to school to learn things.” “It only snows in Winter because that’s the only time it gets cold.” However, these answers rarely satisfy her – they merely open the way for yet more queries to explain the reasons she gets as answers to her previous questions: “Why do I have to learn things?” “Why does it only get cold in the Wintertime?” Her impatient father rarely wins this game; it inevitably ends either with a definitive “Just because!” or, more usually, with a simple “Gee, I don’t know, that’s a very good question! Go ask your mother.”

This behaviour demonstrates more than mere curiosity; and in fact curiosity typically has little to do with it. It is the fun of the game of logical reasoning which motivates her: the pursuit of the absolute, unquestionable premises from which all the other points follow. Her father’s goal in this game, of course, is to identify these premises as quickly as possible. (Her true goal, one can’t help but feel, is to get her father to give up in exasperation.)

It is in our nature as human beings to reason about the world and our existence, to assimilate the knowledge which we accumulate and to make logical deductions based on this knowledge. Despite the fact that we are born with a built-in propensity to apply logical rules to make deductions from our knowledge – if we do something potentially dangerous such as step out into the street without looking for cars, then we may get hurt, and therefore we shouldn’t do such things – it is nonetheless the case that we are very bad at doing this correctly consistently. The problem lies to a great extent with the ambiguities in our language.

In this chapter we shall see how logically correct reasoning manifests itself in a multitude of ways, and we shall learn how to tame our use of

language in order to prevent the types of ambiguities and mismatches which lead to the sorts of invalid logical arguments which all too typically underly system failures. We will see that precise rules of logical reasoning can be written down and mechanically applied like the rules of chess. But, due to their universality as laws of thought, they are much more than a mere formal game. They can be applied to model and reason about a huge variety of systems and situations. In particular, they can be very useful in detecting unexpected misbehaviour or inconsistency of computing systems.

Logic in fact lies at the very core of computing. Historically, the concepts of computation and effective computability have been developed from a logical basis and they were motivated by questions about the mathematical foundations of logic. All computer programming languages rely on logical notions in their specifications, their implementations and their constructs. Logics are also among the most popular and effective methods for specifying and analysing computational systems in formally rigorous ways. And, last but not least, the design and implementation of digital systems is strongly based on logic.

## 1.1

## Propositions and Deductions

Consider the following argument.

1. Either this man is dead or my watch has stopped.
2. My watch is still ticking.

*Therefore*

3. This man is dead.

This is an example of the sort of reasoning which we (mostly unconsciously) perform constantly all day long. If we analyse the structure of the argument, we see the following elements.

- A. The argument involves three *statements*, or *propositions*, by which we mean declarations which are either true or false (but not both). Each of the statements in the argument is declared to be true.
- B. The first statement expresses an *option* between two simpler statements, namely
  - 1a. This man is dead.
  - or*
  - 1b. My watch has stopped.
- C. A *deduction* or *inference* is made to infer the truth of the third statement from the truth of the first two statements. The third statement is referred to as the *conclusion* of the argument, while the first

two statements from which we draw the conclusion are referred to as the *premises* of the argument.

Such arguments can be formalised in propositional logic. The *syntax* (structure) of propositional logic provides a language for modelling systems, situations and arguments. The *semantics* (meaning) of propositional logic gives an interpretation to the symbols of the language. The language of propositional logic starts with *atomic propositions*, such as “**This man is dead**”, and builds up larger compound propositions using a variety of *propositional connectives*, such as “*or*”. Each connective is given a precise prescribed meaning which aims to reflect its everyday use in natural language. The purpose of this formalization is to remove ambiguities which are prevalent in the use of English or any other natural language.

### Example 1.1

The following rules, adapted from those specified by the World Chess Federation FIDE, describe the conditions for castling. *Castling* is a move of the king and either rook of the same colour, counting as a single move of the king, and executed as follows: the king is transferred from its original square two squares towards the rook in question, and then that rook is transferred to the square which the king has just crossed.

1. The right for castling with a particular rook has been lost:
  - (a) *if* the king has already moved; *or*
  - (b) *if* the rook in question has already moved.
2. Castling with a particular rook is prevented:
  - (a) *if* the right for castling with that rook has been lost; *or*
  - (b) *if* there is a piece between the king and the rook in question; *or*
  - (c) *if* the square on which the king stands, *or* the square which it must cross, *or* the square which it is to occupy, is under attack by one or more of the opponent's pieces.

The conditions that permanently or temporarily prevent castling use the propositional connectives “*or*” and “*if*” to express constraints under which castling is prohibited.

Arguments are all about truth. Therefore, not all sentences can take part in arguments, simply because not all sentences express statements which can be true or false. This is the case with questions like “*Is that man dead?*” and requests like “*Bring me a watch that works.*” To be true or false, a sentence must state a potential fact, hence be related to a potential bit of reality. This criterion distinguishes statements or propositions from all other kinds of sentences.

**Exercise 1.1** (Solution on page 405)

Which of the following are statements (propositions)?

1.  $2+3=5$ .
  2.  $2+3=6$
  3. Do your homework, Joel!
  4. Joel didn't do his homework.
  5. Is there life on Mars?
  6. False
  7. What Felix says is false.
  8. What this sentence says is false.
- 

Each atomic statement can, of course, be further analysed with respect to its grammatical structure – *Joel*, for instance, is a subject noun, *do* a verb, and *homework* an object noun – but this is of no relevance to propositional logic. It is concerned solely with the distinction between logical and non-logical components and, correspondingly, with the way in which the truth of simpler statements determines that of more complex ones.

**Exercise 1.2** (Solution on page 405)

Which of the following are valid deductions?

1. If the fire alarm sounds, then everyone must leave the building.  
Everyone is leaving the building.  
Therefore the fire alarm has sounded.
2. If the fire alarm sounds, then everyone must leave the building.  
The fire alarm has sounded.  
Therefore everyone is leaving the building.
3. If the signal is green, then the train may proceed.  
The signal is red.  
Therefore the train must wait.
4. The right for castling with a particular rook has been lost if the king has already moved.  
Both rooks have already moved.  
Therefore the right for castling with a particular rook has been lost.
5. The right for castling with a particular rook has been lost if the king has already moved, or if the rook in question has already moved.  
One of the two rooks has already moved.  
Therefore the right for castling with a particular rook has been lost.

6. It is unlawful for any person to keep more than three dogs and three cats on their property within the city.  
 Charles keeps five dogs (but no cats) on his property in the city.  
 Therefore Charles is breaking the law.

**Exercise 1.3** (Solution on page 406)

Which of the following are valid deductions?

1. Epimenides is a Cretan.  
 All Cretans are liars.  
 Therefore Epimenides is a liar.
2. Epimenides is a Cretan.  
 Epimenides said that "All Cretans are liars."  
 Therefore Epimenides is a liar.
3. Epimenides is a Cretan.  
 Epimenides said that "All Cretans are liars."  
 Therefore all Cretans are liars.
4. Epimenides is a Cretan.  
 Epimenides said that "All Cretans are liars."  
 Therefore not all Cretans are liars.
5. Epimenides is a Cretan.  
 Aristotle said that "All Cretans are liars."  
 Therefore Epimenides is a liar.

## 1.2

## The Language of Propositional Logic

The *syntax* of propositional logic is the formal definition of the language, the *object language* of formal logic. This definition is given in a *meta-language* – natural language in this case – in which we speak *about* the language of propositional logic. The metalanguage itself will use logical notions and reasoning, albeit at an informal level; since the levels can be kept separate, there should be no conceptual confusion.

The definition of syntax has two steps. In the first step, the basic *symbols* of the language are defined. In the second step, the rules for writing *formulae* with these symbols is defined; these represent statements or propositions. The precise definition of a formula will be given at the end of this section; we first introduce the components of this definition informally.

### 1.2.1 Propositional Variables

In propositional logic, the meaning of a particular atomic proposition is given solely by its truth or falsity. We therefore abstract from these propositions and introduce propositional variables instead.

In algebra, variables such as  $x$ ,  $y$  and  $z$  are used to represent unknown numbers. The occurrences of the variable  $x$  in the quadratic equation  $x^2 + 2x - 15 = 0$  are place holders for some value, in this case a number. The equation restricts the admissible values of  $x$  to being either 3 or  $-5$ . That is, if 3 is substituted for every occurrence of  $x$  in the equation, or if  $-5$  is substituted for every occurrence of  $x$  in the equation, then the equation holds; and when any other number is substituted, it doesn't.

We use variables in a similar way in propositional logic. *Propositional variables* such as  $P, Q, R, \dots$  represent unknown propositions. In algebra we may assign a specific value to a variable; for example, we might write "let  $x=3$ " and then interpret every subsequent occurrence of the variable  $x$  by the value 3. Similarly we may let a propositional variable represent a specific proposition, for example writing "let Dead represent the statement: This man is dead." (Following good programming style, we will typically use meaningful words as propositional variables rather than mere letters to obtain more readable statements.)

