

Chapter 2

Operator Algebra

An operator transforms one function into another. One writes and says that A operates on f to produce g . We generally assume that operators are linear which means that for any two functions, f_1 and f_2 ,

$$A(f_1 + f_2) = Af_1 + Af_2. \quad (2.1)$$

When we say that two operators are equal

$$A = B \quad (2.2)$$

one means that for an arbitrary function, $f(x)$,

$$Af = Bf. \quad (2.3)$$

For example, the two operators D^2XD and $XD^3 - 2iD^2$ are equal, because indeed when they operate on an arbitrary function they give the same answer. Specifically

$$D^2xDf(x) = (xD^3 - 2iD^2)f(x). \quad (2.4)$$

If we wanted to verify this we could write it explicitly in terms of derivatives,

$$\frac{1}{i^3} \frac{d^2}{dx^2} x \frac{d}{dx} f(x) = \left(\frac{1}{i^3} x \frac{d^3}{dx^3} - 2i \frac{1}{i^2} \frac{d^2}{dx^2} \right) f(x) \quad (2.5)$$

and carry out the differentiation of both sides to verify the identity. One can also do it in the Fourier domain and verify that indeed

$$p^2 Xp \hat{f}(p) = (Xp^3 - 2ip^2) \hat{f}(p) \quad (2.6)$$

or explicitly

$$p^2 i \frac{d}{dp} p \hat{f}(p) = \left(i \frac{d}{dp} p^3 - 2ip^2 \right) \hat{f}(p). \quad (2.7)$$

However, a much better way is to manipulate the operators algebraically by using the commutation relation, $DX = XD - i$. In particular

$$D^2XD = DDXD = D(XD - i)D = DXD^2 - iD^2 \quad (2.8)$$

$$= (XD - i)D^2 - iD^2 = XD^3 - 2iD^2. \quad (2.9)$$

Of course, in manipulating operator equations, attention must be paid to the fact that operators generally do not commute.

We now give some basic definitions and results regarding operators.

Inverse. The inverse of an operator, A^{-1} , is an operator such that

$$A^{-1}A = AA^{-1} = I \quad (2.10)$$

where I is the identity operator. The inverse of the product of two operators is given by

$$(AB)^{-1} = B^{-1}A^{-1}. \quad (2.11)$$

Adjoint. The adjoint of an operator is another operator, denoted by A^\dagger , which forces the equality

$$\int g^* Af dx = \int f \{A^\dagger g\}^* dx \quad (2.12)$$

where g and f are any two functions. The adjoint of a product of operators is given by

$$(AB)^\dagger = B^\dagger A^\dagger. \quad (2.13)$$

The adjoint is sometimes called the Hermitian adjoint.

Hermitian operator. If the adjoint of an operator equals the operator

$$A = A^\dagger, \quad (2.14)$$

one then says that the operator is a self adjoint or a Hermitian operator. For a Hermitian operator Eq. (2.12) becomes

$$\int g^* Af dx = \int f \{Ag\}^* dx \quad (2.15)$$

which may be taken as the definition of a Hermitian operator. The basic operators X and D are Hermitian operators. Hermitian operators play a particularly important role for many reasons that will be discussed in subsequent sections, but the most important is the eigenvalues of a Hermitian operator are real and the eigenfunctions are orthogonal and complete.

Expressing an arbitrary operator in terms of Hermitian operators. An arbitrary operator, A , can be written as

$$A = \frac{1}{2}(A + A^\dagger) + \frac{1}{2}i(A - A^\dagger)/i. \quad (2.16)$$

One can readily prove that both $(A + A^\dagger)$ and $(A - A^\dagger)/i$ are Hermitian. Eq. (2.16) shows that we can express any operator as a sum of a Hermitian operator plus i times a Hermitian operator. This is the operator analog of writing a complex number in terms of its real and imaginary parts.

Product of two operators. The product of two Hermitian operators is generally not Hermitian. However, writing AB as

$$AB = \frac{1}{2} [A, B]_+ + \frac{i}{2} [A, B]/i \quad (2.17)$$

expresses AB in terms of the commutator and anti-commutator. Moreover $[A, B]_+$ and $[A, B]/i$ are Hermitian.

Unitary Operator. An operator U is said to be unitary if its adjoint is equal to its inverse,

$$U^\dagger = U^{-1}. \quad (2.18)$$

Unitary operators are discussed in Chap. 7 but we mention here that the importance of being unitary is that when it operates on a function, normalization is preserved in the sense that

$$\int |f(x)|^2 dx = \int |Uf(x)|^2 dx. \quad (2.19)$$

An operator of the form

$$U = e^{iA} \quad (2.20)$$

is unitary if A is Hermitian and conversely. This is sometimes known as Stone's theorem.

Functions of operators. There are many ways to define functions of operators and we mention two of them. One way is via a power series. A function $f(A)$ of the operator A shall mean that we expand the ordinary function $f(x)$ in a power series

$$f(x) = \sum_{n=0}^{\infty} f_n x^n \quad (2.21)$$

and then substitute A for x ,

$$f(A) = \sum_{n=0}^{\infty} f_n A^n. \quad (2.22)$$

A second way is by way of the Fourier transform of $f(x)$. Define

$$\widehat{f}(p) = \frac{1}{\sqrt{2\pi}} \int f(x) e^{-ixp} dx, \quad (2.23)$$

we then define $f(A)$ by

$$f(A) = \frac{1}{\sqrt{2\pi}} \int \widehat{f}(p) e^{iAp} dp. \quad (2.24)$$

2.1 Exponential Operator

The exponential operator, e^A , where A is an operator, plays a fundamental role in correspondence rules and in many other branches of science and mathematics [95]. We discuss here some of its basic properties and further developments will be discussed as needed.

