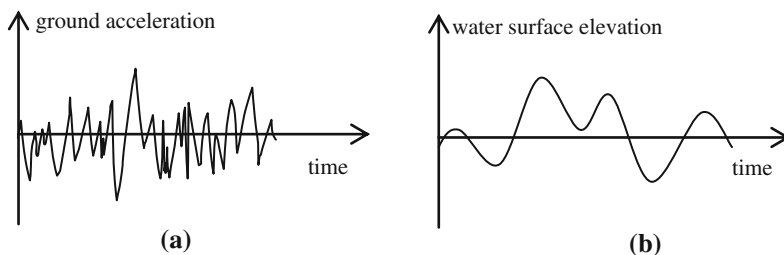


## Chapter 2

# Introduction to Random Vibration and Stochastic Analysis

### 2.1 Introduction

Structures are subjected loading that is mostly time dependent in a weak or strong fashion. Response histories under weakly time-dependent loading may be calculated by using the quasi-static analysis procedure. For moderately or strongly time dependent loading, calculation of response quantities requires a full dynamic analysis procedure as it is presented in the previous chapter assuming that the structure is deterministic and the loading history is fully determined or known, i.e., it obeys a specific rule or a definite function of time such as constant, linear, harmonic, etc. time functions with known properties. Under such a structural and loading case, the corresponding analysis type is called as *the deterministic dynamic analysis* since all necessary parameters of the analysis can be uniquely determined or known. However, the difficulty in the structural dynamic analysis is to determine the loading functions and their properties correctly, such as frequencies, durations, amplitudes, and phases, in practice. Due to lack of sufficient knowledge of dynamic excitations in nature, we possess limited information on loading parameters which is usually obtained from recorded data or observations of occurrences, such as earthquakes and sea waves, which occur in arbitrary fashions. Other examples can be wind loading on high-rise buildings and towers, and traffic loading on bridges and viaducts, which do not follow specific rules. Earthquakes occur periodically in seismic areas with unknown information and sea waves occur continuously with random fluctuation of the sea surface. The only information that we have is based on experiences of past occurrences from which we can predict information of the structural response in a probabilistic manner. When the excitation loading varies arbitrarily in time, the corresponding response will also be arbitrary in time. Such a response process deals with *the random vibration* and its characteristic properties can be determined by using statistical and probabilistic methods. A couple of examples of random loading excitations are shown in Fig. 2.1. For earthquakes, the acceleration of ground motion is recorded



**Fig. 2.1** Examples of excitations of random loadings. **a** An example of earthquake record. **b** An example of random sea wave

in different places as illustrated in Fig. 2.1a, and collected for all occurrences to get statistical information. For the sea wave, which always exists, the water surface elevation displays a random fluctuation as shown in Fig. 2.1b that is used as being the input function of wave loads. The problem of random vibrations has been studied for a long time and explained in many text books, see e.g., [1–8], and also reported in numerous papers, see e.g., [9–20]. An overview of historical developments is presented in [21]. In the random vibration theory of linear systems, statistical and probabilistic information of the input function is determined firstly from recorded or observed data that have been collected in the past. Then, by using a stochastic analysis procedure such as a spectral analysis, response statistical characteristics are calculated to be used further in the determination of structural behaviors in probabilistic terms. This subject is closely related to the probability theory and its applications. This chapter is devoted to introduction of the probability theory and stochastic treatment of structural dynamics which is thought to be useful to understand the essence of a probabilistic analysis. In the following sections, basic definitions of random processes and their statistical properties, which are needed in proceeding chapters, are briefly outlined.

## 2.2 Probability, Random Variables, Stochastic Processes, Probability Distribution and Density Functions

In the theory of random vibrations, some basic principles of the probability theory are applied. They are related to the concept of random phenomena such as random occurrences or outcomes of random experiments. In order to study random vibration, some terminology and definitions from the probability theory are briefly outlined in this section [22, 23].

- *Outcome*. The result of an experiment or occurrence of a natural phenomenon.
- *Random experiment*. An experiment that its outcomes are not predictable in advance.

- *Set*. A collection of individual elements in the domain  $D$ . The universal set  $U$  is defined if it contains every element in  $D$ . The null set  $\emptyset$  is defined if it contains no element.
- *Event*. A set of outcomes to which a probability is assigned.
- *Sample space*. A set of all possible outcomes of an experiment, denoted by  $S$ . Every outcome of the experiment is represented by a point in  $S$  called as a sample point.
- *Union*. The union of two events  $A$  and  $B$ , which is denoted by  $(A \cup B)$  or  $(A \text{ or } B)$ , is the set of all elements that belong to at least one of the sets  $A$  and  $B$ , shown in Fig. 2.2a.
- *Intersection*. The intersection of two events  $A$  and  $B$ , which is denoted by  $(A \cap B)$  or  $(A \text{ and } B)$ , is the set of elements that belong to both sets  $A$  and  $B$ , which is also referred to as a joint event of  $A$  and  $B$ , shown in Fig. 2.2b.
- *Complement*. The complement of an event  $A$ , denoted by  $\bar{A}$ , is the set containing all points in the sample space  $S$ , but not in the set  $A$ , shown in Fig. 2.2c.
- *Mutually exclusive*. Two events  $A$  and  $B$  are said to be mutually exclusive if they do not have common elements, i.e., the intersection of  $A$  and  $B$  is a null set  $(A \cap B = \emptyset)$ .
- *Collectively exhaustive*. The events  $B_1, B_2, \dots, B_n$  are said to be collectively exhaustive if their union covers all the events within the entire sample space, i.e.,  $P(B_1 \cup B_2 \cup \dots \cup B_n) = S$  where  $S$  is the sample space.

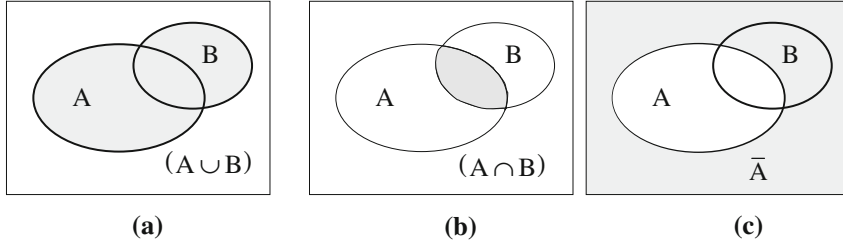
The union and intersection of the events  $A$  and  $B$ , and the compliment of the event  $A$  are shown in the Venn diagram in Fig. 2.2.

### 2.2.1 Probability Measure

The probability is a measure of outcomes of an event  $A$  among all outcomes of the experiment. It is denoted by  $P(A)$  and defined commonly in two ways as the *relative frequency* and the *classical* definitions [22]. In the frequency definition, a random experiment is repeated  $n$  times and if the event  $A$  occurs  $n_A$  times, then the probability of  $A$  is defined as the fraction of occurrences of  $A$  in  $n$  trials. Thus,

$$P(A) = \lim_{n \rightarrow \infty} \left( \frac{n_A}{n} \right) \quad (2.1)$$

The precision of  $P(A)$  depends on the number of trials. If the number of trials  $n$  approaches infinity, then  $P(A)$  tends to a definite limiting value. In the classical definition, all outcomes of the experiment are supposed to be equally likely, i.e., they have the same probability of occurrences. Then, counting the total number  $N$  of possible outcomes of the experiment, and from which the number  $N_A$  of the favorable outcomes to the occurrence of the event  $A$ , the probability of  $A$  is defined as,



**Fig. 2.2** Venn diagrams for two events. **a** Union. **b** Intersection. **c** Complement of A

$$P(A) = \frac{N_A}{N} \quad (2.2)$$

The probability of A,  $P(A)$ , is a number and satisfies the following three axioms [23].

1. The probability of an event A is a number between zero and one, i.e.,  $0 \leq P(A) \leq 1$ .
2. For a certain event S, the probability  $P(S)$  equals 1, i.e.,  $P(S) = 1$ .
3. The probability of the union of a number of mutually exclusive events, i.e., intersections are null sets, is the sum of probabilities of the events, i.e.,

$$P\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n P(A_i), \quad \text{where } P\left(\bigcup_{i=1}^n A_i\right) = P(A_1 \cup A_2 \cup \dots \cup A_n) \quad (2.3)$$

From these axioms it can be concluded [24] that

Probability of null Set:  $\dots \rightarrow P(\{\emptyset\}) = 0$

Probability of Complement:  $\rightarrow P(\bar{A}) = 1 - P(A) \quad (2.4)$

Probability of Union:  $\dots \rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$

One other probability measure having practical importance is the *conditional probability*, which is denoted by  $P(A|B)$ . It is defined as the probability of the event A given that the event B has occurred. In the probability terms, it is

$$\text{Conditional Probability: } \rightarrow P(A|B) = \frac{P(A \cap B)}{P(B)} \quad (2.5)$$

As it can be realized from Eq. (2.5), if the event B is a null (empty) set, i.e.,  $P(B) = 0$ , then a conditional probability is not defined. If the two events, A and B,

are not related in any way they are said to be independent events. The only condition of the independence is

$$\text{Independence, if and only if: } \rightarrow P(A \cap B) = P(A)P(B) \quad (2.6)$$

Using the conditional probability definition, the *total probability theorem* can be derived. This is expressed as, if  $B_1, B_2, \dots, B_n$  are collectively exhaustive events of the sample space  $S$  and  $A$  is an arbitrary event on  $S$ , then the total probability of  $A$  can be stated as the sum of all intersections, i.e.,

$$P(A) = \sum_{i=1}^n P(A \cap B_i) \quad (2.7a)$$

Substituting the intersection from Eqs. (2.5) into (2.7a), the total probability of  $A$  is written as,

$$\text{The total probability theorem: } \rightarrow P(A) = \sum_{i=1}^n P(A|B_i)P(B_i) \quad (2.7b)$$

which is the statement of the total probability theorem. Using the conditional probability, the total probability theorem and the commutative law of the events  $A$  and  $B$ , i.e.,  $P(A \cap B) = P(B \cap A)$ , the well-known *Bayes' theorem* can be stated as,

$$\text{The Bayes' theorem: } \rightarrow P(B_k|A) = \frac{P(A|B_k)P(B_k)}{\sum_{i=1}^n P(A|B_i)P(B_i)} \quad (2.8)$$

The Bayes' theorem helps for making decision under uncertainties that engineers confront frequently in the practice [24] provided that prior probabilistic models of uncertainties are available or determined previously by experiments.

### 2.2.2 Random Variables

In practice, outcomes of all experiments, even under equal conditions, are not unique and show discrepancies in values. For example, to find elasticity modulus  $E$  of a material, say steel, a number of equal samples have to be tested in laboratory. Each sample produces a specific value of  $E$ , which is mostly different than those obtained from other samples, although they may be very close together. As being the design value, we use an average value of  $E$  over the values obtained from all experiments made under the same condition. Here,  $E$  is a random variable which associates a unique numerical value with every outcome of an experiment. Thus, a *random variable (r.v.)* is a finite single valued function  $X(\cdot)$  which associates a real numerical value with every outcome of a random experiment [23]. An r.v.  $X$  can be thought as a measurement of outcomes of the random experiment. Its randomness comes from the

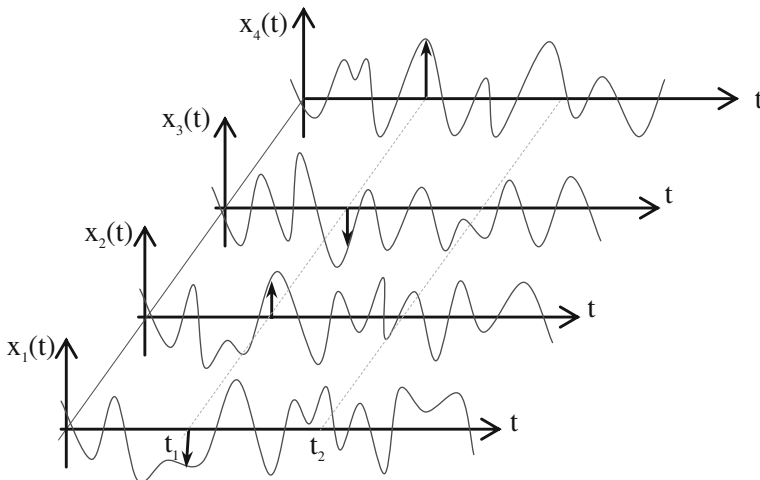
value that depends on the outcome of the experiment, which cannot be predicted exactly before the experiment is carried out. More generally, an r.v.  $X$  is a function that maps the sample space  $S$  into a real line with  $\pm\infty$  thrown in. There are two types of random variables, discrete and continuous. A *discrete random variable* is defined as, if an r.v. can take only a finite number of distinct values, then it is discrete, i.e., a discrete r.v. takes only a countable number of distinct values. A *continuous random variable* is defined as, if an r.v. can take an infinite number of possible values, then it is continuous, i.e., it is not defined at specific values, instead it is defined over an interval of values. A particular outcome of an r.v. is termed as a *random variate*. In the random vibration theory, an r.v.  $X$  is a function of time  $t$ , which means that the outcome of an experiment is time dependent. It is denoted by  $X(x, t)$  where  $x$  is a time-dependent outcome, i.e.,  $x = x(t)$ , which represents an excitation input function or a response function. In the random vibration theory, the probability information of random time functions  $x_i(t)$ , where  $(i = 1, 2, 3, \dots, n)$ , is used to determine statistical characteristics of an event represented by the r.v.  $X(x, t)$ , which involves in an ensemble process explained in the following section.

### 2.2.3 Stochastic Processes

As a short definition, a random process is an infinite collection of realizations of an r.v. In a similar way to the definition of an r.v., a random process is a mapping from the sample space into an ensemble of time functions known as sample functions. The r.v.  $X(x, t)$  for a fixed random  $x$  value, say  $x_1$ , is a specific time signal that it is called as *the realization* of the r.v.  $X(x, t)$  at  $x = x_1$ , which is denoted by  $x_1(t)$ . For a fixed time, say  $t_1$ , the r.v.  $X(x, t_1)$  is a time-independent r.v. that probability principles are applied. For both fixed values of  $x$  and  $t$ , say  $(x = x_1$  and  $t = t_1)$ , the r.v.  $X(x, t)$  will be a mere number with the value of  $X(x_1, t_1)$  [23]. The ensemble of all realizations of a time-dependent r.v. represents *the stochastic process* that we use the notation  $X(t)$  to indicate it, disregarding its dependence on the outcome  $x$ . Such an ensemble, which represents a stochastic process, is shown in Fig. 2.3 with four realizations, or samples,  $x_1(t)$ ,  $x_2(t)$ ,  $x_3(t)$  and  $x_4(t)$ . As indicated above, a stochastic process represents a single number, a time function, an r.v., and a process with time function and r.v. Thus,

1. if  $x$  and  $t$  are both fixed ( $x = x_1$  and  $t = t_1$ ), then  $X(x_1, t_1)$  is a single number,
2. if  $x$  is fixed ( $x = x_1$ ) and  $t$  is a variable, then  $X(x_1, t)$  is a time function as  $x_1(t)$ ,
3. if  $x$  is a variable and  $t$  is fixed ( $t = t_1$ ), then  $X(x, t_1)$  is a random variable at  $t = t_1$ ,
4. if  $x$  and  $t$  are both variables then  $X(x, t)$  is a stochastic process.

If we consider an infinite number of samples, at a specific time, say  $t = t_1$  as shown in Fig. 2.3, the stochastic process will be a continuous r.v. with the outcomes  $x$ , as  $(x_1, x_2, \dots, x_n)$  where  $(n \rightarrow \infty)$ . This r.v. is fully described by its probability characteristics explained in the next section.