In algebra, values (including unknown values represented by variables) can be combined using various operations, such as addition (+), subtraction ( $-$ ), multiplication ( $\times$ ) and division ( $\div$ ). In propositional logic, we may combine propositions using various *propositional connectives*, specifically "not" ( $\neg$ ), "or" ( $\vee$ ), "and" ( $\wedge$ ), "if ... then ..." ( $\Rightarrow$ ), and "... if, and only if, ..." ( $\Leftrightarrow$ ). An informal description of the connectives of propositional logic is given in the follow sections.

### 1.2.2 Negation

The *negation*  $\neg p$  of a statement  $p$ , pronounced "not  $p$ ", is a statement which is true if, and only if,  $p$  is false. This is typically expressed in English in one of the following ways:

- *not  $p$ ; (more precisely, the statement  $p$  with "not" modifying the verb, typically by appearing immediately after it.)*
- *$p$  does not hold / is not true / is false;*
- *it is not the case that  $p$ .*

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**Example 1.3**

If Dead stands for the statement "This man is dead," then  $\neg$ Dead says "It is *not* the case that this man is dead," or, equivalently, "This man is *not*

dead.”

If a proposition is not true, then it must be false; and conversely, if it is not false, then it must be true. In particular then, if a proposition is *not* not true, then it is true:  $\neg\neg p$  is the same as  $p$ . This is referred to as the *Law of Double Negation*.

**Exercise 1.4** (Solution on page 406)

Rewrite the following statements without negations at the start.

1.  $\neg$  “The Earth revolves around the sun.”
2.  $\neg$  “All of my children are boys.”
3.  $\neg(2 + 2 \leq 4)$ .

### 1.2.3 Disjunction

The *disjunction*  $p \vee q$  of two statements  $p$  and  $q$ , pronounced “ $p$  or  $q$ ”, is a statement which is true if, and only if,  $p$  is true or  $q$  is true (or indeed if both are true); that is, at least one of  $p$  and  $q$  is true. This is typically expressed in English in one of the following ways:

- $p$  or  $q$ ;
- $p$  or  $q$  or both;
- $p$  and/or  $q$ ;
- $p$  unless  $q$ .

In the context of the disjunction  $p \vee q$ , the propositions  $p$  and  $q$  are individually referred to as *disjuncts*.

**Example 1.4**

If Dead stands for the statement “This man is dead” and Watch stands for the statement “My watch has stopped,” then  $\text{Dead} \vee \text{Watch}$  says “Either this man is dead or my watch has stopped,” or, equivalently, “If this man is alive, then my watch must have stopped.” This does not preclude the possibility that the man is dead *and* my watch has stopped, in which case  $\text{Dead} \vee \text{Watch}$  will still be true.

**Example 1.5**

In chess, the right for castling with a particular rook has been lost if the king has already moved, or if the rook in question has already moved. This

condition can be formalised as  $\text{KingMoved} \vee \text{RookMoved}$ , where  $\text{KingMoved}$  and  $\text{RookMoved}$  are propositional variables stand for the statements “The king has moved” and “The rook has moved,” respectively. In particular, therefore, one may not castle with a particular rook if *both* the king *and* the rook in question have already been moved.

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Recalling that  $\neg p$  is true if  $p$  is not true, we can note that  $p \vee \neg p$  must always be true regardless of what proposition  $p$  stands for: either  $p$  is true, or it is not true. This fact is referred to as the *Law of the Excluded Middle*: there is no middle ground when it comes to the truth of a propositional formula.

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**Exercise 1.5** (Solution on page 406)

Are the following disjunctions true or false?

1.  $(3 < 2) \vee (3 < 5)$
2.  $(5 < 4) \vee (7 < 5)$
3.  $(5 < 6) \vee (6 < 8)$

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Note that  $p \vee q$  is true if (though not only if) *both*  $p$  and  $q$  are true. In propositional logic, there can be no ambiguity: the “or” is always taken in this *inclusive* sense. In some everyday circumstances, however, “or” is used in the *exclusive* sense: the statement “*Either you be quiet now or you won’t get an ice cream!*” certainly is not supposed to be true in the case in which the child under consideration is quiet but still doesn’t get the ice cream – that would be an unfair trick. Such an “exclusive or” is in fact provided by a different connective from the (inclusive) “or” used in propositional logic; it is written  $\oplus$ , and it has its own different truth conditions:  $p \oplus q$  is true if, and only if, one of  $p$  and  $q$  is true and the other is false; that is, *precisely* one of  $p$  and  $q$  is true. Note that this connective is not formally a part of the definition of propositional logic; however, it can be expressed using the connectives of propositional logic (see Example 1.10 on page 29).

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**Exercise 1.6** (Solution on page 406)

For each of the following disjunctive statements, decide whether you think the speaker intends to use the inclusive or exclusive sense of the disjunction.

1. Joel came in last place in the round-robin competition; so that mean that either Felix beat him or Oskar beat him.
  2. The light is either on or off.
  3. You can have tea or coffee.
-



### 1.2.4 Conjunction

The *conjunction*  $p \wedge q$  of two statements  $p$  and  $q$ , pronounced “ $p$  and  $q$ ”, is a statement which is true if, and only if, both  $p$  and  $q$  are true. This is typically expressed in English in one of the following ways:

- $p$  and  $q$ ;
- $p$  but  $q$ ;
- not only  $p$  but also  $q$ .

In the context of the conjunction  $p \wedge q$ , the propositions  $p$  and  $q$  are individually referred to as *conjuncts*.

#### Example 1.6

If Dead stands for the statement “This man is dead” and Watch stands for the statement “My watch has stopped,” then  $\text{Dead} \wedge \text{Watch}$  says “This man is dead *and* my watch has stopped,” or, equivalently, “Not only is this man dead, but so is my watch!”

Recalling that  $\neg p$  is false if  $p$  is true, we can note that  $p \wedge \neg p$  must always be false regardless of what proposition  $p$  stands for:  $p$  and  $\neg p$  cannot both be true at the same time.

#### Exercise 1.7 (Solution on page 407)

Are the following conjunctions true or false?

1.  $(3 < 2) \wedge (3 < 5)$
2.  $(5 < 4) \wedge (7 < 5)$
3.  $(5 < 6) \wedge (6 < 8)$

### 1.2.5 Implication

Given two statements  $p$  and  $q$ , the *implication*  $p \Rightarrow q$ , pronounced “ $p$  implies  $q$ ”, is a statement which is true if, and only if,  $p$  is false, or  $q$  is true; that is, if  $p$  is true then  $q$  must also be true. In other words,  $p \Rightarrow q$  is *false* if, and only if,  $p$  is true and  $q$  is false. This is typically expressed in English in one of the following ways:

- $p$  implies  $q$ ;
- if  $p$  then  $q$ ;
- $q$  if  $p$ ;
- $p$  only if  $q$ ;

- $q$  whenever  $p$ ;
- $p$  is a sufficient condition for  $q$ ;
- $q$  is a necessary condition for  $p$ .

In the context of the implication  $p \Rightarrow q$ ,  $p$  is referred to as the *premise* and  $q$  is referred to as the *conclusion*.

**Example 1.7**

Let the variable SignalDanger stand for the statement “The signal shows danger,” and let the variable TrainStop stand for the statement “The train stops.” Then  $\text{SignalDanger} \Rightarrow \text{TrainStop}$  stands for the statement “If the signal shows danger then the train stops.”

The only event in which this statement can be false is when the signal shows danger and yet the train does not stop. Hence the rule allows the case that the signal does not show danger and yet the train nevertheless stops.

**Exercise 1.8** (Solution on page 407)

Letting JoelHappy stand for “Joel is happy” and AmandaHappy stand for “Amanda is happy,” each of the following statements translates as either  $\text{JoelHappy} \Rightarrow \text{AmandaHappy}$  or as  $\text{AmandaHappy} \Rightarrow \text{JoelHappy}$ . Determine which in each case.

1. “Joel is happy whenever Amanda is happy.”
2. “Joel is happy only if Amanda is happy.”
3. “Joel is happy unless Amanda is not happy.”

**Exercise 1.9** (Solution on page 407)

On the door of a particular house is the following warning to potential thieves:

*Barking dogs don't bite.*

*My dog doesn't bark.*

Should a potential thief *necessarily* be concerned?

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### 1.2.6 Equivalence

The *equivalence*  $p \Leftrightarrow q$  of two statements  $p$  and  $q$ , pronounced “ $p$  if, and only if,  $q$ ”, is a statement which is true if, and only if, both  $p$  and  $q$  are true, or both  $p$  and  $q$  are false; that is, if  $p$  and  $q$  have the same truth value. This is typically expressed in English in one of the following ways:

- $p$  if, and only if,  $q$ ;
- $p$  is equivalent to  $q$ ;
- $p$  is a necessary and sufficient condition for  $q$ .

