

Chapter 2

Properties of Linear and Nonlinear Operators

2.1 Linear Operators

In this section we point out some fundamental properties of linear operators in Banach spaces. The key assertions presented are the Uniform Boundedness Principle, the Banach–Steinhaus Theorem, the Open Mapping Theorem, the Hahn–Banach Theorem, the Separation Theorem, the Eberlain–Smulyan Theorem and the Banach Theorem. We recall that the collection of all continuous linear operators from a normed linear space X into a normed linear space Y is denoted by $\mathcal{L}(X, Y)$, and $\mathcal{L}(X, Y)$ is a normed linear space with the norm

$$\|A\|_{\mathcal{L}(X, Y)} = \sup \{\|Ax\|_Y : \|x\|_X \leq 1\}.$$

Proposition 2.1.1. *Let Y be a Banach space. Then $\mathcal{L}(X, Y)$ is a Banach space, too. In particular, the space X^* of all linear continuous forms on X is complete.*

Proof. Let $\{A_n\}_{n=1}^\infty$ be a Cauchy sequence in $\mathcal{L}(X, Y)$. Then for any $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ and $x \in X$

$$\|A_n x - A_m x\| \leq \|A_n - A_m\| \|x\| \leq \varepsilon \|x\|.$$

Since Y is complete, the sequence $\{A_n x\}_{n=1}^\infty$ is convergent to a point in Y that can be denoted by Ax . Obviously A is a linear operator from X into Y and

$$\|Ax - A_m x\| = \lim_{n \rightarrow \infty} \|A_n x - A_m x\| \leq \varepsilon \|x\|, \quad m \geq n_0, \quad x \in X.$$

This implies (Proposition 1.2.10) that $A \in \mathcal{L}(X, Y)$ and $\|A - A_m\| \rightarrow 0$. □

The importance of this result can be seen from the following statement.

Proposition 2.1.2. *Let X be a Banach space and $A \in \mathcal{L}(X)$. If $\|A\| < 1$, then the operator $I - A$ is continuously invertible and*

$$(I - A)^{-1} = \sum_{n=0}^{\infty} A^n$$

where the sum is convergent in the $\mathcal{L}(X)$ -norm.

Proof. First we prove the convergence. Let $\varepsilon > 0$ be arbitrary. Put $S_k = \sum_{n=0}^k A^n$. Then

$$\|S_l - S_k\| = \left\| \sum_{n=k+1}^l A^n \right\| \leq \sum_{n=k+1}^l \|A^n\| \leq \sum_{n=k+1}^l \|A\|^n < \varepsilon^1 \quad \text{for } l > k$$

provided k is sufficiently large. By Proposition 2.1.1, the limit of S_k exists in the $\mathcal{L}(X)$ -norm. Denote $B := \lim_{k \rightarrow \infty} S_k = \sum_{n=0}^{\infty} A^n$. We have

$$\begin{aligned} (I - A)B &= \lim_{k \rightarrow \infty} (I - A) \sum_{n=0}^k A^n = \lim_{k \rightarrow \infty} \left(\sum_{n=0}^k A^n - \sum_{n=1}^{k+1} A^n \right) \\ &= \lim_{k \rightarrow \infty} (I - A^{k+1}) = I \end{aligned}$$

since $\lim_{n \rightarrow \infty} A^n = O$. Similarly,

$$B(I - A) = I, \quad \text{i.e.,} \quad B = (I - A)^{-1}. \quad \square$$

If X is a complex Banach space and $A \in \mathcal{L}(X)$, we denote

$$\varrho(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is continuously invertible in } \mathcal{L}(X)\}$$

(the so-called *resolvent set* of A) and

$$\sigma(A) := \mathbb{C} \setminus \varrho(A)$$

(the so-called *spectrum* of A).² The operator valued function

$$\lambda \mapsto (\lambda I - A)^{-1}, \quad \lambda \in \varrho(A),$$

is called the *resolvent* of A .

¹If $A \in \mathcal{L}(X, Y)$, $B \in \mathcal{L}(Y, Z)$, then $BA \in \mathcal{L}(X, Z)$ and $\|BA\|_{\mathcal{L}(X, Z)} \leq \|B\|_{\mathcal{L}(Y, Z)} \|A\|_{\mathcal{L}(X, Y)}$.

²The reason for considering only complex spaces consists in the fact that $\sigma(A) \neq \emptyset$ for all $A \in \mathcal{L}(X)$ in this case. This will be proved later in this section (see the discussion following Example 2.1.20).

Corollary 2.1.3. *Let X be a complex Banach space and $A \in \mathcal{L}(X)$. Then $\varrho(A)$ is an open set and*

$$\{\lambda : |\lambda| > \|A\|\} \subset \varrho(A).$$

Proof. If $|\lambda| > \|A\|$, then

$$\lambda I - A = \lambda \left(I - \frac{A}{\lambda} \right)$$

and $(I - \frac{A}{\lambda})^{-1} \in \mathcal{L}(X)$ according to Proposition 2.1.2. Hence we have

$$(\lambda I - A)^{-1} = \sum_{n=0}^{\infty} \frac{A^n}{\lambda^{n+1}}.^3$$

Similarly, if $\lambda_0 \in \varrho(A)$, then

$$\lambda I - A = (\lambda - \lambda_0)I + (\lambda_0 I - A) = (\lambda_0 I - A)[I - (\lambda_0 - \lambda)(\lambda_0 I - A)^{-1}].$$

For a parameter λ such that $\|(\lambda_0 - \lambda)(\lambda_0 I - A)^{-1}\| < 1$, the inverse operator $B = [I - (\lambda_0 - \lambda)(\lambda_0 I - A)^{-1}]^{-1}$ exists and

$$(\lambda I - A)^{-1} = B(\lambda_0 I - A)^{-1}. \quad \square$$

The next theorem together with Theorems 2.1.8 and 2.1.13 is one of the most significant results in linear functional analysis. For the proofs the interested reader can consult textbooks on functional analysis, e.g., Conway [36], Dunford & Schwartz [53], Rudin [131], Yosida [161]. Here we present only quite elementary proof of Theorem 2.1.4 which appeared recently in Sokal [143].

Theorem 2.1.4 (Uniform Boundedness Principle). *Let X be a Banach space and Y a normed linear space. If $\{A_\gamma\}_{\gamma \in \Gamma} \subset \mathcal{L}(X, Y)$ is such that the sets $\{\|A_\gamma x\|_Y : \gamma \in \Gamma\}$ are bounded for all $x \in X$, then $\{\|A_\gamma\|_{\mathcal{L}(X, Y)} : \gamma \in \Gamma\}$ is also bounded.*

Proof. Let $A \in \mathcal{L}(X, Y)$. Then for any $a \in X$ and $r > 0$, we have

$$\sup_{x \in B(a; r)} \|Ax\|_Y \geq r \|A\|_{\mathcal{L}(X, Y)}. \quad (2.1.1)$$

Indeed, using the triangle inequality in the form $\|\alpha - \beta\| \leq \|\alpha\| + \|\beta\|$, we have

$$\max \{\|A(a+x)\|, \|A(a-x)\|\} \geq \frac{1}{2} [\|A(a+x)\| + \|A(a-x)\|] \geq \|Ax\|.$$

Take supremum over $x \in B(0; r)$ to get (2.1.1). We now proceed to prove the theorem via contradiction. Suppose that $\sup_{\gamma \in \Gamma} \|A_\gamma\|_{\mathcal{L}(X, Y)} = \infty$ and choose $\{A_n\}_{n=1}^{\infty} \subset \{A_\gamma\}_{\gamma \in \Gamma}$ such that $\|A_n\| \geq 4^n$.

³This series actually converges for λ such that $|\lambda| > r(A) := \sup \{|\mu| : \mu \in \sigma(A)\}$ but its proof is more involved. The quantity $r(A)$ is called the *spectral radius* of A .

Set $x_0 = 0$, and for $n \geq 1$ use (2.1.1) above to choose inductively $x_n \in X$ such that

$$\|x_n - x_{n-1}\| \leq \frac{1}{3^n} \quad \text{and} \quad \|A_n x_n\| \geq \frac{2}{3} \cdot \frac{1}{3^n} \|A_n\|. \quad (2.1.2)$$

The sequence $\{x_n\}_{n=1}^\infty$ is Cauchy and X is a Banach space. Hence, there is $x \in X$ such that

$$\lim_{n \rightarrow \infty} \|x_n - x\| = 0.$$

It follows from the first inequality in (2.1.2) that

$$\|x - x_n\| \leq \frac{1}{2} \cdot \frac{1}{3^n}$$

and hence

$$\|A_n x\| \geq \frac{1}{6} \cdot \frac{1}{3^n} \|A_n\| \geq \frac{1}{6} \left(\frac{4}{3}\right)^n \rightarrow +\infty,$$

a contradiction. □

The method of construction of sequence $\{x_n\}_{n=1}^\infty$ in the proof above is called *the gliding hump method* and the reader can find it in the literature also in different contexts.

Uniform Boundedness Principle is a quintessence of several results on approximation of functions in classical analysis and can be used for “modern” proofs of such results. The following example is typical.

Example 2.1.5. There exists a periodic continuous function the Fourier series of which is divergent at zero.⁴ To see this we recall that the n^{th} partial sum of the Fourier series of a function f at 0 is given by

$$s_n(f)(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(0-t)f(t) dt \quad \text{where} \quad D_n(t) = \frac{\sin(n + \frac{1}{2})t}{\sin \frac{t}{2}}, \quad 0 < |t| < \pi$$

(the n^{th} *Dirichlet kernel*). Since $\sigma_n: f \mapsto s_n(f)(0)$ are continuous linear forms on the space $C[-\pi, \pi]$, the sequence of their norms $\{\|\sigma_n\|_{\mathcal{L}(C[-\pi, \pi], \mathbb{R})}\}_{n=1}^\infty$ should be bounded provided $\sigma_n(f)$ is convergent for all $f \in C[-\pi, \pi]$ (Theorem 2.1.4). One can calculate that

$$\|\sigma_n\| = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)| dt,$$

and a careful estimate shows that $\|\sigma_n\|$ is like $\log n$ for large n . ○

As indicated in the previous example, Theorem 2.1.4 is essentially an approximation result. This is clearer from its next variant.

⁴Even divergent at uncountably many points but always of measure zero. The set of such “bad” functions is dense in $C[-\pi, \pi]$.

Corollary 2.1.6 (Banach–Steinhaus). *Let X and Y be Banach spaces and let $\{A_n\}_{n=1}^\infty \subset \mathcal{L}(X, Y)$. Then the limits $\lim_{n \rightarrow \infty} A_n x$ exist for every $x \in X$ if and only if the following conditions are satisfied:*

- (i) *There is a dense set $\mathcal{M} \subset X$ such that $\lim A_n x$ exists for each $x \in \mathcal{M}$.*
- (ii) *The sequence of norms $\{\|A_n\|\}_{n=1}^\infty$ is bounded.*

Moreover, under these conditions

$$Ax := \lim_{n \rightarrow \infty} A_n x$$

exists for all $x \in X$ and $A \in \mathcal{L}(X, Y)$.⁵

The following proposition is also often useful.

Proposition 2.1.7. *Let X be a Banach space and Y a normed linear space. If $B: X \times X \rightarrow Y$ is a bilinear operator (i.e., linear in both variables) and*

- (i) *for every $y \in X$ the mapping $x \mapsto B(x, y)$ belongs to $\mathcal{L}(X, Y)$;*
- (ii) *for every $x \in X$ the mapping $y \mapsto B(x, y)$ belongs to $\mathcal{L}(X, Y)$,*

then there exists a constant c such that

$$\|B(x, y)\|_Y \leq c\|x\|_X\|y\|_X, \quad x, y \in X.$$

In particular, if $x_n \rightarrow x$, $y_n \rightarrow y$, then $B(x_n, y_n) \rightarrow B(x, y)$.

Proof. Denote $B_y: x \mapsto B(x, y)$. By (i), $B_y \in \mathcal{L}(X, Y)$ for all $y \in X$, $\|y\| \leq 1$. By (ii), $\|B_y(x)\| \leq c(x)$. The Uniform Boundedness Principle implies the existence of a constant c such that

$$\sup_{\|x\| \leq 1} \sup_{\|y\| \leq 1} \|B(x, y)\| \leq c. \quad \square$$

Theorem 2.1.8 (Open Mapping Theorem). *Let X, Y be Banach spaces, let $A \in \mathcal{L}(X, Y)$ and let A have a closed range $\text{Im } A$. Then for any open set $\mathcal{G} \subset X$ its image $A(\mathcal{G})$ is an open set in $\text{Im } A$. In particular, if A is, in addition, injective and surjective, then $A^{-1} \in \mathcal{L}(Y, X)$.*

When applied to linear equations

$$Ax = y,$$

Theorem 2.1.8 says that the continuous dependence of a solution on the right-hand side is a consequence of the existence and uniqueness result. Such continuous dependence is important for any reasonable numerical approximation.

Theorem 2.1.8 can be also used in a “negative” sense:

⁵This type of convergence is the so-called *convergence in the strong operator topology*. It is weaker than the norm convergence.

Example 2.1.9. Denote by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt$$

the n^{th} Fourier coefficient of $f \in L^1(-\pi, \pi)$. Since $\hat{f}(n) \rightarrow 0$ for $|n| \rightarrow \infty$ for all trigonometric polynomials which are dense in $L^1(-\pi, \pi)$, we have

$$\hat{f}(n) \rightarrow 0 \quad \text{for all } f \in L^1(-\pi, \pi)$$

(the so-called Riemann–Lebesgue Lemma). In other words, $A: f \mapsto \hat{f}(\cdot)$ is a continuous linear operator from $L^1(-\pi, \pi)$ into

$$c_0(\mathbb{Z}) := \left\{ \{a_n\}_{n \in \mathbb{Z}} : \lim_{|n| \rightarrow \infty} |a_n| = 0 \right\}, \quad \|\{a_n\}\|_{c_0(\mathbb{Z})} = \sup |a_n|.$$

Applications of Fourier series to various problems in analysis (like convolution equations, differential equations, ...) would be much easier if A were a surjective mapping onto $c_0(\mathbb{Z})$. Theorem 2.1.8 shows that this cannot be true for then A^{-1} would be bounded, i.e.,

$$\|f\|_{L^1(-\pi, \pi)} \leq c \sup_{n \in \mathbb{Z}} |\hat{f}(n)| \quad \text{for all } f \in L^1(-\pi, \pi).$$

If $\{D_k\}_{k=1}^{\infty}$ is the sequence of Dirichlet kernels (Example 2.1.5), then

$$\hat{D}_k(n) = \begin{cases} 1, & |n| \leq k, \\ 0, & |n| > k, \end{cases} \quad \text{and} \quad \|D_k\|_{L^1(-\pi, \pi)} \sim \log k,$$

a contradiction. ○

Theorem 2.1.8 also yields a sufficient condition for a linear operator to be continuous. To formulate it we need the notion of a closed operator:

Let X, Y be normed linear spaces. A linear operator

$$A: \text{Dom } A \subset X \rightarrow Y$$

is said to be *closed* if

$$\{x_n\}_{n=1}^{\infty} \subset \text{Dom } A, \quad x_n \rightarrow x, \quad Ax_n \rightarrow y$$

implies that

$$x \in \text{Dom } A \quad \text{and} \quad Ax = y.$$

Equivalently, A is a closed operator if and only if the graph of A , i.e.,

$$\mathcal{G}(A) := \{(x, Ax) : x \in \text{Dom } A\},$$

is a closed linear subspace of $X \times Y$.

Corollary 2.1.10 (Closed Graph Theorem). *Let X, Y be Banach spaces and let A be a closed operator from $\text{Dom } A = X$ into Y . Then A is continuous.*

Proof. If $\mathcal{G}(A)$ denotes the graph of A , then put

$$T(x, Ax) = x.$$

By Theorem 2.1.8, T^{-1} is continuous, and therefore

$$A = \pi_2 \circ T^{-1}$$

is continuous as well (π_2 is the projection of $X \times Y$ onto the second component Y). $\square$

Example 2.1.11. Many differential operators are either closed or have closed extensions. If they are viewed as operators from X into X , then they are only densely defined. A very simple example:

$$\begin{aligned} X &= C[0, 1], & Ax &= \dot{x}, \\ \text{Dom } A &= \{x \in X : \dot{x}(t) \text{ exists for all } t \in [0, 1] \text{ and } \dot{x} \in X\}. \end{aligned}$$

A well-known classical result says that A is a closed operator. But A is not continuous. For $x_n(t) = t^n$ we have $\|x_n\| = 1$, $\|\dot{x}_n\| = n$. $\circ$

Example 2.1.12. Let X be a Banach space and M a linear subspace of X . Let N be an (algebraic) complement of M and let P be the corresponding projection onto M . Then P is continuous if and only if both M and N are closed.

The sufficiency part follows from the Closed Graph Theorem and from an observation that P is closed whenever M and N are closed subspaces. The necessity part is obvious since

$$M = \text{Ker}(I - P), \quad N = \text{Ker } P.$$

This statement should be compared with the Hilbert space case (Corollary 1.2.35).

An important special case is $\text{codim } M < \infty$. By definition, this means that an algebraic direct complement N has a finite dimension ($\text{codim } M := \dim N$) and therefore N is closed (Corollary 1.2.11(i)). If M is closed as well, then any projection onto M is continuous. We postpone the case of $\dim M < \infty$ to Remark 2.1.19.

We note that if X is a Banach space such that there exists a continuous projection P , $\|P\|_{\mathcal{L}(X)} \leq 1$, onto every closed subspace of X , then X has an equivalent norm induced by the scalar product on X (see Kakutani [85]). $\circ$

Now we turn our attention to the *dual space* X^* of all *continuous linear forms* on a normed linear space X . In Section 1.1 we have seen the importance of linear forms. Namely, they allowed us to define an algebraic adjoint operator $A^\#$ and formulate Theorem 1.1.25. The dual space X^* is even more important for a normed linear space X since another topology can be introduced on X with help of X^* which in a certain sense has better properties (Theorem 2.1.25 below).

Surprisingly, the following basic result does not need any topology.