Differentiation of $e^{A(\theta)}$. If $A(\theta)$ is an operator that depends on a parameter θ , then

$$\frac{d}{d\theta} e^{A(\theta)} = \int_0^1 e^{(1-s)A(\theta)} \frac{dA}{d\theta} e^{sA(\theta)} ds = \int_0^1 e^{sA(\theta)} \frac{dA}{d\theta} e^{(1-s)A(\theta)} ds. \quad (2.25)$$

A special case that often arises is the differentiation of $e^{A+\theta B}$ with respect to θ but where A and B are operators that do not depend on θ . In this case

$$\frac{d}{d\theta} e^{A+\theta B} = \int_0^1 e^{(1-s)(A+\theta B)} B e^{s(A+\theta B)} ds. \quad (2.26)$$

Also of interest is the evaluation of Eq. (2.26) at zero,

$$\left. \frac{d}{d\theta} e^{A+\theta B} \right|_{\theta=0} = \int_0^1 e^{(1-s)A} B e^{sA} ds. \quad (2.27)$$

2.2 Manipulating $D^m X^n$ and $X^m D^n$

In manipulating expressions involving X and D the following relations are useful,

$$D^m X^n = \sum_{k=0}^{\min(m,n)} (-i)^k k! \binom{n}{k} \binom{m}{k} X^{n-k} D^{m-k} \quad (2.28)$$

and

$$X^n D^m = \sum_{k=0}^{\min(m,n)} i^k k! \binom{n}{k} \binom{m}{k} D^{m-k} X^{n-k}. \quad (2.29)$$

These expressions were first derived by McCoy [55]. A few other important relations are

$$[X, D^n] = inD^{n-1} \quad (2.30)$$

$$[X^n, D] = inX^{n-1} \quad (2.31)$$

$$[D, f(x)] = \left(\frac{1}{i} \frac{d}{dx} f(x) \right) = (Df(x)). \quad (2.32)$$

In Eq. (2.32) we have put parentheses around $\frac{1}{i} \frac{d}{dx} f(x)$ to emphasize that the differentiation is only on $f(x)$ and not on the function that $[D, f(x)]$ operates on. That is, for an arbitrary function $g(x)$,

$$[D, f(x)]g(x) = \left(\frac{1}{i} \frac{d}{dx} f(x) \right) g(x). \quad (2.33)$$

2.3 Translation Operator

The operator $e^{i\tau D}$ is the translation operator in the x representation

$$e^{i\tau D} f(x) = f(x + \tau). \quad (2.34)$$

This is readily proven. Since

$$e^{i\tau D} f(x) = \sum_{n=0}^{\infty} \frac{(i\tau)^n D^n}{n!} f(x) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \frac{d^n}{dx^n} f(x) \quad (2.35)$$

which is precisely the Taylor expansion of $f(x + \tau)$ and therefore Eq. (2.34) follows. Since D is Hermitian, τD is Hermitian for real τ , the translation operator is hence unitary. For translation in the Fourier domain we have

$$e^{-i\theta X} \hat{\varphi}(p) = e^{\theta \frac{\partial}{\partial p}} \hat{\varphi}(p) = \hat{\varphi}(p + \theta). \quad (2.36)$$

Some other relations that are useful regarding the translation operators are

$$e^{i\tau D} f(x) \varphi(x) = [e^{i\tau D} f(x)] [e^{i\tau D} \varphi(x)]. \quad (2.37)$$

Also,

$$[e^{i\tau D}, x] = \tau e^{i\tau D} \quad (2.38)$$

and more generally

$$[e^{i\tau D}, g(x)] = [g(x + \tau) - g(x)] e^{i\tau D}. \quad (2.39)$$

The adjoint of the translation operator is given by

$$(e^{i\tau D})^\dagger = e^{-i\tau D}. \quad (2.40)$$

This can be proved in the following way. We have

$$\begin{aligned} \int g^* e^{i\tau D} f dx &= \int g^*(x) f(x + \tau) dx = \int g^*(x - \tau) f(x) dx \\ &= \int f(x) [e^{-i\tau D} g(x)]^* dx \end{aligned} \quad (2.41)$$

which shows Eq. (2.40).

2.4 The Operator $e^{i\theta X + i\tau D}$

The simplification and disentanglement of the operator e^{A+B} is very important and also very difficult. We will be discussing this in detail in a later chapter; however, fortunately, there are some special cases where simplification is possible. When the two operators, A and B , both commute with their commutator

$$[A, [A, B]] = [B, [A, B]] = 0, \quad (2.42)$$

then

$$e^{A+B} = e^{-\frac{1}{2}[A,B]} e^A e^B = e^{\frac{1}{2}[A,B]} e^B e^A. \quad (2.43)$$

This is a special case of the Baker-Cambell-Hausdorf formula. See Eq. (15.50).