The symbol for equivalence  $\Leftrightarrow$  looks like the symbol for implies  $\Rightarrow$  pointing in both directions. This is very much by design since, with a bit of thought, it is evident that  $p \Leftrightarrow q$  is true if, and only if,  $p \Rightarrow q$  and  $p \Leftarrow q$  (that is,  $q \Rightarrow p$ ) are both true.

#### Example 1.9

Let the variable `TrainEnter` stand for the statement “The train enters the tunnel,” and let the variable `TunnelClear` stand for the statement “The tunnel is clear.” Then  $\text{TrainEnter} \Leftrightarrow \text{TunnelClear}$  stands for the statement “The train enters the tunnel if, and only if, the tunnel is clear.”

This statement is false if the train enters the tunnel while the tunnel is not clear, or if the tunnel is clear but the train does not enter.

### 1.2.7 The Syntax of Propositional Logic

We can now summarise the above discussion of propositional logic in the following formal definition. A statement written in propositional logic is called a *propositional formula*, and is either:

- an *atomic formula*, typically represented by a variable such as  $P$ ,  $Q$  or  $R$ ; or
- a *compound formula*, in which case it is built up using the above propositional connectives as summarised in Figure 1.1.

There are two special atomic propositional formulæ, *true* (representing the proposition which is always true) and *false* (representing the proposition which is always false).

The above defines the formal *syntax* of the language of propositional formulæ. To emphasise that a propositional formula must be written syntactically correctly according to Figure 1.1, it is also referred to as a *well-formed formula (wff)*.

Note that in Figure 1.1 (as well as throughout this whole chapter) the letters  $p$  and  $q$  are *not* propositional variables, but rather *metavariables* which stand for arbitrary propositions.

If  $p$  and  $q$  are propositional formulæ, then so are the following:

true		<i>truth</i>
false		<i>falsity</i>
$P$		<i>atomic proposition</i>
$\neg p$	<u>not</u> $p$	<i>negation</i>
$p \vee q$	$p$ <u>or</u> $q$	<i>disjunction</i>
$p \wedge q$	$p$ <u>and</u> $q$	<i>conjunction</i>
$p \Rightarrow q$	<u>if</u> $p$ <u>then</u> $q$	<i>implication</i>
$p \Leftrightarrow q$	$p$ <u>if, and only if,</u> $q$	<i>equivalence</i>

Figure 1.1: The formulæ of propositional logic.

### 1.2.8 Parentheses and Precedences

It is common to use parentheses when writing mathematical expressions such as  $(5 + 3) \times 2$ , in order to disambiguate such expressions. Most mathematicians (as well as many hand-held calculators) will calculate  $5 + 3 \times 2 = 11$ , as it is standard to consider multiplication as binding more tightly than addition; that is, multiplications are applied before additions whenever possible. Multiplication is said to have a *higher precedence* than addition. However, with parentheses the meaning of this expression changes dramatically:  $(5 + 3) \times 2 = 16$ . Similarly, we would use parentheses to calculate  $5 - (3 - 1) = 5 - 2 = 3$ , as without them we would naturally apply the subtractions left-to-right and calculate  $5 - 3 - 1 = 2 - 1 = 1$ .

In a similar vein we can and will regularly make use of parentheses within propositional formulæ to ensure that the meaning of our formulæ is clear. For example, the formula  $P \vee Q \Rightarrow R$  can be read either as  $(P \vee Q) \Rightarrow R$  or as  $P \vee (Q \Rightarrow R)$ , so we shall write the formula with parentheses in one of the above ways in order to make sure it is read as intended. We shall thus extend our definition of a well-formed formula to include parentheses which enclose subformulæ.

However, to reduce the need for parentheses, we will consider  $\neg$  as binding more tightly than  $\wedge$ , which will bind more tightly than  $\vee$ , which will bind more tightly than  $\Rightarrow$ , which will bind more tightly than  $\Leftrightarrow$ . Apart from this, the connectives will be applied right-to-left, so that for example an expression of the form

$$p \Rightarrow q \wedge r \Rightarrow s$$

would be interpreted as

$$p \Rightarrow (q \wedge r) \Rightarrow s$$

due to  $\wedge$  binding more tightly than  $\Rightarrow$ , and thus as

$$p \Rightarrow ((q \wedge r) \Rightarrow s)$$

due to the right-to-left application order of the  $\Rightarrow$  connectives.

Omitting parentheses by adopting the above precedence and application orders on connectives will often make formulæ easier to read. However, parentheses can and should still be used despite these conventions in cases when confusions can easily arise. For example, we will typically write

$$p \Rightarrow ((q \wedge r) \Rightarrow s)$$

despite the redundancy of the parentheses.

### Example 1.10

We can express the “exclusive or” operation  $p \oplus q$  – which says that one of  $p$  and  $q$  is true and the other is false – as a simple equivalence, by noting that  $p \oplus q$  says that one of  $p$  and  $q$  is true if, and only if, the other is *not* true. It can thus be defined simply by:

$$p \oplus q = p \Leftrightarrow \neg q$$

or, equivalently, by

$$p \oplus q = \neg p \Leftrightarrow q.$$

Both of these options abide by the hint that  $p \oplus q$  says that one of  $p$  and  $q$  is true if, and only if, the other is not true.

You may be tempted to define it as

$$p \oplus q = (p \Leftrightarrow \neg q) \wedge (q \Leftrightarrow \neg p)$$

which would be correct, but this would be overkill; with a little thought you should realise that  $p \Leftrightarrow \neg q$  is the same as  $q \Leftrightarrow \neg p$ .

### Exercise 1.10

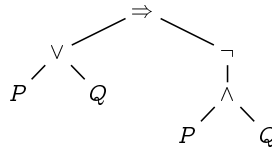
(Solution on page 407)

Express the following connectives using the connectives of propositional logic.

1. The NAND connective  $p | q$  which is true if, and only if,  $p$  and  $q$  are not both true.
2. The NOR connective  $p \downarrow q$  which is true if, and only if, neither  $p$  nor  $q$  are true.
3. The *conditional connective*  $q \triangleleft p \triangleright r$  which is true if, and only if, either  $p$  and  $q$  are both true, or  $\neg p$  and  $r$  are both true. In other words: “*If  $p$  is true then  $q$  must be true; otherwise  $r$  must be true.*”

### 1.2.9 Syntax Trees

It can be helpful to view a well-formed propositional formula as a tree-like diagram, called a *syntax tree*, in which the tree structure reflects the way in which the formula is constructed. For example, the formula  $(P \vee Q) \Rightarrow \neg(P \wedge Q)$  corresponds to the following syntax tree:



To recognise the expression  $(P \vee Q) \Rightarrow \neg(P \wedge Q)$  as a well-formed propositional formula, we need only break it down to its constituent parts, and to reconstruct it from the inside out:

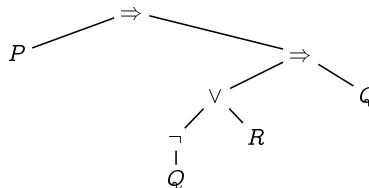
- $P$  and  $Q$ , being propositional variables, are propositional formulæ.
- Since  $P$  and  $Q$  are propositional formulæ, so too are their disjunction  $P \vee Q$  and conjunction  $P \wedge Q$ .
- Since  $P \wedge Q$  is a propositional formula, so too is its negation  $\neg(P \wedge Q)$ .
- Since  $P \vee Q$  and  $\neg(P \wedge Q)$  are propositional formulæ, so too is their implication  $(P \vee Q) \Rightarrow \neg(P \wedge Q)$ .

This decomposition is directly reflected in the syntax tree, and also provides a method for determining whether or not the formula is true.

The syntax tree makes it clear how the expression should be parsed, without the need for parentheses or precedence rules to tell the reader how to interpret the formula. Without the rules of precedence, there are many different ways to read the expression  $P \vee Q \Rightarrow \neg P \wedge Q$ , all of which having completely different meanings and syntax trees.

#### Example 1.11

Consider the expression  $P \Rightarrow \neg Q \vee R \Rightarrow Q$ . According to the precedence rules, it is represented by the following syntax tree:



In order to evaluate this expression – that is, to determine its truth value – we first need to know the truth values of the propositional variables  $P$ ,  $Q$  and  $R$ . We then compute  $\neg Q$ , as  $\neg$  binds more tightly than the other connectives; then  $(\neg Q) \vee R$  is computed, as  $\vee$  binds more closely than  $\Rightarrow$ ; then  $((\neg Q) \vee R) \Rightarrow Q$  is computed followed by  $P \Rightarrow ((\neg Q) \vee R) \Rightarrow Q$ , since the two  $\Rightarrow$  connectives are computed in a left-to-right order.

Fully bracketed, the formula is thus interpreted as

$$P \Rightarrow (((\neg Q) \vee R) \Rightarrow Q).$$

### Example 1.12

The string of symbols  $\neg(P \wedge (Q \vee \neg))$  is *not* a well-formed propositional formula. This can be seen by applying the formation rules in Figure 1.1 backwards.

