Introduction

The chief purpose of this book is to present, in detail, a compilation of proofs of the Cantor-Bernstein Theorem (CBT) published through the years since the 1870s. Over 30 such proofs are surveyed.

CBT, an elementary but not trivial theorem of set theory, is usually stated either in its single-set formulation (if $M'' \subseteq M' \subseteq M \sim M''$ then $M \sim M'$) or in its two-set formulation (if $M \sim N' \subseteq N$ and $N \sim M' \subseteq M$ then $M \sim N$), which are equivalent.

At the turn of the twentieth century, CBT was called the Equivalence Theorem. The veterans, e.g., Bernstein, Zermelo, Hausdorff and Fraenkel, kept using this name. Bernstein (1905 p 121) says that the name was suggested by Cantor.

In Whitehead's 1902 paper, in section III written by Russell, CBT is referenced as "the Bernstein's and Schröder's theorem" (p 369, 383) after the two mathematicians who first published proofs for the theorem, both in 1898. In 1907b (p 355), Jourdain suggested that the name be 'the Schröder-Bernstein theorem'. He thought that Cantor's name should not be included in the name of the theorem because Cantor did not provide a proof of it, a mistake that has been often repeated. As it turned out, the name of Schröder should have been omitted (cf. Tait 2005) because it was found (Korselt 1911) that his proof is erroneous (or rather, as will be demonstrated, senseless). The omission already appears in Poincaré 1906a (p 27). Still, many kept using the Schröder-Bernstein theorem name for CBT. Zermelo mentioned both proofs of Schröder and Bernstein in 1932 (Cantor 1932 p 209 [6]), without any reservation, even noting that Schröder's proof dates to 1896 and that of Bernstein to 1897. Likewise, in the second edition of PM (Principia Mathematica), from 1925, the theorem is still called the Schröder-Bernstein theorem. Similar naming is used in J. Kőnig 1906, Banach 1924, Lindenbaum-Tarski 1926, Whittaker 1927, Hellmann 1961, and even Grattan-Guinness (1971a p 117 reference to #37 in the bibliography). In fact, Schröder's name remains linked to CBT even today, as a quick search of the Internet reveals.

We have not identified when the name 'the Cantor-Bernstein Theorem' was introduced. It seems to be the preferred name today. In the second edition of Fraenkel 1973, edited by A. Levy, the editor systematically changed the original

'Equivalence Theorem' of the 1958 edition, to the Cantor-Bernstein Theorem. For reasons to be detailed later it appears that the proper name for the theorem would be the Cantor-Dedekind-Bernstein Theorem.

The two-set formulation was cached already in Cantor's 1878 *Beitrag* but it was explicitly stated first in Cantor's 1887 *Mitteilungen* paper (p 413) and again in Cantor's 1895 *Beiträge* paper, §2, as Corollary B to the Comparability Theorem for cardinal numbers, presented there as Theorem A.

The single-set formulation was first stated in Cantor's letter to Dedekind of November 5, 1882.[1] In the important 1883 *Grundlagen* paper (§13),[2] which was published shortly after, an instance of the theorem was fully stated and proved. The instance was for sets of the power of (II) – the second number-class. There also the denumerable instance was mentioned. The latter, in a slightly different form, was already stated in Cantor's 1878 *Beitrag* as easy to demonstrate. It was nevertheless proved in Cantor's 1895 *Beiträge* (§6 Theorem B).[3]

Following his statement of CBT for sets of the power of (II) in *Grundlagen*, Cantor said that "this theorem has general validity". However, Cantor never published a demonstration to CBT in its general form. It was perhaps because the theorem was not stated there in full, as in the letter to Dedekind, that Bernstein (1905 p 121) references Cantor's 1887 *Mitteilungen* as the place where CBT was first proposed by Cantor. The single-set formulation was stated again in Cantor's 1895 *Beiträge* paper (§2), as Corollary C to Theorem A.[4] We will present Cantor's proof of the instance for sets of the power of (II) in Chap. 1. In Chap. 2 we will present a generalization of that proof, to sets with power in the scale of number-classes, which we believe is the proof Cantor promised in *Grundlagen*. Our generalization fills a gap in the common belief on the history of Cantor's set theory. It also led us to a new understanding of Cantor's Limitation Principle from the 1883 *Grundlagen*.

