

Chapter 2

Generalizing Cantor's CBT Proof

Following his statement of CBT, in its single-set formulation, for sets of the power of (II), as a corollary to the Fundamental Theorem, Cantor said (Cantor 1932 p 201, Ewald 1996 vol 2 p 912 [12]):

[T]hat this theorem has general validity, regardless of the power of the set M , seems to me highly remarkable. I shall go further into this matter in a later article and then indicate the peculiar interest which attaches to this general theorem.

However, Cantor never came back to fulfill this promise. Moreover, in his 1895 *Beiträge*, Cantor presented CBT as corollary C to the Comparability Theorem for cardinal numbers. Thus it became a generally held view in the literature on early set theory that Cantor never proved CBT directly, for sets with power other than the power of (II). For example see Zermelo in Cantor 1932 p 209 [5], Fraenkel 1966 p 77 footnote 1, Medvedev 1966 p 229f, Levy 1979 p 85, Dauben 1979 p 172, Hallett 1984 p 60 footnote 2, p 74, Ferreirós 1999 p 239, Felscher 1999; Grattan-Guinness 2000 p 94.

Cantor's statements to the contrary, made in the letter to Dedekind of November 5, 1882, and in the above cited passage, are either not mentioned (Dauben, Hallett) or brushed away as a mistake Ferreirós (1999 p 239 footnote 4). Because no generalization to Cantor's proof from *Grundlagen* (see Chap. 1) has been suggested yet,¹ nothing appears to counter this view. In this chapter we hope to correct this misconception by producing a generalization of the proof given in *Grundlagen*.² To this end we will show how the scale of number-classes is defined in Cantor's terms past the second number-class.

One assumption must be made though. In the passage cited, the term 'the power of the set M ' must be interpreted to mean that M can be gauged by Cantor's scale of

¹ Hallett (1984 § 2.2) attempted to obtain such generalization but concluded (p 74) that it was not possible.

² Alone to state a view similar to ours, but with actually no details to support his thesis, is Tait (2005 p 164).

number-classes. It is the generalization of the assumption made for the proof given in the previous chapter that M has the power of (II). This assumption is justified by the well-ordering principle and the numbering principle which Cantor introduced in *Grundlagen* (§ 2). The well-ordering principle states that every set can be well-ordered and the numbering principle states (somewhat obscurely) that every well-ordered set is similar to the segment-set of one of the numbers generated by the generation principles. This number is then called the *Anzahl* of the set. Thus every set has a power in the scale of number-classes (see Chap. 4), to the definition of which we turn.

A caveat regarding the proof of the generalized Union Theorem must also be mentioned: the only evidence that Cantor had a proof of the theorem is Bernstein's testimony mentioned in Sect. 1.4. But then, Jourdain had a proof by 1908a (see Sect. 2.2), so it is not improbable to assume the same of Cantor, nearly 25 years earlier.

2.1 The Scale of Number-Classes

Our first step will be to define Cantor's scale of number-classes in its full generality as envisioned by Cantor. We begin with a definition of notation: Let us denote by (γ) the γ th number-class, by ω_γ the first number in (γ) , the so-called the initial number of (γ) , and by U_γ the union of all (κ) for $(0 < \kappa < \gamma)$. This notation is not compatible with the common notation when γ is a positive integer. Thus, after Cantor, (I), (II), (III) are used for our (1), (2), (3), and for our $\omega_1, \omega_2, \omega_3, \dots$ the common use is 1, ω, ω_1, \dots . We stick to the general notation when we speak in general terms and to the common notation when we speak in particular terms. A similar practice is common in the literature.³

ω_γ is the sequent of U_γ and thus its power is the power of U_γ and is the γ th power. (γ) is the set of all numbers that have the same power as ω_γ . Cantor later (1895 *Beiträge*) denoted by \aleph_0 (aleph zero) the power of ω , by \aleph_1 the power of his ω_1 and so on. Under our generalized notation \aleph_γ denotes the power of ω_γ but again we stick to Cantor's terminology when γ is a positive integer.⁴

Though in *Grundlagen* Cantor did not mention explicitly number-classes for limit index, his view on the matter is clear from a remark which he made in endnote 2 (Cantor 1932 p 205, Ewald 1996 vol 2 p 917):

The number γ which gives the order of a power (in case the number γ has an immediate predecessor) stands to the numbers [*Zahlen*; Ewald has here 'number'] of the number-class that has that power in a relationship of size whose smallness mocks all description; and all the more so, the greater we take γ to be.