- $\neg(P \wedge (Q \vee \neg))$  is a formula only if  $(P \wedge (Q \vee \neg))$  is a formula.
- $(P \wedge (Q \vee \neg))$  is a formula only if  $P$  and  $(Q \vee \neg)$  are formulae.
- $P$  is a propositional variable and is therefore a formula.
- $(Q \vee \neg)$  is a formula only if  $Q$  and  $\neg$  are formulae.
- $Q$  is a propositional variable and is therefore a formula.
- However,  $\neg$  is a logical connective; it is neither a propositional variable nor a compound formula, so it is *not* a formula.
- Therefore,  $\neg(P \wedge (Q \vee \neg))$  is not a well-formed formula.

### Exercise 1.12

(Solution on page 407)

Which of the following are well-formed formulae? Rewrite each well-formed formula using a minimal number of parentheses without changing its meaning, and draw its syntax tree.

1.  $((P \Rightarrow Q) \Leftrightarrow (Q \Rightarrow P)).$
2.  $P \vee Q(\wedge P).$
3.  $(P \vee Q) \wedge P.$
4.  $(P \vee Q) \Leftrightarrow (R \neg S)).$
5.  $(P \vee (Q \wedge R)) \Leftrightarrow (P \vee (Q \wedge (P \vee R))).$

## 1.3 Modelling with Propositional Logic

Propositional logic is very important for modelling real-life scenarios, in which we define propositional variables to represent particular properties which may be true or false. Indeed we have described many such examples already above. We shall here consider a few further such examples.

### Example 1.13

A particular computer program contains the following lines of code:

```
...
if CabinPressure < MinPressure then PrepareForLanding;
if FlightHeight < MinHeight then PrepareForLanding;
...
```

A software engineer assessing this code proposes that it could be optimised as follows:

```
...
if (CabinPressure < MinPressure and FlightHeight < MinHeight)
    then PrepareForLanding;
...
```

Is this correct?

Logically, we can use the variables *Pressure* and *Height* to express the two conditions that signal a need to land; and the variable *Land* to express the execution of *PrepareForLanding*. The program then gives rise to the following propositional formula:

$$(\text{Pressure} \Rightarrow \text{Land}) \wedge (\text{Height} \Rightarrow \text{Land})$$

while the suggested optimisation corresponds to

$$(\text{Pressure} \wedge \text{Height}) \Rightarrow \text{Land}.$$

The formula corresponding to the program is false if, and only if, either *Pressure* is true and *Land* is false, or *Height* is true and *Land* is false; this is the case if, and only if, either *Pressure* or *Height* is true while *Land* is false.

The formula for the suggested optimisation, on the other hand, would only be false if *both* *Pressure* and *Height* are true while *Land* is false; for example, having the cabin pressure drop below its minimum allowed value would wrongly *not* cause the aeroplane to prepare for landing if the aeroplane is cruising above its minimum allowed height.

The correct variant of the propositional formula – one which is equivalent to the formula corresponding to the program – would be

$$(\text{Pressure} \vee \text{Height}) \Rightarrow \text{Land}.$$



That is, the optimised code should have a disjunction (or) in the condition, not a conjunction (and). Of course this logical analysis only confirms our intuition: The aeroplane should prepare for landing if *either* condition is satisfied, not if *both* of them hold.

### Example 1.14

Consider the following four symbols: a white circle, a black circle, a white square, and a black square:



Let  $B$  represent the proposition that the symbol in question is black, and  $C$  represent the proposition that the symbol in question is a circle.

- $B$  is true of the black circle and the black square, but false of the white circle and the white square.
- $\neg B$  is true of the white circle and the white square, but false of the black circle and the black square.
- $B \vee C$  is true of the white circle, the black circle and the black square, but false of the white square.
- $B \wedge C$  is true of the black circle, but false of the white circle, the white square and the black square.
- $B \Rightarrow C$  is true of the white circle, the black circle and the white square, but false of the black square.
- $B \Leftrightarrow C$  is true of the black circle and the white square, but false of the white circle and the black square.

These facts are summarised in the table in Figure 1.2. Almost all of them are self-evident, though you should spend time considering carefully when  $B \Rightarrow C$  is true and when it is not true. Specifically, the only way that it can be false is if the symbol in question is black yet is not a circle.

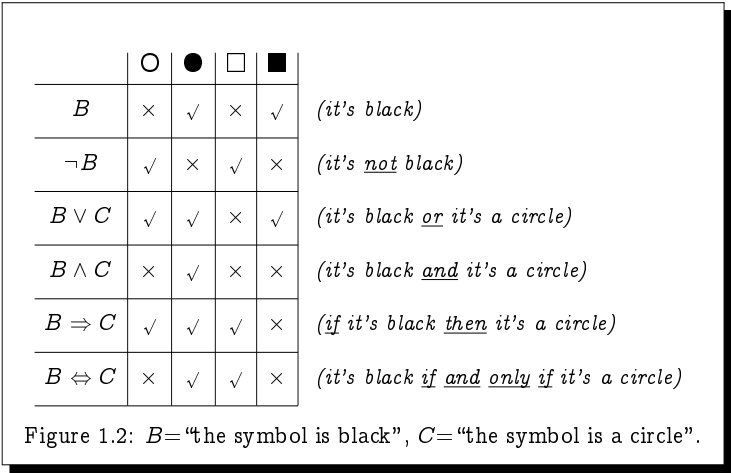
The Oxford mathematician Charles Lutwidge Dodgson (1832-1898), better known as Lewis Carroll, the author of *Alice in Wonderland*, enjoyed inventing puzzles which required careful logical reasoning to solve. The following is a typical example.

### Exercise 1.14

(Solution on page 408)

Lewis Carroll concludes that “Amos Judd loves cold mutton” from the following seven assumptions:

1. All the policemen on this beat sup with our cook.



- 2. No man with long hair can fail to be a poet.
- 3. Amos Judd has never been in prison.
- 4. Our cook's cousins all love cold mutton.
- 5. None but policemen on this beat are poets.
- 6. None but her cousins ever sup with our cook.
- 7. Men with short hair have all been in prison.

Explain how Lewis Carroll can draw his conclusion.

**Exercise 1.15** (Solution on page 410)

Translate the rules for castling in chess presented in Example 1.1 into propositional logic using the following propositional variables:

- RightToCastleLeft / RightToCastleRight:  
You have the right to castle with the rook to the left / right.
- MayCastleLeft / MayCastleRight:  
You may perform a castling move with the rook to the left / right.
- KingMoved: The king has moved.
- LeftRookMoved / RightRookMoved:  
The left / right rook has moved.
- PieceBetweenLeft / PieceBetweenRight:  
There is a piece between the king and the rook to the left / right.
- KingAttack: The king is under attack.

- LeftSquareAttack / RightSquareAttack:  
The square to the left / right of the king is under attack.
- KingMoveLeftAttack / KingMoveRightAttack:  
The square two to the left / right of the king is under attack.

---

The following puzzle may appear hard at first sight, but it becomes surprisingly simple when approached logically.

**Exercise 1.16** (Solution on page 410)

Joel, Felix and Oskar give Amanda the following puzzle. The three of them each write their name on a piece of paper, and then exchange the pieces of paper so that no one has the piece with their own name on it. They then hold these pieces of paper so that Amanda can't see what's on them, but tell her that each has the name of one of the others, and they challenge her to figure out who is holding each name. She is allowed to look at the name written on any one piece of paper.

1. Give a propositional formula which expresses the fact that each boy holds one of the pieces of paper but no one holds the piece of paper with their own name on it. Use the following propositional variables to do this.

JonF: "Joel" is on Felix's paper.  
 JonO: "Joel" is on Oskar's paper.  
 FonJ: "Felix" is on Joel's paper.  
 FonO: "Felix" is on Oskar's paper.  
 OonJ: "Oskar" is on Joel's paper.  
 OonF: "Oskar" is on Felix's paper.

2. Suppose Amanda looks at Joel's paper and sees "Oskar" written on it. Use the formula above to deduce what name is written on the other two pieces of paper.
- 

## 1.4 Ambiguities of Natural Languages

Despite their intentionally obfuscated form, the statements in the Amos Judd puzzle in Exercise 1.14 are precise and unambiguous. There are, however, many common abuses of logical arguments arising from the ambiguities of a natural language such as English. In the following examples we consider particular difficulties which beginning logicians often find problematic.

**Example 1.16**

Children can get very unruly in the back seat of the family car during long drives. In such instances, an increasingly exasperated father in the driving seat might find himself making promises such as the following:

**“Everyone who sits quietly for the next hour  
will get an ice cream when we stop for petrol.”**

What exactly does this statement say? And more importantly, does it express what the father means to say? You might well imagine that he wants to suggest that:

**“Anyone who misbehaves will not get ice cream.”**

However, this does not follow from his statement: the children who get ice cream will include those who sit quietly, but may well include the noisy ones as well. In fact, he knows that even greater problems of retribution will arise later on during the drive if only some of the children get the promised ice cream, so it is always his unspoken intention that all of the children will get ice cream, regardless of their behaviour (within reason).