Theorem 2.1.13 (Hahn–Banach). *Let X be a real linear space and let Y be a linear subspace of X . Assume that f is a linear form on Y which is dominated by a sublinear functional p defined on X .⁶ Then there exists $F \in X^\#$ such that*

- (i) $F(y) = f(y)$ for all $y \in Y$ (extension);
- (ii) $F(x) \leq p(x)$ for all $x \in X$ (dominance).

Proof. The proof is based on an extension of f to a subspace whose dimension is larger by 1 and such that this extension is dominated by the same p (for details see, e.g., references given on page 58), and the use of Zorn's Lemma as an inductive argument, similarly as in the proof of Theorem 1.1.3. $\square$

Remark 2.1.14. If X is a complex linear space, then we need p to satisfy a stronger condition than (2) in footnote 6, namely

$$(2') \quad p(\alpha x) = |\alpha|p(x), \quad \alpha \in \mathbb{C}, \quad x \in X.$$

In this case p is called a *semi-norm*.⁷ The dominance also has to be stronger:

$$|f(x)| \leq p(x).$$

The extension result follows from Theorem 2.1.13 by considering $\operatorname{Re} f$ and $\operatorname{Im} f$ and observing that $\operatorname{Re} f(ix) = -\operatorname{Im} f(x)$.

Corollary 2.1.15. *Let X be a normed linear space and let Y be a linear subspace of X (not necessarily closed). If $f \in Y^*$, then there exists $F \in X^*$ such that*

- (i) $F(y) = f(y)$ for $y \in Y$;
- (ii) $\|F\|_{X^*} = \|f\|_{Y^*}$.

Proof. Put $p(x) = \|f\|\|x\|$, $x \in X$, and apply Theorem 2.1.13 or Remark 2.1.14, respectively. $\square$

Corollary 2.1.16 (Dual Characterization of the Norm). *Let X be a normed linear space. Then*

$$\|x\|_X = \max \{|f(x)| : f \in X^* \text{ with } \|f\|_{X^*} \leq 1\}. \quad (2.1.3)$$

Proof. Put $g_0(\alpha x) = \alpha\|x\|$, $\alpha \in \mathbb{R}$ (or $\alpha \in \mathbb{C}$). Then g_0 is a continuous linear form on $\operatorname{Lin}\{x\}$ and its norm is 1 (provided $x \neq o$). Let f_0 be its extension from Corollary 2.1.15. Then

$$f_0(x) = \|x\|, \quad \|f_0\| = 1, \quad \text{i.e.,} \quad \|x\| \leq \sup \{|f(x)| : f \in X^* \text{ with } \|f\| \leq 1\}.$$

The converse inequality follows from the definition of $\|f\|$. $\square$

⁶ A mapping $p: X \rightarrow \mathbb{R}$ is called *sublinear* if

- (1) $p(x + y) \leq p(x) + p(y)$ for any $x, y \in X$;
- (2) $p(\alpha x) = \alpha p(x)$ for any $x \in X$ and $\alpha \geq 0$.

⁷The difference between a norm and a semi-norm is that a semi-norm need not satisfy the condition: $p(x) = 0 \implies x = o$.

Remark 2.1.17.

- (i) If X is a Hilbert space, then the equality (2.1.3) can be obtained immediately from the Riesz Representation Theorem (Theorem 1.2.40). This theorem can be often used in Hilbert spaces instead of the Hahn–Banach Theorem.
- (ii) A slightly weaker form of (2.1.3) is often used:

If $f(x) = 0$ for all $f \in X^$, then $x = o$.*

The equivalent assertion reads as follows:

X^ separates points of X .*

Corollary 2.1.18 (Separation Theorem). *Let X be a normed linear space and let $\mathcal{C}$ be a nonempty, closed, convex set. If $x_0 \notin \mathcal{C}$, then there exists $F \in X^*$ such that*

$$\sup \{ \operatorname{Re} F(x) : x \in \mathcal{C} \} < \operatorname{Re} F(x_0). \quad (2.1.4)$$

Proof. It is sufficient to give the proof for a real space X and under the additional assumption $o \in \mathcal{C}$. In particular, this assumption means that $x_0 \neq o$. We wish to extend the form f defined on $\operatorname{Lin}\{x_0\}$ by $f(\alpha x_0) = \alpha$, $\alpha \in \mathbb{R}$. To do that we need a suitable dominating functional. Since $d := \operatorname{dist}(x_0, \mathcal{C}) > 0$, there exists a convex neighborhood of $\mathcal{C}$ which does not contain x_0 , e.g.,

$$\mathcal{K} = \left\{ x + y : x \in \mathcal{C}, \|y\| < \frac{d}{2} \right\}.$$

Put

$$p_{\mathcal{K}}(z) := \inf \left\{ \alpha > 0 : \frac{z}{\alpha} \in \mathcal{K} \right\} \quad \text{for } z \in X.^8$$

It is a matter of simple calculation to show that $p_{\mathcal{K}}$ is sublinear, $p_{\mathcal{K}}(x_0) > 1$, and $p_{\mathcal{K}}(z) \leq 1$ for $z \in \mathcal{K}$.

Let F be an extension of f given by Theorem 2.1.13. Since $o \in \mathcal{C}$, we have

$$F(\pm y) \leq p_{\mathcal{K}}(\pm y) \leq 1 \quad \text{for } \|y\| < \frac{d}{2}.$$

This shows that

$$\|F\| \leq \frac{2}{d}, \quad \text{i.e.,} \quad F \in X^*.$$

The inequality (2.1.4) follows from domination: namely, we have

$$F(x) + F(y) \leq p_{\mathcal{K}}(x + y) \leq 1 \quad \text{for } x \in \mathcal{C} \text{ and all } \|y\| < \frac{d}{2},$$

i.e.,

$$F(x) \leq 1 - \sup \left\{ F(y) : \|y\| < \frac{d}{2} \right\} < 1 = F(x_0). \quad \square$$

⁸ $p_{\mathcal{K}}$ is the so-called *Minkowski functional* of the convex set $\mathcal{K}$.

Remark 2.1.19. If $\mathcal{C}$ from Corollary 2.1.18 is a closed linear subspace of X and $F \in X^*$ satisfies (2.1.4), then $F(x) = 0$ for all $x \in \mathcal{C}$. Notice that $F(x_0) = 1$ for F which has been constructed in the proof.

This observation yields the existence of a continuous projection onto a finite-dimensional subspace Y of X . Namely, suppose that $\{y_1, \dots, y_n\}$ is a basis of Y , and denote by Y_k the span of $y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_n$. Then Y_k is a closed linear subspace of X and $y_k \notin Y_k$. Let $F_k \in X^*$ be such that

$$F_k(y_j) = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \quad j = 1, \dots, n.$$

Then

$$Px = \sum_{k=1}^n F_k(x)y_k$$

is a continuous projection onto Y .

Warning. *It is not true that every projection onto Y is continuous even if $\text{codim } Y = 1$ but the construction (i.e., the construction of a noncontinuous linear form) is not obvious!*

Example 2.1.20.

- (i) By Corollary 1.2.11(ii), $(\mathbb{R}^M)^* = (\mathbb{R}^M)^\#$. This means that $(\mathbb{R}^M)^*$ can be identified with $\mathbb{R}^M$.
- (ii) Let $\mathcal{K}$ be a compact subset of $\mathbb{R}^M$. Then for any $F \in [C(\mathcal{K})]^*$ there exists a unique complex Borel measure μ on $\mathcal{K}$ such that

$$F(f) = \int_{\mathcal{K}} f(x) d\mu(x) \quad \text{for every } f \in C(\mathcal{K}),$$

and $\|F\|_{[C(\mathcal{K})]^*} = |\mu|(\mathcal{K})$ where $|\mu|$ is the total variation of μ . A similar statement holds under a more general assumption on $\mathcal{K}$ – for details and the corresponding notions see Dunford & Schwartz [53, Section IV, 6] or Rudin [132, Chapter 6] and, especially, Bourbaki [17]. In the last book the integration theory is developed on the basis of this representation theorem.

- (iii) Let Ω be an open subset of $\mathbb{R}^M$ and let $p \in [1, \infty)$. Then the dual space $[L^p(\Omega)]^*$ can be identified with $L^{p'}(\Omega)$ (p' is the conjugate exponent, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$) in the following sense. For any $F \in [L^p(\Omega)]^*$ there exists a unique $\varphi \in L^{p'}(\Omega)$ such that

$$F(f) = \int_{\Omega} f(x)\varphi(x) dx \quad \text{for every } f \in L^p(\Omega).$$

Moreover, $\|F\|_{[L^p(\Omega)]^*} = \|\varphi\|_{L^{p'}(\Omega)}$. Details can be found in books cited above.

Warning. The dual space $[L^\infty(\Omega)]^*$ is much larger than $L^1(\Omega)!$

- (iv) The dual spaces to Sobolev spaces $W^{k,p}(\mathbb{R}^M)$ can be identified with special subspaces of tempered distributions for example via the Fourier transform. We omit details since their description is beyond the scope of this book. $\circ$

The reader can ask why we are so interested in continuous linear forms. One of the reasons is the following. Suppose that φ is a *vector-valued function* (i.e., a mapping from $\mathbb{R}$ or $\mathbb{C}$ into a normed linear space X). For any $f \in X^*$ the composition $f \circ \varphi$ is a real or complex function of a real or complex variable and therefore results of classical analysis can be applied to $f \circ \varphi$. To be more specific, consider the resolvent (see page 56) of $A \in \mathcal{L}(X)$

$$R(\lambda)x := (\lambda I - A)^{-1}x, \quad \lambda \in \varrho(A),$$

which is an X -valued function for every $x \in X$. Then for any $F \in X^*$, the complex function $\varphi(\lambda) = F[(\lambda I - A)^{-1}x]$ is holomorphic in $\varrho(A)$. For $|\lambda| > \|A\|$ we also have

$$\begin{aligned} |\varphi(\lambda)| &\leq \|F\|_{X^*} \|(\lambda I - A)^{-1}x\|_{\mathcal{L}(X)} \|x\|_X = \|F\| \|x\| \left\| \sum_{n=0}^{\infty} \frac{A^n}{\lambda^{n+1}} \right\| \\ &\leq \|F\| \|x\| \sum_{n=0}^{\infty} \frac{\|A\|^n}{|\lambda|^{n+1}}, \end{aligned}$$

and so

$$\lim_{|\lambda| \rightarrow \infty} |\varphi(\lambda)| = 0.$$

If $\varrho(A) = \mathbb{C}$, φ would be identically zero (by the Liouville Theorem from the complex functions theory). Since this should be true for all $F \in X^*$, we get $(\lambda I - A)^{-1}x = 0$ for all $x \in X$, a contradiction. Therefore, the spectrum $\sigma(A)$ is nonempty for each $A \in \mathcal{L}(X)$. This is a generalization of the existence of an eigenvalue of a linear operator in a finite-dimensional space and therefore also a generalization of the Fundamental Theorem of Algebra (cf. page 15). It is worth mentioning that the Jordan Canonical Form (Theorem 1.1.34) is based on this result.

Warning. It is not true that any $A \in \mathcal{L}(X)$, $\dim X = \infty$, has an eigenvalue! A simple example is

$$X = C[0, 1], \quad Ax(t) := tx(t).$$

Our main reason for considering dual spaces comes from an attempt to find a weaker topology on a normed linear space in which bounded sets would be relatively compact. The importance of this fact will become clear in Chapter 7. We also ask the reader to return to Proposition 1.2.2 for motivation.

Definition 2.1.21. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of elements in a normed linear space X . We say that $\{x_n\}_{n=1}^{\infty}$ *converges weakly* to $x \in X$ (notation $x_n \rightharpoonup x$ or $w\text{-}\lim_{n \rightarrow \infty} x_n = x$) if

$$\lim_{n \rightarrow \infty} f(x_n) = f(x) \quad \text{for every } f \in X^*.$$

Proposition 2.1.22.

- (i) (*uniqueness*) If $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$, then $x = y$.
- (ii) If $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$, then $x_n \rightharpoonup x$.⁹
- (iii) A weakly convergent sequence is bounded. Moreover, if $x_n \rightharpoonup x$, then

$$\|x\| \leq \liminf_{n \rightarrow \infty} \|x_n\|.$$

- (iv) If X is a uniformly convex Banach space,¹⁰ $x_n \rightharpoonup x$ and $\|x_n\| \rightarrow \|x\|$, then $\{x_n\}_{n=1}^{\infty}$ converges to x in the norm topology.

Proof. Assertion (i) follows immediately from Remark 2.1.17(ii) since in this case $f(x) = f(y)$ for every $f \in X^*$.

Assertion (ii) is obvious.

Assertion (iii) is basically a consequence of Theorem 2.1.4, but certain preliminaries are needed: Since X^* is a normed linear space, its dual $X^{**} := (X^*)^*$ is defined. Put

$$\kappa(x): f \mapsto f(x), \quad f \in X^*.$$

Then κ (the so-called *canonical embedding*) is a linear continuous operator from X into X^{**} , and

$$\|\kappa(x)\|_{X^{**}} = \sup_{\|f\|_{X^*} \leq 1} |f(x)| = \|x\|_X$$

(Corollary 2.1.16).¹¹ Since the space X^* is always complete (Proposition 2.1.1), Theorem 2.1.4 can be applied to the sequence $\{\kappa(x_n)\}_{n=1}^{\infty}$. This shows that $\{x_n\}_{n=1}^{\infty}$ is bounded. If $x_n \rightharpoonup x$, we choose $f \in X^*$ such that

$$\|f\| = 1 \quad \text{and} \quad f(x) = \|x\|$$

⁹**Warning.** The converse statement is not true in general (see Exercise 2.1.38)!

¹⁰ A Banach space X is said to be *uniformly convex* if for every $\varepsilon > 0$ there is $\delta > 0$ such that $x, y \in X$, $\|x\| = \|y\| = 1$, $\|x - y\| \geq \varepsilon \implies 1 - \left\| \frac{x+y}{2} \right\| \geq \delta$. Every uniformly convex space is reflexive, see Yosida [161, Chapter V, 2]. Hilbert spaces, $L^p(\Omega)$ -spaces and $W^{1,p}(\Omega)$ -spaces ($1 < p < \infty$) are uniformly convex (for a Hilbert space this follows from the parallelogram identity (1.2.14), for the other two cases see, e.g., Adams [2, Corollary 2.29 and Theorem 3.5]).

¹¹ It is not generally true that κ is surjective. A Banach space X is said to be *reflexive* if κ is surjective. Every Hilbert space and spaces $L^p(\Omega)$, $W^{k,p}(\Omega)$, $1 < p < \infty$, are reflexive (the Riesz Representation Theorem and Example 2.1.20(iii)). Spaces $L^1(\Omega)$, $L^\infty(\Omega)$ and $C(\Omega)$ are not reflexive.

(Corollary 2.1.16). Then

$$\|x\| = f(x) = \lim_{n \rightarrow \infty} f(x_n) \leq \liminf_{n \rightarrow \infty} \|x_n\|.$$

Assertion (iv) is obvious for $x = o$. If $x \neq o$, then we may assume that also $x_n \neq o$ and put $y := \frac{x}{\|x\|}$ and $y_n := \frac{x_n}{\|x_n\|}$. Since $x_n \rightharpoonup x$ and $\|x_n\| \rightarrow \|x\|$, we have

$$f(y_n) = \frac{1}{\|x_n\|} f(x_n) \rightarrow \frac{1}{\|x\|} f(x) = f(y) \quad \text{for any } f \in X^*, \quad \text{i.e., } y_n \rightharpoonup y.$$

If we prove that $\|y_n - y\| \rightarrow 0$, then

$$\|x_n - x\| = \|(y_n \|x_n\| - y \|x\|)\| \leq \|x_n\| \|y_n - y\| + \|y\| |\|x_n\| - \|x\|| \rightarrow 0$$

due to the assumption $\|x_n\| \rightarrow \|x\|$. To prove $y_n \rightarrow y$ we proceed by contradiction using the uniform convexity of X . Suppose that there is $\varepsilon > 0$ such that $\|y_n - y\| \geq \varepsilon$ for infinitely many n . Then, by the uniform convexity of X ,

$$\|y_n + y\| \leq 2(1 - \delta).$$

Let us choose $f_0 \in X^*$, $\|f_0\| = 1$, $f_0(y) = \|y\| = 1$ (see Corollary 2.1.16). Then

$$2(1 - \delta) \geq \limsup_{n \rightarrow \infty} \|y_n + y\| \geq \limsup_{n \rightarrow \infty} f_0(y_n + y) = 2f_0(y) = 2,$$

a contradiction. □

Remark 2.1.23. The weak convergence is the convergence in the *weak topology*. It is convenient to define this topology by systems of neighborhoods of points. We say that $\mathcal{U} \subset X$ is a *weak neighborhood* of a point $x \in X$ if there are $f_1, \dots, f_n \in X^*$ such that

$$\{y \in X : |f_i(y) - f_i(x)| < 1 \text{ for } i = 1, \dots, n\} \subset \mathcal{U}.$$

A subset $\mathcal{G} \subset X$ is *weakly open* (i.e., open in the weak topology) provided it is a weak neighborhood of each of its points. It is easy to see that a weakly open set is also open in the norm topology. The converse is generally true only in finite-dimensional spaces.

As we have mentioned, our aim is to find compact sets in the weak topology.

Remark 2.1.24. The weak topology in an infinite-dimensional space is not metrizable. Therefore two concepts of compactness, namely the sequential and the covering one (see footnote 12 on page 27) are in principle different. It is surprising that they coincide for weak topologies in Banach spaces. This very deep result is known as the Eberlain–Smulyan Theorem (see Dunford & Schwartz [53, Chapter 5]).

