# Propagation Through Trapped Sets and Semiclassical Resolvent Estimates

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Let $P = -h^2\Delta + V(x)$, $V \in C_0^\infty(\mathbb{R}^n)$. We are interested in semiclassical resolvent estimates of the form

$$\|\chi(P - E - i0)^{-1}\chi\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le \frac{a(h)}{h}, \qquad h \in (0, h_0], \tag{1}$$

for $E > 0$, $\chi \in C^\infty(\mathbb{R}^n)$ with $|\chi(x)| \le \langle x\rangle^{-s}$, $s > 1/2$. We ask: how is the function $a(h)$ for which (1) holds affected by the relationship between the support of $\chi$ and the trapped set at energy $E$, defined by

$$K_E = \{\alpha \in T^*\mathbb{R}^n : \exists C > 0, \forall t > 0, |\exp(tH_p)\alpha| \le C\}?$$

Here $p = |\xi|^2 + V(x)$ and $H_p = 2\xi \cdot \nabla_x - \nabla V \cdot \nabla_\xi$.

We have (1) with $\chi(x) = \langle x\rangle^{-s}$ and $a(h) = C$ for all $E$ in a neighborhood of $E_0 > 0$ if and only if $K_{E_0} = \emptyset$ ([6, 7]). For general $V$ and $\chi$, the optimal bound is $a(h) = \exp(C/h)$, but Burq [1] and Cardoso-Vodev [2] prove that for any given $V$, if $\chi$ vanishes on a sufficiently large compact set, for any $E > 0$ there exists $C$ such that (1) holds with $a(h) = C$. In our main theorem we improve the condition on $\chi$ and obtain a shorter proof at the expense of an a priori assumption.

**Theorem 1 ([3]).** *Fix $E > 0$. Suppose that (1) holds for $\chi(x) = \langle x\rangle^{-s}$ with $s > 1/2$ and with $a(h) = h^{-N}$ for some $N \in \mathbb{N}$. Then if we take instead $\chi$ such that $K_E \cap T^* \operatorname{supp} \chi = \emptyset$, we have (1) with $a(h) = C$.*

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In fact our result holds for more general operators, and the cutoff $\chi$ can be replaced by a cutoff in phase space whose microsupport is disjoint from $K_E$. In certain situations it is even possible to take a cutoff whose support overlaps $K_E$: see [3] for more details and references.

The a priori assumption that (1) holds for $\chi(x) = \langle x \rangle^{-s}$ with $a(h) = h^{-N}$ is not present in [1, 2] and is not always satisfied, but there are many examples of hyperbolic trapping where it holds: see e.g. [5, 8].

To indicate the comparative simplicity of our method, we prove a special case of the Theorem, under the additional assumption that $\operatorname{supp} V \subset \{|x| < R_0\}$ and $\operatorname{supp} \chi \subset \{R_0 < |x| < R_0 + 1\}$. In other words, suppose $(P - \lambda)u = f$, with $\operatorname{Re} \lambda = E$, and $\operatorname{supp} f \subset \{R_0 < |x| < R_0 + 1\}$, $\|f\| \le 1$. We must prove that $\|\chi u\| \le C h^{-1}$, uniformly as $\operatorname{Im} \lambda \to 0^+$. Here and below all norms are $L^2$ norms.

Let $S$ denote functions in $C^\infty(T^*\mathbb{R}^n)$ which are bounded together with all derivatives, and for $a \in S$ define

$$\operatorname{Op}(a)u(x) = (2\pi h)^{-n} \int \exp(i(x-y)\cdot\xi/h)a(x,\xi)u(y)dy\,d\xi.$$

Because $P - \lambda$ has a semiclassical elliptic inverse away from $p^{-1}(E)$ (see for example [4, Chap. 4]), we have $\|\operatorname{Op}(a)u\| \le C$ whenever $\operatorname{supp} a \cap p^{-1}(E) = \emptyset$. Consequently it is enough to show that $\|\operatorname{Op}(a)u\| \le C h^{-1}$ for some $a \in S$ with $a$ nowhere vanishing on $T^* \operatorname{supp} \chi \cap p^{-1}(E)$. We will prove this inductively: we will show that if there is $a_1$ with this nowhere vanishing property such that $\|\operatorname{Op}(a_1)u\| \le C h^k$, then there is $a_2$ with the same nowhere vanishing property such that $\|\operatorname{Op}(a_2)u\| \le C h^{k+1/2}$, provided $k \le -3/2$. The base case follows from the a priori assumption that $\|u\| \le h^{-N-1}$, so it suffices to prove the inductive step.