Since Cantor speaks here of successor γ we infer that he had in mind also powers of orders which are limit numbers. Then, if γ is a limit number and we apply Cantor's

³ The convention to denote by ω_γ the initial number of the $(\gamma + 2)$ th number-class (Jourdain 1904b p 295 footnote †) will not do for number-classes from the $(\omega + 1)$ th on.

⁴ The confused notation originated with Cantor in the following quotation.

observation to $\omega_{\gamma+1}$, we are led to conclude that ω_γ must be a singular number. Because if ω_γ is a regular number, it is equal to γ and then $\gamma + 1$ has the power of ω_γ and so it belongs to the number-class (γ) and is not mockingly smaller than the numbers in that number-class. Indeed, in the definition of ω_γ , we assumed that γ is given, namely, defined before ω_γ . Thus ω_γ cannot be equal to γ and ω_γ is a singular number.⁵

In the quoted paragraph Cantor assumes that the γ th power, when γ is a successor number, has the power of the number-class ($\gamma-1$). This is the Limitation Principle applied to successor initial number, which must be proved. As a number in a number-class has the power of the initial number of its number-class, it is necessary to prove compliance with the Limitation Principle only for successor initial numbers, such as Ω . It should be stressed that this is the crux of Cantor's construction. It is in order to prove compliance with the Limitation Principle (rather the Limitation Theorem for successor initial numbers) that the bunch of theorems of § 11, 12 of *Grundlagen* must be generalized.

What about the Limitation Principle for ω_γ with limit γ ? Clearly, such ω_γ is not the power of a number-class. We must conclude that Cantor exempted the singular numbers from the Limitation Principle. In the same endnote from which we quoted above, Cantor said:

every transfinite [*überendliche*]⁶ number, however great, of any of the higher number-classes. . . is followed by an aggregate of numbers and number-classes whose power is not the slightest reduced compared to the entire absolutely infinite aggregate of numbers.

Cantor further equated there this property of the absolutely infinite with the similar property of ω with regard to the positive integers. Thus the absolutely infinite was regarded by Cantor to be regular, just as ω is. Therefore, singular numbers cannot represent the absolutely infinite and no Limitation Principle is necessary for them.

Zermelo's comment (Cantor 1932 p 199, in the text, not in Ewald) that already for forming the ω th number-class Cantor's principles do not suffice, emerged probably because ω_ω is not equivalent to a number-class and not because he thought that the second generation principle is not sufficient to generate ω_ω or because of the lack of a tailored comprehension principle. For some reason Zermelo ignored the possibility that the Limitation Principle is not necessary for limit initial numbers.

⁵ Cantor did not use the terms 'regular number' and 'singular number', introduced by Hausdorff (1914a p 130).

⁶ Cantor is using here the term 'transfinite number' which gained popularity after Jourdain used "transfinite numbers" in the title of his translation (Cantor 1915) of Cantor 1895/7, instead of Cantor's *Transfiniten Mengenlehre*. But Cantor preferred in his 1897 the term 'ordinal number' and in *Grundlagen* he preferred *unendlichen realen Zahlen*, which Ewald translated into 'infinite integers' (Ewald 1996 vol 2 pp 883 [7], 908 [7]). We prefer to use 'infinite numbers' to stress that on the one hand we are attached to the *Grundlagen* presentation and on the other hand to the 'number' attribute of 1897.

Our view on the limit initial numbers is not contradicted by Cantor's proclamation in *Grundlagen* that every power is represented by a number-class (Cantor 1932 pp 167, 181, Ewald 1996 vol 2 pp 884 [14], 895 [1]), for Cantor spoke there only of successor initial numbers.

Our reconstruction of Cantor's scale of number-classes is based solely on evidence taken from *Grundlagen*. Nevertheless it is compatible with Cantor's definition of the scale in his letter to Dedekind of August 3, 1899.⁷ In later chapters we will show evidence that also other results mentioned in that letter, Cantor possessed already when he wrote *Grundlagen*.