In Chap. 3 we will present evidence that Cantor knew of CBT already by the time he wrote his 1878 *Beitrag* and probably had a proof for it at least for the continuum. In Chap. 3 we will also promote the view that Cantor had, by the time of his 1878 *Beitrag*, the idea of his infinite numbers from 1883 *Grundlagen*, as well as the ideas of number-classes and his theorem about the power of the power-set. In Chap. 4 we discuss Cantor's theory of inconsistent sets and in Chap. 5 we explain how this theory provided Cantor with a proof of the Comparability Theorem for cardinal numbers. On this issue too our views are contrary to common belief on Cantor's theory. In Chap. 5 we also explain the difference between that theorem and

---

[1] Cantor-Dedekind 1937 p 55, Cavailles 1962 p 232, Dugac 1976 p 258, Hallett, 1984 p 59, Meschkowski-Nilson 1991 p 85f, Ferreirós 1993 p 353, 1995 p 37, Ewald 1996 vol 2 p 874.

[2] See Ewald 1996 vol 2 p 878–881 for bibliographical background on *Grundlagen*.

[3] As the proof is simple, it is reasonable to assume that Cantor possessed it already when he wrote his 1878 *Beitrag*. Cantor probably proved it in his 1895 *Beiträge* simply because he wanted at that time to present his theory in full detail.

[4] A third formulation of CBT, Corollary E, appears there as well.

Cantor's comparability of sets which leads us to discuss in Chap. 6 the scheme of complete disjunction.

Dedekind was the first to have learned of CBT and the first to provide the theorem with a general proof independent of the power concept. In Chap. 9 we bring Dedekind's proof of CBT with its historical background. However, prior to that we are led to discuss two issues regarding the relationship between Cantor and Dedekind. First (Chap. 7), with regard to the ruptures in their correspondence, we add a new explanation to those discussed in the literature. Then (Chap. 8) we discuss Cantor's criticism of Dedekind's infinite set, in view of his theory of inconsistent sets. Chapter 9 concludes the first part of the book.

In the second part of the book we bring the early published proofs of CBT. We relate to papers published in the period 1898–1901 by Schröder, Borel, Schoenflies and Zermelo. In the chapter on Schröder we also cover Korselt's criticism from 1911. In the chapter on Borel's proof we suggest a reconstruction of Bernstein's original proof upon which Borel based his own. Schoenflies' proof amalgamated the proofs of Borel and Schröder, together with a proof of Cantor communicated to him in a letter. Zermelo's proof unknowingly generalized ideas of Dedekind and stretched out the applicability of the cardinal number notion for a CBT proof. We conclude the second part with Bernstein's proof of a division theorem that we name the Bernstein Division Theorem (BDT). It says that if $k\mathfrak{m} = k\mathfrak{n}$ then $\mathfrak{m} = \mathfrak{n}$, where k is a natural number, $\mathfrak{m}$, $\mathfrak{n}$ cardinal numbers. We also touch upon an inequality version of the theorem: if $k\mathfrak{m} \geq k\mathfrak{n}$ then $\mathfrak{m} \geq \mathfrak{n}$. A research project emerging from BDT and the inequality-BDT plays a significant role in later chapters.

In part III we discuss proofs of CBT that emerged during the period 1902–1912. Most proof are related to the development of the logicist movement and the connected debate between Poincaré and Russell. First we present the CBT proof of Russell from 1902, which was circular. Then we discuss the role of CBT in the derivation of Russell's Paradox. We move to the attempts of Jourdain (1904) to extend Cantor's construction of the scale of number-classes. We point out the many defects of Jourdain's endeavor, which were, one may say, balanced by his ability to identify Cantor's theory of inconsistent sets and his proof of the Comparability Theorem. We cover Harward's criticism (1905) of Jourdain's 1904 papers and the latter's attempts to correct his 1904 exposition with new papers of 1907 and 1908. The papers of Poincaré of 1905–1906 are the centerpiece of part III. Several proofs of CBT are presented in these papers and there the criticism of an impredicative definition emerged. Poincaré's papers provoked several new proofs of CBT, mainly by Peano and Zermelo, who provided impredicative proofs. The latter gave the proof in his axiomatic system. Another proof came from J. Kőnig; it influenced his son, D. Kőnig, into a research project, on the tracks of BDT, which led to several interesting results in graph theory. We also cover Korselt's 1911 proof who applied it to solve a problem of Schröder. This problem initiated another research project through a paper from 1926 by Lindenbaum-Tarski and a paper of Sierpiński from 1947. The period surveyed in part III culminated in the Whitehead-Russell Principia Mathematica, which contains several proofs of CBT that we survey. We end part III with a discussion of Hausdorff's paradox, which we link to BDT.