Of particular interest is the operator $e^{i\theta X + i\tau D}$ with real θ and τ . The relevant commutator is

$$[i\theta X, i\tau D] = -i\theta\tau \quad (2.44)$$

and using Eq. (2.43) we have

$$e^{i\theta X + i\tau D} = e^{i\theta\tau/2} e^{i\theta X} e^{i\tau D} = e^{-i\theta\tau/2} e^{i\tau D} e^{i\theta X}. \quad (2.45)$$

Therefore for an arbitrary function, $f(x)$, we have

$$e^{i\theta X + i\tau D} f(x) = e^{i\theta\tau/2} e^{i\theta X} e^{i\tau D} f(x) = e^{i\theta\tau/2} e^{i\theta x} f(x + \tau). \quad (2.46)$$

In addition,

$$e^{i\theta X} e^{i\tau D} = e^{-i\theta\tau} e^{i\tau D} e^{i\theta X} \quad (2.47)$$

and

$$e^{i\tau D} e^{i\theta X} = e^{i\theta\tau} e^{i\theta X} e^{i\tau D}. \quad (2.48)$$

Also,

$$e^{i\theta X + i\tau D} f(x) g(x) = [e^{i\theta X + i\tau D} f(x)] [e^{i\tau D} g(x)]. \quad (2.49)$$

Characteristic function operator. It is convenient to define the characteristic function operator for the Weyl correspondence

$$\mathcal{M}(\theta, \tau) = e^{i\theta X + i\tau D}. \quad (2.50)$$

The reason why this is called the characteristic function operator will become clear in Chap. 5. It often arises that one has to evaluate the product of two such operators $\mathcal{M}(\theta', \tau')\mathcal{M}(\theta, \tau)$. Explicitly

$$\mathcal{M}(\theta', \tau')\mathcal{M}(\theta, \tau) = e^{i\theta' X + i\tau' D} e^{i\theta X + i\tau D} \quad (2.51)$$

$$= e^{i\theta' \tau' / 2} e^{i\theta \tau / 2} e^{i\theta \tau'} e^{i(\theta + \theta') X} e^{i(\tau' + \tau) D}. \quad (2.52)$$

But using Eq. (2.45) we have that

$$e^{i(\theta + \theta') X} e^{i(\tau' + \tau) D} = e^{-i(\theta + \theta')(\tau + \tau') / 2} e^{i(\theta + \theta') X + i(\tau + \tau') D} \quad (2.53)$$

and therefore we obtain

$$\begin{aligned} & e^{i\theta' \tau' / 2} e^{i\theta \tau / 2} e^{i\theta \tau'} e^{i(\theta + \theta') X} e^{i(\tau' + \tau) D} \\ &= e^{i\theta' \tau' / 2} e^{i\theta \tau / 2} e^{i\theta \tau'} e^{-i(\theta + \theta')(\tau + \tau') / 2} e^{i(\theta + \theta') X + i(\tau + \tau') D} \end{aligned} \quad (2.54)$$

giving

$$e^{i\theta' X + i\tau' D} e^{i\theta X + i\tau D} = e^{i(\theta \tau' - \theta' \tau) / 2} e^{i(\theta + \theta') X + i(\tau + \tau') D} \quad (2.55)$$

or

$$\mathcal{M}(\theta', \tau')\mathcal{M}(\theta, \tau) = e^{i(\theta \tau' - \theta' \tau) / 2} \mathcal{M}(\theta + \theta', \tau + \tau'). \quad (2.56)$$

For future reference we also write

$$e^{i\theta X + i\tau D} e^{i\theta' X + i\tau' D} = e^{i(\theta' \tau - \theta \tau') / 2} e^{i(\theta + \theta') X + i(\tau + \tau') D}. \quad (2.57)$$

One also has

$$\mathcal{M}(\theta - \theta', \tau - \tau')\mathcal{M}(\theta', \tau') = e^{i(\theta \tau' - \theta' \tau) / 2} \mathcal{M}(\theta, \tau). \quad (2.58)$$

Also of interest is the commutator

$$[\theta' X + \tau' D, \theta X + \tau D] = \theta' \tau [X, D] + \theta \tau' [D, X] = i(\theta' \tau - \theta \tau'). \quad (2.59)$$

The adjoint of $e^{i\theta X + i\tau D}$. The adjoint of the characteristic function operator often arises. We designate it by $M^\dagger(\theta, \tau)$ and we now show that it is given by

$$\mathcal{M}^\dagger(\theta, \tau) = \mathcal{M}(-\theta, -\tau) = e^{-i\theta X - i\tau D}. \quad (2.60)$$

To prove this consider

$$\int g^* e^{i\theta X + i\tau D} f dx = \int g^*(x) e^{i\theta \tau / 2} e^{i\theta x} e^{i\tau D} f(x) dx = \int g^*(x) e^{i\theta \tau / 2} e^{i\theta x} f(x + \tau) dx. \quad (2.61)$$

A straightforward change of variables results in

$$\int g^* e^{i\theta X + i\tau D} f dx = \int f(x) \left[e^{i\theta\tau/2} e^{-i\theta x} g(x - \tau) \right]^* dx \quad (2.62)$$

$$= \int f(x) \left[e^{i\theta\tau/2} e^{-i\theta x} e^{-i\tau D} g(x - \tau) \right]^* dx \quad (2.63)$$

$$= \int f(x) \left[e^{-i\theta X - i\tau D} g(x) \right]^* dx \quad (2.64)$$

which proves Eq. (2.60).