In the literature, Cantor's development of the scale of number-classes, as presented in *Grundlagen* and reconstructed here, is generally ignored and Hessenberg's presentation (1906 Chaps. 13 and 14) is adopted. In it, however, CBT is assumed (Chap. 7),⁸ while in Cantor's original development this theorem was proved with the construction of the scale, to which subject we turn next.

2.2 The Induction Step

We will now generalize the theorems of Sect. 1.2 for sets with power in Cantor's scale of number-classes. The proof is by transfinite induction and the theorems must be proved in tandem because of their interdependencies. The base case given in Sect. 1.2 corresponds to $\gamma = 3$: ω_3 is Ω of *Grundlagen*; U_3 is (I) + (II), U_2 is (I)⁹; (2) is (II), (1) is (I). The induction hypothesis is that all the theorems below hold for every γ' , $3 \leq \gamma' < \gamma$. We have to prove that hence they hold for γ . Alone is Lemma 2 that need not be repeated for it holds for any set of numbers by the same arguments used to justify it in Sect. 1.2.

Before we begin we call the reader's attention to the difference between arguments that use Lemma 2, for example in the proof of the Fundamental Theorem below, arguments that use the Sequent Argument, for example in the proof of Theorem 1 below, and arguments that assume theorems in the pack under the induction hypothesis. Despite their differences, all these argument types use transfinite induction, but the first two use what may be called local induction, the inductive properties of the infinite numbers defined up to the induction step, and they do not invoke the induction hypothesis.¹⁰

⁷ Cantor 1932 p 443, van Heijenoort 1967 p 113, Grattan-Guinness 1974 p 128, Ewald 1996 vol 2 p 931, Meschkowski-Nilson 1991 p 407. In the first two sources the letter is dated July 28. See the Grattan-Guinness reference.

⁸ Hessenberg says that his proof follows the proof of Bernstein (namely – Borel, see Sect. 11.2) with some changes. Actually his proof is similar to Peano's first (inductive) proof (see Sect. 20.1), published in March 1906. Another case of simultaneity of proofs.

⁹ U_1 is the empty set. We assume by convention that when we use U_k , we have $\kappa > 1$.

¹⁰ Compare to Zermelo's three principles of complete induction in theorems I, III, V, of his 1909 paper (p 192).

Theorem 1. *If γ is a successor number, the power of $(\gamma-1)$ is different from the power of U_κ for any $\kappa < \gamma$.*

Proof: Otherwise, by Lemma 1 (see below) for $\gamma-1$, assumed by the induction hypothesis, $(\gamma-1)$ would have a sequent in it, a contradiction by the Sequent Argument.

The Fundamental Theorem. *A subset of U_γ is either finite, or has the power of U_κ for some $\kappa \leq \gamma$.*

Proof: Let (α') be the subset; note that it is ordered according to the size of its members. Enumerate (α') by U_γ : let α_1 be the smallest number in (α') (Lemma 2(i)), α_2 be the next larger (Lemma 2(ii)), and so on (Lemma 2(ii)). Having defined all α_ζ with $\zeta < \zeta' \in U_\gamma$, and if (α') is not yet exhausted, $\alpha_{\zeta'}$ is defined to be the first member of (α') not yet selected (Lemma 2(ii)). Now, if (α') is exhausted by this process after a finite number of steps it is finite. If it is exhausted at the ζ step and ζ belongs to (κ) , $\kappa < \gamma$, then the power of (α') is the power of ζ , which is the power of ω_κ , which is the power of U_κ . That ω_κ complies by the Limitation Principle, when κ is a successor number, follows from the Limitation Theorem (see below) for κ , assumed by the induction hypothesis. If (α') is not exhausted at any ζ , then, since for every α' there is a ζ such that α' is α_ζ (if not for $\zeta < \alpha'$ then for $\zeta = \alpha'$; here Lemma 2(ii) is used), the procedure renders the equivalence of (α') and U_γ .

The Limitation Theorem. *If γ is a successor number, the power of $(\gamma-1)$ is equal to the power of U_γ , which is the power of ω_γ , which thus fulfills the Limitation Principle.*

Proof: If γ is a successor number then as $(\gamma-1) \subset U_\gamma$, and as $(\gamma-1)$ is not finite, and is not, by Theorem 1, of the power of some U_κ , $\kappa < \gamma$, $(\gamma-1)$ must be, by the Fundamental Theorem, equivalent to U_γ , so ω_γ fulfills the Limitation Principle.