Part IV of the book deals with results obtained mostly within the Polish school of logic concerning CBT, BDT and the inequality-BDT. The champions of this part are Sierpiński, Banach, Kuratowski, Lindenbaum, the Englishman Whittaker, Knaster, Sikorski, Reichbach and, foremost, Tarski.

The final part of the book includes only three chapters. One handles a proof by Hellmann from 1961. Another chapter concerns the attempts to port CBT to intuitionist context. The last chapter handles porting CBT to category theory.

My special interest in CBT arose when I first studied set theory and used Fraenkel's (1966) book, 'Abstract Set Theory', as textbook. There (p 72–79), two proofs of the theorem, of Poincarè and Whittaker, were given in detail and about a dozen more referenced. This surprised me, for I could not understand at the time, how theorems can have more than one proof and why anyone would be interested in providing a proof of an already proven theorem.

Another surprise came from the comparisons Fraenkel made between the CBT proofs that he brought. Of one of the proofs he said that it used the natural numbers, while the other was more abstract and applied the "union of a set of sets", which he linked to the proof of the Well-Ordering Theorem. Thus I learned that not only are there different proofs of a theorem, but there is an art of comparing proofs and proofs can compare even when they are not proofs of the same theorem! There and then I decided to compile one day all the proofs of CBT and study, through this compilation, how to compare proofs. This book is the outcome of that undertaking and in developing a discussion of proof comparison lies its second purpose.

Fraenkel's comparison of the CBT proofs was focused on the means whereby the proof is won, to which he tried to give a concise description. As I attempted to follow Fraenkel's example with regard to the proofs which I compiled I came to realize that what I was doing was giving metaphors to central aspects of the proof. Thus I obtained the view that proofs can be compared using their metaphors. Metaphors may involve mathematical notions (natural numbers) or template methods (complete induction, Cantor's diagonal method), as well as non-mathematical terms (pushdown an infinite stack). Thus there is room for idiosyncrasy in the generation of metaphors for proofs. The latter are especially of interest as they are rarely pointed out in mathematical texts, but are apparently used in informal discourse among peers.

Fraenkel did not compare the proofs he presented through their metaphors alone. For example, with regard to the proof of CBT of J. Kőnig Fraenkel remarked that it "is distinguished by its lucidity" and "has yielded remarkable generalizations". From the second remark I learned that the value of a proof can be appreciated only through a diachronic study, which I have contemplated in this book. In this I have followed the inspiring example of Lakatos' book, "Proofs and Refutations" (1976), where the history of proofs of Euler's polyhedron formula is presented.

From the first of the above remarks of Fraenkel, I have obtained another significant lesson. I compared J. Kőnig's proof to the first proof which Fraenkel brought. Both proofs looked equally lucid to me, so I wondered why Fraenkel deemed J. Kőnig's proof to be especially lucid. It occurred to me that

Fraenkel perhaps saw the lucidity of J. Kőnig's proof in that it shed new light on the mathematical situation of CBT. I thought that it was as if J. Kőnig identified in the situation of CBT a new gestalt. This then became my observation: proofs are characterized by gestalt, which, like metaphors, may contain non-mathematical attributes.

The gestalt explication, as a dimension in proof comparison, answered my qualm with regard to the reason why there are different proofs of a theorem and why people bother to give a new proof of a proved theorem. A proof is a description, like driving instructions. These are construed under a certain gestalt of the area in which the driving occurs. Different people may have different gestalt of the area and so they give different instructions. People are willing to offer their driving instructions because it is their own, special, dear to them, gestalt that they offer. People are even willing to debate whose instructions are better, with regard to such parameters as distance, complexity, traffic, etc. Same thing with mathematicians. They are eager to offer their gestalt. Mathematicians have an additional justification as it may turn out that one gestalt is more fruitful than another, as it happened with J. Kőnig's gestalt applied in his CBT proof.

With time I realized that I use gestalt with regard to the static elements of a proof, which are in general rather easy to identify, while metaphor I use for the dynamic elements, which, in general, change the gestalt. I will not dwell now on better explicating these notions since I believe that some experience in their use must be gained prior to such discussion.

Metaphors and gestalt ('proof descriptors') need not correspond to the entire proof; they may relate just to some of its parts. Actually, it is through descriptors that the parts of a proof are often discerned. In this capacity descriptors may thus serve as mnemonics. Descriptors are not necessarily verbal; they may include drawings or gestures. Many descriptors can be given to a proof and they need not be coherent or accurate in any objective sense, rather the opposite; informal character is essential to descriptors.