Generalized characteristic function operator. In later chapters, and particularly in Chap. 4, we will study the generalized characteristic function operator given by

$$\mathcal{M}^\Phi(\theta, \tau) = \Phi(\theta, \tau) e^{i\theta X + i\tau D} \quad (2.65)$$

where $\Phi(\theta, \tau)$ is a complex function called the kernel. The adjoint of this operator is

$$\mathcal{M}^{\Phi\dagger}(\theta, \tau) = \Phi^*(\theta, \tau) e^{-i\theta X - i\tau D}. \quad (2.66)$$

2.5 The Operator $e^{\xi H} A e^{-\xi H}$

The operator $e^{\xi H} A e^{-\xi H}$ comes up often in many fields and is particularly important in quantum mechanics. An expansion for this operator is

$$e^{\xi H} A e^{-\xi H} = A + \xi[H, A] + \frac{1}{2!}\xi^2[H, [H, A]] + \frac{1}{3!}\xi^3[H, [H, [H, A]]] + \cdots \quad (2.67)$$

The reason the operator is fundamental in quantum mechanics is because it is the formal solution to the Heisenberg equation of motion for an operator. Heisenberg's equation of motion for a time independent Hamiltonian operator, H , and for an operator A that does not explicitly depend on time is,

$$\frac{dA}{dt} = i[H, A]. \quad (2.68)$$

The formal solution of this equation is

$$A(t) = e^{itH} A(0) e^{-itH} \quad (2.69)$$

as can be readily verified.

An example of Eq. (2.67) is the simplification of $e^{i\beta D} X e^{-i\beta D}$. Since $[D, X] = -i$ the series truncates after the second term and we have that

$$e^{i\beta D} X e^{-i\beta D} = X - i(i\beta) = X + \beta. \quad (2.70)$$

Repeated Commutator. Expressions such as $[H, [H, [H, A]]]$ are called repeated commutators and appear often. A convenient notation to denote them is $[H, A]_n$ where n indicates the number of repetitions of the commutator. Eq. (2.67) may now be written as

$$e^{\xi H} A e^{-\xi H} = A + \sum_{n=1}^{\infty} \frac{1}{n!} \xi^n [H, A]_n. \quad (2.71)$$

An explicit expression for the repeated commutator is [86]

$$[H, A]_n = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} H^k A H^{n-k}. \quad (2.72)$$

Also, we note that

$$[H, A]_{n+1} = H[H, A]_n - [H, A]_n H. \quad (2.73)$$

2.6 Phase-Space Operator Formulas

In dealing with operators in phase-space one comes across integrals that can be written in an operational form that allows one to manipulate operators in very advantageous ways. We list some of the important ones:

i) If $f(\theta, \tau)$ and $\hat{a}(\theta, \tau)$ are arbitrary functions then

$$\iint f(\theta, \tau) \hat{a}(\theta, \tau) e^{i\theta x + i\tau p} d\theta d\tau = f\left(\frac{1}{i} \frac{\partial}{\partial x}, \frac{1}{i} \frac{\partial}{\partial p}\right) a(x, p) \quad (2.74)$$

where the normalization in the Fourier transform, $\hat{a}(\theta, \tau)$, is as in Eq. (1.13).

ii) For any two functions $f(\theta, \tau)$ and $F(x, p)$,

$$\frac{1}{4\pi^2} \iint f(\theta, \tau) e^{i\theta(x'-x) + i\tau(p'-p)} F(x', p') d\theta d\tau dx' dp' = f\left(i \frac{\partial}{\partial x}, i \frac{\partial}{\partial p}\right) F(x, p). \quad (2.75)$$

iii) For any two functions $g(x)$ and $h(x, p)$,

$$g\left(x + \frac{i}{2} \frac{\partial}{\partial p}\right) h(x, p) = \frac{1}{2\pi} \iint g(x') e^{i\theta(x-x')} h(x, p - \frac{1}{2}\theta) d\theta dx' \quad (2.76)$$

$$= \frac{1}{\pi} \iint g(x') e^{2i(p-p')(x-x')} h(x, p') dp' dx'. \quad (2.77)$$

iv) If we have two phase-space functions $a(x, p)$ and $b(x, p)$ then

$$f\left(i \frac{\partial}{\partial x}, i \frac{\partial}{\partial p}\right) a(x, p) b(x, p) = f\left(i \frac{\partial}{\partial x_a} + i \frac{\partial}{\partial x_b}, i \frac{\partial}{\partial p_a} + i \frac{\partial}{\partial p_b}\right) a(x, p) b(x, p). \quad (2.78)$$

In Eq. (2.78) the meaning of $\frac{\partial}{\partial x_a}$ is that it operates only on $a(x, p)$ and similarly for the other partial derivatives.

v) If $f(x, p)$ is a real function then

$$\int a(x, p) f \left(i \frac{\partial}{\partial x}, i \frac{\partial}{\partial p} \right) b(x, p) dx dp = \int b(x, p) f \left(\frac{1}{i} \frac{\partial}{\partial x}, \frac{1}{i} \frac{\partial}{\partial p} \right) a(x, p) dx dp. \quad (2.79)$$

vi) For any function of two variables, say $a(x, p)$,

$$\frac{1}{\pi} \iint a(x' + x, p' + p) e^{i2x'p'} dx' dp' = \exp \left[-\frac{1}{2i} \frac{\partial}{\partial x} \frac{\partial}{\partial p} \right] a(x, p). \quad (2.80)$$

vii) For any two functions $a(x, p)$ and $f(x, p)$

$$a \left(x + \frac{i}{2} \frac{\partial}{\partial p}, p - \frac{i}{2} \frac{\partial}{\partial x} \right) f(x, p) = \iint \hat{a}(\theta, \tau) f(x + \tau/2, p - \theta/2) e^{i\theta x + i\tau p} d\theta d\tau. \quad (2.81)$$