The Limitation Theorem was proved for Ω in the previous chapter separately from the other theorems, because Cantor did not mention it. It is important to realize that without the Limitation Theorem, the scale of numbers, and thus the scale of number-classes, does not exist. Moreover, all the theorems in this section, including CBT, are necessary to establish the Limitation Theorem. Thus, Cantor's proof of CBT is part of his construction of the scale of numbers and number-classes and not a separate theorem that can rely on the existence of the said scale. This point is not noted in the literature on Cantor's set theory.

The Different Alephs Theorem. *The power of U_γ is different from that of any U_κ , $\kappa < \gamma$.*

Proof: If γ is a successor number this follows from Theorem 1 and the Limitation Theorem. Otherwise, we would have that ω_γ (exempt from the Limitation Principle) belongs to (κ) , a contradiction by the Sequent Argument.

The name of the previous theorem was given to it because the power of the U_γ is \aleph_γ and thus the theorem says that all the alephs are different. From the preceding theorems it follows that the power of $(\gamma-1)$, for γ successor, is the next following the power of $U_{\gamma-1}$. From this results the Next-Aleph Theorem follows: $\aleph_{\gamma+1}$ is the next aleph following \aleph_γ .

The Cantor-Bernstein Theorem. *If M is a set of the power of U_γ , and $M'' \subseteq M' \subseteq M \sim M''$, then $M' \sim M$.*

Proof: Without loss of generality we can assume that M is U_γ and thus M' is a subset of U_γ . If M' is not equivalent to M then by the Fundamental Theorem it is either finite or of the power U_κ for some $\kappa < \gamma$. M' cannot be finite as it contains a copy of (I) contained in M'' because $M'' \sim M$. Therefore M'' can, without loss of generality, be regarded as a subset of U_κ and thus either finite (which it is not by the mentioned argument), or, by the Fundamental Theorem for κ , of the power of U_ρ for some $\rho \leq \kappa$. However, this is in contradiction to the Different Alephs Theorem.

The naive reaction of someone educated in college set theory, is that CBT for sets of infinite numbers must be trivial. Our proof shows that this is not the case in the context of early Cantorian set theory.

The enumeration-by method, and in particular the enumeration of the subset by the whole set as utilized in the proof of the Fundamental Theorem, is the metaphor of Cantor's proof of CBT. Cantor's gestalt is that every set can be enumerated. It seems that Cantor's voyage into the infinite began with the maxim "the part is smaller than or equal to the whole" replacing the antique "the part is smaller than the whole" (see Schröder 1898 p 336).¹¹

In view of the Limitation Theorem, the Fundamental Theorem and CBT can be phrased, for γ successor, for subsets of $(\gamma-1)$.

The Union Theorem. *The set of all ordered-pairs of numbers from U_γ (this set is customarily denoted by $\omega_\gamma * \omega_\gamma$) is equivalent to U_γ .*

Proof: We define the order in the set of ordered-pairs as in the proof for (I) + (II) given in the previous chapter and likewise prove that every subset has a first member under this order. Then we enumerate the set of ordered-pairs by U_γ and move to deny that the enumeration exhausts either of the sets before the other. In the definition of the order we rely on the Sum Lemma that the sum of any two numbers from U_γ is in U_γ . We will discuss this lemma below.

First assume that the set of ordered-pairs is exhausted before U_γ . Let ζ be the first number in U_γ to which no corresponding ordered-pair was assigned. $\zeta \in (\kappa)$ for some $\kappa < \gamma$ so ζ is equivalent to U_κ . The set of all $(1, \chi)$, $\chi < \gamma$, ordered-pairs is thus equivalent to a subset of U_κ , so that, by the Fundamental Theorem for U_κ assumed under the induction hypothesis, this set, which is equivalent to U_γ (so obviously is not finite), has the power of $U_{\kappa'}$ for some $\kappa' \leq \kappa$, so $U_\gamma \sim U_{\kappa'}$, contrary to the Different Alephs Theorem.