**Fig. 2.3** Ensemble of time-dependent random variables with four samples

### 2.2.4 Probability Distribution and Density Functions

A probability measure of an event has been outlined in the [Sect. 2.2.1](#). In order to determine this measure, probability distributions and related functions are explained in this section. Let us define an event  $A$  such that all outcomes of the r.v.  $X(x, t_1)$ , in short  $X$ , between  $(x_1 \leq X \leq x_2)$  fall in, where  $x_1$  and  $x_2$  are the lower and upper bounds of the outcomes. The probability of the event  $A$  is sated as

$$P(A) = P(x_1 \leq X \leq x_2) \quad (2.9a)$$

It is the probability that the outcomes of  $X$  fall between the bounds  $x_1$  and  $x_2$ . This probability definition of the r.v.  $X$  can be extended to the probability definition of a stochastic process  $X(t)$ , of which the event is denoted by  $A(t)$ , i.e.

$$P(A(t)) = P(x_1(t) \leq X(t) \leq x_2(t)) \quad (2.9b)$$

will be a time function. If the outcomes of the event  $A(t)$  fall between  $-\infty$  and any realization  $x(t)$  in the region,  $(-\infty \leq x(t) \leq +\infty)$ , of the stochastic process  $X(t)$ , then by definition the probability is called as the *probability distribution function*, or the *cumulative distribution function* (CDF). It is denoted by  $F_X(x, t)$ , i.e.,

$$\text{CDF} : \rightarrow F_X(x, t) = P(-\infty \leq X(t) \leq x(t)) \quad (2.10a)$$

For a time independent r.v.  $X$ , it will be

$$F_X(x) = P(-\infty \leq X \leq x) \quad (2.10b)$$

The probability of the process  $X(t)$ , which is given by Eq.(2.9b), is stated in terms of the probability distributions at  $x_2(t)$  and  $x_1(t)$  as

$$P(x_1(t) \leq X(t) \leq x_2(t)) = F_X(x_2, t) - F_X(x_1, t) \quad (2.10c)$$

The probability distribution function  $F_X(x, t)$  satisfies the following properties.

$$\left. \begin{aligned} F_X(-\infty) &= 0 \\ F_X(+\infty) &= 1 \end{aligned} \right\} \text{ thus } 0 \leq F_X(x, t) \leq 1 \quad (2.11)$$

$$F_X(x_1, t) \leq F_X(x_2, t) \quad \text{for } (x_1(t) \leq x_2(t))$$

One other important definition in the probability theory is the *probability density function* (PDF), which determines probability characteristics of the stochastic process. It is related to the probability that the stochastic process  $X(t)$  lies in the interval  $x(t)$  to  $(x(t) + dx(t))$ . It is stated from Eq. (2.10c)

$$P(x(t) \leq X(t) \leq x(t) + \Delta x(t)) = F_X(x(t) + \Delta x(t)) - F_X(x(t)) \quad (2.12a)$$

and taking the limit of this statement when  $(\Delta x(t) \rightarrow 0)$  it will be

$$P(x(t) \leq X(t) \leq x(t) + dx(t)) = \lim_{\Delta x(t) \rightarrow 0} (F_X(x(t) + \Delta x(t)) - F_X(x(t))) \quad (2.12b)$$

$$\text{or } P(x(t) \leq X(t) \leq x(t) + dx(t)) = f_X(x, t) dx(t) \quad (2.12c)$$

Thus, the probability that  $X(t)$  lies in the interval  $x(t)$  to  $(x(t) + dx(t))$  is the area of a function  $f_X(x, t)$  in the interval  $dx(t)$ . As it may be seen from Eqs. (2.12b) and (2.12c), this function is the derivative of the probability distribution  $F_X(x, t)$ , which is

$$\text{Probability Density Function (PDF): } \rightarrow f_X(x, t) = \frac{\partial F_X(x, t)}{\partial x} \quad (2.13)$$

The function  $f_X(x, t)$  is called as the PDF of the stochastic process  $X(t)$ . For a *continuous process*  $X(t)$ , the probability distribution  $F_X(x, t)$  can be stated from Eq. (2.13) as,

$$F_X(x, t) = \int_{-\infty}^x f_X(\xi, t) d\xi \quad (2.14)$$

With this definition the probability of the process  $X(t)$  in the region of  $x_1(t) \leq X(t) \leq x_2(t)$  is written

$$P(x_1(t) \leq X(t) \leq x_2(t)) = \int_{x_1}^{x_2} f_X(\xi, t) d\xi \quad (2.15)$$

Since the probability distribution of a continuous stochastic process  $X(t)$  is an increasing function and  $F_X(+\infty) = 1$  the PDF satisfies the conditions



$$\begin{aligned}
 f_X(x, t) &\geq 0 \\
 \int_{-\infty}^{\infty} f_X(x, t) dx &= 1
 \end{aligned}
 \tag{2.16}$$

The probability distribution and density functions of a continuous process are smooth function as shown in Fig. 2.4. For discrete and mixed processes, the probability distribution  $F_X(x, t)$  is respectively staircase and discontinuous functions [23]. If the distribution function  $F_X(x, t)$  is discontinues but not staircase then the process is called as a mixed process. The probability density function  $f_X(x, t)$  of a discrete process is in the form of impulses. Each impulse, which is defined as a probability mass, is equal to the corresponding step size of the distribution function as shown in Fig. 2.5, e.g.,  $p_X(x_i) = F_X(x_i) - F_X(x_i - \varepsilon)$  at  $X = x_i$  where  $\varepsilon$  is a small positive number. The sum of all probability masses is equal to 1. The probability mass of a continuous process at  $X = x_i$  can be stated from the definition as

$$\text{Probability Mass: } \rightarrow p_X(x_i, t) = f_X(x_i, t) dx \tag{2.17}$$

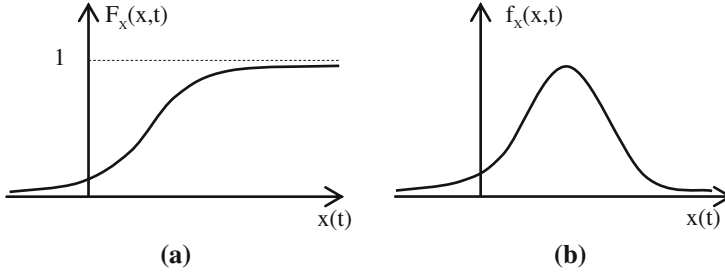
Discrete random processes have the same properties of continuous processes provided that probability masses are used instead of probability density functions and integrations of continuous random processes are replaced with summations for discrete processes. In practical applications, an event such as a structural response contains usually outputs of multiple random variables with joint distributions. This matter and related subjects are outlined briefly in the next section.

### 2.2.4.1 Distribution and Density Functions of Joint Random Variables

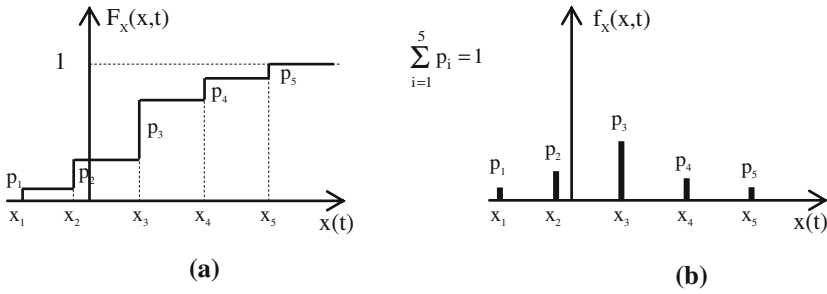
In the previous section, probability distribution and density function of a single random process  $X(t)$  is presented. If an event is a collection of outputs of multiple random variables and processes, the probability distribution and density functions are somewhat different than those of a single variable or process. For simplicity, we use only two random variables  $X$  and  $Y$ . For multiple random variables, the same principles of two random variables can be applied. The joint probability distribution function of multiple random variables are defined as the probability of their intersection for  $(-\infty \leq X \leq x)$  and  $(-\infty \leq Y \leq y)$ , i.e.,

$$\text{Joint CDF: } \rightarrow F_{XY}(x, y) = P(X \leq x \cap Y \leq y) \tag{2.18}$$

In other words, it is the probability that all outcomes of random variables  $X$  and  $Y$  fall in the region  $(X \leq x)$  and  $(Y \leq y)$ , where  $x$  and  $y$  may be either time dependent or time invariant. If they are time dependent the random variables  $X(t)$  and  $Y(t)$  become joint random processes as being similar to a single random



**Fig. 2.4** Probability distribution and density functions of a continuous process  $X(t)$ . **a** Probability distribution function. **b** Probability density function



**Fig. 2.5** Probability distribution and density functions of a discrete process  $X(t)$ . **a** Probability distribution function **b** Probability density function

process explained in the previous section. The joint probability distribution function  $F_{XY}(x, y)$  satisfies the following properties.

$$\begin{aligned}
 F_{XY}(x, y) &\geq 0 \quad \text{for } (-\infty \leq x \leq \infty) \text{ and } (-\infty \leq y \leq \infty) \\
 F_{XY}(-\infty, y) &= F_{XY}(x, -\infty) = 0 \\
 F_{XY}(\infty, \infty) &= 1 \\
 F_Y(y) &= F_{XY}(\infty, y) \quad \text{and} \quad F_X(x) = F_{XY}(x, \infty)
 \end{aligned}
 \tag{2.19}$$

For continuous random variables,  $X$  and  $Y$ , the joint probability density function  $f_{XY}(x, y)$  can be obtained from the derivatives of the joint probability distribution function as written by

$$\text{Joint PDF: } \rightarrow f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y}
 \tag{2.20}$$

It is related to the probability that  $X$  and  $Y$  lie in the intervals  $x$  and  $(x + dx)$ , and  $y$  and  $(y + dy)$  respectively, i.e.,

$$P((x \leq X \leq x + dx) \cap (y \leq Y \leq y + dy)) = f_{XY}(x, y) dx dy
 \tag{2.21}$$

which defines the probability mass at  $x$  and  $y$ . For discrete r.v. at  $X = x_i$  and  $Y = y_j$ , it becomes  $p_{XY}(x_i, y_j)$  which is

$$\text{Joint Probability Mass: } \rightarrow p_{XY}(x_i, y_i) = f_{XY}(x_i, y_i) dx dy \quad (2.22)$$

The joint probability and joint probability distribution function are expressed from the joint density function as

$$P((x_1 \leq X \leq x_2) \cap (y_1 \leq Y \leq y_2)) = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_{XY}(\xi, \eta) d\eta d\xi \quad (2.23)$$

$$F_{XY}(x, y) = \int_{-\infty}^x \int_{-\infty}^y f_{XY}(\xi, \eta) d\eta d\xi$$

The joint probability density function satisfies the following requirements

$$f_{XY}(x, y) \geq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(\xi, \eta) d\eta d\xi = F_{XY}(-\infty, \infty) = 1 \quad (2.24)$$

Two special cases of joint probability distribution and density functions are presented in the following sections.

#### 2.2.4.2 Marginal Probability Distribution and Density Functions

In the case of multiple random variables, the statistics of an individual variable is called as *marginal*. The related probability distribution and density functions are also called as marginal probability distribution and density functions. From Eq. (2.19), the marginal probability distributions of joint random variables  $X$  and  $Y$  can be written

$$\text{Marginal CDF of X: } \rightarrow F_X(x) = F_{XY}(x, \infty) = \int_{-\infty}^x \int_{-\infty}^{\infty} f_{XY}(\xi, \eta) d\eta d\xi \quad (2.25a)$$

$$\text{Marginal CDF of Y: } \rightarrow F_Y(y) = F_{XY}(\infty, y) = \int_{-\infty}^y \int_{-\infty}^{\infty} f_{XY}(\xi, \eta) d\xi d\eta \quad (2.25b)$$

The marginal probability density functions  $f_X(x)$  and  $f_Y(y)$  can be obtained from the derivatives of  $F_X(x)$  and  $F_Y(y)$  or from integrations of the joint density function as stated

$$\text{Marginal PDF of } X: \rightarrow f_X(x) = \frac{\partial F_X(x)}{\partial x} = \int_{-\infty}^{\infty} f_{XY}(x, y) dy \quad (2.26a)$$

$$\text{Marginal PDF of } Y: \rightarrow f_Y(y) = \frac{\partial F_Y(y)}{\partial y} = \int_{-\infty}^{\infty} f_{XY}(x, y) dx \quad (2.26b)$$

Marginal probability density function is used to calculate the probability of particular events required in the *Bayes' theorem* or in the calculation of a conditional probability and in the total probability theorem.

### 2.2.4.3 Conditional Probability Distribution and Density Functions

For two events  $A$  and  $B$ , the conditional probability has been defined in Eq. (2.5). Here, the conditional probability distribution and density function of two joint random variables  $X$  and  $Y$  will be explained. The joint probability distribution (joint cumulative distribution function, CDF) of the random variables  $X$  and  $Y$  has been defined in Eq. (2.18). It is now assumed that one of them, say  $Y$ , takes a specific value, i.e.,  $Y = y$ , and the probability distribution of other one ( $X$ ) is defined as

$$F_X(x | y) = P(X \leq x | Y = y) \quad (2.27)$$

It is the probability that  $(-\infty \leq X \leq x)$  on the condition  $(Y = y)$ , which is called the conditional probability density function. By using the conditional probability given in Eq. (2.5) it can be written

$$\text{Conditional CDF: } \rightarrow \begin{cases} F_X(x | y) = \frac{F_{XY}(x, y)}{F_Y(y)} \\ F_Y(y | x) = \frac{F_{XY}(x, y)}{F_X(x)} \end{cases} \quad (2.28)$$

As an example of the conditional CDF function, let us assume that the r.v.  $Y$  is defined to be a subset of the r.v.  $X$  as  $(Y = x \geq a)$  [23, 24]. The intersection of random variables  $X$  and  $Y$  is defined as

$$(X \leq x) \cap (Y \leq y) \rightarrow (X \leq x) \cap (X \geq a) = \begin{cases} x - a & \text{if } x \geq a \\ 0 & \text{if } x < a \end{cases} \quad (2.29)$$

The marginal distribution of  $Y$  is

$$F_Y(y) = P(X \geq a) = F_X(\infty) - F_X(a) = 1 - F_X(a) \quad (2.30a)$$

and the joint distribution of  $X$  and  $Y$  is obtained by using Eqs. (2.18) and (2.29)

$$F_{XY}(x, y) = \begin{cases} F_X(x) - F_X(a) & \text{if } x \geq a \\ 0 & \text{if } x < a \end{cases} \quad (2.30b)$$

Using Eq. (2.28) the conditional distribution of  $X$ , such that  $(X|X \geq a)$ , is calculated from

$$F_X(x | y) = \frac{F_{XY}(x, y)}{F_Y(y)} \rightarrow F_X(x|y) = \begin{cases} \frac{F_X(x) - F_X(a)}{1 - F_X(a)} & \text{if } x \geq a \\ 0 & \text{if } x < a \end{cases} \quad (2.30c)$$

The conditional probability density functions of joint r.v. are defined in the same manner as the conditional CDF functions

$$\text{Conditional PDF: } \rightarrow f_X(x | y) = \frac{f_{XY}(x, y)}{f_Y(y)} \quad \text{and} \quad f_Y(y | x) = \frac{f_{XY}(x, y)}{f_X(x)} \quad (2.31)$$

By using conditional probability density functions the conditional probability distributions are calculated from

$$F_X(x | y) = \int_{-\infty}^x f_X(\xi | y) d\xi \quad \text{and} \quad F_Y(y | x) = \int_{-\infty}^y f_Y(\eta | x) d\eta \quad (2.32)$$

The conditional density functions satisfy the requirements of an ordinary PDF given in Eq. (2.16), i.e.,

$$\left. \begin{aligned} f_X(x | y) &\geq 0 \\ \int_{-\infty}^{\infty} f_X(x | y) dx &= 1 \end{aligned} \right\} \quad \text{and} \quad \left\{ \begin{aligned} f_Y(y | x) &\geq 0 \\ \int_{-\infty}^{\infty} f_Y(y | x) dy &= 1 \end{aligned} \right. \quad (2.33)$$

If joint random variables are independent, their probability distribution and density functions become products of corresponding marginal functions as similar to the probability of intersection of independent events given in Eq. (2.6). This subject is explained in the following section.

#### 2.2.4.4 Independent Random Variables

Definition of independent events has been given in Eq. (2.6). The only condition is that the probability of their intersection is the product of probabilities of individual events. The same rule also applies for the probability distribution of joint r.v. as to be independent. Thus, for independent random variables  $X$  and  $Y$ , the joint distribution is stated from Eq. (2.18) as written

$$\text{Joint CDF of independent r.v.: } \rightarrow F_{XY}(x, y) = P(X \leq x) P(Y \leq y) \quad (2.34)$$

from which it is seen that the joint probability distribution function is a product of marginal distribution functions, i.e.,

$$F_{XY}(x, y) = F_X(x) F_Y(y) \quad (2.35)$$

For independent random variables, conditional probability distributions become marginal as stated from Eq. (2.28),

$$F_X(x | y) = F_X(x) \quad \text{and} \quad F_Y(y | x) = F_Y(y) \quad (2.36)$$

In a similar way, the joint probability density function of independent continuous random variables  $X$  and  $Y$  becomes a product of marginal PDF, which is written

$$\text{Joint PDF of independent r.v.:} \rightarrow f_{XY}(x, y) = f_X(x) f_Y(y) \quad (2.37)$$

These statements can be extended for multiple independent random variables. The calculation of statistical values of events comprising independent r.v. is simply carried out by integrations, or summations for discrete variables, over individual r.v. It does not require a multiple integrations process, which is usually more time consuming and more complicated. In the following section, calculations of the statistical values of r.v. and functions are presented.

### 2.3 Mean Values, Probability Moments, and Variances of Random Variables and Random Functions

Since outcomes of r.v. are unpredictable and not specific values, the average value, which is also called as the *Mean Value*, of all outcomes of an r.v. is an indicative measure in engineering applications. It is also the *Expected Value* that the most outcomes of the r.v. is likely to occur. It is used as a design value in the deterministic analyses. In the probabilistic analyses, it is one of statistical parameters describing the probability distribution of the r.v. For an r.v.  $X$ , the definition of the mean value is as follows,

$$\text{Mean Value of an r.v.:} \rightarrow m_X = E[X] = \begin{cases} \int_{-\infty}^{\infty} x f_X(x) dx \rightarrow \text{continuous} \\ \sum_{i=1}^n x_i p_X(x_i) \rightarrow \text{discrete} \end{cases} \quad (2.38)$$

where  $m_X$  is the mean value and  $E[.]$  denotes an expected value,  $f_X(x)$  is the probability density function of a continuous r.v.  $X$ ,  $p_X(x_i)$  is the probability mass of a discrete r.v.  $X$ . In Eq. (2.38), having replaced the marginal PDF  $f_X(x)$  by the conditional PDF  $f_X(x|y)$ , a conditional expected value is defined as

$$\text{Conditional Mean Value: } \rightarrow m_{X|Y} = E[X|Y] = \int_{-\infty}^{\infty} x f_X(x|y) dx \quad (2.39)$$

The mean value of an r.v. determines only the gravity center of its PDF. It does not provide full statistical information of the r.v. which is possible only if its probability moments are known. The *Probability Moment* that corresponds to the  $n$ th moment of the area of the PDF of an r.v. with respect to the origin is defined

$$\text{Prob. Moment of an r.v.: } \rightarrow (m_X)_n = E[X^n] = \begin{cases} \int_{-\infty}^{\infty} x^n f_X(x) dx \rightarrow \text{con.} \\ \sum_{i=1}^n x_i^n p_X(x_i) \rightarrow \text{dis.} \end{cases} \quad (2.40)$$

where  $n$  is an integer number indicating the order (degree) of the probability moment. As it is seen from Eq. (2.38) the expected value  $E[X]$  is the first order probability moment. The most statistical informative probability moments are defined with respect to the expected value rather than the origin. They are called as the *Central Probability Moments* which are defined as, for continuous r.v.,

$$\text{Central Prob. Mom.: } \rightarrow (\mu_X)_n = E[(X - m_X)^n] = \int_{-\infty}^{\infty} (x - m_X)^n f_X(x) dx \quad (2.41)$$

As it can be seen from Eq. (2.41) the first order central moment equals zero, i.e.,  $(\mu_X)_1 = 0$ . The second-order central moment is especially important since it defines the variance of the r.v. It provides a measure of the spread or the randomness of an r.v. and it determines the effective width of the PDF. It is defined

$$\text{Variance: } \rightarrow \sigma_X^2 = E[(X - m_X)^2] = \int_{-\infty}^{\infty} (x - m_X)^2 f_X(x) dx \quad (2.42a)$$

The variance can also be stated in terms of the second probability moment and the expected value of the r.v.  $X$ . From Eq. (2.42) it is written

$$\sigma_X^2 = E[(X - m_X)^2] = E[X^2 - 2m_X X + m_X^2] = (m_X)_2 - m_X^2 \quad (2.42b)$$

The square root of the variance is called the *Standard Deviation*,  $\sigma_X$ , and the ratio between  $\sigma_X$  and  $m_X$  is called as the *Coefficient of Variation*, which are

$$\begin{aligned} \text{Standard Deviation (SD)} & \rightarrow \sigma_X = \sqrt{\text{Variance}} \\ \text{Coefficient Of Variation (COV)} & \rightarrow V_X = \frac{\sigma_X}{m_X} \end{aligned} \quad (2.43)$$

One other important measure of an r.v. is the *Coefficient of Skewness*,  $\gamma_1$ , which provides information about the shape of the PDF. It is defined as

$$\text{Coefficient of Skewness: } \rightarrow \gamma_1 = \frac{(\mu_X)_3}{\sigma_X^3} = E\left[(X - m_X)^3 / \sigma_X^3\right] \quad (2.44)$$

If this coefficient is zero, then the PDF is symmetric about its center point, otherwise it is asymmetric as shown in Fig. 2.6. A useful, but less common, measure of an r.v. is the *Coefficient of Kurtosis*,  $\gamma_2$ . It is defined

$$\text{Coefficient of Kurtosis: } \rightarrow \gamma_2 = \frac{(\mu_X)_4}{\sigma_X^4} = E\left[(X - m_X)^4 / \sigma_X^4\right] \quad (2.45)$$

Kurtosis determines the flatness of the r.v. It is a measure of whether the PDF is peaked or flat relative to a normal distribution of which ( $\gamma_2 = 3$ ). The PDF of an r.v. with high kurtosis tends to have a distinct peak near the mean declining rather rapidly and have heavy tails. Low kurtosis indicates that the PDF has a flat top near the mean rather than a sharp peak. The CDF and PDF of an r.v. are usually described by its standard deviation and mean value,  $\sigma_X$  and  $m_X$ . So far, we have presented properties of r.v., but in most practical applications we encounter functions of r.v. The probabilistic properties of such functions are presented in the next section.