His aim in making the statement was to manipulate language to his benefit, as well as to provide a lesson for his children in its logical use. He was being intentionally vague, relying on his children to misinterpret his statement as saying something more than it actually does, namely that any misbehaving children will not get ice cream. When in the end even the misbehaving children get ice cream, those that sat quietly in anticipation of their reward would be mildly upset at the unfairness of it all, but they could not argue with their father's explanation that he did not actually say that the unruly children would lose out. Without a doubt he spoke the truth.

Needless to say, this strategy would not work for very long, as the children will quickly become keen interpreters of any statements that their father makes.

**Example 1.17**

Suppose a menu at a restaurant states the following:

**“You may have coffee or tea with your meal.”**

This clearly expresses a disjunction of two atomic propositions:

**“You may have coffee with your meal  
*or* you may have tea with your meal.”**

However, does it really do this? Clearly the intention is that if you ask for coffee, then you will be served coffee. But consider the following scenarios.

1. Suppose the coffee maker is broken on the day you visit, and only tea is available that day; is the menu wrong in this case? Certainly not logically, assuming that you may still have tea.
2. Suppose the restaurant doesn't have a coffee maker, and never actually serves coffee at all; is the menu wrong in this case? Still as certainly not logically, assuming that it serves tea.

The real intention of the proposition on the menu is something more akin to conjunction rather than disjunction, as follows.

“You may have coffee with your meal  
and you may have tea with your meal.”

However, this is still not true either, as it is unlikely that the restaurant intends to allow you to order both beverages with your meal. The following proposition might be a more accurate interpretation of the intended option on the menu.

“You may have coffee with your meal  
and you may have tea with your meal,  
but not both.”

Are you satisfied with this? There is in fact still something seriously wrong with this proposition. To see this clearly, let us introduce the following two atomic propositions.

$A$  = You may have coffee with your meal.  
 $B$  = You may have tea with your meal.

Then the above proposition is

$(A \wedge B) \wedge \neg(A \vee B)$ .

However, this proposition is of the form  $p \wedge \neg p$ ; and recalling the fact noted after Example 1.6 that no proposition  $p$  (such as  $A \wedge B$ ) can be true at the same time as its negation  $\neg p$ , this means that the menu is giving no option whatsoever!

The problem here is one of *modality*. That we *may* have a coffee, and that we indeed *do* have a coffee, are different propositions, and we need to be careful how we treat such modalities.

To correctly formulate the option, we might introduce the following two atomic propositions.

$C$  = You have coffee with your meal.  
 $T$  = You have tea with your meal.

Then the option stated on the menu would stipulate that one, and only one, of these atomic propositions are true. This can be rendered in many (equivalent) ways, such as

$$(C \vee T) \wedge \neg(C \wedge T)$$

“You have coffee with your meal  
or you have tea with your meal,  
but not both.”

or

$$(C \wedge \neg T) \vee (\neg C \wedge T)$$

“You have coffee but not tea with your meal  
or you have tea but not coffee with your meal.”

But this is still not the end of the story. Perhaps a particular diner drinks neither coffee nor tea. The menu surely doesn't force the diner to accept one of these beverages; the diner surely has the option of having neither. The option on the menu thus is merely stipulating the following

$$\neg(C \wedge T)$$

“You do not have both coffee and tea with your meal.”

or equivalently

$$\neg C \vee \neg T$$

“You do not have coffee with your meal  
or you do not have tea with your meal.”

From this simple English proposition has sprouted a plethora of complications. This is the greatest problem in formulating the design of systems, and hence of getting such designs correct.

### Example 1.18

If  $p$  is false then by definition  $p \Rightarrow q$  is true *regardless of the truth of  $q$* . This observation gives rise not so much to a problem of ambiguity, but to one of misunderstanding and confusion. For example, assuming that Carlos is an ordinary man who is not the King of Spain, the following proposition is false:

“If Carlos is a man, then Carlos is the King of Spain.”

However, the following statement is true:

“If Carlos is a woman, then Carlos is the King of Spain.”

Do not be distracted by the falsity of the conclusion; the only way that the above statement can be false is if the premise is true whilst the conclusion is false. It is unfortunately a common misconception that the above implication is false, as the implication should be as follows:

**“If Carlos is a woman, then Carlos is the *Queen* of Spain.”**

This statement is true as well, for precisely the same reason that the previous one is true: the premise of the implication is false.

Though this is a common confusion, it is well understood and properly applied in several instances of natural language. For example, the statement

**“If I told you once, I told you a hundred times!”**

is meant to convey that you have been told something a hundred times (assuming that you’ve been told once). This statement, of course, is typically false due to an intended use of hyperbole – it is highly unlikely that you have been told something so many times.

As another example, the statement

**“If he ever pays me back, then I’ll be a monkey’s uncle!”**

expresses the doubt (i.e., falsity) that money lent will ever be returned, by concluding an obviously-false conclusion from the premise which is being denied. As I can never be a monkey’s uncle, the only way that this statement can be true is if he never pays me back.

### Example 1.19

Suppose your teacher says the following to you:

**“If you understand implication, then you will pass the exam.”**

There are four scenarios to consider:

1. Suppose you understand implication, and you pass the exam. Clearly you would consider the above statement to be true.
2. Suppose you *don’t* understand implication, and you *fail* the exam. Again you would consider the above statement to be true, and you might even think your teacher to be a wise sage. However, this thought would just go to show that you indeed don’t understand implication. The reason you failed the exam is not (necessarily) because you don’t understand implication. To understand this point, consider the next scenario.
3. Suppose you *don’t* understand implication, but nonetheless you *pass* the exam, because you understand enough of the rest of the material. This does *not* contradict your teacher’s claim; it is still true.
4. Suppose, finally, that you understand implication, but you *fail* the exam nonetheless. In this case you may feel angry towards your teacher, since he was obviously lying to you. (Of course, your teacher would maintain that it is *you* who are lying, in claiming that you understand implication.)

In summary, the *only* way for the teacher's statement to be false is if the premise is true (i.e., you understand implication) while the conclusion is false (you fail the exam).

**Exercise 1.19** (Solution on page 411)

Consider the following four symbols: a white circle, a black circle, a white square, and a black square:



I have in mind one of these four symbols. I will *accept* any symbol which either has the same colour or the same shape (or both) as the one I have in mind, and otherwise I will *reject* it. If I accept the black square, what does this suggest to you about whether I accept or reject the other three symbols?

**Exercise 1.20** (Solution on page 411)

If two's a company and three's a crowd, what's four and five?

## 1.5 Truth Tables

By thinking carefully about the logical connectives, we can informally understand their intended meanings. However, we still need to express these meanings precisely; that is, we need to *define* the meaning of the connectives. In doing this, the *semantics* of propositional logic is formally, rigorously and unambiguously defined.

One way in which we can do this concisely is by explicitly listing out the truth values which a compound formula takes for each of the possible combinations of truth values of its constituent propositions. A table which contains this listing is called a *truth table*.

For example, *negation*  $\neg p$  can be defined by specifying its truth value for each of the two possible truth values of  $p$ : if the truth value of  $p$  is true, then the truth value of  $\neg p$  will be false; and if the truth value of  $p$  is false, then the truth value of  $\neg p$  will be true. For ease of presentation, we shall reserve the symbols **T** for true and **F** for false. The truth table for negation is thus as follows.

$p$	$\neg p$
<b>F</b>	<b>T</b>
<b>T</b>	<b>F</b>



The remaining four connectives are similarly defined by the following truth tables, which all have four rows corresponding to the four distinct combinations of truth values for the two propositions  $p$  and  $q$  being combined using the connectives.

$p$	$q$	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

$p$	$q$	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

$p$	$q$	$p \Rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

$p$	$q$	$p \Leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

Truth tables can also be used to understand far more complicated formulae, such as in the following example.

**Example 1.20**

Consider the statement from Example 1.16 made by a certain father:

“Everyone who sits quietly for the next hour  
will get an ice cream when we stop for petrol.”

Let us define the following atomic propositions.

Quiet = You sit quietly.  
Ice = You get an ice cream.

For you, as a perfectly logical child, the above statement translates to  $\text{Quiet} \Rightarrow \text{Ice}$  – if you remain quiet then you will get an ice cream – which has the following truth table:

Quiet	Ice	$\text{Quiet} \Rightarrow \text{Ice}$
F	F	T
F	T	T
T	F	F
T	T	T

The *only* scenario in which the above statement can be considered false is if Quiet is true and Ice is false – that is, if you do not get an ice cream

despite being quiet; in this instance you would be justified in being angry with your father for lying to you. However, your father, being trustworthy, would never allow this scenario.