If, on the other hand, U_γ is exhausted before the set of ordered-pairs then let (μ', χ') be the first ordered-pair not chosen in the enumeration process and B the set of all smaller ordered-pairs which are the ordered-pairs chosen by that process. B is of the power of U_γ . We have for all (μ, χ) in B , $\mu, \chi \leq \mu' + \chi' + 1$.¹² Denote $\mu' + \chi' + 1$ by ρ (Jourdain 1908a; cf. Lindenbaum-Tarski 1926 p 308f).

¹¹ That the part cannot be greater than the whole is provided by CBT.

¹² The definition of the sum of numbers from U_γ will be discussed below.

Then ρ belongs to some (κ) , $\kappa < \gamma$, by the Sum Lemma (see below), and B is equivalent to a subset of the set of ordered-pairs of numbers smaller than ρ and this set is, under the induction assumption, equivalent to U_κ . By the Fundamental Theorem for κ assumed under the induction hypothesis, B is equivalent to some $U_{\kappa'}$ for some $\kappa' \leq \kappa$, which means that $U_\gamma \sim U_{\kappa'}$ contrary to the Different Alephs Theorem.

Therefore, it turns out that the enumeration process puts the set of ordered-pairs in equivalence with U_γ .

The Sum Lemma used in the above proof is the following:

The Sum Lemma. *If $\alpha, \beta \in U_\gamma$ then $\alpha + \beta$ is also in U_γ . In fact, the power of the sum is equal to the power of the greater of the summands, therewith the sum fulfills the Limitation Principle.*

The Sum Lemma assumes the definition of the sum operation which we will discuss below. Its proof is easy to obtain from the Union Theorem and CBT but as we require it for the proof of the Union Theorem, a direct proof is necessary.

Proof: First we prove that the power of $\alpha + \alpha$ is equal to the power of α . We partition the segment-set of α into two sets: one set contains all the finite even numbers, all the limit numbers δ and all the numbers $\delta + 2n$ ¹³; the other set contains the finite odd numbers and the numbers $\delta + 2n - 1$ ($n > 1$). Each of these sets is equivalent to α : the first by mapping the limit numbers to themselves, finite n to $2n$, and the numbers $\delta + n$ to $\delta + 2n$; the second by mapping finite n to $2n + 1$, the limit numbers to their successors, and the numbers $\delta + n$ to $\delta + 2n + 1$. Hence it is obtained that the power of $\alpha + \alpha$ is equal to the power of α .

For $\alpha + \beta$ (or $\beta + \alpha$), where $\beta < \alpha$, a similar procedure on the segment-set of β as a segment of α will deliver the result.

Interestingly, the Sum Lemma seemingly provides an alternative proof of Theorem 1 which bypasses Lemma 1 and thus AC and the Union Theorem: If we assume that $(\gamma-1) \sim U_\kappa$ for some $\kappa < \gamma$ then by the Sum Lemma, the power of ω_γ is equal to the power of $\omega_{\gamma-1} + \omega_\kappa$ which is the power of $\omega_{\gamma-1}$, so ω_γ would belong to $(\gamma-1)$, a contradiction by the Sequent Argument. Zermelo noticed this proof in his remark [22] on Cantor's 1897 *Beiträge* (Cantor 1932 p 355) referring to Theorem D of § 16, the theorem that the power of (II) is not \aleph_0 , which is the denumerable instance of Theorem 1 (see Sect. 1.2).¹⁴ But this proof has a lacuna: unless κ is a successor number (or ω), we do not know that ω_γ fulfills the Limitation Principle.

Lemma 1. (The Sequent Lemma). *A sequence (α_i) of numbers in (γ) , $i < \kappa \leq \omega_\gamma$, has a sequent in (γ) .*

Proof: If the sequence has a greatest member then its sequent is the required sequent to the sequence. Otherwise, an ascending subsequence is generated from the sequence as follows: let α'_1 be the member of the sequence with smallest index

¹³ Here and in the rest of this paragraph, the $+$ sign does not signify the sum operation but just the stage in which the number is generated by the first generation principle.