2.7 Transforms and the Representation of Functions

The writing of a function in different domains is a fundamental idea that has been developed over the last few hundred years but is particularly crucial in many physical theories and quantum mechanics in particular. Perhaps the first idea along these lines is the Taylor series but certainly the most important historically, is the expression of a function in the Fourier domain. However, there are an infinite number of domains that one can express a function in and in fact any Hermitian operator generates a domain, as we will now discuss. The remarkable historical event that necessitated the development of the expansion of functions in a different domain was the discovery of quantum mechanics. We will describe why this is so later but we first discuss some of the standard reasons. First is that the function may be more simply characterized in another representation and hence gives considerably more insight into the nature of the function. Thus, for example, a messy function in one domain may have a simple expression in the Fourier domain or the Hermite function representation, etc. Secondly, if we want to construct functions with certain characteristics which are described in a certain domain, then clearly we should do the analysis in the representation of that domain and then transform back to the working domain. For example if we want a function that has spatial frequencies only in a band, then clearly the function should be constructed in the frequency domain and then transformed back to the spatial domain.

Representation of functions in domains. Any Hermitian operator generates a domain or representation as we now discuss. One solves the eigenvalue problem for the operator and that results in either continuous or discrete eigenvalues. We write

these two cases as

$$Au(\lambda, x) = \lambda u(\lambda, x) \quad \text{continuous spectrum,} \quad (2.82)$$

$$Au_n(x) = \lambda_n u_n(x) \quad \text{discrete spectrum,} \quad (2.83)$$

where λ or λ_n are the continuous or discrete eigenvalues respectively and where the u 's are the corresponding eigenfunctions. In writing Eqs. (2.82) and (2.83) we have assumed that the operator is expressed in the x representation; however the eigenvalue problem can be solved in any representation. The eigenfunctions thus generated are complete and orthogonal and that allows one to express any function in the representation defined by the operator. We deal with the continuous case and discrete case separately.

Continuous case. By complete and orthogonal one means that the eigenfunctions satisfy

$$\int u^*(\lambda', x) u(\lambda, x) dx = \delta(\lambda - \lambda'), \quad (2.84)$$

$$\int u^*(\lambda, x') u(\lambda, x) d\lambda = \delta(x - x'). \quad (2.85)$$

Eq. (2.84) is called delta function normalization and Eq. (2.85) is called the closure relation. Any function can be expanded as

$$\varphi(x) = \int F(\lambda) u(\lambda, x) d\lambda \quad (2.86)$$

where the inverse transformation is given by

$$F(\lambda) = \int \varphi(x) u^*(\lambda, x) dx. \quad (2.87)$$

The function $F(\lambda)$ is called the *transform* of $f(x)$ and may be considered as the representation of the function in the A representation.

Discrete Case. If the spectrum is discrete the eigenfunctions are orthogonal and it is standard to normalize them so that

$$\int u_m^*(x) u_n(x) dx = \delta_{nm} \quad (2.88)$$

Also, one has that

$$\sum_n u_n^*(x') u_n(x) = \delta(x' - x) \quad (2.89)$$

where the summation runs over all the eigenfunctions. Any function can be expanded as

$$\varphi(x) = \sum_n c_n u_n(x) \quad (2.90)$$

where the coefficients c_n are given by

$$c_n = \int u_n^*(x) \varphi(x) dx. \quad (2.91)$$

The set of coefficients $\{c_n\}$ can be thought of as the function in the u_n representation or as the discrete transform of the function.

Functions of operators operating on an eigenfunction. A particularly important result is that if we have a function of an operator, $f(A)$, then

$$f(A)u(\lambda, x) = f(\lambda)u(\lambda, x) \quad (2.92)$$

where $u(\lambda, x)$ are the eigenfunctions of A . This can be proven as follows. Using Eq. (2.22) for the definition of a function of an operator

$$f(A) = \sum_{n=0}^{\infty} f_n A^n \quad (2.93)$$

we have

$$f(A)u(\lambda, x) = \sum_{n=0}^{\infty} f_n A^n u(\lambda, x) = \sum_{n=0}^{\infty} f_n \lambda^n u(\lambda, x) = f(\lambda)u(\lambda, x). \quad (2.94)$$

This allows one to evaluate the operation of $f(A)$ on an arbitrary function, $\varphi(x)$. Using Eq. (2.86) we write

$$\varphi(x) = \int F(\lambda)u(\lambda, x) d\lambda \quad (2.95)$$

where the transform, $F(\lambda)$, is given by

$$F(\lambda) = \int \varphi(x)u^*(\lambda, x) dx. \quad (2.96)$$

Therefore,

$$f(A)\varphi(x) = \int F(\lambda)f(A)u(\lambda, x) d\lambda = \int F(\lambda)f(\lambda)u(\lambda, x) d\lambda. \quad (2.97)$$

If we further substitute for $F(\lambda)$ then

$$f(A)\varphi(x) = \iint \varphi(x')u^*(\lambda, x')f(\lambda)u(\lambda, x) d\lambda dx' \quad (2.98)$$

which can be written as

$$f(A)\varphi(x) = \int \varphi(x')r(x', x)dx' \quad (2.99)$$

where

$$r(x', x) = \int u^*(\lambda, x') f(\lambda) u(\lambda, x) d\lambda. \quad (2.100)$$