It is tempting to be angry that your noisy siblings *also* get ice cream. However, there is no justification in this based on the above statement. As is clear from the second row of the truth table, the statement is true even in the instance that a noisy child gets an ice cream. It is a common pitfall to interpret  $p \Rightarrow q$  as  $p \Leftrightarrow q$  (that is, to understand from the above statement that you will get an ice cream if, and only if, you are quiet), and to believe that  $p \Rightarrow q$  implies that  $q \Rightarrow p$  (that is, to understand from the above statement that you will *not* get an ice cream if you are not quiet).

The above statement is giving you a guarantee that you will get an ice cream if you are quiet – and therefore you best be quiet. If you are not quiet, then there is no guarantee that you will get an ice cream, but there is no guarantee that you won't!

---

**Exercise 1.21** (Solution on page 411)

Recall the statement from Example 1.19 made by a certain teacher:

**“If you understand implication, then you will pass the exam.”**

Translate this statement into a propositional formula, and use its truth table to justify when it is true or false.

---

**Example 1.21**

Catherine wishes to go to a party tonight, and would be happy to go with either Jim or Jules. However, as she is currently dating both Jim and Jules, she doesn't want to go to the party if they will both be there.

Let us define the following atomic propositions.

Cat = Catherine goes to the party.

Jim = Jim goes to the party.

Jules = Jules goes to the party.

Catherine's predicament then can be formalised as follows.

$$\text{Cat} \Rightarrow \neg(\text{Jim} \wedge \text{Jules}).$$

This proposition states that Catherine goes to the party *only if* Jim and Jules don't both go to the party. We can determine when this proposition is true or false by building up a truth table based on all possible values of the atomic propositions Cat, Jim and Jules, and the values of the constituent propositions which make up the complete proposition. The resulting truth table is as follows.

			$\neg(\text{Jim} \wedge \text{Jules})$		
Cat	Jim	Jules	$\text{Jim} \wedge \text{Jules}$	$\text{Cat} \Rightarrow \neg(\text{Jim} \wedge \text{Jules})$	
F	F	F	F	T	T
F	F	T	F	T	T
F	T	F	F	T	T
F	T	T	T	F	T
T	F	F	F	T	T
T	F	T	F	T	T
T	T	F	F	T	T
T	T	T	T	F	F

The first three columns systematically list out the eight distinct combinations of truth values for the three propositions Cat, Jim and Jules; the next column applies the rules from the truth table for  $\wedge$  to the columns for Jim and Jules; the next column applies the rules for  $\neg$  to the column just constructed; and the final column applies the rules for  $\Rightarrow$  to the columns for Cat and  $\neg(\text{Jim} \wedge \text{Jules})$ . From this we can discover that the proposition is true in all cases *except* when all three atomic propositions are true; that is, it is *false* if, and only if, all three participants in this love triangle go to the party.

As a point of interest, we can build truth tables in a more concise way which entails writing the proposition of interest along the top row of the truth table, and filling in columns defined by the propositional variables and connectives, working from the “inside out.” The truth table for the above example would then be rendered as follows:

Cat	Jim	Jules	$\text{Cat} \Rightarrow \neg(\text{Jim} \wedge \text{Jules})$					
F	F	F	F	T	T	F	F	F
F	F	T	F	T	T	F	F	T
F	T	F	F	T	T	T	F	F
F	T	T	F	T	F	T	T	T
T	F	F	T	T	T	F	F	F
T	F	T	T	T	T	F	F	T
T	T	F	T	T	T	T	F	F
T	T	T	T	F	F	T	T	T
(0)	(0)	(0)	(1)	(4)	(3)	(1)	(2)	(1)

The bottom row of numbers is included in this example to indicate at what stage each column was filled in:

- (0) The three initial columns are filled in, representing all 8 possible combinations of truth values for the atomic propositions Cat, Jim and Jules.
- (1) The columns for the propositional variables are then filled in during the first stage.

- (2) After this the column for  $\text{Jim} \wedge \text{Jules}$  is filled in (under the  $\wedge$  symbol) during the second stage.
- (3) Then the column for  $\neg(\text{Jim} \wedge \text{Jules})$  is filled in (under the  $\neg$  symbol) during the third stage.
- (4) Finally the column for  $\text{Cat} \Rightarrow \neg(\text{Jim} \wedge \text{Jules})$  is filled in (under the  $\Rightarrow$  symbol) during the fourth stage.

Each column is computed by referring to columns which have been computed in earlier stages.

---

**Exercise 1.22** (Solution on page 412)

How many rows will there be in a truth table involving four propositional variables  $P$ ,  $Q$ ,  $R$  and  $S$ ? What if there are five propositional variables? What if there are  $n$  propositional variables?

---

**Exercise 1.23** (Solution on page 412)

Construct truth tables for the following propositions.

1.  $\neg(P \Leftrightarrow \neg Q)$ .
2.  $(P \wedge Q) \vee (\neg P \wedge \neg Q)$ .
3.  $(P \wedge Q) \Rightarrow (\neg R \vee S)$ .

---

**Exercise 1.24** (Solution on page 413)

The “exclusive or” operation  $p \oplus q$  has the following truth table:

$p$	$q$	$p \oplus q$
F	F	F
F	T	T
T	F	T
T	T	F

That is,  $p \oplus q$  is true if, and only if, one of  $p$  and  $q$  is true and the other is false.

Confirm that the formula you gave in Example 1.10 (page 29) for expressing  $p \oplus q$  in propositional logic gives the same truth table.

---

## 1.6 Equivalences and Valid Arguments

We have seen that a given proposition can be expressed as a formula in propositional logic in different yet equivalent ways. As a further example, the formula

$$\text{Cat} \Rightarrow \neg(\text{Jim} \wedge \text{Jules}) \quad \text{“If Cat then not both of Jim and Jules.”}$$

from Example 1.21 is equivalent to the formula

$$\neg(\text{Cat} \wedge \text{Jim} \wedge \text{Jules}) \quad \text{“Cat, Jim and Jules cannot all be true.”}$$

as well as

$$\neg\text{Cat} \vee \neg\text{Jim} \vee \neg\text{Jules}. \quad \text{“One of Cat, Jim or Jules is false.”}$$

To verify that two compound formulae  $p$  and  $q$  are equivalent, we could construct truth tables for  $p$  and  $q$  and observe that they have the same truth values under all interpretations of their respective atomic propositions. Alternatively we could build the truth table for the formula  $p \Leftrightarrow q$  and observe that it is true under all interpretations. If so, the two propositions  $p$  and  $q$  are said to be **logically equivalent**.

A proposition which is true regardless of the truth values of its atomic propositions is called a **tautology**, and the proposition is said to be **valid**. A **contradiction** on the other hand is a proposition which is false regardless of the truth values of its atomic propositions, and is said to be **unsatisfiable**. A proposition which is true under *some* interpretation of its atomic propositions – that is, one that is *not* a contradiction – is said to be **satisfiable**.

### Example 1.24

Any formula of the form  $p \vee \neg p$  is a tautology, while any of the form  $p \wedge \neg p$  is a contradiction. These facts were noted already in Section 1.2, and can be verified formally by constructing the truth tables for these formulae.

$p$	$\neg p$	$p \vee \neg p$
<b>F</b>	<b>T</b>	<b>T</b>
<b>T</b>	<b>F</b>	<b>T</b>

$p$	$\neg p$	$p \wedge \neg p$
<b>F</b>	<b>T</b>	<b>F</b>
<b>T</b>	<b>F</b>	<b>F</b>

Each entry in the column for  $p \vee \neg p$  is true, confirming that  $p \vee \neg p$  is a tautology, while each entry in the column for  $p \wedge \neg p$  is false, confirming that  $p \wedge \neg p$  is a contradiction.

Note that if we take  $p = A \wedge B$ , then the contradiction

$$p \wedge \neg p = (A \wedge B) \wedge \neg(A \wedge B)$$

is precisely the formula which appeared in Example 1.17 (page 37).

**Exercise 1.25**

(Solution on page 413)

Construct truth tables for each of the following formulæ to determine which are tautologies and which are contradictions.

1.  $p \vee (\neg p \wedge q)$ .
2.  $(p \wedge q) \wedge \neg(p \vee q)$ .
3.  $(p \Rightarrow \neg p) \Leftrightarrow \neg p$ .
4.  $(p \Rightarrow q) \Rightarrow p$ .
5.  $p \Rightarrow (q \Rightarrow p)$ .