Theorem 2.1.25 (Eberlain–Smulyan). *Let X be a reflexive space. Then any bounded sequence contains a weakly convergent subsequence.*

Proof. We present a simple proof for the case that X is a Hilbert space. A proof for an arbitrary reflexive space can be found, e.g., in Dunford & Schwartz [53], Fabian et al. [60], Yosida [161]. Let $\{x_n\}_{n=1}^\infty \subset X$ be a bounded sequence, and put

$$Y = \overline{\text{Lin}\{x_1, x_2, \dots\}}$$

(the closure is taken in the norm topology). Since the sequence of scalar products $\{(x_1, x_n)\}_{n=1}^\infty$ is a bounded sequence of numbers (real or complex), there is a subsequence, say $\{x_n^{(1)}\}_{n=1}^\infty$, such that $\{(x_1, x_n^{(1)})\}_{n=1}^\infty$ converges. For the same reason there is a subsequence $\{x_n^{(2)}\}_{n=1}^\infty$ of $\{x_n^{(1)}\}_{n=1}^\infty$ such that $\{(x_2, x_n^{(2)})\}_{n=1}^\infty$ converges, etc. Put $y_k = x_k^{(k)}$ (the diagonal choice). Then $\lim_{k \rightarrow \infty} (x_j, y_k)$ exists for all $j \in \mathbb{N}$, and therefore $\lim_{k \rightarrow \infty} (x, y_k)$ exists for each $x \in \text{Lin}\{x_1, x_2, \dots\}$.

Since the sequence of linear forms $f_k: x \rightarrow (x, y_k)$ is bounded in Y^* , the Banach–Steinhaus Theorem (Corollary 2.1.6) implies the existence of $f \in Y^*$ such that

$$\lim_{k \rightarrow \infty} f_k(x) = f(x) \quad \text{for all } x \in Y.$$

Let P be the orthogonal projection onto Y . Put

$$g(x) = f(Px) \quad \text{for } x \in X.$$

Then $g \in X^*$ and by the Riesz Representation Theorem there is $y \in X$ such that

$$g(x) = (x, y) \quad \text{for } x \in X.$$

Moreover,

$$\lim_{k \rightarrow \infty} (x, y_k) = \lim_{k \rightarrow \infty} (Px, y_k) = f(Px) = (x, y) \quad \text{for all } x \in X.$$

This means that $y_k \rightharpoonup y$. □

Remark 2.1.26. Weak convergence in a dual space X^* is more confusing since two approaches can be used. We say that a sequence $\{f_n\}_{n=1}^\infty \subset X^*$

(i) *converges weakly* to $f \in X^*$ (notation $f_n \rightharpoonup f$ or $\text{w-}\lim_{n \rightarrow \infty} f_n = f$) if

$$\lim_{n \rightarrow \infty} F(f_n) = F(f) \quad \text{for every } F \in X^{**};$$

(ii) *converges weak star* to $f \in X^*$ (notation $f_n \xrightarrow{*} f$ or $\text{w}^*\text{-}\lim_{n \rightarrow \infty} f_n = f$) if

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \quad \text{for every } x \in X.$$

Criteria for weak convergence in L^p -spaces can be found, e.g., in Dunford & Schwartz [53, Chapter IV, 8].

The weak convergence in X^* has obviously the same properties as that in X . Because of the continuous embedding $\kappa: X \rightarrow X^{**}$ (see the proof of Proposition 2.1.22(iii)) the w -convergence implies the w^* -convergence. The converse is true if X is a reflexive space, i.e.,

$$\kappa(X) = X^{**}.$$

Since the w^* -topology is generally weaker than the w -topology there can exist more w^* -compact sets than the w -compact ones. In fact, the following result (the Alaoglu–Bourbaki Theorem, see Conway [36], Dunford & Schwartz [53], Fabian et al. [60]) holds:

If X is a normed linear space, then any closed ball in X^ is w^* -compact. If, moreover, X is separable, then the ball is also sequentially w^* -compact.*

For example, this theorem can be applied to balls in $L^p(\Omega)$, $1 < p \leq \infty$.

In the rest of this section we will examine *adjoint operators*. Suppose that X and Y are normed linear spaces and $A \in \mathcal{L}(X, Y)$. If $g \in Y^*$, then

$$A^*g := g(A) \in X^*.$$

The operator $A^*: Y^* \rightarrow X^*$ is obviously linear, and it is also continuous since

$$|A^*g(x)| = |g(Ax)| \leq \|g\|_{Y^*} \|Ax\|_Y \leq \|g\|_{Y^*} \|A\|_{\mathcal{L}(X, Y)} \|x\|_X.$$

If H_1, H_2 are Hilbert spaces and $A \in \mathcal{L}(H_1, H_2)$ we have another approach to the definition of an adjoint operator, namely the one based on the Riesz Representation Theorem: For $y \in H_2$ the mapping $f: x \mapsto (Ax, y)_{H_2}$ is a continuous linear form on H_1 , and hence there is $z \in H_1$ for which $f(x) = (x, z)_{H_1}$. This z is uniquely determined by y , and we denote for a moment $z = A^+y$, i.e.,

$$(Ax, y)_{H_2} = (x, A^+y)_{H_1}.$$

There is a very slight difference between A^* and A^+ , e.g., $(\alpha A)^* = \alpha A^*$ and $(\alpha A)^+ = \bar{\alpha} A^+$ (see also Example 2.1.28 below). So we will use the same notation, namely A^* , for both concepts. Symmetric matrices have certain special properties (e.g., their canonical forms are diagonal). The same can be expected for their generalization in the Hilbert space setting which is defined as follows:

An operator $A \in \mathcal{L}(H)$ is said to be *self-adjoint* if $A = A^*$, i.e.,

$$(Ax, y) = (x, Ay) \quad \text{for all } x, y \in H.$$

In order to generalize Theorem 1.1.25 to continuous linear operators on infinite-dimensional normed linear spaces we will use the same notation but with a slightly different meaning:

If $\mathcal{M} \subset X$, then

$$\mathcal{M}^\perp := \{f \in X^* : x \in \mathcal{M} \Rightarrow f(x) = 0\}.$$

If $\mathcal{N} \subset X^*$, then

$$\mathcal{N}_\perp := \{x \in X : f \in \mathcal{N} \Rightarrow f(x) = 0\}.$$

We invite the reader to compare these symbols with that for orthogonal complements in Hilbert spaces.

Proposition 2.1.27. *Let X, Y be normed linear spaces and let $A \in \mathcal{L}(X, Y)$. Then*

- (i) *if $x_n \rightharpoonup x$, then $Ax_n \rightharpoonup Ax$;*
- (ii) *if A is, moreover, continuously invertible, then A^* is also continuously invertible and $(A^*)^{-1} = (A^{-1})^*$;*
- (iii) $\text{Ker } A = (\text{Im } A^*)_\perp$;
- (iv) $\overline{\text{Im } A} = (\text{Ker } A^*)_\perp$.

Proof. (i) It is easy with the use of A^* .

(ii) It is sufficient to show that $(A^{-1})^* A^* = I_{Y^*}$ and $A^* (A^{-1})^* = I_{X^*}$. This follows from the more general result

$$(AB)^* = B^* A^*$$

which is easily verified.

(iii) The inclusion $\subset$ is obvious from the definition, for the converse inclusion $\supset$ it is sufficient to use the fact that Y^* separates the points of Y .

(iv) It is easy to see that $(\text{Im } A)^\perp = \text{Ker } A^*$. To get (iv) it suffices to prove that

$$(\mathcal{M}^\perp)_\perp = \overline{\text{Lin } \mathcal{M}} \quad \text{for } \mathcal{M} \subset X.$$

If x_0 belonged to $(\mathcal{M}^\perp)_\perp \setminus \overline{\text{Lin } \mathcal{M}}$, x_0 would be separated from $\overline{\text{Lin } \mathcal{M}}$ by a linear form $f \in X^*$ (Corollary 2.1.18). Since $\overline{\text{Lin } \mathcal{M}}$ is a subspace of X , this separating f would be in $(\overline{\text{Lin } \mathcal{M}})^\perp = \mathcal{M}^\perp$. Therefore $f(x_0) = 0$, and a contradiction is obtained. The converse inclusion $\overline{\text{Lin } \mathcal{M}} \subset (\mathcal{M}^\perp)_\perp$ is obvious. $\square$

Notice that the statement (iv) is not a sufficient condition for solvability of the equation

$$Ax = y$$

since only the closure of $\text{Im } A$ is characterized. There are many operators the range of which is not closed. A simple example is $Ax(t) = \int_0^t x(s) ds$ considered either in $C[0, 1]$ or in $L^2(0, 1)$. It is not an easy task to decide whether an operator has a closed range or not. The following statement is useful in applications.

If X, Y are Banach spaces and $A \in \mathcal{L}(X, Y)$ is injective, then $\text{Im } A$ is closed if and only if there is a positive constant c such that

$$\|Ax\| \geq c\|x\| \quad \text{for all } x \in X.$$

Sufficiency is easy, the necessity part follows from the Open Mapping Theorem.

There is an important subclass of operators with a closed range, namely the so-called *Fredholm operators*. An operator $A \in \mathcal{L}(X)$ is said to be Fredholm if

$$\dim \text{Ker } A < \infty, \quad \text{Im } A \text{ is closed,} \quad \text{and} \quad \text{codim Im } A < \infty$$

(i.e., the dimension of any direct complement of $\text{Im } A$ is finite). We note that

$$\text{codim Im } A = \dim \text{Ker } A^*$$

(this is basically Proposition 2.1.27(iv)). We define

$$\text{ind } A := \dim \text{Ker } A - \dim \text{Ker } A^*$$

and call it the *index of the Fredholm operator*. A special class of Fredholm operators will be examined in the next section.

We have not yet introduced any sufficiently broad family of continuous linear operators. The next example fills this gap.

Example 2.1.28 (Integral operators). Let Ω and $\tilde{\Omega}$ be open subsets of $\mathbb{R}^M$ and $\mathbb{R}^M$, respectively. Assume that $k: \tilde{\Omega} \times \Omega \rightarrow \mathbb{C}$ is a measurable function for which there are constants c_1, c_2 such that

$$\int_{\Omega} |k(t, s)| \, ds \leq c_1 \quad \text{for a.a. } t \in \tilde{\Omega}, \quad \int_{\tilde{\Omega}} |k(t, s)| \, dt \leq c_2 \quad \text{for a.a. } s \in \Omega.$$

Then the operator A defined by

$$Ax(t) = \int_{\Omega} k(t, s)x(s) \, ds \tag{2.1.5}$$

is a linear bounded operator from $L^p(\Omega)$ into $L^p(\tilde{\Omega})$ for $1 \leq p \leq \infty$.¹²

To prove this assertion we have to show that $Ax(t)$ exists for a.a. $t \in \tilde{\Omega}$, is measurable on $\tilde{\Omega}$ and belongs to $L^p(\tilde{\Omega})$. For $1 \leq p < \infty$,¹³ by the Hölder inequality, we get for $\frac{1}{p'} = 1 - \frac{1}{p}$:

$$|Ax(t)| \leq \int_{\Omega} |k(t, s)|^{\frac{1}{p'}} |k(t, s)|^{\frac{1}{p}} |x(s)| \, ds \leq c_1^{\frac{1}{p'}} \left[\int_{\Omega} |k(t, s)| |x(s)|^p \, ds \right]^{\frac{1}{p}}.$$

¹²Conditions on the kernel k which guarantee that $A \in \mathcal{L}(L^p(\Omega), L^p(\tilde{\Omega}))$ can be found in Dunford & Schwartz [53, Chapter VI, 11A].

¹³The case $p = \infty$ is left to the reader.

Set

$$\varphi(t) := \left[\int_{\Omega} |k(t, s)| |x(s)|^p ds \right]^{\frac{1}{p}}.$$

Since the measurable function $(t, s) \mapsto |k(t, s)| |x(s)|^p$ can be approximated by step functions (consider first $\tilde{\Omega} \times \Omega$ bounded), the function $t \mapsto \varphi(t)$ is measurable on $\tilde{\Omega}$. The Fubini Theorem yields

$$\begin{aligned} \int_{\tilde{\Omega}} |\varphi(t)|^p dt &= \int_{\tilde{\Omega}} \left[\int_{\Omega} |k(t, s)| |x(s)|^p ds \right] dt \\ &= \int_{\Omega} \left[\int_{\tilde{\Omega}} |k(t, s)| dt \right] |x(s)|^p ds \leq c_2 \|x\|_{L^p(\Omega)}^p. \end{aligned}$$

In particular, φ is finite a.e. Since $t \mapsto Ax(t)$ is measurable on $\tilde{\Omega}$ (by the same argument as above), we also have

$$\|Ax\|_{L^p(\tilde{\Omega})} = \left[\int_{\tilde{\Omega}} |Ax(t)|^p dt \right]^{\frac{1}{p}} \leq c_1^{\frac{1}{p'}} c_2^{\frac{1}{p}} \|x\|_{L^p(\Omega)}.$$

The Fubini Theorem also yields (we identify $g \in [L^p(\tilde{\Omega})]^*$, $1 \leq p < \infty$, with a function from $L^{p'}(\tilde{\Omega})$ – see Example 2.1.20(iii))

$$\begin{aligned} \int_{\tilde{\Omega}} Ax(t)g(t) dt &= \int_{\tilde{\Omega}} \left[\int_{\Omega} k(t, s)x(s) ds \right] g(t) dt \\ &= \int_{\Omega} \left[\int_{\tilde{\Omega}} k(t, s)g(t) dt \right] x(s) ds = \int_{\Omega} (A^*g)(s)x(s) ds, \end{aligned}$$

i.e.,

$$A^*g: s \mapsto \int_{\tilde{\Omega}} k(t, s)g(t) dt, \quad g \in L^{p'}(\tilde{\Omega}).$$

We note that the adjoint operator to A for $p = 2$ in the sense of the Riesz Representation Theorem is of the form

$$A^*g(s) = \int_{\tilde{\Omega}} \overline{k(t, s)}g(t) dt.$$

In particular, A is self-adjoint if $\Omega = \tilde{\Omega}$ and $k(t, s) = \overline{k(s, t)}$. We will continue the study of integral operators in the next section (Example 2.2.5). $\circ$

In Example 2.1.11 we have mentioned that differential operators on a function space are not continuous and are only densely defined. Therefore we wish to extend the notion of the adjoint operator to this case. Assume that A is a linear operator defined on a dense subspace $\text{Dom } A$ of X with values in Y . Put

$$\begin{aligned} D^* &= \{g \in Y^* : \text{a linear form } x \in \text{Dom } A \mapsto g(Ax) \\ &\quad \text{has a continuous extension } f \text{ to the whole of } X\}. \end{aligned}$$

Obviously, D^* is a linear subspace of Y^* containing o and the extension f is uniquely determined by g . We denote

$$A^*g := f, \quad \text{Dom}(A^*) = D^*$$

and call A^* the *adjoint operator* to A .

Example 2.1.29. The simplest differential operator is defined by

$$Ax(t) = \dot{x}(t).$$

This relation can be considered in various function spaces and also with different domains. If we are interested in its adjoint we should have a good representation of the dual space. This leads to an observation that spaces of integrable functions would be more convenient than spaces of continuous functions. Therefore let $X = L^p(0, 1)$, $1 \leq p < \infty$ and $\text{Dom } A = C^1[0, 1]$. Consider $A: \text{Dom } A \subset X \rightarrow X$. We wish to compute A^* . Assume $g \in \text{Dom}(A^*) \subset L^{p'}(0, 1)$ and $A^*g = f$, i.e.,

$$g(Ax) = \int_0^1 \dot{x}(t)g(t) dt = \int_0^1 x(t)f(t) dt = A^*g(x) \quad \text{for all } x \in \text{Dom } A.$$

In particular, for

$$x \in \mathcal{V} = \{x \in \text{Dom } A : x(1) = 0\} \quad \text{and} \quad F(t) = \int_0^t f(s) ds,$$

the integration by parts¹⁴ yields

$$\int_0^1 x(t)f(t) dt = x(t)F(t)|_0^1 - \int_0^1 \dot{x}(t)F(t) dt = - \int_0^1 \dot{x}(t)F(t) dt.$$

Since the restriction $A|_{\mathcal{V}}$ of A to $\mathcal{V}$ has a dense range in $L^p(0, 1)$ ($\text{Im } A|_{\mathcal{V}} = C[0, 1]$), we have $F + g = o$ in $L^{p'}(0, 1)$. This means that g can be changed on a set of measure zero to have g absolutely continuous and

$$\dot{g} = -f \in L^{p'}(0, 1), \quad \text{i.e.,} \quad g \in W^{1,p'}(0, 1).$$

Moreover, $g(0) = -F(0) = 0$. Taking $F(t) = - \int_t^1 f(s) ds$ we see that also $g(1) = 0$. This proves that

$$\text{Dom}(A^*) \subset \{g \in W^{1,p'}(0, 1) : g(0) = g(1) = 0\} = W_0^{1,p'}(0, 1)^{15}$$

¹⁴If you are not familiar with integration by parts for the Lebesgue integral (notice that $f \in L^{p'}(0, 1) \subset L^1(0, 1)$), you can approximate f by a continuous function to get a standard situation for integration by parts.

and

$$A^*g = -\dot{g}.$$

Integration by parts yields also the converse inclusion, i.e.,

$$\text{Dom}(A^*) = W_0^{1,p'}(0, 1).$$

Notice that $\text{Im } A$ is dense in $L^p(0, 1)$ but not closed while A^* is injective and

$$\text{Im } A^* = \left\{ f \in L^{p'}(0, 1) : \int_0^1 f(t) dt = 0 \right\}$$

is closed but not dense in $L^{p'}(0, 1)$.

Notice also that $\varrho(A) = \varrho(A^*) = \emptyset$ and any $\lambda \in \mathbb{C}$ is an eigenvalue of A . To the contrary, A^* has no eigenvalues. $\circ$

A more general result (due to S. Banach) is stated in the following proposition (see, e.g., Yosida [161]).

Proposition 2.1.30. *Let X, Y be Banach spaces and let A be a closed densely defined linear operator from X into Y . Then $\text{Im } A$ is closed if and only if $\text{Im } A^*$ is closed. Moreover,*

$$\overline{\text{Im } A} = (\text{Ker } A^*)_{\perp} \quad \text{and} \quad \text{Ker } A = (\text{Im } A^*)_{\perp}.$$

Nevertheless, notice that A is not closed in our example. Proposition 2.1.30 can be applied to A^* (A^* is always closed); $\text{Dom}(A^{**}) = W^{1,p}(0, 1)$, $A^{**}x = \dot{x}$.¹⁶ This simple example shows how the domain of a (linear) noncontinuous operator affects its properties.