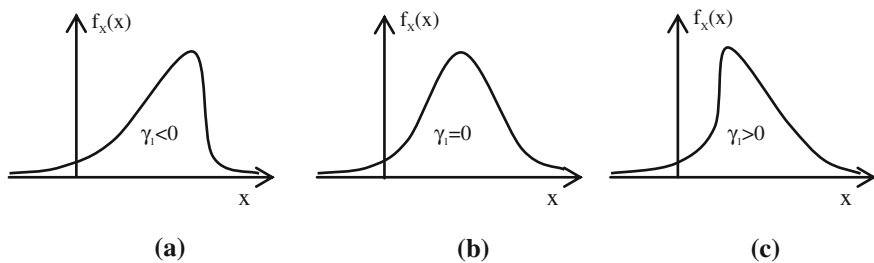
### 2.3.1 Functions of Random Variables

In engineering applications a response quantity is usually a function of r.v. that needs to be determined in probabilistic terms. The r.v. in its content constitute probabilistic input parameters that fully described by their CDF or PDF. Given the random input parameters or variables, the probabilistic and statistical characteristics of random response functions are required in the calculations of their expected values and probabilities of occurrences of specific events under desired conditions, such as probability of failure due to fatigue damage or ultimate loading conditions. The response functions that contain r.v. are denoted by  $Y$  and  $Z$  as defined

$$\begin{aligned} \text{Function of one r.v. } X, \dots, \dots: & \rightarrow Y = g(X) \\ \text{Function of multiple r.v., i.e. } X \text{ and } Y: & \rightarrow Z = g(X, Y) \end{aligned} \quad (2.46)$$

The statistical values of random functions,  $Y = g(X)$  or  $Z = g(X, Y)$ , are defined and calculated in the same manner of r.v. explained in Sect. 2.3. They are explained below for functions of one r.v. and multiple random variables.





**Fig. 2.6** Shapes of PDF with different coefficients of skewness  $\gamma_1$ . **a** Negative skewness. **b** Zero skewness. **c** Positive skewness

### 2.3.1.1 Function of One Random Variable

The function of one r.v. is defined in Eq. (2.46). Its mean value and variance are defined as

$$\text{Mean of a r.f.:} \quad \rightarrow \quad m_{g(X)} = E[g(X)] = \int_{-\infty}^{\infty} g(X) f_X(x) dx \quad (2.47a)$$

$$\text{Variance of a r.f.:} \quad \rightarrow \quad \sigma_{g(X)}^2 = \int_{-\infty}^{\infty} (g(X) - m_{g(X)})^2 f_X(x) dx \quad (2.47b)$$

Having used the series expansion of the function  $g(X)$  at the mean value of  $X$ , it is stated that

$$g(X) = g(m_X) + (X - m_X) g'(m_X) + \frac{(X - m_X)^2}{2} g''(m_X) + \dots \quad (2.48)$$

where  $g^{(n)}(m_X)$ , with  $(n = ', ', \dots)$ , is the  $n$ th derivative of  $g(X)$  evaluated at  $X = m_X$ . Using Eq. (2.48) the mean and variance of the function  $g(X)$  can be calculated approximately [23] from,

$$\begin{aligned} m_{g(X)} &\simeq g(m_X) + \frac{\sigma_X^2}{2} g''(m_X) + \dots + \frac{(\mu_X)_n}{n!} g^{(n)}(m_X) \\ \sigma_{g(X)}^2 &\simeq (g'(m_X))^2 \sigma_X^2 \end{aligned} \quad (2.49)$$

The moments of the random function  $g(X)$  are defined as similar to one r.v.  $X$ ,

$$\text{Prob. Moment of a r.f.:} \quad \rightarrow \quad (m_{g(X)})_n = \int_{-\infty}^{\infty} g^n(X) f_X(x) dx \quad (2.50a)$$

$$\text{Central Prob. Mom. of a r.f.: } \rightarrow \left( \mu_{g(X)} \right)_n = \int_{-\infty}^{\infty} (g(X) - m_Y)^n f_X(x) dx \quad (2.50b)$$

The PDF of a random function  $Y = g(X)$  is calculated in terms of PDF of the r.v.  $X$  [23] from,

$$\text{PDF of a r.f. } Y = g(X): \rightarrow f_Y(y) = \sum_{i=1}^k \frac{f_X(x_i)}{|g'(x_i)|} \quad (2.51)$$

where  $x_i$  ( $i = 1, 2, \dots, k$ ) are the real roots of the equation  $y = g(x)$  in terms of  $y$ ,  $g'(x_i)$  is the derivative of  $g(X)$  evaluated at  $X = x_i$  and |.| denotes the absolute value. As a demonstration, it is assumed that the function  $g(X)$  is given

$$Y = g(X) = ax^{-2} \quad (2.52a)$$

and the PDF of  $Y$  will be calculated. The real roots of  $g(X)$  are calculated for ( $ay > 0$ ) as written

$$y = ax^{-2} \rightarrow x_1 = \sqrt{ay^{-1}} \text{ and } x_2 = -\sqrt{ay^{-1}} \text{ for } (ay > 0) \quad (2.52b)$$

For different signs of  $y$  and  $a$ , i.e., ( $ay < 0$ ), the solution is not real. The derivatives evaluated at the roots are

$$g'(x) = -2ax^{-3} \rightarrow g'(x_1) = -2\sqrt{a^{-1}y^3} \text{ and } g'(x_2) = 2\sqrt{a^{-1}y^3} \quad (2.52c)$$

Using Eq. (2.51) the PDF of  $Y$  is obtained for ( $ay > 0$ ) as written

$$f_Y(y) = \frac{1}{2} \sqrt{\frac{a}{y^3}} \left[ f_X\left(\sqrt{ay^{-1}}\right) + f_X\left(-\sqrt{ay^{-1}}\right) \right] \text{ for } (ay > 0) \quad (2.52d)$$

For ( $ay < 0$ ), the PDF of  $Y$  will be zero, i.e.,  $f_Y(y) = 0$ .

### 2.3.1.2 Functions of Multiple Random Variables

As similar to the case of one continuous r.v.  $X$ , a random function  $Z$  of two, or more, jointly distributed random variables  $X$  and  $Y$  is defined to be  $Z = g(X, Y)$ . Its expected (mean) value is calculated [23, 24] from

$$\text{Mean of a r.f. } Z: \rightarrow E[Z] = E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(X, Y) f_{XY}(x, y) dx dy \quad (2.53)$$

Once the PDF of  $Z$ , i.e.,  $f_Z(z)$ , is known, all statistical values and definitions of a r.v.  $X$  is also valid for the r.f.  $Z$ . However, calculation of the PDF of  $Z$  is not as

simple as that of a function of one r.v. It can be calculated for each function  $Z = g(X, Y)$ . A general calculation can also be obtained through the joint PDF of random functions, say  $Z$  and  $W$ , which are defined

$$Z = g(X, Y) \quad \text{and} \quad W = h(X, Y) \quad (2.54)$$

The joint PDF of the functions  $Z$  and  $W$ ,  $f_{ZW}(z, w)$ , is calculated [23] from

$$\text{Joint PDF of two r.f } Z \text{ and } W: \rightarrow f_{ZW}(z, w) = \sum_{i=1}^k \frac{f_{XY}(x_i, y_i)}{|J(x_i, y_i)|} \quad (2.55)$$

where  $(x_i, y_i) (i = 1, 2, \dots, k)$  are all real solutions of the equations

$$g(x_i, y_i) = z \quad \text{and} \quad h(x_i, y_i) = w \quad (2.56)$$

in terms of  $z$  and  $w$ , and  $|J(x_i, y_i)|$  is the absolute value of the determinant of the *Jacobian* for the transformation given in Eq. (2.54) at the solutions  $(x_i, y_i)$ . It is stated in general as,

$$|J(x, y)| = \begin{vmatrix} \frac{\partial g(x, y)}{\partial x} & \frac{\partial g(x, y)}{\partial y} \\ \frac{\partial h(x, y)}{\partial x} & \frac{\partial h(x, y)}{\partial y} \end{vmatrix} \quad (2.57)$$

Having obtained the joint PDF of the random variables  $Z$  and  $W$  as given in Eq. (2.55) the marginal PDF of  $Z$  can be calculated similarly to Eq. (2.26a) from,

$$\text{Marginal PDF of a r.f. } Z: \rightarrow f_Z(z) = \int_{-\infty}^{\infty} f_{ZW}(z, w) dw \quad (2.58)$$

A useful application of a random function  $Z$  of two random variables  $X$  and  $Y$  is to calculate joint moments of these random variables. If the function  $Z = g(X, Y)$  is defined as

$$Z = g(X, Y) = X^n Y^m \quad (2.59)$$

its expected value will be the joint moments of order  $(n + m)$  of the random variables  $X$  and  $Y$ . From Eq. (2.53) it is written that

$$\text{Joint Mom. of r.v. } X \text{ and } Y: \rightarrow \begin{cases} E[X^n Y^m] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^n y^m f_{XY}(x, y) dx dy \\ (m_{XY})_{nm} = E[X^n Y^m] \end{cases} \quad (2.60)$$

As it is seen from Eq. (2.60), if  $(n = 1, m = 0)$  the marginal mean  $E[X]$  is obtained while for  $(n = 0, m = 1)$  gives the marginal mean  $E[Y]$ . In a similar way to the

joint moments, the central joint moments of order  $(n + m)$  of the random variables  $X$  and  $Y$  are defined

$$E[(X - m_X)^n (Y - m_Y)^m] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - m_X)^n (y - m_Y)^m f_{XY}(x, y) dx dy \quad (2.61)$$

$$(\mu_{XY})_{nm} = E[(X - m_X)^n (Y - m_Y)^m]$$

It can be seen from this statement that, if  $(n = 2, m = 0)$  the variance  $\sigma_X^2$  and if  $(n = 0, m = 2)$  the variance  $\sigma_Y^2$  are obtained. The joint central moment for  $(n = 1, m = 1)$  is called as the *Covariance* of the joint random variables  $X$  and  $Y$ , which is defined

$$\text{Covariance of Joint r.v. } X \text{ and } Y : \rightarrow \sigma_{XY} = E[(X - m_X)(Y - m_Y)]$$

$$\sigma_{XY} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - m_X)(y - m_Y) f_{XY}(x, y) dx dy = (m_{XY} - m_X m_Y) \quad (2.62)$$

The ratio between the covariance and the product of marginal standard deviations is called as the *Correlation Coefficient*, which is defined

$$\text{Correlation Coefficient of r.v. } X \text{ and } Y : \rightarrow \rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} \quad (2.63)$$

The correlation coefficient is the normalized version of the covariance and it satisfies the condition of  $(-1 \leq \rho_{XY} \leq 1)$  [24]. It is a measure of dependency between the random variables  $X$  and  $Y$ . If they are not correlated, then the correlation coefficient becomes zero and, for a full correlation, it becomes  $\rho_{XY} = \pm 1$ . It can be seen from Eq. (2.62) that, for independent random variables  $X$  and  $Y$ , the covariance will be zero and accordingly the correlation coefficient will also be zero, therefore, the independent random variables are also uncorrelated. But, uncorrelated random variables are not necessarily independent, i.e.,

$$\begin{aligned} \text{Uncorrelated r.v. } X \text{ and } Y : & \rightarrow \rho_{XY} = 0, \text{ but not necessarily independent} \\ \text{Independent r.v. } X \text{ and } Y : & \rightarrow \rho_{XY} = 0, \text{ thus uncorrelated} \end{aligned} \quad (2.64)$$

Once the PDF of random variables are determined their statistical values can be calculated as explained above. In the following section, some useful probability distributions that used frequently in practice are presented.

### 2.3.2 Some Useful Probability Distributions

In this section, some probability distributions of continuous r.v. and their properties, which are used in practical applications mostly, are presented briefly. More information about the distributions and a more complete list of distribution types can be found in text books of statistical distributions, e.g., [25–27].

#### 2.3.2.1 Normal (Gaussian) Distribution

The *Normal* or *Gaussian* univariate probability density function of a r.v.  $X$  is the one mostly used in practice. It is defined in general as

$$\text{Normal PDF: } \rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right] \quad (2.65a)$$

in which  $m$  and  $\sigma$  are respectively the mean and standard variation of  $X$ . The corresponding CDF is calculated from

$$\text{Normal CDF: } \rightarrow F_X(x) = \int_{-\infty}^x f_X(\xi) d\xi = F_Y(x) = \Phi\left(\frac{x-m}{\sigma}\right) \quad (2.65b)$$

where  $\Phi(\cdot)$  is called as the *Standard Normal Distribution* function and its PDF is denoted by  $\varphi(\cdot)$ , which are defined

$$\text{Standard Normal PDF: } \rightarrow \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad (2.65c)$$

$$\text{Standard Normal CDF: } \rightarrow \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2} du \quad (2.65d)$$

The central moments of  $X$  and  $|X|$  are calculated [23] from

$$\text{Central Mom. of } X: \rightarrow \begin{cases} \mu_n = 1.3 \dots (n-1) \sigma^n & \text{for } n \text{ even} \\ \mu_n = 0 & \text{for } n \text{ odd} \end{cases} \quad (2.65e)$$

$$\text{Central Mom. of } |X|: \rightarrow \begin{cases} (\mu_{|X|})_n = 1.3 \dots (n-1) \sigma^n & \text{for } n = 2k \\ (\mu_{|X|})_n = \sqrt{\frac{2}{\pi}} 2^k k! \sigma^n & \text{for } n = 2k + 1 \end{cases} \quad (2.65f)$$

Using Eq. (2.65e) the coefficients of skewness  $\gamma_1$  and kurtosis  $\gamma_2$  of a normal r.v. are calculated from Eqs. (2.44) and (2.45) as written

$$\begin{aligned}\text{Coefficient of Skewness of a Normal r.v.: } &\rightarrow \gamma_1 = 0 \\ \text{Coefficient of Kurtosis of a Normal r.v.: } &\rightarrow \gamma_2 = 3\end{aligned}\quad (2.65h)$$

If multivariate normal variables are involved in a process, then a multivariate normal PDF will be required. In this case, a vector process is used and the multivariate normal PDF is stated as,

$$\text{Multivariate Normal PDF: } \rightarrow f_{\tilde{X}}(\tilde{x}) = \left(\frac{1}{2\pi}\right)^{\frac{p}{2}} \frac{1}{\sqrt{|\rho|}} \exp\left(-\frac{\chi^2}{2}\right) \quad (2.66a)$$

where  $\tilde{X}$  is a vector of p-dimensional r.v.,  $\tilde{x}$  is a vector of their realizations and  $\chi^2$  is a scalar calculated from the product

$$\text{Scalar: } \rightarrow \chi^2 = (\tilde{x} - \tilde{m})^T \rho^{-1} (\tilde{x} - \tilde{m}) \quad (2.66b)$$

in which  $\tilde{m}$  is a vector of mean values and  $\rho$  is the covariance matrix of  $\tilde{X}$  and  $|\rho|$  in Eq. (2.66a) denotes the determinant of  $\rho$ . These definitions are written

$$\begin{aligned}\text{Vector of Multivariate r.v.: } &\rightarrow \tilde{X} = \{X_1, X_2, \dots, X_p\}^T \\ \text{Vector of realizations of } \tilde{X}: &\rightarrow \tilde{x} = \{x_1, x_2, \dots, x_p\}^T\end{aligned}\quad (2.66c)$$

$$\text{Vector of mean values of } \tilde{X}: \rightarrow \tilde{m} = \{m_1, m_2, \dots, m_p\}^T \quad (2.66d)$$

The covariance matrix  $\rho$  is defined as

$$\text{Covariance matrix of } \tilde{X}: \rightarrow \rho = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdot & \sigma_{1p} \\ \sigma_{21} & \sigma_2^2 & \cdot & \sigma_{2p} \\ \cdot & \cdot & \cdot & \cdot \\ \sigma_{p1} & \sigma_{p2} & \cdot & \sigma_p^2 \end{bmatrix} \quad (2.66e)$$

As it is seen from Eq. (2.66e), the diagonal terms of this matrix are the variances of the r.v.  $X_i$ . For uncorrelated r.v., the off diagonal terms will be zero and the matrix becomes diagonal.