The process by which descriptors are generated from a proof, I call 'proof-processing'. I will be proof-processing the proofs presented in order to extract descriptors for their comparison.[5] With regard to the proofs of CBT I am after the differences in proof descriptors; however, I also compare proofs of different theorems that demonstrate affinity in terms of gestalt and metaphor. In such cases I believe that proof-processing of the older proof was behind the achievement of the newer proof. Mainly I will consider proofs of BDT, which received through the years several proofs related through proof descriptors to proofs of CBT.

Here Lakatos' 1976 book should be mentioned again. His book introduced me to a new subject: heuristic – patterns in the development of mathematics (Lakatos 1976 p 93, 143n4). Lakatos discerned the patterns of concept stretching, proofs and refutations, change of dominant theory, proof-analysis, hidden lemma. As a tribute

[5] However, I will not exaggerate in this and I will not undertake to extract proof descriptors from every proof presented or all proof descriptors of a proof analyzed.

to his influence on our work, for I believe that proof-processing is also a pattern for the development of mathematics,[6] I will note along our discussion, corroborating instances to elements of his theory.

The book is intended for researchers in the history and philosophy of set theory or in the methodology of the development of mathematics. Perhaps it will be of interest also to graduate students, who wish to take an excursion into the developmental area of mathematical research, which precedes textbooks.

Besides the concise historical remarks on CBT given by Fraenkel (1966), two general surveys of the history of the theorem were previously offered. First, there was a survey by Medvedev (1966, in Russian). It contains mainly anecdotal information on the early proofs of the theorem, up to 1906.[7] Second, there was a survey by Mańka-Wojciechowska (1984, in Polish). It covers not only the early proofs but also the later proofs of the 1920s that had a topological context and topological consequences. The second survey includes several remarks of a proof-processing nature. We will reference these works where appropriate.

Our main references, however, are the original papers and monographs of the mathematicians whose work we cover. In addition, we will often reference survey books, especially such that concern Cantorian set theory and its period. These include mainly Dauben 1979, Moore 1982, Hallett 1984, Meschkowski-Nilson 1991, Ferreiròs 1999, Grattan-Guinness 2000, Ebbinghaus 2007, Reference will also be made to many papers written by historians of mathematics including Grattan-Guinness, Moore, Peckhaus, Purkert, Ferreiròs, to name only a few. Occasionally we will reference some mathematics text books, such as Fraenkel 1966, Rogers 1967, Levy 1979.

The original material appeared in German, French or English. When we quote passages from texts in German or French for which available translations exist, we quote from those translations. Otherwise, the translations are ours.

Many mathematicians are mentioned in the book. Most of them are well known and information on them can easily be found on the internet. For the others we provide whatever details we happened to gather.

The book consists of 39 chapters divided into five parts, with an Introduction and a Conclusion. Most chapters are devoted to one source. The chapters are generally in chronological order, which, however, could not always be maintained because certain developments took place in parallel. Internal references are made by chapter or section numbers.

In the presentation we tried to balance between the desire to cite the original text and the desire to be concise. With regard to notation, we tried to bring the original notation but changed it when it is no longer current or if the presentation can be greatly improved by such change, which is always noted. We took special care to point out mistakes, even if only typos, in the original papers, figuring that they are now only rarely visited so that any such comment has historical value. In certain

[6] We will expand on this in the Conclusion to the book.

[7] A survey on Medvedev is given in Anellis 1994.

cases, when a lacuna in the original proof is obvious, we suggested how it could be completed. Any short-comings in these suggestions are our responsibility and reflect nothing on the original writer. The principle example here is our suggestion how Bernstein's proof of BDT for general *k* can be completed.

Unfortunately, the collection below is not complete. Omitted are papers published after 1973, which generally aim to port CBT to various mathematical structures that emerged following developments in the 1950s. They are quite numerous. Moreover, there is no warranty that all the relevant papers from before 1973 are included, even if mentioned in Fraenkel 1966.[8] Yet we believe that we have addressed the most known proofs, which were most influential.

Finally, I enjoyed writing this book; I hope someone will enjoy reading in it.

[8] Thus we have not addressed the results concerning CBT in the context of families of sets, such as Otchan 1942. and Bruns-Schmidt 1958.

http://www.springer.com/978-3-0348-0223-9

Proofs of the Cantor-Bernstein Theorem
A Mathematical Excursion
Hinkis, A.
2013, XXIII, 429 p. 24 illus., 3 illus. in color., Hardcover
ISBN: 978-3-0348-0223-9
A product of Birkhäuser Basel