Example 2.1.31. Put

$$Ax = -\ddot{x} \quad \text{with} \quad \text{Dom } A = \{x \in C^2(a, b) : x(a) = x(b) = 0\}.$$

If the equation

$$Ax = \lambda x$$

has a nonzero solution $w \in (\text{Dom } A)$, then λ is called an eigenvalue and w a corresponding eigenfunction of A . Simple calculation shows that $\frac{k^2\pi^2}{(b-a)^2}$ are all eigenvalues of A ,¹⁷ and $\sin \frac{k\pi}{b-a}(t-a)$ are the corresponding eigenfunctions. Consider now the boundary value problem

$$\begin{cases} -\ddot{x}(t) = \lambda x(t) + f(t), & t \in (a, b), \\ x(a) = x(b) = 0. \end{cases} \quad (2.1.6)$$

¹⁵The last equality should be proved. A deeper insight into these Sobolev spaces will be given in Chapter 7, cf. also Exercise 1.2.48.

¹⁶Notice that A^{**} is an extension of A and, moreover, the graph of A^{**} is the closure of the graph of A (it is also said that A^{**} is the closure of A).

¹⁷The sign minus in the definition of A is conventional; it is introduced to obtain positive eigenvalues.

Let φ_1, φ_2 be a fundamental system for the differential equation $-\ddot{x} - \lambda x = 0$. The Variation of Constants Formula shows that

$$x(t) = c_1\varphi_1(t) + c_2\varphi_2(t) + \int_a^t \frac{\varphi_1(s)\varphi_2(t) - \varphi_1(t)\varphi_2(s)}{W(s)} f(s) ds \quad (2.1.7)$$

is a solution to $-\ddot{x} - \lambda x = f$. Here W is the Wronski determinant of φ_1, φ_2 (notice that for this equation we always can choose φ_1, φ_2 such that $W \equiv 1$). We wish to find constants c_1, c_2 such that x given by (2.1.7) satisfies the boundary conditions $x(a) = x(b) = 0$. The number λ is not an eigenvalue if and only if

$$\det \begin{pmatrix} \varphi_1(a) & \varphi_2(a) \\ \varphi_1(b) & \varphi_2(b) \end{pmatrix} \neq 0.$$

In this case the formula (2.1.7) shows that for any $f \in C[a, b]$ the problem (2.1.6) has a unique solution in $\text{Dom } A$ ¹⁸ which is called a *classical solution*. This means that $\lambda \in \varrho(A)$. Suppose now that λ is an eigenvalue. Then we can take φ_1 as a corresponding eigenfunction and get

$$x(a) = c_2\varphi_2(a), \quad \text{i.e.,} \quad c_2 = 0$$

($\varphi_2(a) \neq 0$ since φ_1, φ_2 are linearly independent), and

$$x(b) = \varphi_2(b) \int_a^b \varphi_1(s)f(s) ds = 0, \quad \text{i.e.,} \quad \int_a^b \varphi_1(s)f(s) ds = 0 \quad (2.1.8)$$

since $\varphi_2(b) \neq 0$ (by the same argument as above). Notice that (2.1.8) is also a necessary condition for solvability of (2.1.6).

We will return to this example in the next section (see Example 2.2.17). $\circ$

Example 2.1.32. Linear differential operators of the second order with nonconstant coefficients are more complicated. To simplify our exposition we consider a differential expression

$$Lx := p_0\ddot{x} + p_1\dot{x} + p_2x$$

where p_0, p_1, p_2 are continuous functions on a closed bounded interval $[a, b]$ and $p_0 < 0$ on this interval (the so-called regular case). Let $X = L^p(a, b)$, $1 \leq p < \infty$ and

$$\mathcal{D} = \{x \in W^{2,p}(a, b) : x(a) = x(b) = 0\}.$$

Put

$$Ax = Lx, \quad x \in \mathcal{D} = \text{Dom } A$$

and consider

$$A: \text{Dom } A \subset X \rightarrow X.$$

¹⁸If $f \in L^p(a, b)$, then it is possible to show that the function $x = x(t)$ given by (2.1.7) belongs to $W^{2,2}(a, b)$, $x(a) = x(b) = 0$, and the equation in (2.1.6) is satisfied a.e. in (a, b) . Such a solution is called a *strong solution*.

A solution of $Ax = f$ is therefore a strong solution of

$$\begin{cases} Lx(t) = f(t), & t \in (a, b), \\ x(a) = x(b) = 0. \end{cases}$$

It can be proved that A is injective provided $p_2 > 0$ in $[a, b]$. (Assume by contradiction that $\text{Ker } A \neq \{0\}$ and show that there is $x_0 \in \text{Ker } A$ which has a negative minimum at an interior point $c \in (a, b)$. Deduce that $Lx_0(c) < 0$.) The Variation of Constants Formula shows that the operator A is also surjective and A^{-1} is an integral operator

$$A^{-1}f(t) = \int_a^b G(t, s)f(s) ds \quad (2.1.9)$$

where G is the so-called *Green function* of L . The Green function is nonnegative on $[a, b] \times [a, b]$ and satisfies the estimates from Example 2.1.28. Therefore $A^{-1} \in \mathcal{L}(X)$. In order to calculate the adjoint A^* it is convenient to consider the so-called formal adjoint expression to L , i.e.,

$$My = (p_0 y)'' - (p_1 y)' + p_2 y$$

which is obtained by integrating by parts in the integral $\int_a^b Lx(t)y(t) dt$ and omitting the boundary terms. Put

$$By = My \quad \text{for } y \in \mathcal{D} = \text{Dom } B.$$

The same integration as above shows that $B \subset A^*$. The proof of the equality $A^* = B$ needs a more careful calculation.

The interested reader can consult the books Coddington & Levinson [35, Chapter 9], Edmunds & Evans [55] or Dunford & Schwartz [54], in particular Chapter XIII, for details and also for more complicated singular cases which are important in applications, e.g., in Quantum Mechanics (the Schrödinger equation). $\circ$

Exercise 2.1.33. Let X, Y be Banach spaces. If $A \in \mathcal{L}(X, Y)$ has a continuous inverse $A^{-1} \in \mathcal{L}(Y, X)$ and $B \in \mathcal{L}(X, Y)$ is such that

$$\|B - A\| < \frac{1}{\|A^{-1}\|},$$

then B is also continuously invertible and

$$\|B^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\|B - A\|}, \quad \|B^{-1} - A^{-1}\| \leq \frac{\|A^{-1}\|^2}{1 - \|A^{-1}\|\|B - A\|} \|B - A\|.$$

Hint. Examine the proof of Corollary 2.1.3 and write $A^{-1}B = A^{-1}(B - A) + I$.

Exercise 2.1.34. Let X be a Banach space. Prove that the spectrum depends continuously on an operator in the following sense: For any $A \in \mathcal{L}(X)$ and an open set $\mathcal{G}$ containing $\sigma(A)$ there is $\delta > 0$ such that for $B \in \mathcal{L}(X)$, $\|B - A\| < \delta$, the spectrum $\sigma(B) \subset \mathcal{G}$.

Hint. First show that $\|(\lambda I - A)^{-1}\|$ is bounded on $\mathbb{C} \setminus \mathcal{G}$ and then use Exercise 2.1.33.

Exercise 2.1.35. Show that

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$$

is well defined for all $t \in \mathbb{R}$, $A \in \mathcal{L}(X)$, provided X is a Banach space, and, moreover, the vector function

$$\varphi: t \mapsto e^{tA}x_0$$

solves the differential equation

$$\dot{x}(t) = Ax(t)$$

and satisfies the initial condition $\varphi(0) = x_0$. (See also the end of Section 1.1.)

Exercise 2.1.36. Let K be a continuous real function on $[a, b] \times [a, b]$ and let $h \in C[a, b]$ be fixed. Let

$$M = \max_{(t, \tau) \in [a, b] \times [a, b]} |K(t, \tau)|$$

and let $\lambda \in \mathbb{R}$ be such that

$$|\lambda| < \frac{1}{M(b-a)}.$$

Prove that the integral equation

$$x(t) = \lambda \int_a^b K(t, \tau)x(\tau) d\tau + h(t)$$

has a unique solution $x \in C[a, b]$.

Exercise 2.1.37. Let $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ be sequences in a Hilbert space H such that $x_n \rightharpoonup x$, $y_n \rightarrow y$. Then

$$(x_n, y_n) \rightarrow (x, y).$$

Hint. Use Proposition 2.1.22(iii).

Exercise 2.1.38. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal sequence in a Hilbert space. Show that

$$e_n \rightharpoonup 0.$$

Hint. Use the Bessel inequality (1.2.17).

Exercise 2.1.39. Prove assertion (iv) of Proposition 2.1.22 for a Hilbert space X .

Hint. Use the relation between the scalar product and the norm in X .

Exercise 2.1.40. Show that a convex set (in particular a subspace) of a normed linear space is weakly closed if and only if it is closed in the norm topology.

Hint. Suppose by contradiction that $\mathcal{C}$ is a norm-closed convex set which is not weakly closed. Then there is $x_0 \in \overline{\mathcal{C}}^w \setminus \mathcal{C}$. Use the Separation Theorem (Corollary 2.1.18) to obtain a contradiction.

Exercise 2.1.41. Prove that actually

$$\|A^*\|_{\mathcal{L}(Y^*, X^*)} = \|A\|_{\mathcal{L}(X, Y)}.$$

Hint. The inequality $\|A^*\| \leq \|A\|$ follows from the calculation after Remark 2.1.26. For the converse inequality use the dual characterization of the norm $\|Ax\|$.

2.2 Compact Linear Operators

In this section we present a class of continuous linear operators the properties of which are closely related to the properties of finite-dimensional linear operators. The key assertions presented concern the Riesz–Schauder Theory and the Hilbert–Schmidt Theorem.

Definition 2.2.1. Let X and Y be normed linear spaces. A linear operator $A \in \mathcal{L}(X, Y)$ is called a *compact operator* if the image of a ball in X is relatively compact in Y . The set of all compact operators from X into Y is denoted by $\mathcal{C}(X, Y)$.

Remark 2.2.2.

- (i) Every compact linear operator is continuous.
- (ii) The compactness condition is mostly used in the following equivalent form:
For any bounded sequence $\{x_n\}_{n=1}^\infty \subset X$ there is a subsequence $\{x_{n_k}\}_{k=1}^\infty$ such that Ax_{n_k} converge in the norm topology of Y .
- (iii) Replacing the norm topology in Y by the weak topology a *weakly compact operator* can be defined. If either X or Y is reflexive, then any $A \in \mathcal{L}(X, Y)$ is weakly compact. This follows from the Eberlain–Smulyan Theorem (Remark 2.1.24) and the observation that $A \in \mathcal{L}(X, Y)$ maps a weakly convergent sequence into a weakly convergent one (cf. Proposition 2.1.27(i)).

Example 2.2.3.

- (i) If $A \in \mathcal{L}(X, Y)$ and $\dim \operatorname{Im} A < \infty$ (the so-called *operator of finite rank*), then $A \in \mathcal{C}(X, Y)$.

(ii) Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis in a Hilbert space H . Put

$$Ae_n = \lambda_n e_n$$

and extend A by linearity to the dense set $\mathcal{D} := \text{Lin}\{e_1, \dots\}$ in H . The operator A is bounded on $\mathcal{D}$ (and therefore it can be uniquely extended to a continuous operator on H) if and only if $\{\lambda_n\}_{n=1}^\infty$ is a bounded sequence. In addition,

$$\|A\| = \sup_n |\lambda_n|.$$

This follows immediately from the identity

$$\|Ax\|^2 = \sum |\lambda_n|^2 |(x, e_n)|^2 \quad \text{for every } x \in H.$$

Moreover, A is a compact operator on H if and only if

$$\lim_{n \rightarrow \infty} \lambda_n = 0.$$

This is an easy consequence of Proposition 1.2.39. ○

Proposition 2.2.4. *Let X, Y and Z be normed linear spaces. Then*

- (i) *if $A \in \mathcal{C}(X, Y)$, $B \in \mathcal{L}(Y, Z)$, then $BA \in \mathcal{C}(X, Z)$;*
- (ii) *if $A \in \mathcal{C}(Y, Z)$, $B \in \mathcal{L}(X, Y)$, then $AB \in \mathcal{C}(X, Z)$;*
- (iii) *if $A \in \mathcal{C}(X, Y)$ and a sequence $\{x_n\}_{n=1}^\infty \subset X$ converges weakly to $x \in X$, then*

$$\lim_{n \rightarrow \infty} \|Ax_n - Ax\| = 0.$$

- (iv) *Assume that Y is a Banach space and a sequence $\{A_n\}_{n=1}^\infty \subset \mathcal{C}(X, Y)$ converges to $A \in \mathcal{L}(X, Y)$ in the norm operator topology. Then $A \in \mathcal{C}(X, Y)$.*

Proof. The assertions (i) and (ii) are obvious.

To prove (iii) assume by contradiction that there is a subsequence $\{x_{n_k}\}_{k=1}^\infty$ such that

$$\|Ax_{n_k} - Ax\| \geq c > 0.$$

The sequence $\{x_n\}_{n=1}^\infty$ is bounded (Proposition 2.1.22(iii)), and hence there exists a subsequence $\{x_{n_{k_l}}\}_{l=1}^\infty$ and $y \in Y$ such that

$$\|Ax_{n_{k_l}} - y\| \rightarrow 0.$$

Since

$$f(Ax_n) = A^*f(x_n) \rightarrow A^*f(x) = f(Ax) \quad \text{for every } f \in X^*,$$

we have $y = Ax$, and hence a contradiction.

- (iv) Let $B(o; 1)$ be the unit ball. By Proposition 1.2.3 it suffices to show that for any $\varepsilon > 0$ there is a finite ε -net of $A(B(o; 1))$. We choose n such that $\|A_n - A\| < \frac{\varepsilon}{2}$, and a finite $\frac{\varepsilon}{2}$ -net for $A_n(B(o; 1))$. By the triangle inequality, this is the desired ε -net for $A(B(o; 1))$. □

Example 2.2.5.

- (i) Let k be a continuous function on the Cartesian product $[a, b] \times [a, b]$. Then the operator

$$Ax: t \in [a, b] \mapsto \int_a^b k(t, s)x(s) \, ds$$

is compact as an operator from $C[a, b]$ into itself.¹⁹ We give two proofs of this assertion.

The first is based on the use of the Arzelà–Ascoli Theorem (Theorem 1.2.13). Its assumptions are satisfied for $\mathcal{F} = A(B(o; 1))$ where $B(o; 1)$ is the unit ball in $C[a, b]$. The equicontinuity of $\mathcal{F}$ follows from the uniform continuity of k on $[a, b] \times [a, b]$.

The second proof uses Proposition 2.2.4(iv). Put

$$\mathcal{A} = \{(t, s) \mapsto x(t)y(s) : x, y \in C[a, b]\}.$$

It is easy to see that $\mathcal{A}$ is a subalgebra of $C([a, b] \times [a, b])$ which satisfies the assumptions of the real or complex Stone–Weierstrass Theorem (Theorem 1.2.14). Hence there are sequences $\{q_n\}_{n=1}^\infty, \{r_n\}_{n=1}^\infty$ in $C[a, b]$ such that

$$q_n(t)r_n(s) \rightrightarrows k(t, s) \quad \text{uniformly in } [a, b] \times [a, b].$$

In particular, this means that the operators

$$A_n x: t \mapsto q_n(t) \int_a^b r_n(s)x(s) \, ds$$

converge in the operator norm to A . Since $\text{Im } A_n \subset \text{Lin}\{q_n\}$, all A_n are compact and, therefore, A is compact.

- (ii) Let Ω be a measurable subset of $\mathbb{R}^M$ and let $k \in L^2(\Omega \times \Omega)$. Then the operator

$$Ax(t) = \int_\Omega k(t, s)x(s) \, ds$$

is compact as an operator from $L^2(\Omega)$ into itself.

We present again two proofs of this statement. The first will be a typical Hilbert space proof, the second will use the reflexivity of $L^2(\Omega)$ and we will show how it could be used to get compactness of an integral operator on $L^p(\Omega)$.

The first proof is based on the following observation:

¹⁹This is true under more general assumptions, e.g., if the interval $[a, b]$ is replaced by a compact topological space K , μ is a Borel measure on K and A is defined by $Ax(t) = \int_K k(t, s)x(s) \, d\mu(s)$.

Let $\{e_k\}_{k=1}^\infty, \{f_k\}_{k=1}^\infty$ be two orthonormal bases in a separable Hilbert space H . Let $B \in \mathcal{L}(H)$. By the Parseval equality we have

$$\|B\|_2^2 := \sum_{k,n=1}^\infty |(Be_k, f_n)|^2 = \sum_{k=1}^\infty \|Be_k\|^2 = \sum_{n=1}^\infty \|B^* f_n\|^2 \leq \infty.$$

This shows that the quantity $\|B\|_2^2$ depends only on B and not on the particular choice of bases. Moreover, if $\|B\|_2^2 < \infty$, then B is called a *Hilbert-Schmidt operator* and $B \in \mathcal{C}(H)$.

To see this take $n_\varepsilon \in \mathbb{N}$ such that $\sum_{n=n_\varepsilon+1}^\infty \|B^* f_n\|^2 < \varepsilon$ and define

$$B_\varepsilon x = \sum_{n=1}^{n_\varepsilon} (Bx, f_n) f_n.$$

Then $\dim \operatorname{Im} B_\varepsilon < \infty$ and

$$\|B_\varepsilon x - Bx\|^2 = \sum_{n=n_\varepsilon+1}^\infty |(Bx, f_n)|^2 \leq \|x\|^2 \sum_{n=n_\varepsilon+1}^\infty \|B^* f_n\|^2 \leq \varepsilon \|x\|^2.$$

The compactness of B follows from Proposition 2.2.4(iv).