### 2.3.2.2 Lognormal Distribution

One other commonly used distribution in practice is the *Lognormal Distribution*. If the r.v.  $X$  has a Normal distribution with the mean and variance,  $m_X$  and  $\sigma_X^2$ , then the r.v.  $Y = e^X$  is said to be log normally distributed. It is written as

$$\text{Exponential Function of } X: \rightarrow Y = e^X \quad \text{and} \quad X = \ln Y \quad (2.67a)$$

Using Eq. (2.51), the PDF of the r.v.,  $Y = e^X$ , can be obtained as written

$$\text{Lognormal PDF: } \rightarrow f_Y(y) = \frac{1}{\sigma_X \sqrt{2\pi} y} e^{-\frac{1}{2} \left( \frac{\ln y - m_X}{\sigma_X} \right)^2} \text{ for } (y > 0) \quad (2.67b)$$

In the region of  $(y \leq 0)$ , PDF of the r.v.  $Y$  will be zero, i.e.,  $f_Y(y) = 0$  for  $(y \leq 0)$ . The mean and variance of a Lognormal r.v. are calculated from

$$\begin{aligned} \text{Mean of the r.v. } Y = e^X: & \rightarrow m_Y = e^{m_X} e^{\sigma_X^2/2} \\ \text{Variance of the r.v. } Y = e^X: & \rightarrow \sigma_Y^2 = m_Y^2 (e^{\sigma_X^2} - 1) \end{aligned} \quad (2.67c)$$

If  $m_Y$  and  $\sigma_Y$  are given, then the variance and mean of  $X$  are calculated from the following statements

$$\sigma_x^2 = \ln \left[ 1 + (\sigma_Y/\mu_Y)^2 \right] \quad \text{and} \quad \mu_x = (\ln \mu_Y - \sigma_x^2/2) \quad (2.67d)$$

### 2.3.2.3 Uniform Distribution

The *Uniform Distribution* is also used in practice to describe an r.v. that is equally likely to take any value in the interval  $a$  and  $b$ , i.e.,  $(a \leq x \leq b)$ . Its density function is constant in the specified region and zero outside the region as defined

$$\text{Uniform PDF: } \rightarrow f_X(x) = \begin{cases} \frac{1}{(b-a)} & \text{in the interval } (a \leq x \leq b) \\ 0 & \text{outside the interval } (a \leq x \leq b) \end{cases} \quad (2.68a)$$

The mean and standard deviation of a Uniform r.v. are calculated from

$$\begin{aligned} \text{Mean:} & \rightarrow m_X = (a+b)/2 \\ \text{Stand. Dev.:} & \rightarrow \sigma_X = (b-a)/\sqrt{12} \end{aligned} \left\} \rightarrow \begin{cases} a = m_X - \sqrt{3} \sigma_X \\ b = m_X + \sqrt{3} \sigma_X \end{cases} \quad (2.68b)$$

### 2.3.2.4 Exponential Distribution

The *Exponential Distribution* can be used to describe intermittent variation of an r.v., such as an event occurring continuously and independently at a constant rate. Its distribution and density functions are defined in general as

$$\begin{aligned}
\text{Exponential Distribution, CDF: } \rightarrow F_X(x) &= \begin{cases} 1 - e^{-\lambda(x-\alpha)} & \text{for } (x \geq \alpha, \lambda > 0) \\ 0 & \text{for } (x < \alpha) \end{cases} \\
\text{Exponential Distribution, PDF: } \rightarrow f_X(x) &= \begin{cases} \lambda e^{-\lambda(x-\alpha)} & \text{for } (x \geq \alpha, \lambda > 0) \\ 0 & \text{for } (x < \alpha) \end{cases}
\end{aligned} \tag{2.69a}$$

The mean and variance of an Exponential distribution are calculated from

$$\text{Mean} \rightarrow m_X = \alpha + \sigma_X \quad \text{and} \quad \text{Variance} \rightarrow \sigma_X^2 = 1/\lambda^2 \tag{2.69b}$$

As an example, if it is assumed that a number of occurrences of an event per year is  $(n)$  and the event is described by the r.v.  $X$ , then, for independent occurrences, the annual probability distribution of  $X$  is stated

$$\text{Annual CDF of } X: \rightarrow F_X^n(x) = \left(1 - e^{-\lambda(x-\alpha)}\right)^n \tag{2.69c}$$

The density function of the annual probability is obtained from the derivative of  $F_X^n(x)$  as written

$$\text{Annual PDF of } X: \rightarrow f_{Xn}(x) = \frac{\partial F_X^n(x)}{\partial x} \rightarrow f_{Xn}(x) = n F_X^{n-1}(x) f_X(x) \tag{2.69d}$$

### 2.3.2.5 Gamma Distribution

The *Gamma Distribution* represents the sum of  $r$  independent exponentially distributed r.v., and r.v. that take always positive values. Its PDF and CDF functions are defined as written

$$\text{Gamma Dist., PDF: } \rightarrow f_X(x) = \begin{cases} \frac{\lambda}{\Gamma(r)} (\lambda x)^{r-1} e^{-\lambda x} & \text{if } (x \geq 0, \lambda > 0) \\ 0 & \text{if } x \leq 0 \end{cases} \tag{2.70a}$$

$$\text{Gamma Dist., CDF: } \rightarrow F_X(x) = 1 - \sum_{k=0}^{r-1} \frac{1}{k!} e^{-\lambda x} (\lambda x)^k \quad \text{for } (r = +\text{int.}) \tag{2.70b}$$

in which  $\Gamma(\cdot)$  represents a Gamma function [28], which is defined

$$\text{Gamma Function: } \rightarrow \Gamma(x) = \int_0^{\infty} e^{-u} u^{(x-1)} du \tag{2.70c}$$



The mean and variance of the Gamma distribution are calculated to be

$$\text{Mean: } \rightarrow m_X = \frac{r}{\lambda} \quad \text{and} \quad \text{Variance: } \rightarrow \sigma_X^2 = \frac{r}{\lambda^2} \quad (2.70d)$$

The parameters  $r$  and  $\lambda$  are respectively the shape and scale parameters of the distribution. For different values of  $r$  and  $\lambda$ , different type of distributions are obtained. When ( $r = 1$ ), it gives the exponential distribution. If ( $r < 1$ ), then the distribution is exponentially shaped and asymptotic to both horizontal and vertical axes. If ( $r > 1$ ), its shape is unimodal and skewed with the mode (location of the peak of the PDF) equals  $(x_m = (r - 1)/\gamma)$ . The skewness reduces with increasing value of  $r$  as it is seen from the coefficient of skewness, ( $\gamma_1 = 2/\sqrt{r}$ ). If ( $r = s/2$ ) and ( $\gamma = 1/2$ ), then the Gamma distribution becomes the  $\chi^2$  (*Chi-square*) distribution with  $s$  degrees of freedom. In engineering applications, Gamma distributions occur frequently in models of failure analysis and, in Meteorology, for rainfall studies since the variables are always positive and the results are unbalanced.

### 2.3.2.6 Rayleigh Distribution

The Rayleigh Distribution is used as a probability model describing the distribution of wind speed over 1-year period. It is often used for the probability model of the absolute value of two components of a random field, e.g., if  $X$  and  $Y$  are two independent normally distributed random variables, both with zero mean and variance equal to  $\sigma^2$  and if we define a function  $Z = \sqrt{X^2 + Y^2}$  then this function has a Rayleigh distribution with the parameter  $\sigma$  [23]. It also describes the probability distribution of maxima of a narrow band random process with Normal distribution. The PDF and CDF of the Rayleigh distribution are given as

$$\begin{aligned} \text{Rayleigh PDF: } \rightarrow f_X(x) &= \begin{cases} \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.71a) \\ \text{Rayleigh CDF: } \rightarrow F_X(x) &= 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right) \end{aligned}$$

in which  $\sigma$  is the only parameter of the distribution, which is equal to the standard deviations of the independent random variables  $X$  and  $Y$  with Normal distributions and zero means. The mean and variance of the Rayleigh distribution are calculated to be

$$\begin{aligned} \text{Mean: } \rightarrow m_X &= \sigma \sqrt{\frac{\pi}{2}} \\ \text{Variance: } \rightarrow \sigma_X^2 &= 2\sigma^2 \left(1 - \frac{\pi}{4}\right) \end{aligned} \quad (2.71b)$$

### 2.3.2.7 Weibull Distribution

The *Weibull Distribution* with three parameters is the general form of its family, which is also known as the *Extreme Value Type III* distribution. It is originally used in the probability description of strength of materials and fatigue analysis. This distribution is related to extreme value analysis and mostly used in the reliability engineering and failure analysis, survival analysis, industrial engineering, weather forecasting for wind speed variations, extreme value theory. The PDF and CDF of the Weibull distribution are defined as written

$$\text{Weibull PDF: } \rightarrow f_X(x) = \begin{cases} \left(\frac{\alpha}{\beta}\right) \left(\frac{x-\gamma}{\beta}\right)^{(\alpha-1)} \exp\left[-\left(\frac{x-\gamma}{\beta}\right)^\alpha\right] & \text{if } (x \geq \gamma) \\ 0 & \text{otherwise} \end{cases} \quad (2.72a)$$

$$\text{Weibull CDF: } \rightarrow F_X(x) = 1 - \exp\left[-\left(\frac{x-\gamma}{\beta}\right)^\alpha\right] \quad \text{for } (\alpha > 0, \beta > 0) \quad (2.72b)$$

in which  $\alpha$  is the shape parameter,  $\beta$  is the scale parameter and  $\gamma$  is the location parameter. The mean and variance of the Weibull distribution are calculated in terms of its parameters as stated

$$\begin{aligned} \text{Mean: } & \rightarrow m_X = \gamma + \beta \Gamma\left(1 + \frac{1}{\alpha}\right) \\ \text{Variance: } & \rightarrow \sigma_X^2 = \beta^2 \left[ \Gamma\left(1 + \frac{2}{\alpha}\right) - \Gamma^2\left(1 + \frac{1}{\alpha}\right) \right] \end{aligned} \quad (2.72c)$$

If the location parameter is zero, i.e., ( $\gamma = 0$ ), the distribution is called as two parameters Weibull distribution. If, however, the scale parameter  $\beta$  is taken as a constant ( $\beta = C$ ) and ( $\gamma = 0$ ), then the distribution is called as one parameter Weibull distribution. As being special cases, for ( $\alpha = 1$ ), the Weibull distribution becomes the exponential distribution and, for ( $\gamma = 0$ ) and ( $\alpha = 2$ ), it becomes as the Rayleigh distribution.

### 2.3.2.8 Gumbel Distribution

The *Gumbel Distribution* is usually used to model the distribution of the maximum, or the minimum, of a number of samples or various distributions. It can also be used to find the probability that an extreme event, such as earthquake, flood or other natural disaster, will occur. The Gumbel distribution is also known as the *Extreme Value Type I Distribution*. It has two forms as one is for extreme maximum (*Extreme Value Largest I*) and one is for extreme minimum (*Extreme Value Smallest I*), which are respectively defined below.

$$\text{Gumbel (EV Largest-I): } \rightarrow \begin{cases} f_X(x) = \alpha e^{-\alpha(x-\beta)} e^{-\exp(-\alpha(x-\beta))} \\ F_X(x) = e^{-\exp(-\alpha(x-\beta))} \end{cases} \quad \text{for } (-\infty < x < \infty) \quad (2.73a)$$

$$\text{Gumbel (EV Smallest-I): } \rightarrow \begin{cases} f_X(x) = \alpha e^{\alpha(x-\beta)} e^{-\exp(\alpha(x-\beta))} \\ F_X(x) = 1 - e^{-\exp(\alpha(x-\beta))} \end{cases} \quad \text{for } (-\infty < x < \infty) \quad (2.73b)$$

in which  $\beta$  is the location parameter and  $\alpha$  is the scale parameter, which is defined ( $\alpha > 0$ ). The Gumbel distribution supports the range of outcomes of the r.v.  $X$  between  $(-\infty < x < \infty)$ . The means and variances of both *Largest-I* and *Smallest-I* distributions are calculated from

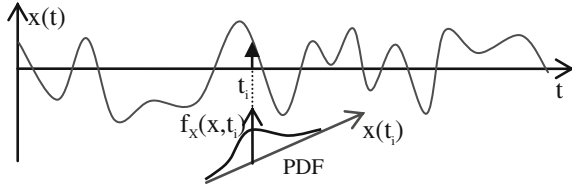
$$\begin{aligned} \text{Mean: } & \rightarrow \begin{cases} m_X = \beta + \frac{0.57722156649}{\alpha} & (\text{Largest-I}) \\ m_X = \beta - \frac{0.57722156649}{\alpha} & (\text{Smallest-I}) \end{cases} \quad (2.73c) \\ \text{Variance: } & \rightarrow \sigma_X^2 = \frac{\pi^2}{6\alpha^2} \quad (\text{Largest-I and Smallest-I}) \end{aligned}$$

The value (0.57722156649) in Eq. (2.73c) is the Euler's constant. More information about the extreme value distributions can be found in text books, e.g., [29–31]. Having presented probability and statistical descriptions of r.v., in the following section, random processes and their probability and statistical descriptions are presented briefly.

## 2.4 Random Processes, Ensemble Averages, Expected Values, Stationary and Ergodic Processes

Random variables and their probability descriptions are explained in previous sections. In this section, random processes and their properties will be outlined briefly. An r.v.  $X$  is a variable which takes values at random and can be thought of as a measurement of outcomes of a random experiment that cannot be predicted beforehand. A random process is a sequence of r.v. that vary in time or 1D space. If the number of random variables are limited then the process is said to be discrete, and if they are infinitely large, a continuous process is obtained. The random process can be thought of as a collection, or ensemble, of functions of time or one dimensional space. Such an ensemble is as shown in Fig. 2.3. If an associated r.v.  $X$  is a function of time, i.e.,  $X(t)$ , then the process is called as the stochastic process. Each possible outcome of a stochastic process is called as the realization, or sample, of the process and all realizations constitute the ensemble of

**Fig. 2.7** One particular realization of the stochastic process  $X(t)$  at  $(t = t_i)$  with its PDF



the process. A single realization  $x(t_i)$  of the process  $X(t)$  at  $(t = t_i)$  is shown in Fig. 2.7 with the PDF while the ensemble is as shown in Fig. 2.3. The value of an observed sample of the ensemble at a particular time, say  $t_i$ , which is shown in Fig. 2.7, is an r.v.  $x(t_i)$  that its probability and statistical characteristics have been explained in previous sections. For a stochastic process, we show explicitly the time dependence of the PDF and CDF in the notation, i.e.,  $f_X(x, t)$  and  $F_X(x, t)$ , as presented in Sect. (2.2.4). In this section, ensemble averages and various forms of the stochastic process are explained.

### 2.4.1 Ensemble Averages and Expected Values

In Sect. 2.3, mean values (or expected values) of r.v. and random functions are explained. In this section, the mean values (or expected values) of stochastic processes will be presented. Since a stochastic process is a collection of time-dependent r.v.,  $x_i(t)$  where  $(i = 1, 2, \dots, n)$  which are realizations of the process  $X(t)$  as shown in Fig. 2.3, its average value, or sample mean, at any time  $t$  is called as the *Ensemble Average*. Assuming that the r.v.  $x_i(t)$  are independent with identical probability distributions, which are equal to the probability distribution of the r.v.  $X(t)$ , then the ensemble average is defined as

$$\text{Ensemble Average: } \rightarrow \bar{X}(t_j) = \frac{1}{n} \sum_{i=1}^n x_i(t_j) = E[X(t)] \quad (\text{for } n \rightarrow \infty) \quad (2.74a)$$

which is equal to the expected value of the process  $X(t)$  when  $n$  approaches infinity according to the law of large numbers [23]. If we have infinite number of realizations, then we can say that the process  $X(t)$  is continuous. In this case, Eq. (2.74a) becomes an integral form and the expected value of the process  $X(t)$  can be written similarly to the statement of the expected value of an r.v. Thus, for a continuous process  $X(t)$ , the expected value is

$$\text{Expected (Mean) Value: } \rightarrow E[X(t)] = \int_{-\infty}^{\infty} x(t) f_X(x, t) dx \quad (2.74b)$$

which is a time function unlike that of an r.v. All other expectations and probability moments presented in Sect. 2.3 are also valid for stochastic processes

provided that they are all functions of time  $t$ . The probability distribution  $F_X(x, t)$  and probability density function  $f_X(x, t)$  of a stochastic process  $X(t)$  are defined respectively in Eqs. (2.10a) and (2.13) at any time  $t$ , which are called as the *First-order* distribution and density functions of the process  $X(t)$ , which are sufficient to define the range of amplitudes of the process  $X(t)$  in a probabilistic sense. In order to observe the variation of the same realization  $x(t)$  at different time stations, say  $t_1$  and  $t_2$ , the joint distribution of  $x(t_1)$  and  $x(t_2)$ , which is called as the *Second-order* joint distribution of  $X(t)$ , is defined as

$$\text{Second-order CDF: } \rightarrow F_X(x_1, t_1; x_2, t_2) = P(X(t_1) \leq x_1 \cap X(t_2) \leq x_2) \quad (2.75a)$$

and the corresponding joint PDF is obtained from the derivation

$$\text{Second-order PDF: } \rightarrow f_X(x_1, t_1; x_2, t_2) = \frac{\partial^2 F_X(x_1, t_1; x_2, t_2)}{\partial x_1 \partial x_2} \quad (2.75b)$$

If two different stochastic processes,  $X(t)$  and  $Y(t)$ , are involved, their second-order joint distribution and density functions at different time stations,  $t_1$  and  $t_2$ , are defined similarly to those of a single stochastic process as written

$$\text{Second-order joint CDF: } \rightarrow F_{XY}(x, t_1; y, t_2) = P(X(t_1) \leq x \cap Y(t_2) \leq y) \quad (2.75c)$$

$$\text{Second-order joint PDF: } \rightarrow f_{XY}(x, t_1; y, t_2) = \frac{\partial^2 F_{XY}(x, t_1; y, t_2)}{\partial x \partial y} \quad (2.75d)$$

In a similar way, the marginal and conditional distributions of stochastic processes can also be defined as in the case of r.v. These are not presented here and attention is paid further to commonly used definitions that have practical importance. These are first moments (expected values) of the joint distributions defined above. The expected values of the joint r.v.  $x(t_1)$  and  $x(t_2)$ , and  $x(t_1)$  and  $y(t_2)$  are respectively defined as the *Auto-Correlation* and *Cross-Correlation* functions. Hence, the auto-correlation function is

$$\text{Auto-Correlation: } \rightarrow \begin{cases} R_{XX}(t_1, t_2) = E[X(t_1) X(t_2)] & \text{or} \\ R_{XX}(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_{XX}(x_1, t_1; x_2, t_2) dx_1 dx_2 \end{cases} \quad (2.76a)$$

and the cross-correlated function is

$$\text{Cross-Correlation: } \rightarrow \begin{cases} R_{XY}(t_1, t_2) = E[X(t_1) Y(t_2)] & \text{or} \\ R_{XY}(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x y f_{XY}(x, t_1; y, t_2) dx dy \end{cases} \quad (2.76b)$$

As in the case of r.v. presented in the [Sect. 2.3](#), the first central moments of the joint random variables  $x(t_1)$  and  $x(t_2)$ , and  $x(t_1)$  and  $y(t_2)$  are respectively defined as the *Auto-Covariance* and *Cross-Covariance* functions, which are

$$\text{Auto-Covariance: } \rightarrow C_{XX}(t_1, t_2) = \begin{cases} E[(X(t_1) - E[X(t_1)])(X(t_2) - E[X(t_2)])] \\ R_{XX}(t_1, t_2) - E[X(t_1)] E[X(t_2)] \end{cases} \quad (2.77a)$$

$$\text{Cross-Covariance: } \rightarrow C_{XY}(t_1, t_2) = \begin{cases} E[(X(t_1) - E[X(t_1)])(Y(t_2) - E[Y(t_2)])] \\ R_{XY}(t_1, t_2) - E[X(t_1)] E[Y(t_2)] \end{cases} \quad (2.77b)$$

The correlation and covariance information of stochastic processes is important and frequently used in the spectral analysis of structures. The above mentioned definitions of the correlation and covariance functions are general. For special processes, such as stationary and ergodic processes, they are simplified as presented in the following section.