If the eigenfunctions are discrete as in Eq. (2.83) then

$$f(A)u_n(x) = f(\lambda_n)u_n(x) \quad (2.101)$$

and

$$f(A)\varphi(x) = \int \sum_{n=0}^{\infty} u_n^*(x') f(\lambda_n) u_n(x) \varphi(x') dx' \quad (2.102)$$

$$= \int \varphi(x') r(x', x) dx' \quad (2.103)$$

where now

$$r(x', x) = \sum_{n=0}^{\infty} u_n^*(x') f(\lambda_n) u_n(x). \quad (2.104)$$

Example. For the D operator the eigenfunctions are

$$u(\lambda, x) = \frac{1}{\sqrt{2\pi}} e^{i\lambda x} \quad (2.105)$$

and hence for any function $f(D)$ we have

$$f(D)e^{i\lambda x} = f(\lambda)e^{i\lambda x}. \quad (2.106)$$

Furthermore, suppose we have a function, $\varphi(x)$, then, using Eq. (2.99) and Eq. (2.100)

$$f(D)\varphi(x) = \frac{1}{2\pi} \iint \varphi(x') f(\lambda) e^{-i\lambda(x'-x)} d\lambda dx' \quad (2.107)$$

$$= \frac{1}{\sqrt{2\pi}} \int \varphi(x') \hat{f}(x' - x) dx' \quad (2.108)$$

where, as usual, the Fourier transform is given by

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int f(\lambda) e^{-i\lambda x} d\lambda. \quad (2.109)$$

2.8 Delta Function

We will be using the delta function freely and it is worthwhile to list some of its basic properties. The fundamental representation of the delta function is

$$\delta(x - \xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\pm iy(x-\xi)} dy \quad (2.110)$$

and its basic property is that, for a function $f(x)$,

$$\int_{-\infty}^{\infty} f(x) \delta(x - \xi) dx = f(\xi). \quad (2.111)$$

If the integration is one sided in Eq. (2.110) then

$$\int_0^{\infty} e^{iy(x-\xi)} dy = \pi \delta(x - \xi) + \frac{i}{x - \xi} \quad (2.112)$$

where the integration implies taking the principle part. The delta function has the following symbolic relations:

$$\delta(x) = \delta(-x), \quad (2.113)$$

$$x\delta(x) = 0, \quad (2.114)$$

$$f(x)\delta(x - \xi) = f(\xi)\delta(x - \xi), \quad (2.115)$$

$$\delta(\xi x) = \xi^{-1} \delta(x), \quad \xi > 0, \quad (2.116)$$

$$\int_{-\infty}^{\infty} f(x) \frac{d^n}{dx^n} \delta(x - \xi) dx = (-1)^n \frac{d^n}{d\xi^n} f(\xi). \quad (2.117)$$

A particularly important relation is that

$$\delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|} \delta(x - x_i) \quad (2.118)$$

where x_i are the zeros of $g(x)$ and $g'(x_i)$ is the derivative $g(x)$ evaluated at the zero's. By a symbolic relation we mean that if both sides of the relation are multiplied by an arbitrary function and integrated from $-\infty$ to ∞ , an identity is obtained. For example, when we say that $\delta(x) = \delta(-x)$ what it means is that for an arbitrary function $f(x)$,

$$\int_{-\infty}^{\infty} f(x) \delta(x) dx = \int_{-\infty}^{\infty} f(x) \delta(-x) dx. \quad (2.119)$$

Two other important representations of the delta function are

$$\delta(x) = \frac{1}{2} \frac{d^2}{dx^2} |x| \quad (2.120)$$

and

$$\delta(x) = \frac{d}{dx} \eta(x) \quad (2.121)$$

where $\eta(x)$ is the step function

$$\eta(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0. \end{cases} \quad (2.122)$$

We point out that

$$x\delta(x-\xi) = \xi\delta(x-\xi) \quad (2.123)$$

and hence $\delta(x-\xi)$ are the eigenfunctions of x , with eigenvalues ξ , which range from $-\infty$ to ∞ . In fact it was to solve the eigenvalue problem for x as exemplified by Eq. (2.123) that Dirac invented the delta function. In quantum mechanics this is paramount because x is the position operator and the eigenvalues are the measurable quantities. What Eq. (2.123) shows is that the measurable values for position, the eigenvalues ξ , are continuous since ξ may be any real number. Therefore one says that position is not quantized.

2.9 Intensity, Probability, and Averages

The transform $F(\lambda)$ as given by Eq. (2.87) gives an indication of how important a particular value of λ is for the function $\varphi(x)$. In particular, one takes the density or intensity of λ to be $|F(\lambda)|^2$. In the discrete case one takes $|c_n|^2$ to be the intensity for λ_n . In quantum mechanics $|F(\lambda)|^2$ is the probability distribution for λ and if the spectrum is discrete then $|c_n|^2$ is the probability for obtaining λ_n .

For this interpretation to be consistent one has to show that intensity is preserved. In particular for the continuous case we have

$$\int |\varphi(x)|^2 dx = \int |F(\lambda)|^2 d\lambda. \quad (2.124)$$

This is easily proven. Now consider the average defined by

$$\langle \lambda \rangle = \int \lambda |F(\lambda)|^2 d\lambda. \quad (2.125)$$

It is a remarkable fact that $\langle \lambda \rangle$ can be calculated in the x domain directly by way of

$$\langle \lambda \rangle = \int \varphi^*(x) A \varphi(x) dx \quad (2.126)$$

and furthermore for a real function, $g(\lambda)$, its average is

$$\int g(\lambda) |F(\lambda)|^2 d\lambda = \int \varphi^*(x) g(A) \varphi(x) dx. \quad (2.127)$$

For the discrete case we have

$$\int |\varphi(x)|^2 dx = \sum_n |c_n|^2 \quad (2.128)$$

and

$$\langle \lambda \rangle = \sum_n \lambda_n |c_n|^2 = \int \varphi^*(x) A \varphi(x) dx. \quad (2.129)$$

Furthermore $\langle g(\lambda) \rangle$ is given by

$$\langle g(\lambda) \rangle = \sum_n g(\lambda_n) |c_n|^2 = \int \varphi^*(x) g(A) \varphi(x) dx. \quad (2.130)$$

The proof of these important statements will be given in Chap. 12.