In order to apply this statement to the integral operator A choose an orthonormal basis $\{e_n\}_{n=1}^\infty$ in $L^2(\Omega)$ and notice that

$$\varphi_{m,n}(t, s) := e_m(t) \overline{e_n(s)}$$

is an orthonormal set in $L^2(\Omega \times \Omega)$ (use Corollary 1.2.36). Since

$$(Ae_n, e_m)_{L^2(\Omega)} = (k, \varphi_{m,n})_{L^2(\Omega \times \Omega)},$$

we have

$$\|A\|_2 = \|k\|_{L^2(\Omega \times \Omega)}.$$

Now we give the second proof. Let $\{x_n\}_{n=1}^\infty$ be a bounded set in $L^2(\Omega)$. Since $L^2(\Omega)$ as a Hilbert space is reflexive, there is a subsequence – denote it again by $\{x_n\}_{n=1}^\infty$ – which is weakly convergent to an x in $L^2(\Omega)$. In particular,

$$\int_\Omega k(t, s) x_n(s) \, ds \rightarrow \int_\Omega k(t, s) x(s) \, ds \quad \text{for a.a. } t \in \Omega$$

(the Fubini Theorem shows that $k(t, \cdot) \in L^2(\Omega)$ for a.a. $t \in \Omega$). Since

$$\begin{aligned} |Ax_n(t) - Ax(t)| &\leq \int_\Omega |k(t, s)| |x_n(s) - x(s)| \, ds \\ &\leq \|x_n - x\|_{L^2(\Omega)} \left[\int_\Omega |k(t, s)|^2 \, ds \right]^{\frac{1}{2}} \leq c \left[\int_\Omega |k(t, s)|^2 \, ds \right]^{\frac{1}{2}}, \end{aligned}$$

the Lebesgue Dominated Convergence Theorem yields

$$\|Ax_n - Ax\|_{L^2(\Omega)} \rightarrow 0. \quad \bigcirc$$

Proposition 2.2.6. *Let H be a Hilbert space and $A \in \mathcal{L}(H)$. Then A is a compact operator if and only if there is a sequence $\{A_n\}_{n=1}^\infty \subset \mathcal{L}(H)$ of operators of finite rank which converges to A in the operator norm topology.*

Proof. Because of Proposition 2.2.4 only the necessity part is left to be proved. Let $B(o; 1)$ be the unit ball in H . Since $\overline{A(B(o; 1))}$ is compact, it is a separable metric space, and therefore

$$Y = \overline{\text{Lin } A(B(o; 1))}$$

is a separable Hilbert space. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis in Y . Put

$$A_n x = \sum_{k=1}^n (Ax, e_k) e_k.$$

Then A_n has finite rank and

$$\|A_n x - Ax\|^2 = \sum_{k=n+1}^{\infty} |(Ax, e_k)|^2 < \varepsilon \quad \text{for every } x \in B(o; 1)$$

provided n is sufficiently large (Proposition 1.2.39). $\square$

Remark 2.2.7. The proof of the preceding proposition indicates that the result holds also in a Banach space X with a Schauder basis $\{e_n\}_{n=1}^\infty$ (see page 41). The famous conjecture of S. Banach was that any separable Banach space has a Schauder basis. The first counterexample was constructed by P. Enflo. He found a compact operator in a separable Banach space which cannot be approximated by operators of finite rank. We notice that separable Banach spaces of functions like $C(\overline{\Omega})$, $L^p(\Omega)$, $W^{k,p}(\Omega)$ ($1 \leq p < \infty$) have a Schauder basis.

One of our goals in this section is to generalize the Fredholm alternative (see footnote 6 on page 14). As we have seen in Section 1.1 the notion of the adjoint operator is very important.

Proposition 2.2.8 (Schauder). *Let X, Y be Banach spaces and assume that $A \in \mathcal{L}(X, Y)$. Then A is compact if and only if A^* is compact.*

Proof.

Step 1 (the “only if” part). Suppose that $A \in \mathcal{C}(X, Y)$ and $\{g_n\}_{n=1}^\infty \subset Y^*$, $\|g_n\|_{Y^*} \leq 1$. It is easy to verify the assumptions of the Arzelà–Ascoli Theorem (Theorem 1.2.13) for the sequence of functions

$$g_n: \mathcal{K} := \overline{A(B(o; 1))} \rightarrow \mathbb{R} \quad (\text{or } \mathbb{C})$$

($B(o; 1)$ is the unit ball in X). By this theorem there is a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ which is uniformly convergent on $\mathcal{K}$. Since

$$|A^*g_{n_k}(x) - A^*g_{n_l}(x)| \leq \sup_{y \in \mathcal{K}} |g_{n_k}(y) - g_{n_l}(y)| \quad \text{for each } x \in B(o; 1)$$

and X^* is complete, the sequence $\{A^*g_{n_k}\}_{k=1}^{\infty}$ is convergent in X^* .

Step 2 (the “if” part). Assume now that $A^* \in \mathcal{C}(Y^*, X^*)$. We embed X into X^{**} and Y into Y^{**} with help of the canonical isometrical embeddings κ_X and κ_Y (see the proof of Proposition 2.1.22(iii)). Since A^* is compact, A^{**} is compact by the first part of the proof. It suffices to show that

$$\kappa_Y(Ax) = A^{**}\kappa_X(x) \quad \text{for } x \in X$$

and we leave that to the reader. □

If $A \in \mathcal{C}(X, Y)$, then the equation

$$Ax = y \tag{2.2.1}$$

is scarcely ever well-posed²⁰ as follows from the first part of the next theorem. This is the reason why we are interested rather in equations of the type

$$x - Ax = y. \tag{2.2.2}$$

Theorem 2.2.9 (Riesz–Schauder Theory). *Let X be a Banach space and $A \in \mathcal{C}(X)$. Then*

- (i) *if $\text{Im } A$ is closed, then $\dim \text{Im } A < \infty$;*
- (ii) *$\dim \text{Ker}(I - A) < \infty$;*
- (iii) *$\text{Im}(I - A)$ is closed;*
- (iv) *(the Fredholm alternative)*

$$\text{Im}(I - A) = X \quad \text{if and only if} \quad \text{Ker}(I - A) = \{o\};$$

- (v) $\dim \text{Ker}(I - A) = \dim \text{Ker}(I^* - A^*)$.

Proof. (i) If $Y = \text{Im } A$ is closed, then $A: X \rightarrow Y$ is an open mapping (Theorem 2.1.8). This means that a certain ball $B(o; \delta)$ in Y is contained in the relatively compact set $A(B(o; 1))$, i.e., $B(o; \delta)$ itself is relatively compact. By Proposition 1.2.15, $\dim Y < \infty$.

²⁰An equation (2.2.1) is said to be *well-posed* if A is injective and A^{-1} is continuous. If A is an integral operator, then (2.2.1) is called an *integral equation of the first kind*. The equation (2.2.2) is called an *integral equation of the second kind*. The research of these equations carried out by I. Fredholm is supposed to be one of the starting points in the development of functional analysis.

(ii) For the rest of the proof we put

$$T := I - A \quad \text{and} \quad Y := \text{Ker } T.$$

Then the restriction of A to the Banach space Y maps Y onto Y . By (i), $\dim Y < \infty$.

(iii) Because of (ii) there exists a continuous projection P of X onto Y (Remark 2.1.19). Denote

$$Z := \text{Ker } P, \quad \text{i.e.,} \quad X = Y \oplus Z$$

and both Y and Z are Banach spaces. Since T is injective on Z , $\text{Im } T$ is closed provided there is a positive constant c such that

$$\|Tz\|_Y \geq c\|z\|_Z \quad \text{for each } z \in Z,$$

see page 71. Suppose by contradiction that such c does not exist, i.e., there are $z_n \in Z$ such that

$$\|z_n\|_Z = 1 \quad \text{and} \quad \|Tz_n\|_Y < \frac{1}{n}\|z_n\|_Z.$$

Then one can find a subsequence $\{z_{n_k}\}_{k=1}^\infty$ for which Az_{n_k} converges to a y . Since $Tz_{n_k} \rightarrow o$, we have $\lim_{n \rightarrow \infty} z_{n_k} = y \in Z$. This means that

$$Ty = o, \quad \text{i.e.,} \quad y \in Y \cap Z, \quad \text{and thus} \quad y = o.$$

This is a contradiction since $z_{n_k} \rightarrow y$ implies that $\|y\|_Y = 1$.

(iv) We will prove the necessity part by way of contradiction. Put

$$Y_k := \text{Ker } T^k.$$

Then

$$Y_1 \subsetneq Y_2 \subsetneq \cdots \subsetneq Y_k \subsetneq \cdots$$

since for $x_1 \in \text{Ker } (I - A)$, $x_1 \neq o$, there is x_2 such that $x_1 = Tx_2$, i.e., $x_2 \in Y_2 \setminus Y_1$, etc. It follows from the construction in the proof of Proposition 1.2.15 that there are $y_k \in Y_k$, $\|y_k\|_{Y_k} = 1$, such that $\text{dist}(y_{k+1}, Y_k) \geq \frac{1}{2}$. For $k > l$ we have

$$\|Ay_k - Ay_l\|_{Y_k} = \|y_k - (y_l - Ty_l + Ty_k)\|_{Y_k} \geq \text{dist}(y_k, Y_{k-1}) \geq \frac{1}{2}.$$

This means that there is no convergent subsequence of $\{Ay_k\}_{k=1}^\infty$, a contradiction.

The sufficiency part is now easy: It follows from Proposition 2.2.8 and the previous part (iii) that $\text{Im } T^*$ is closed. Assume that $\text{Ker } T = \{o\}$. By Proposition 2.1.27(iii),

$$\text{Im } T^* = (\text{Ker } T)^\perp = X^*.$$

According to the first part of this proof, $\text{Ker } T^* = \{o\}$ and, again by (iii) and Proposition 2.1.27(iv),

$$\text{Im } T = (\text{Ker } T^*)^\perp = X.$$

(v) As in the proof of (iii), $X = Y \oplus Z$ and the corresponding projection P of X onto Y is continuous. It can be shown that a direct complement W of $\text{Im } T$ in X is isomorphic to $\text{Ker } T^*$.²¹ This means that

$$\dim W = \dim \text{Ker } T^* < \infty.$$

Denote

$$\dim \text{Ker } T = n \quad \text{and} \quad \dim \text{Ker } T^* = n^*.$$

We shall prove that $n = n^*$. Assume that

$$n > n^*.$$

In particular, this means that there is a surjective linear operator $\Phi \in \mathcal{L}(Y, W)$. Such Φ cannot be injective (see Corollary 1.1.15), i.e., there is $x_0 \in Y$, $x_0 \neq o$, for which $\Phi(x_0) = o$. Put now

$$B := A + \Phi P.$$

Since $P \in \mathcal{C}(X)$, we have $B \in \mathcal{C}(X)$ and

$$Bx_0 = Ax_0 + o = x_0, \quad \text{i.e.,} \quad \text{Ker } (I - B) \neq \{o\}.$$

By the Fredholm alternative (iv), $\text{Im } (I - B) \neq X$. But

$$(I - B)(Z) = \text{Im } T \quad \text{and} \quad (I - B)(Y) = \Phi(Y) = W,$$

i.e.,

$$\text{Im } (I - B) = \text{Im } T + W = X,$$

a contradiction. This proves the inequality $n \leq n^*$.

By interchanging T and T^* we similarly obtain $n^* \leq n$.²² □

Remark 2.2.10. The proof of the following statement is similar to that of Lemma 1.1.31(i).

If $A \in \mathcal{C}(X)$ and $1 \in \sigma(A)$, then there is $k \in \mathbb{N}$ such that

$$X = \text{Ker } (I - A)^k \oplus \text{Im } (I - A)^k.$$

Moreover, both the spaces on the right-hand side are A -invariant, and $\dim \text{Ker } (I - A)^k < \infty$.²³

²¹This is clear for X being a Hilbert space, since $\text{Im } T$ is closed and the orthogonal complement $(\text{Im } T)^\perp$ is equal to $\text{Ker } T^*$ (Proposition 2.1.27(iv)). In a general Banach space we can use the factor space $X|_{\text{Im } T}$ which is algebraically isomorphic to a direct complement W of $\text{Im } T$ and for $g \in (X|_{\text{Im } T})^*$ put $f(x) = g([x])$. It remains to show that the correspondence $g \rightarrow f$ is an (isometric) isomorphism onto $(\text{Im } T)^\perp = \text{Ker } T^*$.

²²We recommend to the reader to do that carefully to see that no reflexivity of X is needed.

²³This dimension is called the *multiplicity of the eigenvalue 1*.

Remark 2.2.11. Theorem 2.2.9 can be generalized to operators $A \in \mathcal{L}(X)$ for which there is $k \in \mathbb{N}$ such that $A^k \in \mathcal{C}(X)$.

Another way of generalization is connected with perturbations of Fredholm operators. Notice that the statement (v) of Theorem 2.2.9 says that $I - A$ is a Fredholm operator of index zero provided $A \in \mathcal{C}(X)$. The following theorem states the stability of index.

Theorem 2.2.12. *Let X, Y be Banach spaces and let $A \in \mathcal{L}(X, Y)$ be a Fredholm operator. Then*

(i) *if $B \in \mathcal{C}(X, Y)$, then $A + B$ is Fredholm and*

$$\text{ind } A = \text{ind } (A + B); \quad (2.2.3)$$

(ii) *the set of Fredholm operators in $\mathcal{L}(X, Y)$ is an open subset of $\mathcal{L}(X, Y)$; furthermore, ind is a continuous function on this open set.*

Proof. The proofs and further results can be found, e.g., in Kato [87, § IV.5]. $\square$

Corollary 2.2.13. *Let X be a complex Banach space and let $A \in \mathcal{C}(X)$. Then*

(i) *$\sigma(A) \setminus \{0\}$ is a countable set of eigenvalues of finite multiplicity;*

(ii) *if $\dim X = \infty$, then $0 \in \sigma(A)$, and if λ is an accumulation point of $\sigma(A)$, then $\lambda = 0$.*

Proof. (i) If $\lambda \neq 0$, then $\lambda I - A = \lambda \left(I - \frac{A}{\lambda} \right)$ and Theorem 2.2.9 can be applied. In particular, if such λ belongs to $\sigma(A)$, then λ is an eigenvalue of finite multiplicity. It remains to show that for any $r > 0$ the set $\varrho = \{\lambda \in \sigma(A) : |\lambda| > r\}$ is finite.

Assume by way of contradiction that there is a sequence of mutually different points $\{\lambda_n\}_{n=1}^{\infty} \subset \varrho$ and let x_n be the corresponding nonzero eigenvectors. Put

$$W_n = \text{Lin}\{x_1, \dots, x_n\}.$$

It is easy to see by induction that $x_1, \dots, x_n$ are linearly independent. So we can find $y_{n+1} \in W_{n+1}$ such that

$$\|y_{n+1}\| = 1 \quad \text{and} \quad \text{dist}(y_{n+1}, W_n) \geq \frac{1}{2}.$$

Now for $k > l$ we have

$$\|Ay_k - Ay_l\| = \|\lambda_k y_k - [(\lambda_k I - A)y_k + (\lambda_l I - A)y_l - \lambda_l y_l]\| \geq |\lambda_k| \text{dist}(y_k, W_{k-1}) \geq \frac{r}{2}$$

and this contradicts the compactness of A .

(ii) The statement on accumulation points follows immediately from the proof of (i). To see that 0 is a point of $\sigma(A)$ provided $\dim X = \infty$ it is sufficient to realize that $\sigma(A)$ cannot be a finite set of nonzero numbers $\lambda_1, \dots, \lambda_n$. Indeed, with help of Remark 2.2.10 we get

$$X = \text{Ker}(\lambda_1 I - A)^{k_1} \oplus \dots \oplus \text{Ker}(\lambda_n I - A)^{k_n} \oplus V \quad (2.2.4)$$

where V is a nontrivial closed A -invariant subspace of X . Therefore the spectrum $\sigma(A|_V)$ of the restriction $A|_V$ of A to V is a subset of $\sigma(A)$. Since $\sigma(A|_V) \neq \emptyset$ (see the discussion following Example 2.1.20), we have

$$\{\lambda_1, \dots, \lambda_n\} \neq \sigma(A). \quad \square$$

Example 2.2.14. Consider

$$Ax(t) := \int_0^t x(s) \, ds \quad \text{on the space } L^2(0, 1).$$

This is a special class of operators which have been examined in Example 2.2.5(ii):

$$k(t, s) = \begin{cases} 1 & \text{for } 0 \leq s \leq t \leq 1, \\ 0 & \text{for } 0 \leq t < s \leq 1. \end{cases}$$

Therefore $A \in \mathcal{C}(L^2(0, 1))$.

If $\lambda \neq 0$ were an eigenvalue of A with an eigenfunction x , then

$$x(t) = \frac{1}{\lambda} \int_0^t x(s) \, ds,$$

i.e., x is absolutely continuous and

$$\dot{x} = \frac{1}{\lambda} x, \quad x(0) = 0.$$

This implies that $x = 0$ in $[0, 1]$. Since $\sigma(A)$ cannot be empty, $\sigma(A) = \{0\}$, and 0 is no eigenvalue of A .

We notice that the same statement (with a more complicated proof) is valid for any *Volterra integral operator*

$$Ax(t) = \int_0^t k(t-s)x(s) \, ds, \quad x \in L^2(0, 1),$$

provided, e.g., $k \in L^2(0, 1)$. See also Example 2.3.7. ○

Corollary 2.2.13 can be significantly strengthened in the case that X is a Hilbert space and A is a compact, self-adjoint operator. To see this we need some technicalities.

Proposition 2.2.15. *Let H be a Hilbert space and A a self-adjoint continuous operator on H . Then*

- (i) $\|A\| = \sup_{\|x\|=1} |(Ax, x)|$;
- (ii) $m := \inf_{\|x\|=1} (Ax, x)$ and $M := \sup_{\|x\|=1} (Ax, x)$ belong to the spectrum of A ;

- (iii) $\|A\| = \sup \{|\lambda| : \lambda \in \sigma(A)\};$
- (iv) $\sigma(A) \subset \mathbb{R};$
- (v) if $Ax = \lambda x$, $Ay = \mu y$, $\lambda \neq \mu$, then $(x, y) = 0$.