## 2.4.2 Stationary and Ergodic Processes

Two special forms of a general stochastic process, which are the *Stationary* and *Ergodic* processes, are frequently encountered or assumed in probabilistic analyses of structures. These special forms are outlined in this section.

### 2.4.2.1 Stationary Process

If the statistical properties of a random process are invariant in time, then the process is said to be *stationary* in the *strict sense* [23]. This implies that the statistics of a stationary process is not affected by a shift in the time origin. Thus, the processes  $X(t)$  and  $X(t + \tau)$  have the same statistics for any  $\tau$  value. Similarly, the processes  $X(t)$  and  $Y(t)$  are *jointly stationary* if their joint statistics are not affected by a shift in the time origin, i.e., if  $X(t)$  and  $Y(t)$ , and  $X(t + \tau)$  and  $Y(t + \tau)$ , have the same joint statistics for any  $\tau$  value. A random process is said to be *stationary* in the *wide sense* or *weakly* if its expected value is a constant and its auto-correlation is a function of the time difference ( $\tau = t_2 - t_1$ ) which means that it is not affected by a shift in the time origin. From these definitions, it can be stated that a weakly stationary Normal process is also strictly stationary since all statistics of the Normal process are uniquely determined in terms of its mean and auto-correlation. For a stationary process, the statistical characteristics of a general process presented in [Sect. 2.4.1](#) become as written

$$\text{Mean Value:} \quad \rightarrow \quad \mu_X = E[X(t_1)] = E[X(t_2)] = \text{constant} \quad (2.78a)$$

$$\text{Auto-Correlation:} \quad \rightarrow \quad R_{XX}(\tau) = E[X(t)X(t+\tau)] \quad (2.78b)$$

$$\text{Auto-Covariance:} \quad \rightarrow \quad C_{XX}(\tau) = R_{XX}(\tau) - \mu_X^2$$

$$\text{Cross-Correlation:} \quad \rightarrow \quad R_{XY}(\tau) = E[X(t)Y(t+\tau)] \quad (2.78c)$$

$$\text{Cross-Covariance:} \quad \rightarrow \quad C_{XY}(\tau) = R_{XY}(\tau) - \mu_X\mu_Y$$

or, in the integral form, the mean value, auto- and cross-correlation function are expressed as

$$\text{Mean Value:} \quad \rightarrow \quad \mu_X = \int_{-\infty}^{\infty} x(t)f_X(x,t) dx = \text{constant} \quad (2.78d)$$

$$\text{Auto-Correlation:} \quad \rightarrow \quad R_{XX}(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)f_X(x,t) dx$$

$$\text{Cross-Correlation:} \quad \rightarrow \quad R_{XY}(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t)y(t+\tau)f_{XY}(x,y,t) dx dy \quad (2.78e)$$

Since auto- and cross-correlations functions,  $R_{XX}(\tau)$  and  $R_{XY}(\tau)$ , are independent of an absolute time  $t_i$  and functions only the time difference  $\tau$ , it can be verified that these functions become even functions of  $\tau$ , i.e., they are symmetric with respect to the vertical (function) axis. This property is stated as

$$R_{XX}(\tau) = R_{XX}(-\tau) \quad \text{and} \quad R_{XY}(\tau) = R_{YX}(-\tau) \quad (2.79)$$

An example auto-correlation function  $R_{XX}(\tau)$  is shown in Fig. 2.8b. It can be seen from Eq. (2.78) that, when  $\tau = 0$ , the auto- and cross-covariances of the random processes  $X(t)$  and  $Y(t)$  become respectively the variance and covariance, i.e.,

$$\text{Variance:} \quad \rightarrow \quad \sigma_X^2 = C_{XX}(0) = R_{XX}(0) - \mu_X^2 \quad (2.80a)$$

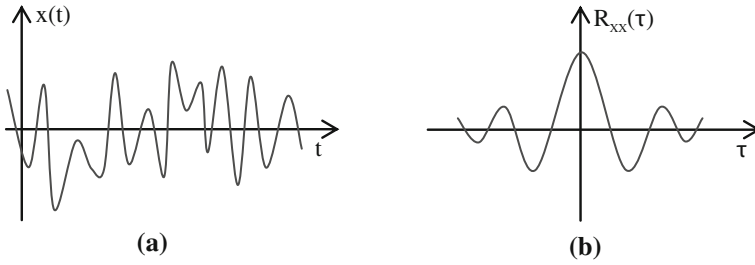
$$\text{Covariance:} \quad \rightarrow \quad \sigma_{XY} = C_{XY}(0) = R_{XY}(0) - \mu_X\mu_Y$$

It can also be shown that the auto-correlation function is maximum at  $\tau = 0$ , i.e.,

$$R_{XX}(0) \geq R_{XX}(\tau) \quad (2.80b)$$

### 2.4.2.2 Ergodic Process

One important concept in the stochastic analysis is the *ergodic* properties of the process. When the time average of a stationary process is equal to the ensemble average, it is said that this stationary process is ergodic. For an ergodic process,



**Fig. 2.8** A sample r.v.  $x(t)$  and the corresponding autocorrelation function  $R_{xx}(\tau)$ . **a** A sample r.v.  $x(t)$ . **b** Autocorrelation function  $R_{xx}(\tau)$

any statistic calculated by averaging over all samples (members) of an ensemble at a fixed time is equal to that calculated by averaging over all time on a single sample of the ensemble. Thus, only one sample time function of the ensemble represents the whole ensemble. For continuous ergodic processes, the mean, auto- and cross-correlation functions are calculated from the time averages

$$\text{Mean Value: } \rightarrow \tilde{\mu}_X = E[X(t_1)] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T X(t) dt \quad (2.81a)$$

$$\text{Auto-Correlation: } \rightarrow \tilde{R}_{XX}(\tau) = E[X(t)X(t+\tau)] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T X(t)X(t+\tau) dt$$

$$\text{Cross-Correlation: } \rightarrow \tilde{R}_{XY}(\tau) = E[X(t)Y(t+\tau)] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T X(t)Y(t+\tau) dt \quad (2.81b)$$

### 2.4.2.3 Complex Stationary Processes

In practice, complex random variables and processes can be frequently encountered. A complex random process  $Z(t)$  is defined as

$$\text{A complex process: } \rightarrow Z(t) = X(t) + iY(t) \quad (2.82a)$$

where  $X(t)$  is the real part and  $Y(t)$  is the imaginary part of the process. If both the real and imaginary parts are stationary, then the complex process  $Z(t)$  is said to be stationary. The conjugate of the complex process  $Z(t)$  is defined

$$\text{Conjugate of } Z(t): \rightarrow Z^*(t) = X(t) - iY(t) \quad (2.82b)$$

The mean and auto-correlation function of a complex process are defined as



$$\text{Complex process } Z(t) : \rightarrow \begin{cases} \text{Mean:} & \rightarrow m_Z = E[Z(t)] = m_X + i m_Y \\ \text{Auto-correlation:} & \rightarrow R_{ZZ}(\tau) = E[Z(t) Z^*(t + \tau)] \end{cases} \quad (2.83a)$$

If two complex stationary processes are defined as  $Z(t)$  and  $V(t)$ , then their cross-correlation function  $R_{ZV}(\tau)$  is defined similarly to auto-correlation function as

$$\text{Cross-correlation of } Z(t) \text{ and } V(t): \rightarrow R_{ZV}(\tau) = E[Z(t) V^*(t + \tau)] \quad (2.83b)$$

Using the properties of real processes  $X(t)$  and  $Y(t)$  given in Eq. (2.79), the following properties of complex processes can be written

$$R_{ZZ}(\tau) = R_{ZZ}^*(-\tau) \quad \text{and} \quad R_{ZV}(\tau) = R_{VZ}^*(-\tau) \quad (2.84)$$

### 2.4.3 Differentiation of Stochastic Processes

In practice, the derivatives of a stochastic process are also of interest in structural dynamics. They are stochastic processes that properties can be obtained simply using the source process. Stochastic displacements, velocities and accelerations are examples of such processes. Let us assume two general stochastic processes to be  $X(t)$  and  $Y(t)$ . Their time derivatives of the order  $n$  and  $m$  are called as the *derived processes* which are defined as

$$\text{Derived Processes:} \rightarrow X^{(n)}(t) = \frac{\partial^n X(t)}{\partial t^n} \quad \text{and} \quad Y^{(m)}(t) = \frac{\partial^m Y(t)}{\partial t^m} \quad (2.85)$$

Their mean values and correlation functions are calculated [23] from

$$\text{Mean Value:} \rightarrow E[X^{(n)}(t)] = \frac{\partial^n E[X(t)]}{\partial t^n} \quad (2.86a)$$

$$\text{Auto-correlation:} \rightarrow R_{X^{(n)}X^{(n)}}(t_1, t_2) = E[X^{(n)}(t_1)X^{(n)}(t_2)] \quad (2.86b)$$

$$\text{Cross-correlation:} \rightarrow R_{X^{(n)}Y^{(m)}}(t_1, t_2) = E[X^{(n)}(t_1)Y^{(m)}(t_2)]$$

Having introduced  $X^{(n)}(t)$  and  $Y^{(m)}(t)$  from Eqs. (2.85) into (2.86b) and taking the average values the auto-correlation function is obtained as written by

$$\text{Auto-correlation:} \rightarrow R_{X^{(n)}X^{(n)}}(t_1, t_2) = \begin{cases} \frac{\partial^{2n} E[X(t_1)X(t_2)]}{\partial t_1^n \partial t_2^n} \\ \frac{\partial^{2n} R_{XX}(t_1, t_2)}{\partial t_1^n \partial t_2^n} \end{cases} \quad (2.86c)$$

ant the cross-correlation function is obtained as written by

$$\text{Cross-correlation: } \rightarrow R_{X^{(n)}Y^{(m)}}(t_1, t_2) = \begin{cases} \frac{\partial^{(n+m)} E[X(t_1)Y(t_2)]}{\partial t_1^n \partial t_2^m} \\ \frac{\partial^{(n+m)} R_{XY}(t_1, t_2)}{\partial t_1^n \partial t_2^m} \end{cases} \quad (2.86d)$$

where  $n$  and  $m$  denote the order of derivatives. For jointly stationary processes the cross-correlation function of the derived processes are stated from Eq. (2.86d) as

$$\left. \begin{array}{l} \text{Cross-correlation of} \\ \text{Stationary derived processes} \end{array} \right\} \rightarrow R_{X^{(n)}Y^{(m)}}(\tau) = (-1)^n \frac{\partial^{n+m} R_{XY}(\tau)}{\partial \tau^{n+m}} \quad (2.87a)$$

In Eq. (2.87a), if the process  $Y(t)$  is replaced with the process  $X(t)$ , then the auto-correlation function of the derived process can be obtained as written

$$\left. \begin{array}{l} \text{Auto-correlation of} \\ \text{Stationary derived processes} \end{array} \right\} \rightarrow R_{X^{(n)}X^{(m)}}(\tau) = (-1)^n \frac{\partial^{n+m} R_{XX}(\tau)}{\partial \tau^{n+m}} \quad (2.87b)$$

In structural dynamics, the first two time derivatives are mostly used, which are denoted by  $\dot{X}(t)$  and  $\ddot{X}(t)$ . Using Eq. (2.87b) the auto-correlations of  $\dot{X}(t)$  and  $\ddot{X}(t)$ , and the cross-correlations of  $X(t)$ ,  $\dot{X}(t)$  and  $\ddot{X}(t)$  are stated as

$$\text{Auto-correlations of derived processes: } \rightarrow \begin{cases} R_{\dot{X}\dot{X}}(\tau) = -\frac{\partial^2 R_{XX}(\tau)}{\partial \tau^2} \\ R_{\ddot{X}\ddot{X}}(\tau) = \frac{\partial^4 R_{XX}(\tau)}{\partial \tau^4} \end{cases} \quad (2.88a)$$

$$\text{Cross-correlations of derived processes: } \rightarrow \begin{cases} R_{X\dot{X}}(\tau) = \frac{\partial R_{XX}(\tau)}{\partial \tau} \\ R_{\dot{X}\ddot{X}}(\tau) = -\frac{\partial^3 R_{XX}(\tau)}{\partial \tau^3} \end{cases} \quad (2.88b)$$

## 2.5 Spectral Analysis of Stochastic Processes

Correlation functions of stochastic processes are used to get information about their characteristics, but they do not provide direct information about their frequency contents. In structural dynamics, the frequency content of a stochastic process, e.g., a stochastic loading on the structure, plays an important role in the design of the structure such that, for a safe design, fundamental natural frequencies of the structure should be far away from the peak frequencies of the loading process. The frequency content and statistical characteristics, under certain

conditions, of a stochastic process can be determined from its *Power Spectrum* or *Spectral Density* function. The power spectrum of a process, which is denoted in general by  $S(\omega)$ , is simply defined as being the *Fourier transform* of its correlation function  $R(\tau)$ . Thus, the power spectrum and correlation functions are the Fourier transform pairs

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau \quad \leftrightarrow \quad R(\tau) = \int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega \quad (2.89)$$

Similar to the correlation functions, auto- and cross-power spectral functions are defined using Eq. (2.89). Assuming that  $X(t)$  and  $Y(t)$  are two stationary stochastic processes. Their correlation and spectral functions are written as, for auto-correlation and auto-spectral functions,

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(\omega) e^{i\omega\tau} d\omega \quad \leftrightarrow \quad S_{XX}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau \quad (2.90a)$$

and for cross-correlation and cross-spectral functions,

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} S_{XY}(\omega) e^{i\omega\tau} d\omega \quad \leftrightarrow \quad S_{XY}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-i\omega\tau} d\tau \quad (2.90b)$$

It can be proved that the power spectrum  $S_{XY}(\omega)$  is the expectation of the product  $(X^*(\omega) Y(\omega))$  where  $X^*(\omega)$  is the complex conjugate of  $X(\omega)$ . For the proof, let us write the Fourier transform pairs of the stationary processes  $X(t)$  and  $Y(t)$ ,

$$\left. \begin{aligned} X(t) &= \int_{-\infty}^{\infty} X(\omega) e^{i\omega t} d\omega \\ Y(t) &= \int_{-\infty}^{\infty} Y(\omega) e^{i\omega t} d\omega \end{aligned} \right\} \leftrightarrow \left\{ \begin{aligned} X(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(t) e^{-i\omega t} dt \\ Y(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(t) e^{-i\omega t} dt \end{aligned} \right. \quad (2.91)$$

Using Eq. (2.91) the cross-correlation function  $R_{XY}(\tau)$  can be written as

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] = E \left[ \int_{-\infty}^{\infty} X(\omega) e^{i\omega t} d\omega \int_{-\infty}^{\infty} Y(\omega) e^{i\omega(t+\tau)} d\omega \right] \quad (2.92a)$$

and substituting  $X(\omega)$  from Eqs. (2.91) into (2.92a) it can be stated

$$R_{XY}(\tau) = \left\{ \begin{aligned} &E \left[ \int_{-\infty}^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} X(t) e^{-i\omega t} dt \right) e^{i\omega t} d\omega \int_{-\infty}^{\infty} Y(\omega) e^{i\omega(t+\tau)} d\omega \right] \\ &E \left[ \int_{-\infty}^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} X(t) e^{i\omega t} dt \right) d\omega \int_{-\infty}^{\infty} Y(\omega) e^{i\omega\tau} d\omega \right] \end{aligned} \right. \quad (2.92b)$$

Since  $\left( \frac{1}{2\pi} \int_{-\infty}^{\infty} X(t) e^{i\omega t} dt = X^*(\omega) \right)$  the cross-correlation function can be written as

$$R_{XY}(\tau) = E \left[ \int_{-\infty}^{\infty} X^*(\omega) Y(\omega) e^{i\omega\tau} d\omega \right] = \int_{-\infty}^{\infty} E[X^*(\omega) Y(\omega)] e^{i\omega\tau} d\omega \quad (2.92c)$$

From comparison of Eqs. (2.90b) and (2.92c) it is seen that

$$S_{XY}(\omega) = E[X^*(\omega) Y(\omega)] \quad (2.92d)$$

By taking its complex conjugate we can obtain the cross-spectral function  $S_{YX}(\omega)$  as stated

$$S_{XY}^*(\omega) = (E[X^*(\omega) Y(\omega)])^* = E[Y^*(\omega) X(\omega)] = S_{YX}(\omega) \quad (2.92e)$$

The power spectrum of the stationary process  $X(t)$  can readily be obtained from Eq. (2.92d) as stated

$$S_{XX}(\omega) = E[X^*(\omega) X(\omega)] = E[|X(\omega)|^2] \quad (2.93)$$

from which it is apparent that the power spectrum  $S_{XX}(\omega)$  is a real-valued even function. This can also be verified by using the properties of correlation functions given in Eq. (2.84). The properties of spectral functions are summarized as follows.

$$\begin{aligned} \text{Auto-spectrum: } & \rightarrow S_{XX}(\omega): \text{ even real, } S_{XX}(\omega) = S_{XX}(-\omega) \\ \text{Cross-spectrum: } & \rightarrow S_{YX}(\omega) = S_{XY}^*(\omega) \end{aligned} \quad (2.94)$$

When  $(\tau = 0)$ , i.e.,  $R_{XX}(0)$  and  $R_{XY}(0)$ , the variance and covariance of the processes  $X(t)$  and  $Y(t)$  can be expressed in terms of spectral functions by using Eq. (2.80a) as written

$$\text{Variance: } \rightarrow \sigma_X^2 = \int_{-\infty}^{\infty} S_{XX}(\omega) d\omega - \mu_X^2 \quad (2.95a)$$

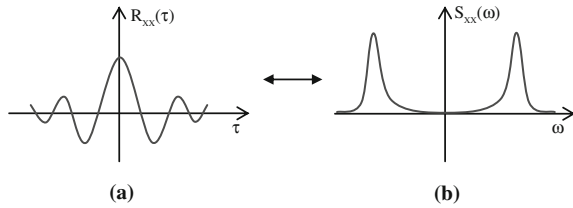
$$\text{Covariance: } \rightarrow \sigma_{XY} = \int_{-\infty}^{\infty} S_{XY}(\omega) d\omega - \mu_X \mu_Y \quad (2.95b)$$

For zero-mean processes, these statistical values become

$$\sigma_{XX}^2 = \int_{-\infty}^{\infty} S_{XX}(\omega) d\omega \quad \text{and} \quad \sigma_{XY} = \int_{-\infty}^{\infty} S_{XY}(\omega) d\omega \quad (2.95c)$$

A typical correlation function and its counterpart spectral function are shown in Fig. 2.9.