Probability interpretation. Suppose we assume that $|\varphi(x)|^2$ is the probability for x and that it is appropriately normalized so that

$$\int |\varphi(x)|^2 dx = 1. \quad (2.131)$$

Notice now that we also have

$$\int |F(\lambda)|^2 d\lambda = 1 \quad (2.132)$$

and therefore $|F(\lambda)|^2$ can be interpreted as the probability of measuring λ . This is the probabilistic interpretation of quantum mechanics where $\varphi(x)$ is called the wave function or state function and the λ 's are the numerical values for the physical quantity represented by the operator A . For the discrete case we have that

$$\sum_n |c_n|^2 = 1 \quad (2.133)$$

and then we say that $|c_n|^2$ is the probability of obtaining λ_n . Since λ_n are discrete one says that the numerical values possible for the physical quantity represented by the operator A are quantized.

Notice that the left hand side of Eq. (2.127) and Eq. (2.130) are the standard definitions of averages in standard probability theory for the continuous and discrete case respectively. That these quantities can be calculated by the right hand side of the respective equations is something that is proved and we will do so in Chap. 12. What is remarkable is that these methods of calculating averages arose naturally in quantum mechanics.

2.10 Simplifying Functions of Operators

An important issue is the consideration of a function of operators, say $H(X, D)$, where one requires the simplification of $G(H(X, D))$ where G is another function. For example, suppose we want to simplify $(X + D)^n$. In our notation $H(X, D) = X + D$ and $G(H) = H^n$. There are many ways to do this and brute force, using the commutation relation, often works. We give here one method [14, 95] that depends on solving the eigenvalue problem

$$H(X, D) u(\lambda, x) = \lambda u(\lambda, x). \quad (2.134)$$

The solution to this eigenvalue problem gives rise to a complete set of eigenfunctions, as given by Eq. $u(\lambda, x)$ and hence for any function $f(x)$,

$$f(x) = \int u(\lambda, x) F(\lambda) d\lambda \quad (2.135)$$

with

$$F(\lambda) = \int u^*(\lambda, x) f(x) dx. \quad (2.136)$$

Now consider

$$G(H(X, D))f(x) = G(H(X, D)) \int u(\lambda, x) F(\lambda) d\lambda \quad (2.137)$$

$$= \int G(H(X, D)) u(\lambda, x) F(\lambda) d\lambda \quad (2.138)$$

$$= \int G(\lambda) u(\lambda, x) F(\lambda) d\lambda. \quad (2.139)$$

Substituting for $F(\lambda)$ we have

$$G(H(X, D))f(x) = \iint G(\lambda) u^*(\lambda, x') u(\lambda, x) f(x') dx' d\lambda \quad (2.140)$$

which can be written as

$$G(H(X, D))f(x) = \int K(x', x) f(x') dx' \quad (2.141)$$

with

$$K(x', x) = \int G(\lambda) u^*(\lambda, x') u(\lambda, x) d\lambda. \quad (2.142)$$

If the spectrum is discrete then the same steps lead to Eq. (2.141) where now

$$K(x', x) = \sum_n G(\lambda_n) u_n^*(x') u_n(x). \quad (2.143)$$

Example. Suppose we seek the simplification of $e^{i\theta X + i\tau D}$ as was discussed in Sec. 2.4. As per Eq. (2.134) we have to solve the eigenvalue problem

$$(\theta X + \tau D)u(\lambda, x) = \lambda u(\lambda, x) \quad (2.144)$$

or explicitly in the x representation,

$$\left(\theta x - i\tau \frac{d}{dx} \right) u(\lambda, x) = \lambda u(\lambda, x). \quad (2.145)$$

The solution normalized to a delta function is

$$u(\lambda, x) = \frac{1}{\sqrt{2\pi\tau}} e^{i(\lambda x - \theta x^2/2)/\tau}. \quad (2.146)$$

For a function, $\varphi(x)$, the transform, $F(\lambda)$, is then

$$F(\lambda) = \frac{1}{\sqrt{2\pi\tau}} \int \varphi(x) e^{-i(\lambda x - \theta x^2/2)/\tau} dx \quad (2.147)$$

and the inverse transformation is

$$\varphi(x) = \frac{1}{\sqrt{2\pi\tau}} \int F(\lambda) e^{i(\lambda x - \theta x^2/2)/\tau} d\lambda. \quad (2.148)$$