¹⁴ Strangely, Zermelo did not refer in his comment also to the proof in *Grundlagen*.

that is greater than α_1 , α'_2 be the one with the smallest index that is larger than α'_1 , and so on. If the first σ members of the subsequence have been so selected and there are still members in the sequence that are greater than all the members already selected for the subsequence, define α'_σ as the member with the smallest index among them. Lemma 2(i) is used in the generation of the subsequence. It is easy to see that every member of the sequence is surpassed by a member of the subsequence (here Lemma 2(ii) is used). The subsequence will be indexed by a number not greater than κ so also not greater than ω_γ . Adding to the subsequence all the numbers that lie between any two consecutive members of it and the numbers preceding its first member, a succession is obtained. If we can demonstrate that the succession has the power of ω_γ then its sequent would fulfill the Limitation Principle and be a member of (γ) . Observing that the succession is composed of up to ω_γ partitions, each with up to ω_γ numbers, we can conclude that the succession is equivalent to a subset of the set of all ordered-pairs of numbers smaller than ω_γ , the power of which is equal to the power of ω_γ by the Union Theorem for U_γ proved above. The axiom of choice is necessary here to choose a mapping from each partition to ω_γ . Noting that the succession contains at least one subset of power ω_γ (the partition defined by the first number of the subsequence), the desired result follows by CBT for ω_γ proved above.

Note that in the proof of Lemma 1 for γ , the Union Theorem for γ and CBT for γ are used as well as AC. In the proof of the Union Theorem for γ the Fundamental Theorem for $\kappa < \gamma$ (induction hypothesis) and the Different Alephs Theorem for γ are used. Also the Sum Lemma is used and it does not require the induction hypothesis directly, only that the numbers in U_γ comply with the Limitation Principle. The latter holds by the Limitation Theorem for $\kappa < \gamma$ which is the induction hypothesis. In the proof of CBT for γ the Different Alephs Theorem for γ and the Fundamental Theorem for γ are used. In the proof of the Different Alephs Theorem for γ the Limitation Theorem for γ , Theorem 1 for γ and the Sequent Argument are used. In the proof of the Limitation Theorem for γ the Fundamental Theorem for γ and Theorem 1 for γ are used. The proof of Theorem 1 for γ is established by Lemma 1 for $\gamma-1$ (induction hypothesis) and the Sequent Argument. The proof of the Fundamental Theorem for γ is established by Lemma 2 and the Limitation Theorem for all $\kappa < \gamma$ (induction hypothesis). These are the interdependencies of the theorems involved.

It is remarkable how heavy Cantor's proof of CBT is, in contrast to the meager, albeit insightful, mathematics used by most mathematicians who later provided proofs for that theorem. The reason is that Cantor's basic gestalt, by 1883, was that sets are gauged by the powers in his scale of number-classes and he guided his proof by this beacon; the later proofs typically used geometric gestalt. The metaphor of the proofs for the bunch of theorems presented above can be described as helical transfinite induction (cf. Ferreiro's 1995 p 40).

2.3 The Declaration of Infinite Numbers

In Cantor's letter to Dedekind of November 5, 1882, Cantor expressed in a dramatic tone his latest findings:

[J]ust after our latest visit in Harzburg and Eisenach¹⁵ God almighty saw to it that I attained the most peculiar, unexpected results in the theory of manifolds¹⁶ and in the theory of numbers – or rather have found something which has been fermenting in me for years, and which I have long sought. – It is not a question of the general definition of a point-continuum about which we have spoken and in which I think I have made further progress, but rather about something much more general, and therefore more important.–

You remember I told you in Harzburg that I could not prove the following theorem:

If M' is a part of a manifold M , M'' part of M' , and if M and M' can be reciprocally correlated one-to-one [by a 1–1 mapping] (i.e., M and M'' have the same power) then M' has the same power as M and M'' .

Now I have found the source of this theorem and can prove it rigorously and with necessary generality; and this fills a large gap in the theory of manifolds.

I arrive at this result through a natural extension or continuation of the sequence of real integers, ...

Cantor continued the letter with a summary of his theory of infinite numbers as developed in *Grundlagen*.