Proof. (i) Denote the right-hand side by α . Obviously $\alpha \leq \|A\|$. To prove the converse inequality take

$$o \neq x \in H, \quad y = Ax.$$

Then for any $t > 0$, using (1.2.14), we have

$$\begin{aligned} \|Ax\|^2 &= \left(A(tx), \frac{1}{t}Ax \right) = \left(A(tx), \frac{1}{t}y \right) \\ &= \frac{1}{4} \left[\left(A \left(tx + \frac{1}{t}y \right), tx + \frac{1}{t}y \right) - \left(A \left(tx - \frac{1}{t}y \right), tx - \frac{1}{t}y \right) \right] \\ &\leq \frac{\alpha}{4} \left[\left\| tx + \frac{1}{t}y \right\|^2 + \left\| tx - \frac{1}{t}y \right\|^2 \right] = \frac{\alpha}{2} \left[t^2\|x\|^2 + \frac{1}{t^2}\|y\|^2 \right]. \end{aligned}$$

Now we choose t such that

$$t^2\|x\|^2 + \frac{1}{t^2}\|y\|^2 = 2\|x\|\|y\|, \quad \text{i.e.,} \quad \left(t\|x\| - \frac{1}{t}\|y\| \right)^2 = 0.$$

Hence

$$t = \left(\frac{\|y\|}{\|x\|} \right)^{\frac{1}{2}} \quad \text{and} \quad \|Ax\|^2 \leq \alpha\|x\|\|y\| \quad \text{follows.}$$

(ii) By taking $A + \|A\|I$ instead of A , we can assume that $0 \leq m \leq M = \|A\|$ (the last equality follows from (i)). Let $\{x_n\}_{n=1}^{\infty}$ be a sequence such that

$$\|x_n\| = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} (Ax_n, x_n) = M.$$

Then

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|Ax_n - Mx_n\|^2 &= \limsup_{n \rightarrow \infty} [(Ax_n, Ax_n) - 2M(Ax_n, x_n) + M^2] \\ &\leq \limsup_{n \rightarrow \infty} [2M^2 - 2M(Ax_n, x_n)] = 0. \end{aligned}$$

If $M \in \varrho(A)$, then there is a constant $c > 0$ such that

$$\|Ax - Mx\| \geq c\|x\|.$$

The previous calculation shows that this cannot be true.

The assertion on m is obtained by replacing A by $-A$.

(iii) This is a consequence of (i) and (ii) and Corollary 2.1.3.

(iv) Let $\lambda = \alpha + i\beta$, $\beta \neq 0$. A simple calculation yields that

$$\|\lambda x - Ax\|^2 \geq |\beta|^2\|x\|^2 \quad \text{for every} \quad x \in H.$$

This inequality shows that both

$$\lambda I - A \quad \text{and} \quad \bar{\lambda} I - A^* = \bar{\lambda} I - A$$

are injective and $\text{Im}(\lambda I - A)$ is closed. By Proposition 2.1.27(iv) and Corollary 1.2.35,

$$\text{Im}(\lambda I - A) = [\text{Ker}(\lambda I - A)^*]^\perp = [\text{Ker}(\bar{\lambda} I - A)]^\perp = H.$$

Therefore $\lambda \in \varrho(A)$.

(v) We have

$$\lambda(x, y) = (Ax, y) = (x, Ay) = (x, \mu y) = \mu(x, y)$$

(by (iv), $\mu \in \mathbb{R}$). Since $\lambda \neq \mu$, we conclude that $(x, y) = 0$. $\square$

Theorem 2.2.16 (Hilbert–Schmidt). *Let H be a separable Hilbert space and A a self-adjoint compact operator. Then there exists an orthonormal basis $\{e_n\}_{n=1}^\infty$ where e_n are the eigenvectors of A .*

If

$$Ae_n = \lambda_n e_n \quad \text{and} \quad x = \sum_{n=1}^{\infty} (x, e_n) e_n,$$

then

$$Ax = \sum_{n=1}^{\infty} \lambda_n (x, e_n) e_n.$$

Proof. Let $\{\lambda_n\}_{n=1}^\infty$ be the sequence of all nonzero and pairwise distinct eigenvalues of A . Choose an orthonormal basis $e_1^{(k)}, \dots, e_{n_k}^{(k)}$ of

$$N_k := \text{Ker}(\lambda_k I - A).$$

Remember that $N_k \perp N_{k+1}$ (Proposition 2.2.15(v)). Let us align the collection

$$\bigcup_k \{e_1^{(k)}, \dots, e_{n_k}^{(k)}\}$$

into a sequence $\{e_1, e_2, \dots\}$. This sequence is an orthonormal basis of

$$H_1 := \overline{\text{Lin}\{e_1, e_2, \dots\}}.$$

If $H_1 = H$, the proof is complete. Assume therefore that $H \neq H_1$. The orthogonal complement $H_1^\perp$ is A -invariant. This means that the restriction

$$B := A|_{H_1^\perp}$$

is a self-adjoint operator on the Hilbert space $H_1^\perp$. Since $\sigma(B) \subset \sigma(A)$, $\sigma(B)$ cannot contain any nonzero number (Corollary 2.2.13(i)). As $\sigma(B) \neq \emptyset$, we have $\sigma(B) = \{0\}$ and, by Proposition 2.2.15(iii),

$$B = O \quad \text{on} \quad H_1^\perp.$$

Hence 0 is an eigenvalue of B as well as of A . By adding an orthonormal basis of $H_1^\perp$ to $\{e_1, e_2, \dots\}$ we obtain an orthonormal basis of H . $\square$

Example 2.2.17.²⁴ We have found that the inverse operator to

$$Ax = -(p\dot{x})' + qx,^{25} \quad x \in \text{Dom } A = \{x \in W^{2,2}(a, b) : x(a) = x(b) = 0\},$$

exists provided $\dot{p}, q \in C[a, b]$ and $p, q > 0$ on $[a, b]$. Moreover, A^{-1} is an integral operator

$$A^{-1}f(t) = \int_a^b G(t, s)f(s) \, ds$$

where G is the Green function of the differential expression. From the construction of G it follows that $G \in C([a, b] \times [a, b])$, in particular, $G \in L^2(a, b)$, and G is a real symmetric function ($G(t, s) = G(s, t)$), see, e.g., Walter [156].

By Example 2.2.5(ii), A^{-1} is a compact, self-adjoint²⁶ operator in the real space $L^2(a, b)$ and Theorem 2.2.16 can be applied to obtain an orthonormal basis of $L^2(a, b)$ formed by the eigenfunctions $\{e_n\}_{n=1}^\infty$ of A^{-1} , i.e., by the eigenfunctions of A . Since

$$(Ax, x)_{L^2(a, b)} = \int_a^b [p(t)\dot{x}^2(t) + q(t)|x(t)|^2] \, dt > 0 \quad \text{for all } x \in \text{Dom } A, x \neq 0,$$

all eigenvalues are positive. If λ is an eigenvalue of A (equivalently $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}), then

$$\dim \text{Ker} (\lambda I - A) = 1$$

since the equation

$$(p\dot{x})' + (q - \lambda)x = 0$$

cannot have two linearly independent solutions satisfying the initial condition $x(a) = 0$. Let the eigenvalues λ_n of A be arranged into a sequence so that

$$0 < \lambda_1 < \lambda_2 < \dots$$

From the properties of compact operators (Corollary 2.2.13) it follows that $\lambda_n \rightarrow \infty$. It is sometimes important to know how quickly λ_n tend to infinity. A simple

²⁴A continuation of Example 2.1.32.

²⁵This operator is called a *Sturm–Liouville operator*.

²⁶We restrict our attention to a special differential operator A in contrast to the general operator from Example 2.1.32 in order to get a self-adjoint inverse A^{-1} .

estimate can be obtained with help of the quantity $\|A^{-1}\|_2$ (Example 2.2.5(ii)), namely

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} = \|A^{-1}\|_2^2 < \infty$$

(see Exercise 2.2.22). However, this result is far from being optimal. We remark here that a variational approach to an eigenvalue problem for compact, self-adjoint operators will be briefly described in Section 7.8.

Consider now the equation

$$Ax = \lambda x + f \quad (2.2.5)$$

or, equivalently (cf. Exercise 2.2.21),

$$\sum_{n=1}^{\infty} (\lambda_n - \lambda)(x, e_n)e_n = \sum_{n=1}^{\infty} (f, e_n)e_n, \quad \text{i.e., } (\lambda_n - \lambda)(x, e_n) = (f, e_n) \quad \text{for } n \in \mathbb{N}.$$

If λ is no eigenvalue of A , then $\inf_n |\lambda_n - \lambda| > 0$ (since $\lambda_n \rightarrow \infty$) and

$$x = \sum_{n=1}^{\infty} \frac{(f, e_n)}{\lambda_n - \lambda} e_n$$

is a unique solution of (2.2.5). (Notice that this series is convergent.) If $\lambda = \lambda_n$, then the condition

$$(f, e_n) = 0$$

is a necessary and sufficient condition for solvability of (2.2.5) (see also Example 2.1.31).

If we examined singular differential operators, e.g., on the interval $[0, \infty)$, we would meet with many difficulties arising for example from the fact that A^{-1} is no more compact and, therefore, its spectrum is more complicated. The interested reader can consult the book Dunford & Schwartz [54]. $\circ$

Remark 2.2.18. The Hilbert–Schmidt Theorem allows us to introduce a functional calculus for compact, self-adjoint operators similarly as it has been done for matrices in Theorem 1.1.38:

Let A be a compact, self-adjoint operator on a Hilbert space H . Then there exists a unique mapping

$$\Phi: C(\sigma(A)) \rightarrow \mathcal{L}(H)^{27}$$

with the following properties:

- (i) Φ is an algebra homomorphism (i.e., Φ preserves operations on $C(\sigma(A))$ and $\mathcal{L}(H)$);

²⁷If $\sigma(A) = \{0\} \cup \{\lambda_n\}_{n=1}^{\infty}$, then $f \in C(\sigma(A))$ if and only if $\lim_{n \rightarrow \infty} f(\lambda_n) = f(0)$.

- (ii) Φ is a continuous mapping from $C(\sigma(A))$ into $\mathcal{L}(H)$ with the operator topology;
- (iii) if $P(x) = \sum_{k=0}^m a_k x^k$, then $\Phi(P) = \sum_{k=0}^m a_k A^k$;
- (iv) if $w \notin \sigma(A)$ and $f(x) = \frac{1}{w-x}$, then $\Phi(f) = (wI - A)^{-1}$;
- (v) $\sigma(\Phi(f)) = f(\sigma(A))$ for every $f \in C(\sigma(A))$.

If $Ax = \sum_{n=1}^{\infty} \lambda_n(x, e_n)e_n$, then it is easy to verify properties (i)–(v) for

$$\Phi(f)x := \sum_{n=1}^{\infty} f(\lambda_n)(x, e_n)e_n.$$

In particular if $A \in \mathcal{C}(H)$ then $|A|(x) := \sum_{n=1}^{\infty} |\lambda_n|(x, e_n)e_n$, where $\{\lambda_n^2\}_{n=1}^{\infty}$ is a sequence of eigenvalues of A^*A .

We omit the proof of uniqueness.

It is worth mentioning that we can introduce a functional calculus for a linear operator A which has a compact, self-adjoint resolvent $(\lambda_0 I - A)^{-1}$. We leave this easy construction to the interested reader. Example 2.2.17 shows a class of such operators.

Exercise 2.2.19. Consider a special case of the Sturm–Liouville operator

$$Ax = -\ddot{x}$$

in the space $L^2(0, \pi)$ with the boundary conditions

- (i) $x(0) = x(\pi) = 0$ (Dirichlet boundary conditions),
- (ii) $\dot{x}(0) = \dot{x}(\pi) = 0$ (Neumann boundary conditions),
- (iii) $\alpha_0 x(0) + \beta_0 \dot{x}(0) = 0$, $\alpha_1 x(\pi) + \beta_1 \dot{x}(\pi) = 0$ (mixed or Newton–Robin boundary conditions),
- (iv) $x(0) = x(\pi)$, $\dot{x}(0) = \dot{x}(\pi)$ (periodic conditions).

Find Green functions, eigenvalues and eigenfunctions. What follows from the Hilbert–Schmidt Theorem? Compare this result with that of Example 1.2.38(i).

Exercise 2.2.20. Define e^{tA} for the operator A from Exercise 2.2.19 (see Remark 2.2.18). Take $x \in \text{Dom } A$ and show that the function

$$u(t, \xi) := (e^{tA}x)(\xi), \quad t \geq 0,$$

is a solution to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \xi^2}$$

satisfying the initial condition $u(0, \cdot) = x(\cdot)$ and the boundary conditions given by $u(t, \cdot) \in \text{Dom } A$. Do not forget to define the notion of a solution.

Exercise 2.2.21. Let A be as in Example 2.2.17. Prove that

$$\text{Dom } A = \left\{ x = \sum_{n=1}^{\infty} (x, e_n) e_n : \sum_{n=1}^{\infty} |\lambda_n|^2 |(x, e_n)|^2 < \infty \right\}$$

and

$$Ax = \sum_{n=1}^{\infty} \lambda_n (x, e_n) e_n.$$

Exercise 2.2.22. Denote $\mathcal{L}_2(H)$ the set of all Hilbert-Schmidt operators on a Hilbert space H . Prove that

- (i) $\mathcal{L}_2(H)$ is an *ideal* in $\mathcal{L}(H)$, i.e., for $A \in \mathcal{L}(H)$, $B \in \mathcal{L}_2(H)$ we have AB and BA belong to $\mathcal{L}_2(H)$;
- (ii) $A \in \mathcal{L}_2(H) \Leftrightarrow |A| := (A^*A)^{\frac{1}{2}} \in \mathcal{L}_2(H)$. (For the definition of $|A|$ see Remark 2.2.18). If $\lambda_1, \dots$ are eigenvalues of $|A|$ then $\|A\|_2^2 = \sum_n \lambda_n^2$. For more information see, e.g., Schatten [137].

Exercise 2.2.23. Suppose that $A \in \mathcal{L}(X, Y)$ maps a weakly convergent sequence into a strongly convergent one. Prove that A is compact provided X is reflexive.

Exercise 2.2.24. Prove the assertion from Remark 2.2.10 and the decomposition (2.2.4).

2.3 Contraction Principle

The previous four sections have been devoted to some basic facts in the linear theory. It is now time to start with nonlinear problems, especially with the solution of the nonlinear equation

$$f(x) = a \quad \text{for} \quad f: X \rightarrow X. \quad (2.3.1)$$

The basic assertions in this section are fixed point theorems for contractible and non-expansive mappings. If X is a linear space, (2.3.1) is equivalent to the equation

$$F(x) := a - f(x) + x = x.$$

The solution of this equation is called a *fixed point* of F . In the case that

$$f(x) = x - Ax \quad (F(x) = Ax + a)$$

where $A \in \mathcal{L}(X)$, we succeeded in solving this equation in Section 2.1 (cf. Proposition 2.1.2) by applying the iteration process

$$x_0 = a, \quad x_n = a + Ax_{n-1} \quad \text{provided} \quad \|A\| < 1.$$

This idea can be easily generalized to the following result which is often attributed to S. Banach. Notice that in the forthcoming assertion we do not need the linear structure of the space at all.

Theorem 2.3.1 (Contraction Principle). *Let M be a complete metric space and let $F: M \rightarrow M$ be a contraction, i.e., there is $q \in [0, 1)$ such that*

$$\varrho(F(x), F(y)) \leq q\varrho(x, y) \quad \text{for every } x, y \in M.$$

Then there exists a unique fixed point $\tilde{x}$ of F in M . Moreover, if

$$x_0 \in M, \quad x_n = F(x_{n-1}),$$

then the sequence $\{x_n\}_{n=1}^\infty$ converges to $\tilde{x}$ and the estimates

$$\varrho(x_n, \tilde{x}) \leq \frac{q^n}{1-q} \varrho(x_1, x_0) \quad (\text{a priori estimate}), \quad (2.3.2)$$

$$\varrho(x_n, \tilde{x}) \leq \frac{q}{1-q} \varrho(x_n, x_{n-1}) \quad (\text{a posteriori estimate}) \quad (2.3.3)$$

hold.

Proof. We prove that $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence. Indeed, for $m > n$ we have

$$\begin{aligned} \varrho(x_m, x_n) &\leq \varrho(x_m, x_{m-1}) + \cdots + \varrho(x_{n+1}, x_n) \\ &= \varrho(F(x_{m-1}), F(x_{m-2})) + \cdots + \varrho(F(x_n), F(x_{n-1})) \\ &\leq q[\varrho(x_{m-1}, x_{m-2}) + \cdots + \varrho(x_n, x_{n-1})] \\ &\leq (q^{m-1} + \cdots + q^n) \varrho(x_1, x_0) \leq \frac{q^n}{1-q} \varrho(x_1, x_0). \end{aligned}$$

Since $q < 1$, the right-hand side is arbitrarily small for sufficiently large n . The Cauchy sequence $\{x_n\}_{n=1}^\infty$ has a limit $\tilde{x}$ in the complete space M , and for this limit the estimate (2.3.2) holds. Being a contraction, F is a continuous mapping, and therefore

$$\tilde{x} = \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} F(x_{n-1}) = F\left(\lim_{n \rightarrow \infty} x_{n-1}\right) = F(\tilde{x}).$$

Uniqueness of a fixed point is even easier: If $\tilde{x} = F(\tilde{x})$, $\tilde{y} = F(\tilde{y})$, then

$$\varrho(\tilde{x}, \tilde{y}) = \varrho(F(\tilde{x}), F(\tilde{y})) \leq q\varrho(\tilde{x}, \tilde{y}), \quad \text{i.e.,} \quad \varrho(\tilde{x}, \tilde{y}) = 0 \quad (q < 1).$$

The a posteriori estimate also follows from the above estimate of $\varrho(x_m, x_n)$. $\square$

The fixed point of F the existence of which has been just established often depends on a parameter. The following result is useful in investigating this dependence.

Corollary 2.3.2. *Let M be a complete metric space and A a topological space. Assume that $F: A \times M \rightarrow M$ possesses the following properties:*

(i) *There is $q \in [0, 1)$ such that*

$$\varrho(F(a, x), F(a, y)) \leq q\varrho(x, y) \quad \text{for all } a \in A \quad \text{and } x, y \in M.$$

(ii) For every $x \in M$ the mapping $a \mapsto F(a, x)$ is continuous on A .