According to Eq. (2.142) we have

$$e^{i\theta X + i\tau D} \varphi(x) = \int K(x', x) \varphi(x') dx' d\lambda \quad (2.149)$$

with

$$K(x', x) = \int e^{i\lambda} u^*(\lambda, x') u(\lambda, x) d\lambda dx' \quad (2.150)$$

$$= \frac{1}{2\pi\tau} \int e^{i\lambda} e^{-i(\lambda x' - \theta x'^2/2)/\tau} e^{i(\lambda x - \theta x^2/2)/\tau} d\lambda \quad (2.151)$$

$$= \delta(x' - \tau - x) e^{i\theta(x'^2 - x^2)/2\tau}. \quad (2.152)$$

Therefore

$$e^{i\theta X + i\tau D} \varphi(x) = \int \delta(x' - \tau - x) e^{i\theta(x'^2 - x^2)/2\tau} \varphi(x') dx' \quad (2.153)$$

$$= e^{i\theta(\tau^2 + 2\tau x)/2\tau} \varphi(\tau + x) \quad (2.154)$$

$$= e^{i\theta\tau/2} e^{i\theta x} \varphi(\tau + x) \quad (2.155)$$

$$= e^{i\theta\tau/2} e^{i\theta x} e^{i\tau D} \varphi(x). \quad (2.156)$$

Hence, we can write

$$e^{i\theta X + i\tau D} = e^{i\theta\tau/2} e^{i\theta X} e^{i\tau D} \quad (2.157)$$

which is Eq. (2.48)

2.11 Rearrangement of Operators

If we have a function of operators $G(X, D)$ we may want to rearrange it so that all the D factors are to the right of the X factors or the other way around. A

procedure that can be used is the following [14, 95]. Write Eq. (2.140) as

$$G(X, D)\varphi(x) = \iint G(\lambda) u^*(\lambda, x') u(\lambda, x) \varphi(x') dx' d\lambda \quad (2.158)$$

$$= \iint G(\lambda) u^*(\lambda, x' + x) u(\lambda, x) \varphi(x' + x) dx' d\lambda \quad (2.159)$$

$$= \iint G(\lambda) u^*(\lambda, x' + x) u(\lambda, x) e^{ix'D} \varphi(x) dx' d\lambda \quad (2.160)$$

and therefore we have

$$G(X, D) = \iint G(\lambda) u^*(\lambda, x' + X) u(\lambda, X) e^{ix'D} dx' d\lambda. \quad (2.161)$$

If the spectrum is discrete then

$$G(X, D) = \sum_n G(\lambda_n) u_n^*(x' + X) u_n(X) e^{ix'D} dx'. \quad (2.162)$$

Notice that in Eq. (2.161) all the X factors are to the left of the D factors. Therefore one way to evaluate the integral in Eq. (2.161) is to replace X and D by ordinary variables, say x and p ,

$$G(x, p) = \sum_n G(\lambda_n) u_n^*(x' + x) u_n(x) e^{ix'p} dx' \quad (2.163)$$

do the integral and then arrange the expression so that all the x factors are to the left of the p factors; then substitute X and D for x and p respectively. This procedure will be illustrated in subsequent chapters. Here we give one example. Suppose we want to expand

$$G(X, D) = (X + D)^n \quad (2.164)$$

and rearrange it so that all the X factors are to the left of the D factors. That can be done by brute force using the commutator relations but one would soon be entangled in laborious algebra. We now show how that can be done using the above method. The eigenvalue problem is

$$\left(x - i \frac{d}{dx}\right) u(\lambda, x) = \lambda u(\lambda, x) \quad (2.165)$$

and the solutions are

$$u(\lambda, x) = \frac{1}{\sqrt{2\pi}} e^{i(\lambda x - x^2/2)}. \quad (2.166)$$

Therefore using Eq. (2.161) we have

$$G(x, p) = \frac{1}{2\pi} \iint \lambda^n e^{-i(\lambda(x'+x) - (x'+x)^2/2)} e^{i(\lambda x - x^2/2)} e^{ix'p} dx' d\lambda \quad (2.167)$$

$$= \frac{1}{2\pi} \iint \lambda^n e^{-ix'(\lambda - x - p) + ix'^2/2} dx' d\lambda \quad (2.168)$$

which simplifies to

$$G(x, p) = \frac{(\sqrt{2i})^n}{\sqrt{\pi}} \int (iy + \frac{1}{\sqrt{2i}}(x+p))^n e^{-y^2} dy. \quad (2.169)$$

Now, the Hermite polynomials, $H_n(x)$, are

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-2k)!} x^{n-2k} \quad (2.170)$$

where $[n/2]$ is the greatest integer function. Pertinent to our considerations is that

$$H_n(x) = \frac{2^n}{\sqrt{\pi}} \int (x + iy)^n e^{-y^2} dy. \quad (2.171)$$

Therefore we have

$$G(x, p) = \left(\frac{i}{2}\right)^{n/2} H_n\left(\frac{1}{\sqrt{2i}}(x+p)\right) \quad (2.172)$$

and

$$G(x, p) = \left(\frac{i}{2}\right)^{n/2} \sum_{k=0}^{[n/2]} \frac{(-1)^k n! \left(\frac{2(x+p)}{\sqrt{2i}}\right)^{n-2k}}{k!(n-2k)!}. \quad (2.173)$$

Expanding $(x+p)^{n-2k}$ in a binomial series one finally obtains that

$$G(x, p) = \sum_{k=0}^{[n/2]} \sum_{l=0}^{n-2k} \left(\frac{1}{2i}\right)^k \frac{n!}{l!k!(n-2k-l)!} x^{n-2k-l} p^l. \quad (2.174)$$

Now all the x factors are to the left of the p factors and we can write

$$G(X, D) = (X + D)^n = \sum_{k=0}^{[n/2]} \sum_{l=0}^{n-2k} \left(\frac{1}{2i}\right)^k \frac{n!}{l!k!(n-2k-l)!} X^{n-2k-l} D^l. \quad (2.175)$$



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