Tautologies are important in ascertaining the validity of arguments. Consider, for example, our first argument from Section 1.1 (page 18):

1. Either this man is dead or my watch has stopped.
2. My watch is still ticking.

*Therefore*

3. This man is dead.

This argument is valid if the conjunction of the two premises implies the conclusion, that is, if the following implication is valid:

$$(\text{Dead} \vee \text{Watch}) \wedge \neg \text{Watch} \Rightarrow \text{Dead}$$

Again, this means that the proposition is a tautology, that it is true regardless of the truth values of its atomic propositions. We can easily confirm this by constructing a truth table for this proposition:

Dead	Watch	$(\text{Dead} \vee \text{Watch}) \wedge \neg \text{Watch} \Rightarrow \text{Dead}$						
F	F	F	F	F	F	T	F	F
F	T	F	T	T	F	F	T	F
T	F	T	T	F	T	T	F	T
T	T	T	T	T	F	F	T	T

In contrast, consider the argument suggested by Exercise 1.9 (page 26):

1. If my dog barks, then my dog doesn't bite.
2. My dog doesn't bark.

*Therefore*

3. My dog bites.

Its formalisation yields the following truth table:

Barks	Bites	$(\text{Barks} \Rightarrow \neg \text{Bites}) \wedge \neg \text{Barks} \Rightarrow \text{Bites}$							
F	F	F	T	T	F	T	T	F	F
F	T	F	T	F	T	T	T	F	T
T	F	T	T	T	F	F	F	T	F
T	T	T	F	F	T	F	F	T	T

The first row of this truth table shows that the proposition – and hence the argument it represents – is not valid. It presents a scenario in which the proposition may be false: a dog that neither barks nor bites satisfies both premises, but not the conclusion. Such a dog provides a *counterexample* to the validity of the argument.

**Exercise 1.26** (Solution on page 414)

In Example 1.13 we represented a piece of computer program in propositional logic as:

$$p = (\text{Pressure} \Rightarrow \text{Land}) \wedge (\text{Height} \Rightarrow \text{Land}).$$

We also considered two optimisations of this program represented as

$$q = \text{Pressure} \wedge \text{Height} \Rightarrow \text{Land};$$

$$r = \text{Pressure} \vee \text{Height} \Rightarrow \text{Land}.$$

Of course, an optimisation is only correct if the representation of the optimised program code is equivalent to the original one. Explain which of the two optimisations is correct and which is not.



**1.7**

## Algebraic Laws for Logical Equivalences

Using truth tables to prove properties about propositions, specifically that two propositions are equivalent, can quickly become tedious. However, we can avoid relying on truth tables by reasoning equationally much as we would do in algebra and arithmetic.

For example, we might conclude that  $3 \times (4+5) = 27$  in the following way:

$$\begin{aligned} 3 \times (4 + 5) &= (3 \times 4) + (3 \times 5) \\ &= 12 + 15 = 27. \end{aligned}$$

In the first line of this calculation we used the algebraic law that says that multiplication distributes over addition:  $a(b+c) = ab+ac$ ; and in the second

line we used the principle that we can replace equals by equals: if  $a = b$  and  $c = d$  then  $a + c = b + d$ .

A similar kind of reasoning is possible with propositional logic, with equivalence  $\Leftrightarrow$  playing the role of equality  $=$ . Once we have determined that two propositions  $p$  and  $q$  are equivalent, that  $p \Leftrightarrow q$ , we can then replace one with the other. First, though, we need to know what equivalences we can use as our "algebraic laws". A large number of these are given as follows.

### *Commutativity Laws*

$$p \vee q \Leftrightarrow q \vee p$$

$$p \wedge q \Leftrightarrow q \wedge p$$

### *Associativity Laws*

$$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$$

$$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$$

### *Idempotence Laws*

$$p \vee p \Leftrightarrow p$$

$$p \wedge p \Leftrightarrow p$$

### *Distributivity Laws*

$$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$$

$$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$$

### *De Morgan's Laws*

$$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$$

$$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$$

### *Double Negation Law*

$$\neg \neg p \Leftrightarrow p$$

### *Tautology Laws*

$$p \vee \text{true} \Leftrightarrow \text{true}$$

$$p \wedge \text{true} \Leftrightarrow p$$

### *Contradiction Laws*

$$p \vee \text{false} \Leftrightarrow p$$

$$p \wedge \text{false} \Leftrightarrow \text{false}$$

### *Excluded Middle Laws*

$$p \vee \neg p \Leftrightarrow \text{true}$$

$$p \wedge \neg p \Leftrightarrow \text{false}$$

### *Absorption Laws*

$$p \vee (p \wedge q) \Leftrightarrow p$$

$$p \wedge (p \vee q) \Leftrightarrow p$$

### *Implication Law*

$$p \Rightarrow q \Leftrightarrow \neg p \vee q$$



**Contrapositive Law**

$$p \Rightarrow q \Leftrightarrow \neg q \Rightarrow \neg p$$

**Equivalence Law**

$$p \Leftrightarrow q \Leftrightarrow (p \Rightarrow q) \wedge (q \Rightarrow p)$$

You can (and should) show that all of the above laws are valid tautologies by constructing appropriate truth tables. However, some laws can be shown to be valid by using laws that have already been previously confirmed. For example, we can verify the validity of the Contrapositive Law as follows:

$$\begin{aligned} p \Rightarrow q &\Leftrightarrow \neg p \vee q && \text{(Implication)} \\ &\Leftrightarrow q \vee \neg p && \text{(Commutativity)} \\ &\Leftrightarrow \neg \neg q \vee \neg p && \text{(Double Negation)} \\ &\Leftrightarrow \neg q \Rightarrow \neg p && \text{(Implication)} \end{aligned}$$

Of course, this derivation relies on the Implication, Commutativity and Double Negation Laws being verified first.

More importantly, we can use the above equivalences to derive ever more equivalences, bypassing the need to construct truth tables to justify them.

**Example 1.26**

We can derive the equivalence  $p \vee (\neg p \wedge q) \Leftrightarrow p \vee q$  using the following sequence of steps:

$$\begin{aligned} p \vee (\neg p \wedge q) &\Leftrightarrow (p \vee \neg p) \wedge (p \vee q) && \text{(Distributivity)} \\ &\Leftrightarrow \text{true} \wedge (p \vee q) && \text{(Excluded Middle)} \\ &\Leftrightarrow (p \vee q) \wedge \text{true} && \text{(Commutativity)} \\ &\Leftrightarrow p \vee q && \text{(Tautology)} \end{aligned}$$

We can equally use this technique to verify that a proposition  $p$  is a tautology by demonstrating that  $p \Leftrightarrow \text{true}$ .

**Example 1.27**

We can demonstrate that  $(p \Rightarrow q) \vee (q \Rightarrow r)$  is a tautology as follows:

$$\begin{aligned}
& (p \Rightarrow q) \vee (q \Rightarrow r) \\
& \Leftrightarrow (\neg p \vee q) \vee (\neg q \vee r) && \text{(Implication, twice)} \\
& \Leftrightarrow \neg p \vee ((q \vee \neg q) \vee r) && \text{(Associativity, twice)} \\
& \Leftrightarrow \neg p \vee (\text{true} \vee r) && \text{(Excluded Middle)} \\
& \Leftrightarrow (\neg p \vee r) \vee \text{true} && \text{(Commutativity, Associativity)} \\
& \Leftrightarrow \text{true} && \text{(Tautology)}
\end{aligned}$$

As in algebra, we will usually not mention applications of associativity and commutativity, and write formulæ like  $p \vee q \vee r$  instead of  $p \vee (q \vee r)$  or  $(p \vee q) \vee r$ . This allows us to represent the above calculation in a more compact way as follows:

$$\begin{aligned}
& (p \Rightarrow q) \vee (q \Rightarrow r) \\
& \Leftrightarrow \neg p \vee q \vee \neg q \vee r && \text{(Implication, twice)} \\
& \Leftrightarrow \neg p \vee \text{true} \vee r && \text{(Excluded Middle)} \\
& \Leftrightarrow \text{true} && \text{(Tautology)}
\end{aligned}$$

---

**Exercise 1.27** (Solution on page 415)

Give derivations of the following equivalences.

1.  $p \wedge (\neg p \vee q) \Leftrightarrow p \wedge q.$
  2.  $\neg(p \Rightarrow q) \Leftrightarrow p \wedge \neg q.$
  3.  $p \Rightarrow (q \vee r) \Leftrightarrow (p \Rightarrow q) \vee (p \Rightarrow r).$
  4.  $p \Rightarrow (q \wedge r) \Leftrightarrow (p \Rightarrow q) \wedge (p \Rightarrow r).$
  5.  $(p \wedge q) \Rightarrow r \Leftrightarrow (p \Rightarrow r) \vee (q \Rightarrow r).$
  6.  $(p \vee q) \Rightarrow r \Leftrightarrow (p \Rightarrow r) \wedge (q \Rightarrow r).$
- 

## 1.8 Additional Exercises

1. Which of the following are statements?
  - (a) "17 is an odd integer."
  - (b) "Manchester is the capital of Great Britain."
  - (c) "Unload the dishwasher if it has completed its washing cycle."
  - (d) "Are all roses red?"