Take $\varphi = \varphi(|x|) \ge 0$ a smooth function such that $\varphi = 1$ when $|x| \le R_0$, $\varphi = 0$ when $|x| \ge R_0 + 1$, $\varphi' = -\psi^2$ with $\psi$ smooth. We require further that $T^* \operatorname{supp} \psi$ be contained in the set where $a_1$ is nonvanishing, and in the end we will take $a_2 = \psi$. We will now use a positive commutator argument with $\varphi$ as the commutant:

$$i\langle [P,\varphi]u, u\rangle = i\langle u, \varphi f\rangle - i\langle \varphi f, u\rangle - 2\operatorname{Im}\lambda\|u\|^2 \ge -C\|\psi u\|\|f\|, \tag{2}$$

where we used first $(P - \lambda)u = f$ and then $\operatorname{Im} \lambda \ge 0$ and $\operatorname{supp} f \subset \{\psi \ne 0\}$. The semiclassical principal symbol of $i[P, \varphi]$ is

$$hH_p\varphi = 2h\rho\varphi' = -2h\rho\psi^2,$$

where $\rho$ is the dual variable to $|x|$ in $T^*\mathbb{R}^n$.

We now define an open cover and partition of unity of $T^* \operatorname{supp} \chi$ according to the regions where this commutator does and does not have a favorable sign (the favorable sign is $H_p\varphi < 0$, because of the direction of the inequality in (2)). Take $c > 0$ small enough that for $\rho < 2c$, $|x| > R_0$, $t < 0$ we have $x + 2\rho t \notin \operatorname{supp} V$. Let $K$ be a neighborhood of $p^{-1}(E) \cap T^* \operatorname{supp} \chi$ with compact closure in $T^*\{R_0 < |x| < R_0 + 1\}$, and let $O$ be a neighborhood of $K$ with compact closure in $T^*\{R_0 < |x| < R_0 + 1\}$, and let

$$U_+ = \{\alpha \in O\colon \rho > c\}, \qquad U_- = \{\alpha \in O\colon \rho < 2c\} \cup (T^*\mathbb{R}^n \setminus K).$$

Take $\phi_\pm \in C_0^\infty(O)$ with $\phi_+^2 + \phi_-^2 = 1$ on $T^* \operatorname{supp} \chi$ and with $\operatorname{supp} \phi_\pm \subset U_\pm$. Then

$$H_p\varphi = -b^2 - 2\rho\psi^2\phi_-^2, \qquad \text{where } b = \sqrt{2\rho}\psi\phi_+,$$

and if $B = \operatorname{Op}(b)$ and $\Phi_- = \operatorname{Op}(\phi_-)$

$$i[P,\varphi] = -hB^*B + h\Phi_- R_1 \Phi_- + h^2 R_2 + O(h^\infty),$$

where $R_{1,2} = \operatorname{Op}(r_{1,2})$ for $r_{1,2} \in S$ with $\operatorname{supp} r_{1,2} \subset \operatorname{supp} \psi$. Combining with (2), and using $L^2$ boundedness of $R_1$, we obtain

$$h\|Bu\|^2 \le Ch\|\Phi_- u\|^2 + h^2\langle R_2 u, u\rangle + C\|\psi u\|\|f\| + O(h^\infty).$$

Since $\langle R_2 u, u\rangle \le Ch^{2k}$ by inductive hypothesis, we have

$$\begin{aligned}\|Bu\|^2 &\le C(\|\Phi_- u\|^2 + h^{2k+1} + h^{-1}\|\psi u\|\|f\|)\\ &\le C(\|\Phi_- u\|^2 + h^{2k+1} + \delta^{-1}h^{-2} + \delta\|\psi u\|^2),\end{aligned}$$

where we used $\|f\| \le 1$, and where $\delta > 0$ will be specified presently. Since at least one of $B$ and $\Phi_-$ is elliptic at each point in the interior of $T^* \operatorname{supp} \psi$, we have

$$\|\psi u\|^2 \le C(\|\Phi_- u\|^2 + \|Bu\|^2), \tag{3}$$

from which we conclude that, if $\delta$ is sufficiently small,

$$\|Bu\|^2 \le C_\delta(\|\Phi_- u\|^2 + h^{-2} + h^{2k+1}). \tag{4}$$

Because $c$ was chosen small enough that all backward bicharacteristics through $\operatorname{supp} \phi_-$ stay in $T^*\{|x| > R_0\}$, where $P = -h^2\Delta$, we have

$$\|\Phi_- u\| \le Ch^{-1},$$

by standard nontrapping estimates (see, for example, [3, Sect. 6]). This, combined with (3) and (4), gives

$$\|\psi u\|^2 \le C_\delta(h^{-2} + h^{2k+1}),$$

after which taking $a_2 = \psi$ completes the proof of the inductive step.

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