Then for each $a \in A$ there is a unique $\varphi(a) := \tilde{x}$ such that

$$F(a, \tilde{x}) = \tilde{x}.$$

Moreover, φ is continuous on A .

Proof. The existence of φ follows directly from Theorem 2.3.1. The estimates

$$\begin{aligned} \varrho(\varphi(a), \varphi(b)) &= \varrho(F(a, \varphi(a)), F(b, \varphi(b))) \\ &\leq \varrho(F(a, \varphi(a)), F(b, \varphi(a))) + \varrho(F(b, \varphi(a)), F(b, \varphi(b))) \\ &\leq \varrho(F(a, \varphi(a)), F(b, \varphi(a))) + q\varrho(\varphi(a), \varphi(b)) \end{aligned}$$

yield

$$\varrho(\varphi(a), \varphi(b)) \leq \frac{1}{1-q} \varrho(F(a, \varphi(a)), F(b, \varphi(a))),$$

and the continuity of φ follows. $\square$

Remark 2.3.3. Notice that φ is Lipschitz continuous provided $a \mapsto F(a, x)$ is Lipschitz continuous uniformly with respect to x (and, of course, A is a metric space).

There is an enormous amount of applications of the Contraction Principle. The proof of the existence theorem for the initial value problem for ordinary differential equations belongs to standard applications. However, the historical development went in the opposite direction. The following theorem had been proved (by iteration) about thirty years before the Contraction Principle was formulated in its full generality. Another application will be given in Section 4.1.

Theorem 2.3.4 (Picard). Let $\mathcal{G}$ be an open set in $\mathbb{R} \times \mathbb{R}^N$ and let $f: (t, x_1, \dots, x_N) \in \mathcal{G} \rightarrow \mathbb{R}^N$ be continuous and locally Lipschitz continuous with respect to the x -variables, i.e., for every $(s, y) \in \mathcal{G}$ there exist $\delta > 0$, $\hat{\delta} > 0$, $L > 0$ such that

$$\|f(t, x_1) - f(t, x_2)\| \leq L\|x_1 - x_2\| \quad \text{whenever } |t - s| < \delta, \|x_i - y\| < \hat{\delta}, i = 1, 2.$$

Then for any $(t_0, \xi_0) \in \mathcal{G}$ there exists $\delta > 0$ such that the equation

$$\dot{x} = f(t, x) \tag{2.3.4}$$

has a unique solution on the interval $(t_0 - \delta, t_0 + \delta)$ satisfying the initial condition

$$x(t_0) = \xi_0. \tag{2.3.5}$$

Proof. First we rewrite the initial value problem (2.3.4), (2.3.5) into an equivalent fixed point problem for an integral operator F defined by

$$F(x): t \mapsto \xi_0 + \int_{t_0}^t f(s, x(s)) ds, \quad t \in (t_0 - \delta, t_0 + \delta).^{28} \tag{2.3.6}$$

This equivalence is easy to establish (by integration and by differentiation with respect to t). Therefore we wish to solve the equation

$$F(x) = x$$

in a complete metric space M . We choose M to be a closed subset of the Banach space $C[t_0 - \delta, t_0 + \delta]$ for a certain small $\delta > 0$.

We need two properties of F and M , namely that F maps M into M and F is a contraction on M . Choose first $\delta_1, \hat{\delta}_1$ such that

$$\mathcal{R}_1 := [t_0 - \delta_1, t_0 + \delta_1] \times \{x \in \mathbb{R}^N : \|x - \xi_0\| \leq \hat{\delta}_1\} \subset \mathcal{G}.$$

This set $\mathcal{R}_1$ is compact, and therefore f is bounded and uniformly Lipschitz continuous on it, i.e., there are constants K, L such that

$$\|f(s, x)\| \leq K, \quad \|f(s, x) - f(s, y)\| \leq L\|x - y\| \quad \text{for } (s, x), (s, y) \in \mathcal{R}_1.$$

Put

$$M = \{x \in C[t_0 - \delta, t_0 + \delta] : \|x(t) - \xi_0\| \leq \hat{\delta}_1 \forall t \in [t_0 - \delta, t_0 + \delta]\} \text{ for a } \delta \leq \delta_1.$$

Then

$$\sup_{t \in \mathcal{I}_\delta} \|F(x(t)) - \xi_0\| \leq \delta K, \quad \sup_{t \in \mathcal{I}_\delta} \|F(x(t)) - F(y(t))\| \leq \delta L \sup_{t \in \mathcal{I}_\delta} \|x(t) - y(t)\|$$

where

$$\mathcal{I}_\delta := [t_0 - \delta, t_0 + \delta].$$

If we choose δ so small that $\delta K \leq \hat{\delta}_1$ and $\delta L \leq \frac{1}{2}$, then F maps M into itself (the first condition) and is a contraction with $q = \frac{1}{2}$ (the second condition). By the Contraction Principle, F has a unique fixed point y in M and this is a solution of (2.3.4), (2.3.5) on the interval $(t_0 - \delta, t_0 + \delta)$. If $\tilde{x}$ is a solution of (2.3.4), (2.3.5) on the interval $(t_0 - \delta, t_0 + \delta)$, then $\tilde{x} \in M$ (prove it!), i.e., $y = \tilde{x}$, and the uniqueness follows. $\square$

Remark 2.3.5. The mapping F defined by (2.3.6) depends actually not only on x but also on t_0, ξ_0 . By taking smaller δ we can prove that F is also Lipschitz continuous with respect to the initial conditions and Corollary 2.3.2 yields that the solution $x(t; \cdot, \cdot)$ of (2.3.4), (2.3.5) is also Lipschitz continuous with respect to the initial conditions. Moreover, if $f \in C^1(\mathcal{G})$ then the solution $x(t; \cdot, \cdot)$ is also continuously differentiable with respect to the initial conditions. We can use for $F(x; t_0, \xi_0)$ the Implicit Function Theorem to obtain that $x(t; t_0, \xi_0)$ is differentiable with respect to all variables (Example 4.2.5).

²⁸If $t < t_0$, then we define $\int_{t_0}^t f(s, x(s)) ds = -\int_t^{t_0} f(s, x(s)) ds$, and $\int_{t_0}^{t_0} f(s, x(s)) ds = 0$.

Remark 2.3.6. A simple example $\dot{x} = x^2$ shows that solutions need not be global. On the other hand, if $\mathcal{G} = (\alpha, \beta) \times \mathbb{R}^N$ and the right-hand side f is *sublinear*, i.e., there are nonnegative functions $a \in L^1_{\text{loc}}(\alpha, \beta)$, $b \in C[\alpha, \beta]$ such that

$$\|f(t, x)\| \leq a(t) + b(t)\|x\|, \quad (t, x) \in (\alpha, \beta) \times \mathbb{R}^N, \quad (2.3.7)$$

then for any initial condition the equation (2.3.4) has a unique solution on (α, β) . To prove this fact we have to show that the local solution x given by Theorem 2.3.4 cannot "explode" inside (α, β) and, therefore, it can be continued to the entire interval (α, β) . This method is based on the estimate of x which follows from the Gronwall inequality (see Proposition 2.3.9 below):

$$\begin{aligned} \|x(t)\| &\leq \|\xi_0\| + \int_{t_0}^t \|f(s, x(s))\| ds \leq \|\xi_0\| + \int_{t_0}^t a(s) ds \\ &\quad + \int_{t_0}^t b(s)\|x(s)\| ds, \quad t \geq t_0. \end{aligned}$$

By this inequality,

$$\|x(t)\| \leq A(t) + \int_{t_0}^t A(s)b(s)e^{\int_s^t b(\sigma)d\sigma},$$

where $A(t) = \|\xi_0\| + \int_{t_0}^t a(s)ds$. If x were defined on $[t_0, \tau)$, $\tau < \beta$, then the last inequality shows that x is bounded and therefore $t \mapsto f(t, x(t))$ is bounded on $[t_0, \tau)$ as well. Hence, x is uniformly continuous on this interval and, in particular, the limit $\lim_{t \rightarrow \tau-} x(t) = \eta$ exists and is finite. So, we can "glue" together the solution x with a solution satisfying the initial condition $x(\tau) = \eta$.

We emphasize the special case of a system of linear differential equations

$$\dot{x} = \mathbf{A}(t)x + g(t)$$

with a continuous matrix $\mathbf{A}$ and a continuous vector function g on some interval (α, β) that satisfies the estimate (2.3.7). However, in this case we can avoid the use of Contraction Principle at all and prove the convergence of iterations directly. The following example demonstrates this approach.

Example 2.3.7. Let k be a bounded measurable function on the set

$$\mathcal{M} = \{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq 1\}.$$

Then for any $f \in L^1(0, 1)$ and $\lambda \neq 0$ there is a unique solution to the integral equation

$$x(t) - \lambda \int_0^t k(t, s)x(s) \, ds = f(t). \quad (2.3.8)$$

To prove this assertion, denote

$$Ax(t) = \int_0^t k(t, s)x(s) \, ds.$$

Then $A \in \mathcal{L}(L^1(0, 1))$ (Example 2.1.28). Put

$$x_0 = f, \quad x_n = f + \lambda Ax_{n-1}.$$

Due to the completeness of $L^1(0, 1)$ the sequence $\{x_n\}_{n=1}^\infty$ is convergent in $L^1(0, 1)$ provided the sum $\sum_{n=1}^\infty \|x_n - x_{n-1}\|_{L^1(0,1)}$ is convergent. We have

$$x_n - x_{n-1} = \lambda^n A^n f \quad \text{and} \quad A^n f(t) = \int_0^t k_n(t, s)f(s) \, ds$$

where

$$k_1 = k \quad \text{and} \quad k_n(t, s) = \int_s^t k_{n-1}(t, \sigma)k(\sigma, s) \, d\sigma$$

(check this relation). It is easy to prove by induction that

$$|k_n(t, s)| \leq \|k\|_{L^\infty(\mathcal{M})}^n \frac{(t-s)^{n-1}}{(n-1)!}, \quad (t, s) \in \mathcal{M},$$

and hence

$$\begin{aligned} \|x_n - x_{n-1}\|_{L^1(0,1)} &\leq |\lambda|^n \int_0^1 \left| \int_0^t k_n(t, s)f(s) \, ds \right| dt \\ &\leq |\lambda|^n \int_0^1 |f(s)| \left[\int_s^1 |k_n(t, s)| \, dt \right] ds \leq \frac{|\lambda|^n \|k\|^n}{n!} \|f\|_{L^1(0,1)}. \end{aligned}$$

Since the series $\sum_{n=1}^\infty \frac{a^n}{n!}$ is convergent for any $a \in \mathbb{R}$ the limit $\lim_{n \rightarrow \infty} x_n = \tilde{x} \in L^1(0, 1)$ exists and $\tilde{x}$ is a solution to (2.3.8). In fact $\tilde{x}$ is a unique solution (see Exercise 2.3.21). Moreover, $\tilde{x}$ depends continuously on f , which means that $\sigma(A) = \{0\}$.²⁹ This result holds also for $k \in C(\mathcal{M})$ in the space $C[0, 1]$. The proof is the same. $\square$

²⁹Actually, we have proved that $\mathbb{C} \setminus \{0\} \subset \varrho(A)$, i.e., $\sigma(A) \subset \{0\}$. Since $A \in \mathcal{L}(L^1(0, 1))$, $\sigma(A) \neq \emptyset$, we have $\sigma(A) = \{0\}$.

Example 2.3.8. Assume $E \in C^1(\mathbb{R}^N)$, $\lim_{\|x\| \rightarrow \infty} E(x) = \infty$, and denote the vector of its partial derivatives at x as $\nabla E(x)$ (gradient of E). Let $f \in C([0, T]; \mathbb{R}^N)$. Consider the following *gradient system*

$$\dot{x}(t) + \nabla E(x(t)) = f(t), \quad x(0) = x_0. \quad (2.3.9)$$

This system need not satisfy the assumptions of Theorem 2.3.4 but the weaker existence theorem (Proposition 5.2.7 below) can be used to yield a maximal solution to (2.3.9) on certain interval $[0, \tau)$. We show that this solution is actually global, i.e., $\tau = T$. To this end assume $\tau < T$ and multiply (2.3.9) by $\dot{x}$ and integrate over $[0, t)$:

$$\begin{aligned} \int_0^t \|\dot{x}(s)\|^2 ds + \int_0^t (\nabla E(x(s)), \dot{x}(s)) ds &= \int_0^t (f(s), \dot{x}(s)) ds \\ &\leq \frac{1}{2} \int_0^t \|f(s)\|^2 ds + \frac{1}{2} \int_0^t \|\dot{x}(s)\|^2 ds. \end{aligned}$$

Since

$$(\nabla E(x(s)), \dot{x}(s)) = \frac{d}{ds} E(x(s)),$$

we have

$$E(x(t)) \leq \frac{1}{2} \int_0^t \|\dot{x}(s)\|^2 ds + E(x(0)) \leq E(x_0) + \frac{1}{2} \int_0^\tau \|f(s)\|^2 ds.$$

This inequality shows that x is bounded on $[0, \tau)$ due to the assumption

$$\lim_{\|x\| \rightarrow \infty} E(x) = \infty.$$

Similarly as in Remark 2.3.6 we deduce that there exists a finite $\lim_{t \rightarrow \tau-} x(t)$ and x can be continued as a solution beyond τ . But this contradicts the maximality of x . $\circ$

Proposition 2.3.9 (Gronwall Lemma). Let $a \in C^1[\alpha, \beta)$, $b \in C[\alpha, \beta)$ and let $b \geq 0$ on the interval $[\alpha, \beta)$. If z is a continuous function satisfying the inequality

$$z(t) \leq a(t) + \int_\alpha^t b(s)z(s)ds, \quad t \in [\alpha, \beta) \quad (2.3.10)$$

then

$$z(t) \leq a(t) + \int_\alpha^t a(s)b(s)e^{\int_s^t b(\xi)d\xi} ds \quad (2.3.11)$$

for $t \in [\alpha, \beta)$.

Proof. Denote the right-hand side in (2.3.10) as $y(t)$. Then

$$\dot{y}(t) = \dot{a}(t) + b(t)z(t) \leq \dot{a}(t) + b(t)y(t).$$

Multiplying by $e^{-B(t)}$, where $B(t) = \int_{\alpha}^t b(s)ds$, we obtain

$$\left[y(t)e^{-B(t)} \right]' \leq \dot{a}(t)e^{-B(t)} = \left[\int_{\alpha}^t \dot{a}(s)e^{-B(s)}ds \right]'$$

Therefore $y(\alpha) \geq y(t)e^{-B(t)} - \int_{\alpha}^t \dot{a}(s)e^{-B(s)}ds$. Integrating by parts we get (2.3.11). $\square$

Using this proposition for $z(t) := \|x(t)\|$, where x is a local solution of the linear equation $\dot{x} = \mathbf{A}(t)x + g(t)$, we find a function φ which depends only on $\mathbf{A}$ and g and it is defined on the interval (a, b) such that the inequality $z(t) \leq \varphi(t)$ holds on the domain of a solution x . This means that the continuation process can be prolonged (by the uniform continuity of x) to the whole interval (a, b) . Similarly, this continuation process works for the equation (2.3.4) provided f is globally Lipschitz continuous with respect to the x -variables and $\mathcal{G} = \mathbb{R} \times \mathbb{R}^N$.

Example 2.3.10. Find sufficient conditions for the existence of a classical solution (cf. Example 2.1.31) of the boundary value problem

$$\begin{cases} \ddot{x}(t) = \varepsilon f(t, x(t)), & t \in (0, 1), \\ x(0) = x(1) = 0. \end{cases} \quad (2.3.12)$$

Theorem 2.3.4 suggests the assumption that f is continuous with respect to t and Lipschitz continuous with respect to the x -variable on a certain rectangle $[0, 1] \times [-r, r]$. Denote a Lipschitz constant on this interval by $L(r)$. We wish to rewrite the problem (2.3.12) as a fixed point problem. To reach this goal suppose that we have a solution y and let $g(t) := \varepsilon f(t, y(t))$. Then y solves also the equation

$$\ddot{y} = g$$

and satisfies $y(0) = y(1) = 0$.

It is easy to see that this problem has exactly one solution which is given by

$$y(t) = \int_0^1 G(t, s)g(s)ds := \int_0^t (t-1)s g(s)ds + \int_t^1 t(s-1)g(s)ds$$

(G is the Green function – see Example 2.1.32). Therefore, we are looking for a continuous function x which solves the integral equation

$$x(t) = \varepsilon \int_0^1 G(t, s)f(s, x(s))ds. \quad (2.3.13)$$

Denote

$$F(\varepsilon, x) := \varepsilon \int_0^1 G(t, s) f(s, x(s)) \, ds.$$

We can solve (2.3.13) by applying the Contraction Principle in

$$M := \{x \in C[0, 1] : \|x\| \leq r\} \quad \text{for an appropriate choice of } r.$$

For $x \in M$ we have

$$|f(s, x(s))| \leq |f(s, 0)| + |f(s, x(s)) - f(s, 0)| \leq K + L(r)r$$

where $K > 0$ is a constant such that $|f(s, 0)| \leq K$, $s \in [0, 1]$, and

$$\|F(\varepsilon, x)\| \leq \frac{|\varepsilon|}{8}(K + L(r)r).$$

This estimate shows that F maps M into itself if

$$q := \frac{|\varepsilon|}{8}L(r) < 1 \quad \text{and} \quad r \geq \frac{|\varepsilon|K}{8} \frac{1}{1-q}.$$

Then F is also a contraction on M with the constant q . We can conclude that for a given r there is $\varepsilon_0 > 0$ such that for $|\varepsilon| \leq \varepsilon_0$ both the above conditions³⁰ are satisfied and (2.3.13) has a solution.

Now we have to show that a continuous solution x of (2.3.13) is actually a classical solution of the boundary value problem (2.3.12). Since we know the explicit form of the Green function G , it is obvious that $x(0) = x(1) = 0$ and it is also easy to differentiate twice the right-hand side of (2.3.13) (taking into account that x is continuous).