- (e) "All roses are red."
2. Negate each of the items from above that you determine to be statements.
  3. Which of the following are valid deductions?
    - (a) Mammals are warm-blooded animals.  
Whales are mammals.  
Therefore whales are warm-blooded animals.
    - (b) Mammals are warm-blooded animals.  
Fish are not mammals.  
Therefore fish are not warm-blooded animals.
    - (c) Some doctors are surgeons.  
Some women are doctors.  
Therefore some women are surgeons.
    - (d) All horses are animals.  
Therefore all horses' heads are animal heads.
    - (e) Some girls are better than others.  
Therefore some girls' mothers are better than other girls' mothers.
  4. Formalise the following statement of Sherlock Holmes in propositional logic:
 

*"If I'm not mistaken Watson, that was the Dore and Totley tunnel through which we have just come, and if so we shall be in Sheffield in a few minutes."*
  5. Let  $E$  and  $T$  and  $W$  represent the following propositions.
 

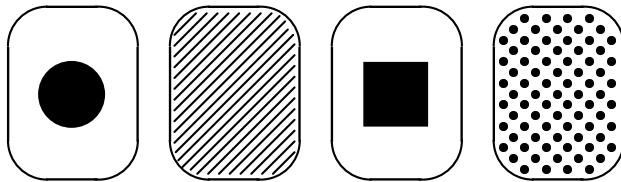
$E$ : Your laptop's warranty has expired.  
 $T$ : You have tampered with the electronics in your laptop.  
 $W$ : Your laptop is covered by its warranty.

    - (a) Translate the following statements into propositional logic.
 

$W_1$ : Your laptop is covered by its warranty as long as the warranty has not expired and you have not tampered with the laptop's electronic components.

$W_2$ : Your laptop is not covered by its warranty if the warranty has expired or if you have tampered with the laptop's electronic components.
    - (b) How do these two statements differ? Which one would you prefer to see on the warranty of your new laptop?
  6. Given that  $P$  and  $R$  are true while  $Q$  is false, determine the truth values of the following formulæ. Verify these by building truth tables for the given formulæ.

- (a)  $P \wedge (Q \vee R)$
  - (b)  $(P \wedge Q) \vee R$
  - (c)  $\neg(P \wedge Q) \wedge R$
  - (d)  $\neg P \vee \neg(\neg Q \wedge R)$
7. Write each of the following statements symbolically in the form  $P \Rightarrow Q$  (using the suggested propositional variables), and then express them in English in the form “If ... then ...”
- (a) I will play golf tomorrow ( $G$ ) unless it rains ( $R$ ).
  - (b) I'll do it ( $D$ ) if you ask me nicely ( $N$ ).
  - (c) Ann cries ( $C$ ) every time she watches *The Titanic* ( $W$ ).
  - (d) I never leave the house ( $L$ ) without locking the door ( $D$ ).
  - (e) A rectangle is a square ( $S$ ) only if all four of its sides are the same length ( $L$ ).
  - (f) A rectangle is a square ( $S$ ) if all four of its sides are the same length ( $L$ ).
8. Letting CatAway stand for “The cat's away” and MicePlay stand for “The mice will play,” translate each of the following into propositional logic.
- (a) “The mice will play whenever the cat's away.”
  - (b) “The mice will play only if the cat's away.”
  - (c) “The mice will play unless the cat's not away.”
9. Suppose I lay the following four cards on the table, each of which has a shape on one side (either a circle or a square) and a pattern on the other side (either stripes or dots).



I claim that:

**“Every card with a circle on one side  
always has stripes on the other side.”**

Which card(s) do you need to turn over in order to be certain that I am telling the truth?

This exercise is known as a **Wason Selection Test** after the psychologist Peter Wason who first described it in 1966. Be careful with your answer: studies rarely result in a reported success rate of over 20%!

10. Explain the difference between the following three offers:

- (a) You can watch TV if you tidy your room.
- (b) You can watch TV only if you tidy your room.
- (c) You can watch TV if, and only if, you tidy your room.

Which offer should a logical parent make to their children?

11. Give the truth tables defining the NAND, NOR and conditional connectives  $p \mid q$ ,  $p \downarrow q$  and  $q \triangleleft p \triangleright r$  defined in Exercise 1.10, and show that these are the same as the truth tables for the formulæ you gave in Exercise 1.10 for these connectives.

12. Propositional Logic is based on the three connectives  $\neg$ ,  $\vee$  and  $\wedge$ ; the *Implication Law* and the *Equivalence Law* show that the two connectives  $\Rightarrow$  and  $\Leftrightarrow$  can be defined in term of the other three.

- (a) Show how to express  $\neg p$ ,  $p \vee q$  and  $p \wedge q$  using only the NAND connective  $\mid$ .
- (b) Show how to express  $\neg p$ ,  $p \vee q$  and  $p \wedge q$  using only the NOR connective  $\downarrow$ .

13. A friend proposes the following game to you. You keep tossing a coin over and over until one of the following two things happens:

- if two heads occur in a row, then the game ends; you win, and your friend will give you £2;
- if a tail occurs followed immediately by a head, then the game ends; your friend wins, and you must give your friend £1.

Is it worth playing this game?

14. In a certain country, every inhabitant is either a truth teller who always tells the truth, or a liar who always lies. While travelling in this country, you meet two people, Abe and Ben. Abe says, "Ben and I are both liars." Is Abe a truth teller or a liar? What about Ben?

15. Argue that Superman doesn't exist. To do this, start by making the following four assumptions:

$X_1$ : If Superman were able and willing to prevent evil, he would do so.

$X_2$ : Superman does not prevent evil.

$X_3$ : If Superman were unable to prevent evil, he would be impotent; and if he were unwilling to prevent evil, he would be malevolent.

$X_4$ : If Superman exists, he is neither impotent nor malevolent.

Argue as follows. First introduce the following variables:

A: "Superman is able to prevent evil."  
W: "Superman is willing to prevent evil."  
I: "Superman is impotent."  
M: "Superman is malevolent."  
P: "Superman prevents evil."  
E: "Superman exists."

- (a) The first assumption translates into the following formal logical statement:

$$X_1 : (A \wedge W) \Rightarrow P.$$

Translate the remaining assumptions  $X_2$ ,  $X_3$  and  $X_4$  into formal logical statements.

- (b) Use assumptions  $X_1$  and  $X_2$  to argue that  $\neg A \vee \neg W$ .  
(c) Use assumption  $X_3$ , and the fact from (b), to argue that  $I \vee M$ .  
(d) Use assumption  $X_4$ , and the fact from (c), to draw your conclusion.

16. Which of the following statements is true?

- (a) All of the below.  
(b) None of the below.  
(c) All of the above.  
(d) One of the above.  
(e) None of the above.  
(f) None of the above.

17. The following famous puzzle is referred to as the *Einstein Riddle* as Albert Einstein is sometimes credited with inventing it as a boy. He is also credited with claiming that only two percent of the world's population can solve it.

You are given the following information about five houses sitting in a row on some street which are each painted a different colour, and whose inhabitants are of different nationalities, own different pets, drink different beverages, and smoke different brands of American cigarettes. In statement (e), right refers to the reader's right.

- (a) The Englishman lives in the red house.  
(b) The Spaniard owns the dog.  
(c) Coffee is drunk in the green house.  
(d) The Ukrainian drinks tea.  
(e) The green house is immediately to the right of the ivory house.  
(f) The Old Gold smoker owns snails.

- (g) Kools are smoked in the yellow house.
- (h) Milk is drunk in the middle house.
- (i) The Norwegian lives in the first house.
- (j) Chesterfields are smoked next door to the man with the fox.
- (k) Kools are smoked next door to the house where the horse is kept.
- (l) The Lucky Strike smoker drinks orange juice.
- (m) The Japanese smokes Parliaments.
- (n) The Norwegian lives next to the blue house.

The question is: Who drinks water? Who owns the zebra?

18. Verify the Laws of Equivalence from Section 1.7, either directly by using truth tables, or by deriving them from previous laws which have already been verified.
19. Verify the following laws for implication and equivalence.

- (a)  $p \Rightarrow p$
- (b)  $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$ .
- (c)  $(p \Rightarrow q) \Rightarrow (p \vee r \Rightarrow q \vee r)$ .
- (d)  $(p \Rightarrow q) \Rightarrow (p \wedge r \Rightarrow q \wedge r)$ .
- (e)  $(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$ .
- (f)  $p \Leftrightarrow p$ .
- (g)  $(p \Leftrightarrow q) \Rightarrow (q \Leftrightarrow p)$ .
- (h)  $(p \Leftrightarrow q) \wedge (q \Leftrightarrow r) \Rightarrow (p \Leftrightarrow r)$ .
- (i)  $(p \Leftrightarrow q) \Rightarrow (p \vee r \Leftrightarrow q \vee r)$ .
- (j)  $(p \Leftrightarrow q) \Rightarrow (p \wedge r \Leftrightarrow q \wedge r)$ .
- (k)  $(p \Leftrightarrow q) \Leftrightarrow (\neg p \Leftrightarrow \neg q)$ .



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