**Fig. 2.9** An example correlation function and its corresponding spectral function. **a** A correlation function  $R_{XX}(\tau)$ . **b** Corresponding spectral function  $S_{XX}(\omega)$



### 2.5.1 Spectral Moments, Variances of Derived Processes, Spectral Bandwidth

In engineering practice, *Spectral Moments* and *Spectral Bandwidth* are used to calculate statistical characteristics of stochastic processes and probability distributions of their extreme values and mean periods.

**Spectral moments** are defined as calculated from the following integration of the spectrum:

$$\text{Spectral Moments: } \rightarrow m_n = \int_{-\infty}^{\infty} \omega^n S_{XX}(\omega) d\omega \quad (2.96)$$

where  $m_n$  is called as the  $n$ th spectral moment. The first three even moments are used to describe probability distributions of extremes and mean frequencies, or periods, of the process. They are written as

$$m_0 = \int_{-\infty}^{\infty} S_{XX}(\omega) d\omega, \quad m_2 = \int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega) d\omega, \quad m_4 = \int_{-\infty}^{\infty} \omega^4 S_{XX}(\omega) d\omega, \quad (2.97)$$

**Correlation functions of derived processes** are calculated using Eqs. (2.87a) and (2.87b) in general, and Eqs. (2.88a) and (2.88b) in particular for the first and second derivatives. By using the spectral counterparts of these functions from Eqs. (2.90a) and (2.90b) they can be stated for the processes  $X(t)$  and  $Y(t)$  as

$$R_{X^{(n)}Y^{(m)}}(\tau) = (-1)^n i^{(n+m)} \int_{-\infty}^{\infty} \omega^{(n+m)} S_{XY}(\omega) e^{i\omega\tau} d\omega \quad (2.98)$$

and, for the first and second derivatives, they are, e.g.,

$$R_{\dot{X}\dot{X}}(\tau) = i \int_{-\infty}^{\infty} \omega S_{XX}(\omega) e^{i\omega\tau} d\omega \quad \text{and} \quad R_{\ddot{X}\ddot{X}}(\tau) = \int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega) e^{i\omega\tau} d\omega \quad (2.99a)$$

$$R_{\ddot{X}\dot{X}}(\tau) = i \int_{-\infty}^{\infty} \omega^3 S_{XX}(\omega) e^{i\omega\tau} d\omega \quad \text{and} \quad R_{\ddot{X}\ddot{X}}(\tau) = \int_{-\infty}^{\infty} \omega^4 S_{XX}(\omega) e^{i\omega\tau} d\omega \quad (2.99b)$$

When ( $\tau = 0$ ), these functions are more meaningful since they are closely related to the variance and covariance of the derived processes of  $X(t)$ . Thus,

$$\left. \begin{aligned} R_{X\dot{X}}(0) &= i \int_{-\infty}^{\infty} \omega S_{XX}(\omega) d\omega = 0 \\ R_{\dot{X}\dot{X}}(0) &= \int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega) d\omega = m_2 \end{aligned} \right\} \text{ and } \left\{ \begin{aligned} R_{\dot{X}\ddot{X}}(0) &= i \int_{-\infty}^{\infty} \omega^3 S_{XX}(\omega) d\omega = 0 \\ R_{\ddot{X}\ddot{X}}(0) &= \int_{-\infty}^{\infty} \omega^4 S_{XX}(\omega) d\omega = m_4 \end{aligned} \right. \quad (2.100)$$

which are spectral moments of the source process  $X(t)$ . As it can be seen from these statements that, since  $S_{XX}(\omega)$  is an even function, odd spectral moments are all zero. This implies that the related derived processes are uncorrelated, e.g., the processes  $X(t)$  and  $\dot{X}(t)$  are uncorrelated since ( $E[X(t)\dot{X}(t)] = R_{X\dot{X}}(0) = 0$ ). Thus, the variances and covariances of derived processes can be stated as, for the first two derivatives,

$$\left. \begin{aligned} \sigma_{X\dot{X}} &= -m_X m_{\dot{X}} \\ \sigma_{\dot{X}}^2 &= m_2 - m_X^2 \end{aligned} \right\} \text{ and } \left\{ \begin{aligned} \sigma_{\dot{X}\ddot{X}} &= -m_{\dot{X}} m_{\ddot{X}} \\ \sigma_{\ddot{X}}^2 &= m_4 - m_{\dot{X}}^2 \end{aligned} \right. \quad (2.101)$$

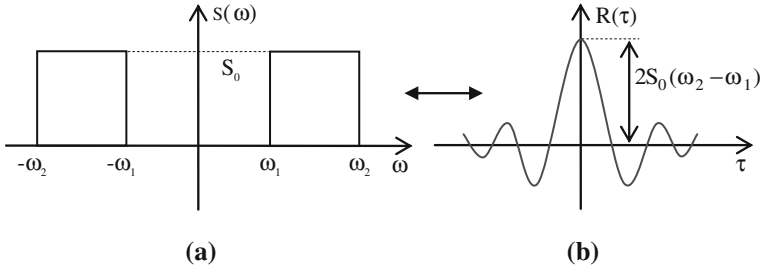
**Spectral Bandwidth** is a useful informative parameter which defines the relative width of a spectral function. It appears in the formulation of probability distributions of derived processes, and accordingly, in the formulation of extreme value statistics. It is defined in terms of spectral moments, or mean frequencies, or mean periods, of the process as written by

$$\text{Spectral Bandwidth: } \rightarrow \varepsilon = \sqrt{1 - \frac{m_2^2}{m_0 m_4}} = \sqrt{1 - \frac{\omega_0^2}{\omega_m^2}} = \sqrt{1 - \frac{T_m^2}{T_0^2}} \quad (2.102)$$

where  $m_0$ ,  $m_2$ , and  $m_4$  are the spectral moments,  $\omega_0$  and  $\omega_m$  are mean frequencies of zero-crossings and maxima of the process, whereas  $T_0$  and  $T_m$  are the corresponding mean periods.

### 2.5.2 Band-Limited, Narrow-Band and Broad-Band Processes

A *band-limited process* is defined if its spectrum has a uniform magnitude  $S_0$  over a frequency band between  $\omega_1$  and  $\omega_2$  as shown in Fig. 2.10. This spectrum is also called as the band-limited white noise. Its correlation function is calculated by using Eq. (2.89). It is stated as



**Fig. 2.10** Spectrum of a band-limited process and its corresponding correlation function. **a** A band-limited spectrum. **b** Corresponding correlation function

$$R(\tau) = \begin{cases} \int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega = S_0 \left( \int_{-\omega_2}^{-\omega_1} e^{i\omega\tau} d\omega + \int_{\omega_1}^{\omega_2} e^{i\omega\tau} d\omega \right) \\ \frac{2S_0}{\tau} (\sin \omega_2 \tau - \sin \omega_1 \tau) = \frac{4S_0}{\tau} \left( \cos\left(\frac{\omega_2 + \omega_1}{2}\right) \tau \sin\left(\frac{\omega_2 - \omega_1}{2}\right) \tau \right) \end{cases} \quad (2.103)$$

which is shown in Fig. 2.10. If this spectrum belongs to the process  $X(t)$ , its mean square value is calculated from

$$E[X^2(t)] = R(0) = 2S_0(\omega_2 - \omega_1) \quad (2.104)$$

The 0th, 2nd, and 4th spectral moments are calculated using Eq. (2.97) and the spectral bandwidth parameter  $\varepsilon$  is calculated from Eq. (2.102). The results are

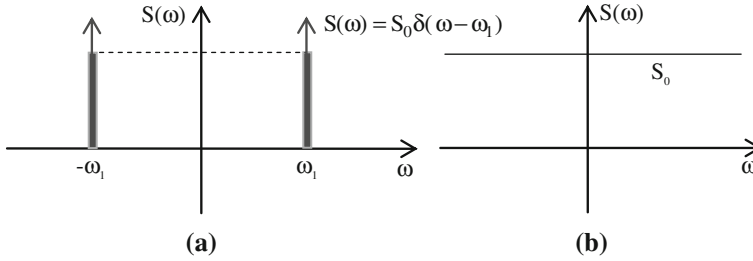
$$\left. \begin{aligned} m_0 &= 2S_0(\omega_2 - \omega_1) \\ m_2 &= 2S_0(\omega_2^3 - \omega_1^3)/3 \\ m_4 &= 2S_0(\omega_2^5 - \omega_1^5)/5 \end{aligned} \right\} \rightarrow \varepsilon = \sqrt{1 - \frac{5}{9} \frac{(\omega_2^3 - \omega_1^3)^2}{(\omega_2 - \omega_1)(\omega_2^5 - \omega_1^5)}} \quad (2.105)$$

If  $(\omega_1 = 0)$ , the correlation function, spectral moments and the spectral bandwidth parameter  $\varepsilon$  are calculated as to be

$$R(\tau) = \frac{2S_0}{\tau} \sin(\omega_2 \tau) \quad \text{and} \quad \left. \begin{aligned} m_0 &= 2S_0\omega_2 \\ m_2 &= 2S_0\omega_2^3/3 \\ m_4 &= 2S_0\omega_2^5/5 \end{aligned} \right\} \rightarrow \varepsilon = \frac{2}{3} \quad (2.106)$$

A **narrow-band process** is defined if its spectrum is a spike with infinite height and zero width so that the area remains finite. Such a spectrum is shown in Fig. 2.11a and represented [2] by the Dirac's delta function which is defined as

$$\text{Delta function: } \rightarrow \delta(\omega - \omega_1) = \begin{cases} 1 & \text{at } (\omega = \omega_1) \\ 0 & \text{elsewhere} \end{cases} \quad (2.107a)$$



**Fig. 2.11** Narrow-band and white noise (broad-band) spectra. **a** A narrow-band spectrum. **b** A white noise spectrum

The delta function has the following property

$$\int_{-\infty}^{\infty} \delta(\omega - \omega_1) f(\omega) d\omega = f(\omega_1) \quad (2.107b)$$

where  $f(\omega)$  is any function of  $\omega$ . The narrow-band process can be obtained from the band-limited process when the frequency  $\omega_2$  approaches  $\omega_1$ , i.e.,  $(\omega_2 \rightarrow \omega_1)$ , except that the spectral moments are calculated using the property of the delta function given in Eq. (2.107b). The spectral moments and bandwidth parameter  $\varepsilon$  are calculated to be

$$m_0 = 2S_0, \quad m_2 = 2S_0\omega_1^2, \quad m_4 = 2S_0\omega_1^4 \quad \text{and} \quad \varepsilon = 0 \quad (2.108)$$

As it is seen from Eq. (2.108), the spectral bandwidth parameter  $\varepsilon$  of a narrow-band process equals zero, which means that the period of zero-crossings is equal to the period of maxima. The correlation function is calculated as stated below.

$$R(\tau) = \int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega = \int_{-\infty}^{\infty} S_0 \delta(\omega - \omega_1) e^{i\omega\tau} d\omega \quad (2.109a)$$

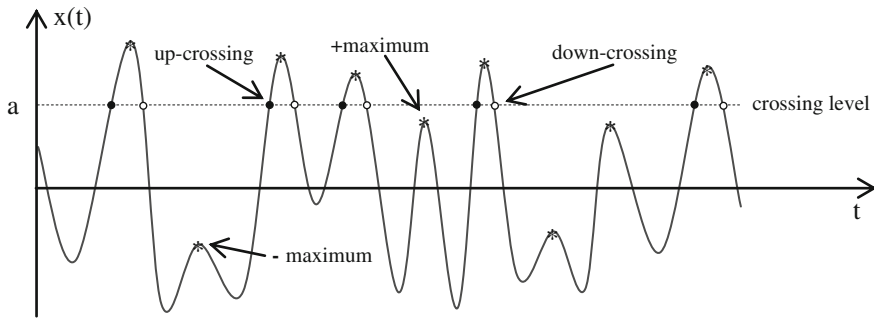
having carried out the integration it is obtained as written by

$$R(\tau) = S_0 (e^{i\omega_1\tau} + e^{-i\omega_1\tau}) = 2S_0 \cos(\omega_1\tau) \quad (2.109b)$$

which is a cosine function of  $(\omega_1\tau)$ . If the frequency  $\omega_1$  approaches zero, then  $R(\tau)$  approaches a constant value, and for  $(\omega_1 = 0)$ ,  $R(\tau) = 2S_0$  is obtained, i.e., the correlation function is constant for all frequency ranges with the value of  $2S_0$ . Here, the coefficient 2 arises from two spikes at the origin, one is from (+) region and other one is from (−) region of the frequency.

**A broad-band process** is defined if its spectrum is extended over a wide frequency range with smooth variation that does not display sharp peaks, i.e., it produces a spectral bandwidth parameter  $\varepsilon$  is much greater than zero ( $0 < \varepsilon < \infty$ ). In the case of a large frequency range, the band-limited process is an example of a broad-band process. In the extreme case, when  $(\omega_1 = 0)$  and  $(\omega_2 \rightarrow \infty)$ , it covers all frequency ranges between  $-\infty$  and  $\infty$ , i.e.  $(-\infty < \omega < \infty)$ . This special case of





**Fig. 2.12** A sample record of time signals, crossing level, maxima, up- and downcrossings

the spectral shape is called as the *white noise spectrum* which is shown in Fig. 2.11b. The correlation function of the band-limited white noise has been given in Eq. (2.106), which displays a peak for large  $\omega_2$  at  $\tau = 0$  and diminishes rapidly with increasing  $\tau$  values. As  $(\omega_2 \rightarrow \infty)$ , this peak becomes a spike with a constant area of  $2\pi S_0$  so that the correlation function of the white noise will be

$$R(\tau) = 2\pi S_0 \delta(\tau) \quad (2.110)$$

which can be proved by back calculation of  $S(\omega)$  from the Fourier transform of  $R(\tau)$  to obtain  $S_0$  that has been assumed.

### 2.5.3 Crossing Analysis and Probability Distributions of Maxima

The *crossing analysis* is the central issue to determine average frequencies of a stochastic process and also to determine probability distribution of its peak values as outlined in the following subsections.