It was Cantor's tone in this passage that compelled me to believe that indeed he had a proof of CBT. Here is how we explicate his statements: Cantor may have completed the design of the helical proof presented in the previous section, which at once gives the construction of the scale of number-classes and CBT.¹⁷ In addition, he may have formulated the well-ordering and numbering principles which enable CBT for any set, not just for sets of numbers. Moreover, these principles, on the one hand, provide meaning to the infinity symbols (see the next chapter) of a number-class, which turn out to represent all the possible well-orderings that a set of a certain power can obtain. On the other hand, the principles enable the definition of the arithmetic operations for the infinite numbers, thereby granting them indeed the status of numbers.¹⁸ The background to Cantor's discovery of well-ordering was his realization that a set can obtain many orders. He mentioned this with regard to the continuum in the letter to Dedekind of September 15, 1882, (Cavailles 1962 p 230ff, Dugac 1976 p 255, Ewald 1996 vol 2 p 872). The letter was sent on the

¹⁵ Cantor met Dedekind twice during September 1882; first in Harzburg, their favorite vacation retreat, and then in Eisenach, in a gathering of mathematicians.

¹⁶ Manifold (*Mannigfaltig*) was Cantor's term for set (*Menge*) before *Grundlagen*.

¹⁷ The crucial step was probably a scheme for the proof of the Union Theorem.

¹⁸ It does not make much sense to assume that the entire construction of *Grundlagen* was developed in that month, as some writers have suggested, because of the wealth of the ideas and technical details in *Grundlagen*. Cf. Ferreiro's 1995 p 41, Meschkowski-Nilson 1991 p 90 (3).

day Cantor traveled to Eisenach where he met Dedekind, as he recounts in the passage cited above.

The importance of the definition of the arithmetic operations, comes out in the above letter to Dedekind. In about half-way through the letter Cantor added these words regarding the infinite numbers:

Perhaps you are surprised by my boldness in calling the things $\omega, \omega + 1, \dots, \alpha, \dots$ integers, and even real integers of the second class, while I gave them the more modest title infinity symbols when I used them previously.¹⁹

But my freedom is explained by the remark that among the conceptual things α that I call real integers of the second class there are relations that can be reduced to the basic operations.—

The point is made clearer in Cantor's 1884 letter to Kronecker (Meschkowski-Nilson 1991 p 199, Ferreirós 1995 p 38):

[the opinion that the transfinite numbers] have to be conceived as numbers is based on the possibility of determining concretely the [arithmetical] relations among them, and on the fact that they can be conceived under a common viewpoint with the familiar finite numbers.

It is clear that Cantor was ready to call his infinity symbols numbers only after he had generalized for them the arithmetic operations. A similar approach Cantor exhibited when he introduced the irrationals by way of the fundamental sequences: he first introduced them as symbols (*Zeichen*) and called them numbers only after defining the arithmetic operations on them (Dauben 1979 p 38f). The operations between infinite numbers Cantor defined in *Grundlagen* § 3, after he had introduced the well-ordering principle and the numbering principle (§ 2).

For the definition of $\alpha + \beta$ Cantor took two sets M, M_1 which have the *Anzahlen* α, β and generated the set $M + M_1$ in which the members of M are ordered before the members of M_1 . Conveniently, M and M_1 can be assumed disjoint. It is easy to see that $M + M_1$ is well-ordered and its *Anzahl* is then taken to be the sum $\alpha + \beta$. This definition gives the basis for the Sum Lemma which in turn proves that the sum complies with the Limitation Principle if α, β do. The well-ordering principle is not necessary for the definition since the well-ordering of $M + M_1$ can be proved directly, but the numbering principle is necessary to obtain the *Anzahl* of this set. After the definition by abstraction of the ordinal numbers was introduced, the numbering principle was no longer required.

For the definition of multiplication Cantor proceeded in a similar way²⁰:

If one takes a succession, determined by a number β , of various sets which are similar and which are similarly ordered [and pairwise disjoint] and such that each has an *Anzahl* of its

¹⁹ Reference is here made to Cantor's 1880 paper, 1882 paper and 1883 paper, parts 2, 3, 4, in the series *Ueber unendliche, lineare Punktmannichfaltigkeiten*, Cantor 1932 pp 145, 149, 157, respectively.

²⁰ In 1883 *Grundlagen* Cantor denoted the result of the following definition by $\beta\alpha$, but he reversed the notation in 1895 *Beiträge*. We use the latter convention that prevailed.

elements equal to α , then one obtains a new well-ordered set whose corresponding *Anzahl* supplies the definition of the product $\alpha\beta$.

Cantor does not argue why the product complies with the Limitation Principle. By the Union Theorem it can be established that the product has the power of the greater of the powers of α , β . Again, it is the numbering principle that is necessary for this definition, until the context is switched to ordinals.

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