We remark that we have not used all properties of the integral operator with the kernel G . In particular, such an operator is compact (Example 2.2.5(i)) and this property has not been used. This property will be significant in Chapter 5. $\circ$

The a posteriori estimate (2.3.3) shows that the convergence of iterations may be rather slow. It can be sometimes desirable to have faster convergence at the expense of more restrictive assumptions. The classical Newton Method for solving an equation

$$f(x) = 0, \quad f: \mathbb{R} \rightarrow \mathbb{R},$$

is illustrated in [Figure 2.3.1](#).

In order to generalize this method we need the notion of a derivative of $f: X \rightarrow X$. This will be the main subject of the next chapter.

³⁰Notice that for a fixed ε these conditions are antagonistic, namely the first requires small r and the other large r . This situation is typical in applications of the Contraction Principle.

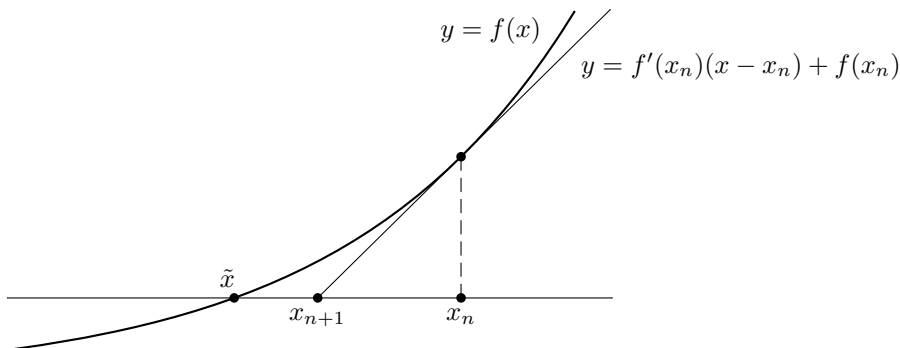


Figure 2.3.1.

There are many generalizations of the Contraction Principle. One of them concerns the assumption $q < 1$. A mapping $F: M \rightarrow M$ is called *non-expansive* if

$$\varrho(F(x), F(y)) \leq \varrho(x, y) \quad \text{for all } x, y \in M.$$

A simple example $F(x) = x + 1$, $x \in \mathbb{R}$, shows that F may have no fixed point. This can be caused by the fact that F does not map any bounded set into itself. However, there are non-expansive mappings which map the unit ball into itself and do not possess any fixed point either. See the following example or Exercise 2.3.20.

Example 2.3.11 (Beals). Let M be the space of all sequences with zero limit with the sup norm (this space is usually denoted by c_0) and let

$$F(x) = (1, x_1, x_2, \dots) \quad \text{for } x = (x_1, x_2, \dots) \in M.$$

Then F is a non-expansive map of the unit ball into itself without any fixed point. $\circ$

This example indicates that some special properties of the space are needed. We formulate the following assertion in a Hilbert space and use the Hilbert structure essentially in its proof. The statement is true also in uniformly convex spaces but the proof is more involved (see, e.g., Goebel [72]). Let us note an interesting fact that the validity of Proposition 2.3.12 in a reflexive Banach space is an open problem.

Proposition 2.3.12 (Browder). Let $\mathcal{M}$ be a bounded closed and convex set in a Hilbert space H . Let F be a non-expansive mapping from $\mathcal{M}$ into itself. Then there is a fixed point of F in $\mathcal{M}$. Moreover, if

$$x_0 \in \mathcal{M}, \quad x_n = F(x_{n-1}) \quad \text{and} \quad y_n = \frac{1}{n} \sum_{k=0}^{n-1} x_k,$$

then the sequence $\{y_n\}_{n=1}^{\infty}$ is weakly convergent to a fixed point.

Proof. The existence result is not difficult to prove.³¹ So we will prove a more interesting result which yields also a numerical method for finding a fixed point. The proof consists of four steps, the last one is crucial and has a variational character.

Step 1. Since $\mathcal{M}$ is bounded, closed and convex, $y_n \in \mathcal{M}$ and there is a subsequence $\{y_{n_k}\}_{k=1}^\infty$ weakly convergent to an $\tilde{x} \in \mathcal{M}$ (Theorem 2.1.25 and Exercise 2.1.40). Fix such a weakly convergent subsequence $\{y_{n_k}\}_{k=1}^\infty$ and its weak limit $\tilde{x} \in \mathcal{M}$.

Step 2. We have $\lim_{n \rightarrow \infty} \|F(y_n) - y_n\| = 0$. Indeed,

$$\begin{aligned} \|F^k(x_0) - F(y_n) + F(y_n) - y_n\|^2 &= \|F^k(x_0) - F(y_n)\|^2 + \|F(y_n) - y_n\|^2 \\ &\quad + 2 \operatorname{Re}(F^k(x_0) - F(y_n), F(y_n) - y_n) \end{aligned}$$

where

$$F^k(x_0) = F(F^{k-1}(x_0)).$$

Summing up this equality from $k = 0$ to $k = n - 1$ and dividing by n we get

$$\begin{aligned} \frac{1}{n} \sum_{k=0}^{n-1} \|F^k(x_0) - y_n\|^2 &= \frac{1}{n} \sum_{k=0}^{n-1} (\|F^k(x_0) - F(y_n)\|^2 \\ &\quad + \|F(y_n) - y_n\|^2 + 2 \operatorname{Re}(y_n - F(y_n), F(y_n) - y_n)) \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \|F^k(x_0) - F(y_n)\|^2 - \|F(y_n) - y_n\|^2. \end{aligned}$$

Since F is non-expansive, we conclude from this equality that

$$\begin{aligned} \|F(y_n) - y_n\|^2 &\leq \frac{1}{n} \sum_{k=1}^{n-1} \|F^{k-1}(x_0) - y_n\|^2 + \frac{1}{n} \|x_0 - F(y_n)\|^2 \\ &\quad - \frac{1}{n} \sum_{k=0}^{n-1} \|F^k(x_0) - y_n\|^2 \\ &= \frac{1}{n} \|x_0 - F(y_n)\|^2 - \frac{1}{n} \|F^{n-1}(x_0) - y_n\|^2 - \frac{1}{n} \|x_0 - y_n\|^2 \rightarrow 0 \end{aligned}$$

(all sequences belong to $\mathcal{M}$, and hence they are bounded).

Step 3. The element $\tilde{x}$ is a fixed point of F . To see this, observe that the inequality

$$\begin{aligned} (z - F(z) - (y_{n_k} - F(y_{n_k})), z - y_{n_k}) \\ = (z - y_{n_k}, z - y_{n_k}) - (F(z) - F(y_{n_k}), z - y_{n_k}) \geq \|z - y_{n_k}\|^2 - \|z - y_{n_k}\|^2 = 0 \end{aligned}$$

³¹It is possible to assume that $o \in \mathcal{M}$. For any $t \in (0, 1)$ the mapping $F_t(x) := tF(x)$ is a contraction. Letting $t \rightarrow 1$ we obtain a sequence $\{x_n\}_{n=1}^\infty \subset \mathcal{M}$ for which $x_n - F(x_n) \rightarrow o$. Therefore it is sufficient to show that $(I - F)(\mathcal{M})$ is closed. This needs a trick which is typical for monotone operators (Section 6.1). Notice that $I - F$ is monotone provided F is non-expansive.

holds for any $z \in \mathcal{M}$. By Exercise 2.1.37 and Step 2, the limit of the left-hand side is $(z - F(z), z - \tilde{x})$, i.e., the inequality

$$(z - F(z), z - \tilde{x}) \geq 0 \quad (2.3.14)$$

is also true. Now take $t \in (0, 1)$ and put

$$z = (1 - t)\tilde{x} + tF(\tilde{x}) \quad (z \in \mathcal{M}).$$

For $t \rightarrow 0$, the inequality (2.3.14) divided by t yields

$$\|\tilde{x} - F(\tilde{x})\|^2 \leq 0.$$

Step 4. If x is a fixed point of F , then

$$\|x_n - x\|^2 = \|F(x_{n-1}) - F(x)\|^2 \leq \|x_{n-1} - x\|^2$$

and, therefore, the limit $\varphi(x) := \lim_{n \rightarrow \infty} \|x_n - x\|^2$ exists. By Step 3, $\tilde{x}$ is also a fixed point, and we get

$$\varphi(\tilde{x}) \leq \|\tilde{x} - x_k\|^2 = \|\tilde{x} - v\|^2 + \|v - x_k\|^2 + 2\operatorname{Re}(\tilde{x} - v, v - x_k) \quad \text{for any } v \in H.$$

Summing up from $k = 0$ to $k = n - 1$ and dividing by n we arrive at

$$\varphi(\tilde{x}) \leq \|\tilde{x} - v\|^2 + \frac{1}{n} \sum_{k=0}^{n-1} \|v - x_k\|^2 + 2\operatorname{Re}(\tilde{x} - v, v - y_n). \quad (2.3.15)$$

Let v be a weak limit of a subsequence $\{y_{n_l}\}_{l=1}^\infty \subset \{y_n\}_{n=1}^\infty$, possibly different from $\{y_{n_k}\}_{k=1}^\infty$. Then v is a fixed point of F by virtue of the previous steps. Set $n = n_k$ and take the limit for $k \rightarrow \infty$ in (2.3.15). We finally obtain³²

$$\varphi(\tilde{x}) \leq \varphi(v) - \|\tilde{x} - v\|^2,$$

and $v = \tilde{x}$ follows. In particular, the limit of any weakly convergent subsequence of $\{y_n\}_{n=1}^\infty$ coincides with $\tilde{x}$, and therefore the whole sequence $\{y_n\}_{n=1}^\infty$ weakly converges to $\tilde{x}$. $\square$

Remark 2.3.13. We have mentioned in footnote 31 on page 103 that $I - F$ is a monotone operator whenever F is non-expansive. The converse statement is not true even in $\mathbb{R}$. Consider, e.g., $F(x) = -2x$. Proposition 2.3.12 should be compared with Theorem 6.1.4.

Exercise 2.3.14. Let $\{a_n\}_{n=1}^\infty$ be a sequence. Prove that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n a_k = \lim_{n \rightarrow \infty} a_n$$

provided the right-hand side exists.

³²Observe that $\lim_{k \rightarrow \infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \|v - x_j\|^2 = \lim_{n \rightarrow \infty} \|v - x_n\|^2 = \varphi(v)$, cf. Exercise 2.3.14.

In the following three exercises we briefly show other modifications of the Contraction Principle.

Exercise 2.3.15. If M is a complete metric space, $F: M \rightarrow M$ and there is a function $V: M \rightarrow \mathbb{R}^+$ such that

$$V(F(x)) + \varrho(x, F(x)) \leq V(x), \quad x \in M, \quad (2.3.16)$$

then for arbitrary

$$x_0 \in M, \quad x_n = F(x_{n-1}),$$

the sequence $\{x_n\}_{n=1}^\infty$ is convergent in M to an $\tilde{x}$. Moreover, if the graph of F is closed in $M \times M$, then

$$F(\tilde{x}) = \tilde{x}.$$

Hint. Show that $\{V(x_n)\}_{n=1}^\infty$ is a decreasing sequence; this implies that $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence.

Remark 2.3.16. The condition (2.3.16) is suitable for a vector-valued mapping F and plays an important role in the game theory. For details see, e.g., Aubin & Ekeland [11, Chapter VI].

Exercise 2.3.17. Let M be a complete metric space and let $F: M \rightarrow M$. If there is $n \in \mathbb{N}$ such that F^n is a contraction, then F has a unique fixed point in M .

Hint. Let $\tilde{x}$ be a fixed point of

$$G := F^n, \quad \tilde{x} = \lim_{k \rightarrow \infty} G^k(x_0).$$

Estimate $\varrho(F(G^k(x_0)), G^k(x_0))$. It is possible to show that

$$\tilde{x} = \lim_{k \rightarrow \infty} F^k(x_0).$$

Remark 2.3.18. The power $n \in \mathbb{N}$ need not be the same for all $x, y \in M$, i.e., if there is $q \in [0, 1)$ such that for every $x, y \in M$ there exist $n(x), n(y) \in \mathbb{N}$ and

$$\varrho(F^{n(x)}(x), F^{n(y)}(y)) \leq q\varrho(x, y),$$

then F also has a unique fixed point (Sehgal [139]). The proof is similar to the previous one.

Exercise 2.3.19 (Edelstein). Let M be a compact metric space and let $F: M \rightarrow M$ satisfy the condition

$$\varrho(F(x), F(y)) < \varrho(x, y) \quad \text{for all } x, y \in M, x \neq y.$$

Then F has a unique fixed point in M .

Hint. Only existence has to be proved: By compactness there is a convergent subsequence $F^{n_k}(x_0) \rightarrow \tilde{x}$. Now show that the sequence

$$\alpha_n := \varrho(F^n(x_0), F^{n+1}(x_0))$$

is decreasing and

$$\lim_{n \rightarrow \infty} \alpha_n = \varrho(\tilde{x}, F(\tilde{x})) = \varrho(F(\tilde{x}), F^2(\tilde{x})), \quad \text{i.e.,} \quad F(\tilde{x}) = \tilde{x}.$$

Exercise 2.3.20. Let

$$\mathcal{K} = \{x \in C[0, 1] : 0 \leq x(t) \leq 1, x(0) = 0, x(1) = 1\}, \quad F: x(t) \mapsto tx(t).$$

Then $F(\mathcal{K}) \subset \mathcal{K}$, F is non-expansive and there is no fixed point of F in $\mathcal{K}$! Prove these facts and explain their relation to Proposition 2.3.12.

Exercise 2.3.21. Let $x \in L^1(0, 1)$ be a solution of

$$x(t) = \lambda \int_0^t k(t, s)x(s) \, ds$$

where λ and k are as in Example 2.3.7. Prove that $x = 0$ a.e. in $(0, 1)$.

Hint. First show that $x \in L^\infty(0, 1)$. From the equation we have

$$\|x\|_{L^\infty(0, t)} \leq |\lambda|t\|k\|_{L^\infty(\mathcal{M})}\|x\|_{L^\infty(0, t)}, \quad t \in (0, 1).$$

Now deduce $x = 0$ a.e. in $(0, 1)$.

Exercise 2.3.22. Prove Corollary 2.1.3 using Theorem 2.3.1.

Exercise 2.3.23. Let $f \in C[0, 1]$. Prove that there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$ the boundary value problem

$$\begin{cases} \ddot{x}(t) - x(t) + \varepsilon \arctan x(t) = f(t), & t \in (0, 1), \\ x(0) = x(1) = 0, \end{cases}$$

has a unique solution $x \in C^2[0, 1]$.

Exercise 2.3.24. Let K be a continuous real function on $[a, b] \times [a, b] \times \mathbb{R}$ and assume there exists a constant $N > 0$ such that for any $t, \tau \in [a, b]$, $z_1, z_2 \in \mathbb{R}$, we have

$$|K(t, \tau, z_1) - K(t, \tau, z_2)| \leq N|z_1 - z_2|.$$

Let $h \in C[a, b]$ be fixed and let $\lambda \in \mathbb{R}$ be such that

$$|\lambda| < \frac{1}{N(b-a)}.$$

Prove that the integral equation

$$x(t) = \lambda \int_a^b K(t, \tau, x(\tau)) \, d\tau + h(t)$$

has a unique solution $x \in C[a, b]$.

Exercise 2.3.25. Let $\mathbf{A}: (a, b) \rightarrow \mathbb{R}^{M \times M}$ be a continuous matrix-valued function and let $\alpha \in (a, b)$, $\xi \in \mathbb{R}^M$.

- (i) Modify the procedure from Example 2.3.7 to prove that the initial value problem

$$\begin{cases} \dot{x}(t) = \mathbf{A}(t)x(t), \\ x(\alpha) = \xi, \end{cases}$$

has a unique solution which is defined on (a, b) .

- (ii) Prove that the equation

$$\dot{x}(t) = \mathbf{A}(t)x(t) \quad (2.3.17)$$

has M linearly independent solutions $\varphi^1, \dots, \varphi^M$ on the interval (a, b) and any solution of (2.3.17) is a linear combination of $\varphi^1, \dots, \varphi^M$. The matrix $\Phi = (\varphi_i^j)_{i,j=1,\dots,M}$ is called a fundamental matrix of (2.3.17).

- (iii) Let $\mathbf{A}$ be continuous on $\mathbb{R}$ and T -periodic ($T > 0$). Denote $\mathbf{C} = \Phi(T)$ where Φ is a fundamental matrix, $\Phi(0) = \mathbf{I}$. Suppose that $\mathbf{B}$ is a solution of the equation

$$e^{TB} = \mathbf{C}$$

(see Exercise 1.1.43). Prove that

$$\mathbf{Q}(t) := \Phi(t)e^{-tB}$$

is regular for all $t \in \mathbb{R}$ and T -periodic. Moreover, x is a solution to (2.3.17) if and only if

$$y(t) := \mathbf{Q}^{-1}(t)x(t)$$

is a solution of the equation

$$\dot{y} = \mathbf{B}y$$

which has constant coefficients. Find a condition in terms of $\sigma(\mathbf{C})$ for the existence of a nontrivial kT -periodic solution to (2.3.17) ($k \in \mathbb{N}$).

- (iv) Let $f: \mathbb{R} \rightarrow \mathbb{R}^M$ be a continuous and T -periodic mapping. Is there any relation between the existence of a nontrivial T -periodic solution to (2.3.17) and the existence of a T -periodic solution to the equation

$$\dot{x}(t) = \mathbf{A}(t)x(t) + f(t)?$$

Hint. Use the Variation of Constant Formula and (iii).

Exercise 2.3.26. Which of equations $\dot{x} = \pm x^3$ agrees with Example 2.3.8.

Exercise 2.3.27. If E from Example 2.3.8 is convex on $\mathbb{R}^N$ then the problem (2.3.9) has a unique solution on $[0, T]$. Prove this statement.

Hint. Write the equation which satisfies the difference of possible two solutions, $z(t) = x_1(t) - x_2(t)$. Multiply this equation by z . Use the convexity of E to show that $\|z(\cdot)\|$ is decreasing.



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