#### 2.5.3.1 Average Frequencies of Crossing Levels, Frequency of Maxima

A typical crossing and definition of some terms that used in the crossing analysis are shown in Fig. 2.12. It is required that the average frequency of a stationary process  $X(t)$  with amplitudes greater than a crossing level ( $X = a$ ) is to be calculated. An up-crossing, which is shown in Fig. 2.12, occurs whenever the realization  $X(t)$  passes through the crossing level ( $X = a$ ) with positive slope. We wish to determine the probability of this up-crossing in the time interval  $t$  and  $(t + dt)$ . The r.v. at  $t$  and  $(t + dt)$  are  $X(t)$  and  $X(t + dt)$ , respectively. The variable at  $(t + dt)$  is calculated using the Taylor expansion [28]. The conditions that an up-crossing occurs in the interval  $X(t)$  and  $X(t + dt)$  are

$$\begin{aligned} \text{Crossing condition: } & \rightarrow X(t) \leq a \text{ and } X(t + dt) \geq a \\ \text{Up-crossing condition: } & \rightarrow \frac{\partial X(t)}{\partial t} \geq 0 \end{aligned} \quad (2.111a)$$

which can also be written as

$$\begin{aligned} \text{Crossing condition: } & \rightarrow \begin{cases} X(t) \leq a \text{ and } (X(t) + \dot{X}(t)dt) \geq a \\ \text{or } (a - \dot{x}dt) \leq X \leq a \end{cases} \\ \text{Up-crossing condition: } & \rightarrow \dot{X} \geq 0 \end{aligned} \quad (2.111b)$$

Since  $X$  and  $\dot{X}$  are derived processes with a joint PDF of  $f_{X\dot{X}}(x, \dot{x})$ , by using Eq. (2.23), the probability that an up-crossing occurs between the time interval  $t$  and  $(t + dt)$  is written as

$$dP = P((a - \dot{x}dt \leq X \leq a) \cap (0 \leq \dot{X} \leq \infty)) = \int_0^\infty \left( \int_{(a - \dot{x}dt)}^a f_{X\dot{X}}(x, \dot{x}) dx \right) d\dot{x} \quad (2.112)$$

The integration in the brackets (.) with respect to  $x$  is the area of  $f_{X\dot{X}}(x, \dot{x})$  over the band  $(a - (a - \dot{x}dt) = \dot{x}dt)$  and the values of  $f_{X\dot{X}}(x, \dot{x})$  at  $(X = a \text{ and } a - \dot{x}dt)$  are the same as  $(dt \rightarrow 0)$ . Thus, the area will be  $(f_{X\dot{X}}(a, \dot{x})\dot{x}dt)$ . Having substituted this value into Eq. (2.112), the probability of the up-crossing is obtained as written by,

$$\text{Probability of up-crossing at } (X = a): \rightarrow dP = \left( \int_0^\infty \dot{x} f_{X\dot{X}}(a, \dot{x}) d\dot{x} \right) dt \quad (2.113)$$

This probability will be equal to the average number of up-crossings at  $X = a$  in time  $dt$  [2]. If we assume that the average frequency of up-crossings in cycle per second at  $X = a$  is  $v_a^+$ , then the average number of up-crossings in  $dt$  will be  $(v_a^+ dt)$  and equalizing this to Eq. (2.113) it can be stated that

$$\text{Average frequency of up-crossings at } (X = a): \rightarrow v_a^+ = \int_0^\infty \dot{x} f_{X\dot{X}}(a, \dot{x}) d\dot{x} \quad (2.114)$$

Since  $X$  and  $\dot{X}$  are derived processes, they are uncorrelated. Thus, the joint PDF can be stated as  $f_{X\dot{X}}(a, \dot{x}) = f_X(a)f_{\dot{X}}(\dot{x})$ . Once these PDF are known, then the average frequency of up-crossings at  $X = a$  can be calculated from Eq. (2.114). We now assume that the processes  $X$  and  $\dot{X}$  have Normal probability distributions

with zero means and standard deviations of  $\sigma_X$  and  $\sigma_{\dot{X}}$ . Using their PDF from Eqs. (2.65a) in (2.114) it can be obtained that

$$\text{Up-crossing frequency: } \rightarrow \begin{cases} v_a^+ = \frac{1}{2\pi} \frac{\sigma_{\dot{X}}}{\sigma_X} \exp[-a^2/2\sigma_X^2] & \text{at } (X = a) \\ v_0^+ = \frac{1}{2\pi} \frac{\sigma_{\dot{X}}}{\sigma_X} \rightarrow \omega_0 = \frac{\sigma_{\dot{X}}}{\sigma_X} & \text{at } (X = 0) \end{cases} \quad (2.115)$$

where  $v_0^+$  is the mean zero-crossings frequency in (cyc/sec) and  $\omega_0$  is that in (rad/sec) of a Normal process. For a zero-mean process, the mean zero-crossings frequency can be obtained using Eq. (2.101 for  $\sigma_{\dot{X}}$  as

$$\text{Up-crossing frequency at } (X = 0): \rightarrow \omega_0 = \sqrt{m_2/m_0} \quad (\text{rad/sec}) \quad (2.116)$$

As similar to the average frequency of up-crossings, the average frequency of local maxima of a stationary process can be determined. The maxima of a process is obtained if its first derivative is zero and second derivative (curvature) is negative in all time, i.e., if the conditions,  $(\dot{X} = 0)$  and  $(\ddot{X} < 0)$ , are satisfied in all time, then maxima of the process  $X$  are obtained. The conditions of  $(\dot{X} = 0)$  and  $(\ddot{X} < 0)$  implies that the average frequency of maxima of  $X$  is equal to the average frequency of down-crossings, see Fig. 2.12, of the derived process  $\dot{X}$  at  $X = 0$ . Thus, using  $\dot{X}$  and  $\ddot{X}$  instead of  $X$  and  $\dot{X}$  in Eq. (2.115), the average frequency of maxima of the process  $X$  can be obtained as written

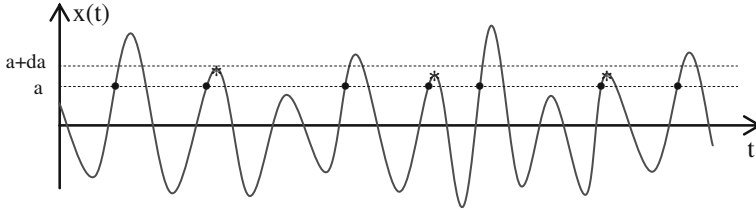
$$\text{Average frequency of maxima: } \rightarrow \omega_m = \sigma_{\ddot{X}}/\sigma_{\dot{X}} = \sqrt{m_4/m_2} \quad (2.117)$$

The average frequencies of zero-crossings and maxima are used to determine the spectral bandwidth parameter  $\varepsilon$ , given in Eq. (2.102), which is a good measure to give information about the spectral shape of a stochastic process.

### 2.5.3.2 Probability Distribution of Maxima

Maxima of a stochastic process are important values in engineering analyses as they are closely related to amplitudes or highest values of the process. Wave heights and maximum stresses are examples of maxima. In the probabilistic and reliability analyses, their probability distributions are required, which can be determined under certain conditions and probability model of the process.

For a stationary narrow-band process the probability distribution of maxima, or peak values, can be determined easily, while, for a broad-band process, it is relatively complicated. For a narrow-band process, which is shown in Fig. 2.13, the probability that the maxima between  $(a \leq X(t) \leq a + da)$  is  $f_m(a)da$ , where  $f_m(a)$  is the PDF of the maxima. The probability of maxima exceeding  $(X = a)$  is written by,



**Fig. 2.13** A narrow-band process with maxima between  $(a \leq X(t) \leq a + da)$

$$\text{Probability of maxima exceeding } (X = a): \rightarrow P_m = \int_a^\infty f_m(a) da \quad (2.118)$$

This probability is equal to  $(N_a/N_0)$  where  $N_a$  is the number of up-crossings at the level  $X = a$  and  $N_0$  is that at  $X = 0$ . For a period  $T$ , these numbers of up-crossings and the probability  $P_m$  are calculated from

$$(N_a = v_a^+ T \text{ and } N_0 = v_0^+ T) \rightarrow P_m = \frac{v_a^+ T}{v_0^+ T} \rightarrow P_m = \frac{v_a^+}{v_0^+} = 1 - F_m(a) \quad (2.119)$$

in which  $F_m(a)$  is the cumulative probability distribution of maxima. Having derived Eq. (2.119) with respect to  $a$ , the PDF of maxima of a stationary narrow-band process can be obtained as written by,

$$\text{PDF of maxima of a narrow-band process } X(t): \rightarrow f_m(a) = -\frac{1}{v_0^+} \frac{\partial v_a^+}{\partial a} \quad (2.120)$$

For a Normal stationary narrow-band process the crossings frequencies have been given in Eq.(2.115), from which and Eq.(2.120) the PDF of the maxima of a narrow-band Normal process  $X(t)$  can be obtained as

$$\text{PDF of peaks of a Normal process } X(t): \rightarrow f_m(a) = \frac{a}{\sigma_X^2} \exp(-a^2/2\sigma_X^2) \quad (2.121)$$

This is a Rayleigh distribution that explained in Sect. 2.3.2.6.

For a general stationary process, calculation of probability distribution is not so easy. If the process is *stationary* and *Normal*, then the PDF of its maxima can be determined in terms of the spectral bandwidth parameter  $\varepsilon$ . It is calculated using the joint PDF of  $X(t)$ ,  $\dot{X}(t)$  and  $\ddot{X}(t)$ , i.e., from  $f(x, \dot{x}, \ddot{x})$ , which is assumed to be Normal. For the maxima above  $(X = a)$ , the general conditions are

$$\text{Condition of maxima of a process } X(t): \rightarrow \begin{cases} X \geq a \\ \dot{X} = 0 \text{ and } \ddot{X} < 0 \end{cases} \quad (2.122)$$

The probability that  $X$  lies in the band  $x$  to  $(x + dx)$  above  $(X \geq a)$  is written as

$$\text{Prob. of up-crossings above } (X = a): \rightarrow P_a = \int_a^\infty \left( \int_{-\infty}^0 f_m(x, \dot{x}, \ddot{x}) d\ddot{x} d\dot{x} \right) dx \quad (2.123a)$$

Since  $\dot{X} = 0$  and  $d\dot{x} \simeq |\ddot{x}|dt$ , the probability of a maximum above  $(X = a)$  can be written from Eq. (2.123a) as

$$\text{Prob. of maxima above } (X = a): \rightarrow P_m = \int_a^\infty \left( \int_{-\infty}^0 |\ddot{x}| f_m(x, 0, \ddot{x}) d\ddot{x} \right) dx dt \quad (2.123b)$$

which is equal to the number of cycles of maxima in  $dt$  above  $(X = a)$ , i.e.,  $(n_a = v_m(a) dt)$  so that the frequency of maxima above  $(X = a)$  can be stated as

$$\text{Freq. of maxima above } (X = a): \rightarrow v_m(a) = \int_a^\infty \left( \int_{-\infty}^0 |\ddot{x}| f_m(x, 0, \ddot{x}) d\ddot{x} \right) dx \quad (2.124a)$$

In a similar way, the frequency of maxima above a given crossing, say  $(X = b)$  is written as

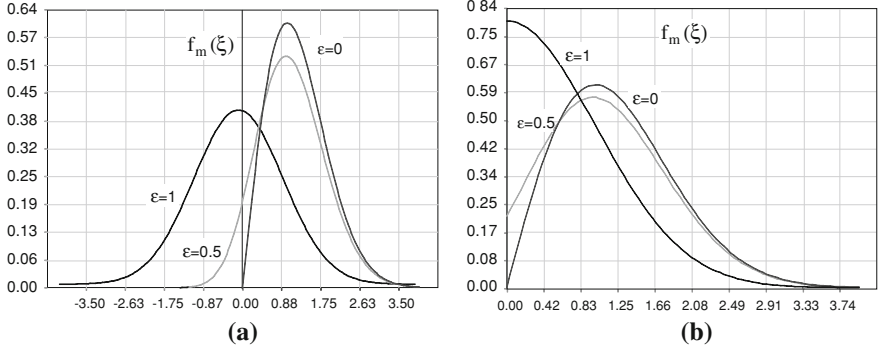
$$\text{Frequency of maxima above } (X = b): \rightarrow v_m(b) = \int_b^\infty \left( \int_{-\infty}^0 |\ddot{x}| f_m(x, 0, \ddot{x}) d\ddot{x} \right) dx \quad (2.124b)$$

In a period of time  $T$ , the probability of maxima for  $(X > a)$  can be stated in terms of numbers of maxima, which leads to

$$\int_a^\infty f_m(a) da = 1 - F_m(a) = \frac{v_m(a)}{v_m(b)} \rightarrow f_m(a) = -\frac{1}{v_m(b)} \frac{\partial v_m(a)}{\partial a} \quad (2.125)$$

Having substituted  $v_m(a)$  and  $v_m(b)$  from Eqs. (2.124a) and (2.124b) into (2.125), the PDF of maxima above a crossing level  $(X = b)$  can be obtained as written by

$$\text{PDF of maxima above } (X = b): \rightarrow f_m(a) = \frac{\int_{-\infty}^0 |\ddot{x}| f_m(a, 0, \ddot{x}) d\ddot{x}}{\int_b^\infty \left( \int_{-\infty}^0 |\ddot{x}| f_m(x, 0, \ddot{x}) d\ddot{x} \right) dx} \quad (2.126)$$



**Fig. 2.14** Probability density functions of maxima of a general process, where  $\xi = a/\sqrt{m_0}$ . **a** PDF of all maxima between  $(-\infty$  and  $\infty)$ . **b** PDF of +maxima between  $(0$  and  $\infty)$

In Eq. (2.126), if  $(b = 0)$ , then all positive maxima, i.e. maxima above  $(X = 0)$ , are considered, and if  $(b = -\infty)$ , then all maxima (positive and negative) of the process  $X(t)$  are considered. The PDF of these two cases, which are of practical interest, are given [30] in the following.

PDF of all maxima, i.e. for  $(-\infty \leq \xi \leq \infty)$ :

$$f_m(\xi) = \frac{\varepsilon}{\sqrt{2\pi}} e^{-\xi^2/2\varepsilon^2} + \sqrt{1-\varepsilon^2} \xi e^{-\xi^2/2} \Phi\left(\frac{\sqrt{1-\varepsilon^2}}{\varepsilon} \xi\right) \quad (2.127a)$$

PDF of positive maxima, i.e. for  $(0 \leq \xi \leq \infty)$ :

$$f_m(\xi) = \frac{2}{1 + \sqrt{1-\varepsilon^2}} \left[ \frac{\varepsilon}{\sqrt{2\pi}} e^{-\xi^2/2\varepsilon^2} + \sqrt{1-\varepsilon^2} \xi e^{-\xi^2/2} \Phi\left(\frac{\sqrt{1-\varepsilon^2}}{\varepsilon} \xi\right) \right] \quad (2.127b)$$

in which  $(\xi = a/\sqrt{m_0})$ , where  $a$  is the *amplitude* and  $m_0$  is the zeroth spectral moment,  $\Phi(\cdot)$  is the *Standard Normal Distribution* function given in Eq. (2.65c). The plots of these two PDF are shown in Figs. 2.14a and 2.14b, respectively. For the limiting case for  $\varepsilon = 0$  (narrow-band process) and  $\varepsilon = 1$  (extreme broad-band process), these distributions become respectively the Rayleigh and Normal distributions. As it is seen from Fig. 2.14b, for  $\varepsilon > 0$ , the distributions obtained from Eq. (2.127b) are truncated at the zero value of the variable  $(\xi = 0)$ . For the spectral bandwidth parameter  $\varepsilon$  between 0 and 1,  $(0 < \varepsilon < 1)$ , the distributions have mixed forms of the Normal and Rayleigh distributions. As it is expected, for an infinitely narrow-band process  $(\xi = 0)$ , the PDF of the +maxima and all maxima will be the same since, for a narrow-band process, all maxima are located in the positive side of the process as shown in Fig. 2.13.

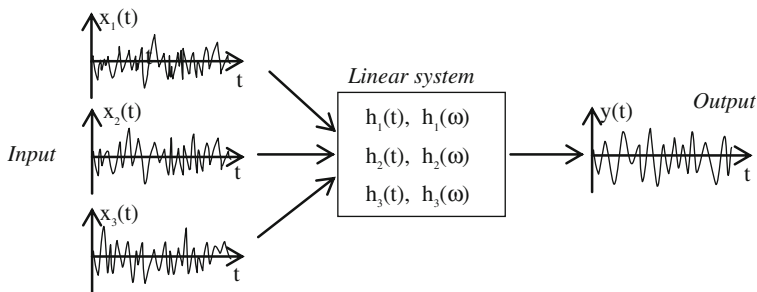


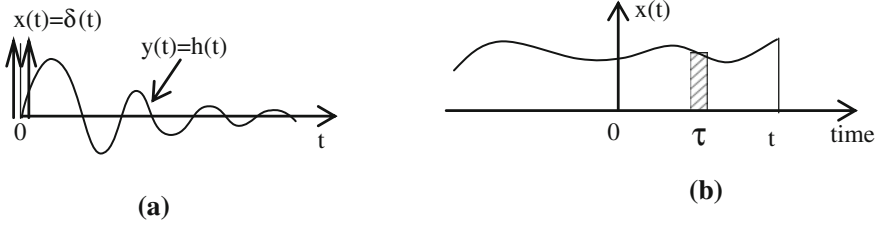
Fig. 2.15 A linear system with three input excitations and one output response

## 2.6 Input-Output Relations of Stochastic Processes, Transfer Functions

In the stochastic analysis of structures, the loadings exerted by natural phenomena are mostly stochastic processes such as wave, earthquake, and wind loadings. The stochastic loadings produce structural responses that will be also stochastic. We assume that the loading process is fully determined stochastically and probabilistically. Our purpose is to determine a response process under this loading condition which is only known or available information that we can use. In the stochastic structural analysis, the loading is the *input* and a response to be calculated is the *output*. The inputs are also called *excitations* of a response process. Multiple input (loadings) processes can be applied to produce one response (output) process or samples of responses. The main issue in this context is that the input and output relation is assumed to be linear, and they form a *linear system* together as shown schematically in Fig. 2.15 where  $h_i(t)$  and  $h_i(\omega)$  with  $(i = 1, 2, 3)$  are respectively impulse response functions and frequency response functions [2]. A linear system in structural dynamics is defined if the response function is related to the input by a linear differential equation [2] in the form of

$$\text{A linear system: } \rightarrow \sum_{k=0}^n a_k \frac{\partial^k y}{\partial t^k} = \sum_{k=0}^{n1} b_{1k} \frac{\partial^k x_1}{\partial t^k} + \sum_{k=0}^{n2} b_{2k} \frac{\partial^k x_2}{\partial t^k} + \dots + \sum_{k=0}^{nr} b_{rk} \frac{\partial^k x_r}{\partial t^k} \quad (2.128)$$

which is composed of one output and superposition of  $(nr)$  number of inputs. The dynamic equilibrium equation, given by Eq. (1.86) in Chap. 1, is a simple example of a linear system in the time domain, and that given by Eq. (1.258) in the frequency domain. The output process of a linear system follows the type of the input process [2], which means that, if the input process is stationary, then the output process becomes also stationary, and if it is ergodic, then the corresponding



**Fig. 2.16** An impulse response function and an arbitrary input consisted of infinite impulses. **a** An impulse response function. **b** an arbitrary input function

output process is also ergodic. The *impulse response* function  $h(t)$  is the response of the system at time  $t$  due to a unit impulse applied at time  $(t = 0)$  [2, 32] as shown in Fig. 2.16a. For an arbitrary input  $x(t)$  shown in Fig. 2.16b, the response at time  $t$  can be considered as infinite collections of impulse response functions due to impulses applied at  $(t = \tau)$ , where  $(-\infty \leq \tau \leq t)$ . Thus, the response function  $y(t)$  at time  $t$  can be written as

$$\text{Response to an arbitrary input } x(t): \rightarrow y(t) = \begin{cases} \int_{-\infty}^t h(t - \tau)x(\tau) d\tau \\ \int_{-\infty}^{\infty} h(t - \tau)x(\tau) d\tau \end{cases} \quad (2.129a)$$

The integration is extended from  $t$  to  $+\infty$ , since  $h(t - \tau) = 0$  for  $(\tau > t)$  and  $y(t)$  remains unchanged. In the frequency domain, the response to an arbitrary input  $x(\omega)$  can be stated [2] as

$$\text{Response to an arbitrary input } x(\omega): \rightarrow y(\omega) = h(\omega)x(\omega) \quad (2.129b)$$

where  $h(\omega)$  is called as the *frequency response* function. It can be shown that, see e.g., [2], the impulse and frequency response functions,  $h(t)$  and  $h(\omega)$ , are the Fourier transform pairs. Our intention is to determine the response spectral function  $S_Y(\omega)$  in terms of the input spectrum  $S_X(\omega)$ . Since, in the structural analysis, we deal mostly with multi degrees-of-freedom systems, henceforth we use vector processes instead of scalar to determine spectral outputs of structural responses. The output of a stationary input vector process in the time domain is written from Eq. (2.129a) as,

$$\text{Response to input } \{x(t)\}: \rightarrow \{y(t)\} = \begin{cases} \int_{-\infty}^{\infty} [h(\theta)]\{x(t - \theta)\} d\theta \\ \text{where } (\theta = t - \tau) \end{cases} \quad (2.130a)$$



and, in the frequency domain from Eq. (2.129b), it is

$$\text{Response to input}\{x(\omega)\}: \rightarrow \{y(\omega)\} = [h(\omega)]\{x(\omega)\} \quad (2.130b)$$

In general, let us determine the cross-correlation matrix  $[R_{YZ}(\tau)]$  of two output stationary vector processes,  $\{Y(t)\}$  and  $\{Z(t)\}$ , from which the auto-correlation matrix can be obtained by equalizing the two processes. Using the relation given in Eq. (2.78c) it is written that

$$\text{Cross-correlation of } Y(t) \text{ and } Z(t): \rightarrow [R_{YZ}(\tau)] = E\left[\{Y(t)\}\{Z(t+\tau)\}^T\right] \quad (2.131a)$$

or using the relation of input-output written in Eq. (2.130a), it is stated as

$$[R_{YZ}(\tau)] = E\left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [H_Y(\theta_Y)]\{X_Y(t-\theta_Y)\}\{X_Z(t-\theta_Z+\tau)\}^T [H_Z(\theta_Z)]^T d\theta_Z d\theta_Y\right] \quad (2.132a)$$

By using a variable transformation of  $(t_1 = t - \theta_Y)$  and taking the ensemble average, Eq. (2.132a) can be written as

$$[R_{YZ}(\tau)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [H_Y(\theta_Y)][R_{X_Y X_Z}(\theta_Y - \theta_Z + \tau)][H_Z(\theta_Z)]^T d\theta_Y d\theta_Z \quad (2.132b)$$

in which  $[R_{X_Y X_Z}(\theta_Y - \theta_Z + \tau)]$  is the cross-correlation matrix of the input processes  $\{X_Y(t)\}$  and  $\{X_Z(t)\}$ . In order to calculate the cross-spectral matrix of the output processes  $\{Y(t)\}$  and  $\{Z(t)\}$ , the Fourier transform of Eq. (2.132b) will be taken. By using the relation written in Eq. (2.90b), the cross-spectral matrix of the outputs  $\{Y(t)\}$  and  $\{Z(t)\}$  can be obtained as written by

$$\text{Cross-spectral matrix of outputs: } \rightarrow [S_{YZ}(\omega)] = [H_Y^*(\omega)][S_{X_Y X_Z}(\omega)][H_Z(\omega)]^T \quad (2.133)$$

in which  $(*)$  denotes a complex conjugate and  $[S_{X_Y X_Z}(\omega)]$  is the cross spectral matrix of the inputs  $\{X_Y(t)\}$  and  $\{X_Z(t)\}$ . The output processes in the frequency domain,  $\{Y(\omega)\}$  and  $\{Z(\omega)\}$ , are linked to input processes  $\{X_Y(\omega)\}$  and  $\{X_Z(\omega)\}$  by the following relations

$$\text{Input-Output relations: } \rightarrow \begin{cases} \{Y(\omega)\} = [H_Y(\omega)]\{X_Y(\omega)\} \\ \{Z(\omega)\} = [H_Z(\omega)]\{X_Z(\omega)\} \end{cases} \quad (2.134)$$

which means that the inputs  $\{X_Y(\omega)\}$  and  $\{X_Z(\omega)\}$  are transferred to the outputs  $\{Y(\omega)\}$  and  $\{Z(\omega)\}$  by means of frequency dependent matrices  $[H_Y(\omega)]$  and  $[H_Z(\omega)]$ . These matrices are called as the *transfer function matrices* of linear

stochastic processes. Eq. (1.259) in Chap. 1, is a good example of the input–output relation in structural random vibrations, which is written below for convenience.

$$\{D(\omega)\} = [H(\omega)]_{DP} \{P(\omega)\} \rightarrow [H(\omega)]_{DP} = ([K] + i\omega[C] - \omega^2[M])^{-1} \quad (2.135)$$

where  $\{D(\omega)\}$  is the structural displacements vector,  $\{P(\omega)\}$  is the applied load vector and  $[H(\omega)]_{DP}$  is the transfer function matrix between the load and displacement vectors. The spectral matrix of displacements can be readily determined using the spectral relation given in Eq. (2.133) as

$$\text{Spectral matrix of displacements: } \rightarrow [S_D(\omega)] = [H(\omega)]_{DP}^* [S_P(\omega)] [H(\omega)]_{DP}^T \quad (2.136)$$

in which  $[S_P(\omega)]$  is the spectral matrix of applied loads. As it can be realized from Eq. (2.136),  $[S_D]$  is the auto-spectral matrix of the displacements.

## 2.7 Examples

Three stationary processes are defined as written

$$X(t) = \sin(\omega t + x), \quad Y(t) = \cos^2(\omega t + y) \quad \text{and} \quad Z(t) = X(t) + Y(t) \quad (2.137)$$

in which  $x$  and  $y$  are random phase angles with a uniform joint distribution between  $(-\pi \leq x \leq \pi)$  and  $(-\pi \leq y \leq \pi)$ . The following items are asked to be calculated:

1. Expected values of  $X(t)$ ,  $Y(t)$  and  $Z(t)$
2. Auto- and cross-correlation functions of  $X(t)$ ,  $Y(t)$  and  $Z(t)$
3. Show that these processes are also ergodic.

*Solution*

1. In order to use the ensemble averages, the PDF function of the random variables  $x$  and  $y$  need to be determined. Since a uniform distribution for both  $x$  and  $y$  is used, the joint PDF is defined as

$$f_{XY}(x, y) = \begin{cases} \frac{1}{4\pi^2} & \rightarrow (-\pi \leq x \leq \pi) \text{ and } (-\pi \leq y \leq \pi) \\ 0 & \text{elsewhere} \end{cases} \quad (2.138a)$$

The marginal PDF of  $x$  and  $y$  are calculated from

$$f_X(x) = \int_{-\pi}^{\pi} f_{XY}(x, y) dy = \frac{1}{2\pi} \quad \text{and} \quad f_Y(y) = \int_{-\pi}^{\pi} f_{XY}(y) dx = \frac{1}{2\pi} \quad (2.138b)$$

The expected values of  $X(t)$ ,  $Y(t)$  are calculated from

$$\begin{aligned} \mu_X = E[X(t)] &= \int_{-\pi}^{\pi} x f_X(x) dx \rightarrow \mu_X = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(\omega t + x) dx = 0 \\ \mu_Y = E[Y(t)] &= \int_{-\pi}^{\pi} y f_Y(y) dy \rightarrow \mu_Y = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos^2(\omega t + y) dy = \frac{1}{2} \end{aligned} \quad (2.139a)$$

and the expected value of  $Z(t)$  is calculated from the sum of two processes

$$\mu_Z = E[Z(t)] = E[X(t) + Y(t)] = E[X(t)] + E[Y(t)] = \frac{1}{2} \quad (2.139b)$$

2. The auto- and cross-correlation functions of the processes  $X(t)$  and  $Y(t)$  are calculated from, for auto-correlation of  $X(t)$ ,

$$R_{XX}(\tau) = \begin{cases} E[X(t)X(t+\tau)] = \int_{-\pi}^{\pi} x(t)x(t+\tau)f_X(x)dx \\ \frac{1}{4\pi} \left| x \cos(\omega\tau) - \frac{1}{2} \sin(2\omega t + 2x + \omega\tau) \right|_{-\pi}^{\pi} = \frac{1}{2} \cos(\omega\tau) \end{cases} \quad (2.140a)$$

and for auto-correlation of  $Y(t)$ ,

$$R_{YY}(\tau) = \begin{cases} E[X(t)Y(t+\tau)] = \int_{-\pi}^{\pi} y(t)y(t+\tau)f_Y(y)dy \\ \frac{1}{16\pi} \left| y \cos(2\omega\tau) + \frac{1}{4} \sin(4\omega t + 4y + 2\omega\tau) + \dots \right. \\ \left. \dots + \sin(2\omega t + 2y) + \sin(2\omega t + 2\omega\tau + 2y) + 2y \right|_{-\pi}^{\pi} \\ \frac{1}{4} \left( \cos^2(\omega\tau) + \frac{1}{2} \right) \end{cases} \quad (2.140b)$$

The cross-correlations of  $X(t)$  and  $Y(t)$  are calculated from

$$R_{XY}(\tau) = \begin{cases} E[X(t)Y(t+\tau)] = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} x(t)y(t+\tau)f_{XY}(x,y) dx dy \\ \frac{1}{4\pi^2} \left| \cos(\omega t + x) \left( \frac{1}{2} \cos(\omega t + \omega\tau + y) \sin(\omega t + \omega\tau + y) + \dots \right) \right|_{-\pi}^{\pi} = 0 \end{cases} \quad (2.140c)$$

$$R_{YX}(\tau) = \begin{cases} E[Y(t)X(t+\tau)] = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} y(t)x(t+\tau)f_{XY}(x,y) dx dy \\ -\frac{1}{8\pi^2} \left| \cos(\omega t + \omega\tau + x)(\cos(\omega t + y) \sin(\omega t + y) + \omega t + y) \right|_{-\pi}^{\pi} = 0 \end{cases} \quad (2.140d)$$

The auto-correlation and cross-correlation functions,  $R_{ZZ}(\tau)$ ,  $R_{XZ}(\tau)$  and  $R_{YZ}(\tau)$ , are calculated using the sum of two processes

$$R_{ZZ}(\tau) = \begin{cases} E[(X(t) + Y(t))(X(t+\tau) + Y(t+\tau))] \\ R_{XX}(\tau) + R_{YY}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) = \frac{1}{4} \left( 2 \cos(\omega\tau) + \cos^2(\omega\tau) + \frac{1}{2} \right) \end{cases} \quad (2.141a)$$

the cross-correlation function of  $X(t)$  and  $Z(t)$  is calculated from

$$R_{XZ}(\tau) = \begin{cases} E[X(t)(X(t+\tau) + Y(t+\tau))] = R_{XX}(\tau) + R_{XY}(\tau) \\ \frac{1}{2} \cos(\omega\tau) \end{cases} \quad (2.141b)$$

and the cross-correlation function of  $Y(t)$  and  $Z(t)$  is calculated from

$$R_{YZ}(\tau) = \begin{cases} E[Y(t)(X(t+\tau) + Y(t+\tau))] = R_{YX}(\tau) + R_{YY}(\tau) \\ \frac{1}{4} \left( \cos^2(\omega\tau) + \frac{1}{2} \right) \end{cases} \quad (2.141c)$$

3. In order to show that the processes  $X(t)$ ,  $Y(t)$  and  $Z(t)$  are also ergodic, we need to calculate time averages. If the statistics are equal to the ensemble averages calculated above, these processes are said to be ergodic. The statistics of  $X(t)$  and  $Y(t)$  are sufficient to prove ergodicity of the processes  $X(t)$ ,  $Y(t)$  and  $Z(t)$ . Using Eq. (2.81a) the mean values are calculated from

$$\tilde{\mu}_X = \begin{cases} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T X(t) dt = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \sin(\omega t + x) dt \\ \lim_{T \rightarrow \infty} \left| -\frac{1}{2\omega T} (\cos(\omega t + x)) \right|_{t=-T}^T = 0 \end{cases} \quad (2.142a)$$

$$\tilde{\mu}_Y = \begin{cases} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T Y(t) dt = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \cos^2(\omega t + y) dt \\ \lim_{T \rightarrow \infty} \left| \frac{1}{4\omega T} (\cos(\omega t + y) \sin(\omega t + y) + \omega t + y) \right|_{t=-T}^T = \frac{1}{2} \end{cases}$$

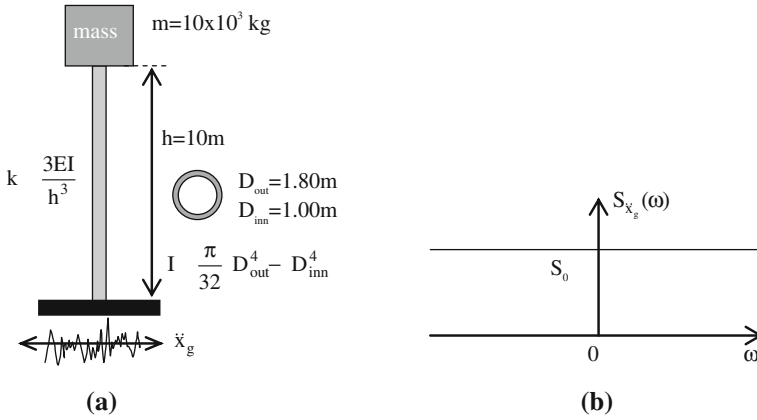
and the correlation functions are calculated from

$$\tilde{R}_{XX}(\tau) = \begin{cases} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \sin(\omega t + x) \sin(\omega t + \omega\tau + x) dt \\ \lim_{T \rightarrow \infty} \left| -\frac{1}{8\omega T} (-2t\omega \cos(\omega\tau) + \sin(2\omega t + 2x + \omega\tau)) \right|_{t=-T}^T = \frac{1}{2} \cos(\omega\tau) \end{cases} \quad (2.142b)$$

$$\tilde{R}_{YY}(\tau) = \begin{cases} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \cos^2(\omega t + y) \cos^2(\omega t + \omega\tau + y) dt \\ \lim_{T \rightarrow \infty} \left| \frac{1}{64\omega T} \left( 4t\omega \cos(2\omega\tau) + \sin(4\omega t + 4y + 2\omega\tau) + \dots \right. \right. \\ \left. \left. \dots + 4 \sin(2\omega t + 2y) + 4 \sin(2\omega t + 2\omega\tau + 2y) + 8\omega t \right) \right|_{t=-T}^T \\ \frac{1}{4} \left( \cos^2(\omega\tau) + \frac{1}{2} \right) \end{cases} \quad (2.142c)$$

$$\tilde{R}_{XY}(\tau) = \begin{cases} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \sin(\omega t + x) \cos^2(\omega t + \omega\tau + y) dt \\ \lim_{T \rightarrow \infty} \left| -\frac{1}{24\omega T} \left( \cos(3\omega t + x + 2\omega\tau + 2y) - \dots \right. \right. \\ \left. \left. \dots - 3 \cos(\omega t - x + 2\omega\tau + 2y) + 6 \cos(\omega t + x) \right) \right|_{t=-T}^T = 0 \end{cases} \quad (2.142d)$$

$$\tilde{R}_{YX}(\tau) = \begin{cases} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \cos^2(\omega t + y) \sin(\omega t + \omega\tau + x) dt \\ \lim_{T \rightarrow \infty} \left| -\frac{1}{24\omega T} \left( \cos(3\omega t + \omega\tau + x + 2y) - \dots \right. \right. \\ \left. \left. \dots - 3 \cos(\omega t - \omega\tau - x + 2y) + 6 \cos(\omega t + \omega\tau + x) \right) \right|_{t=-T}^T = 0 \end{cases} \quad (2.142e)$$



**Fig. 2.17** A tower subjected to random earthquake motion, and the white noise spectrum  $\ddot{x}_g$ . **a** A tower with input information. **b** White noise spectrum of  $\ddot{x}_g$

It seen from these statements that the statistics calculated from the time averages are same as those calculated from the ensemble averages, which proves that the processes are ergodic.

#### Exercise 1

The following processes are defined

$$X(t) = \sin(\omega t + x/2), \quad Y(t) = \cos(\omega t + y/2) \quad (2.143)$$

in which  $x$  and  $y$  are independent random phase angles with a uniform distributions defined in  $(-\pi \leq x \leq \pi)$  and  $(-2\pi \leq y \leq 2\pi)$ .

1. Show that the process  $X(t)$  is not stationary.
2. Show that the process  $Y(t)$  is stationary in the wide sense and also ergodic.

#### Exercise 2

A water tower is given as shown in Fig. 2.17a. It is made of reinforced concrete with hollow circular cross-section of  $D_{\text{out}}$  (outer diameter) and  $D_{\text{inn}}$  (inner diameter), and the height  $h$  with numeric values given in Fig. 2.17a. The mass at the top is denoted by  $m$  and the stiffness  $k$  of the tower is calculated as written in Fig. 2.17a. It is assumed that the mass of the tower is neglected and only the mass at the top is considered with the value written in Fig. 2.17a. The tower is subjected to a random earthquake motion with an acceleration of  $\ddot{x}_g$ , which is Normally distributed with a zero mean. It is assumed that the earthquake acceleration has a white noise spectrum with a magnitude of  $S_0$  as shown in Fig. 2.17b. The elasticity modulus  $E$  of the reinforced concrete is assumed to be random with a Lognormal probability distribution presented in Sect. 2.3.2.2. It is further assumed that the r.v.  $E$  (elasticity modulus) and  $\ddot{x}_g$  (acceleration of the ground) are independent. The mean and coefficient of variation of the elasticity modulus  $E$  are assumed to be,

$$\begin{aligned}
 \text{Mean of Elasticity modulus, } E &: \rightarrow \mu_E = 30.0 \text{ GPa} \quad (30.0 \times 10^9 \text{ N/m}^2) \\
 \text{COV of Elasticity modulus } (\sigma_E/\mu_E): &\rightarrow V_E = 0.05
 \end{aligned}
 \tag{2.144}$$

The damping ratio of the material is assumed to be ( $\xi = 0.04$ ). Under these conditions, the following items are required to be calculated and determined.

1. Natural frequency of the tower as being function of the elasticity modulus  $E$  of the concrete, which is a random function
2. Mean value and variance of the natural frequency
3. Correlation function and spectrum of the horizontal displacement  $d_{\text{top}}$  at the top as depending on the spectrum of ground acceleration
4. Variance of the ground acceleration in terms of the magnitude  $S_0$  of its spectrum, which is shown in Fig. 2.17b
5. Spectral bandwidth parameter  $\varepsilon$  of the horizontal response displacement at the top, and its mean frequencies of zero-crossings and maxima.
6. Probability density function of the amplitude (maxima) of the horizontal displacement at the top
7. Mean value and variance of the amplitude of the displacement  $d_{\text{top}}$ .

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Karadeniz, H.

2013, XIV, 410 p. With online files/update., Hardcover

ISBN: 978-1-84996-189-9