

Preface

The acquaintance of a newcomer with real interpolation often starts (and fairly often also finishes) with the Marcinkiewicz theorem about the Lorentz scale $L^{p,q}$, see, e.g., [SW], [Z], or the introductory material in [BL]. It can be seen from the classical proof that this subtle result is related to careful splitting of a function in two parts. More advanced expositions show (though this may be hidden behind technicalities; see, e.g., [BL] except the introductory material) that, in fact, not too many splittings are required. Specifically, the *entire* content of the Marcinkiewicz theorem stems from the solution of *only one* extremal problem.

Problem 0.1. Suppose $0 < p < \infty$, $f \in L^p$, and $t > 0$. Find the distance from f to the t -ball of L^∞ in the L^p -metric, i.e., the quantity

$$\text{dist}_{L^p}(f, B_t(L^\infty)) = \inf\{\|f - g\|_{L^p} : g \in L^\infty, \|g\|_{L^\infty} \leq t\}.$$

Luckily, this problem can be resolved, moreover, a *minimizer*, i.e., a function g at which the infimum is attained, can be exhibited immediately:

$$g = \frac{f}{\max(1, t^{-1}|f|)}. \quad (0.1)$$

The splitting mentioned above is going to be $f = g + (f - g)$.

In general, real interpolation is in intimate relationship with similar extremal problems, but for more or less arbitrary couples of spaces (X, Y) in place of (L^p, L^∞) . However, for general couples the solution may be not as easy. Typically, there is no way to explicitly find a minimizer for the *distance functional* $\text{dist}_X(f, B_t(Y))$. In this book we shall show that, in recompense, *explicit* construction of *near-minimizers*, i.e., of functions at which the distance in question is attained roughly (up to a multiplicative constant), is often possible.

Largely, a near-minimizer gives as much information as an exact minimizer, and, if well-chosen, it may behave even better. We shall describe a certain unified algorithm for finding near-minimizers with important additional properties. The method applies to surprisingly many couples (X, Y) of function spaces (and for many of them no explicit near-minimizers were available previously), and is based on some far-reaching refinements of a procedure invented by Calderón and

Zygmund [CZ] for the needs of the theory of singular integral operators. Vaguely speaking, these refinements are related to the case when Y is a space of *smooth* functions, e.g., a Sobolev or Lipschitz class. In this case, we say that a function undergoes a *smooth version of the Calderón–Zygmund decomposition*. As in the classical case, a system of cubes adjusted to one another in some intricate way is required for this procedure.

Usually, families of cubes with special properties arise from *covering theorems*. We discuss classical covering theorems (in particular, those due to Besicovitch and Whitney), and then prove several *joint* refinements of these results. These refinements apply to families of cubes possessing a certain type of smoothness, and yield coverings with the entire list of nice features available in the two classical statements named above. Moreover, a numerical characteristic (the so-called α -capacity) of the outcoming covering is controlled in terms of a similar quantity for the initial family of cubes.

This being done, smooth Calderón–Zygmund decompositions and, with them, near-minimizers can be written out almost immediately. It should be mentioned that, even in the scarce situations where an exact minimizer is available, very often it does not fit any longer if we deviate even slightly from the initial setting. This is the case of the function given by (0.1): though simple, this formula is “too rigid”. Near-minimizers constructed by our methods are much more flexible: we prove that they remain near-minimizers after the action of a singular integral operator (see the Introduction below for more details).

Our principal aim in this book is to give a complete, unified, and polished treatment to this stuff, covered before only partly by journal publications (see, e.g., [K1], [K0], [KK]). However, in Part 1 we present a substantial background material (whence its title). For instance, classical covering theorems are discussed there, and very basic facts about spaces of smooth functions and singular integrals are exposed. This material is included only to the extent required for more or less independent presentation of the main topic. Some original results pertaining to this main topic appear already in Part I, but, generally, this part is more easy reading than Part II, where the principal line is developed fully.

It should be noted, however, that we did not aim at an anywhere near detailed exposition of interpolation theory when discussing the background. When talking about “interpolation”, we mostly mean a careful decomposition of a function in two parts, and leave interpolation spaces (and operators acting in them) apart. More advanced results about interpolation are employed from time to time, so that the reader familiar with them will feel more comfort, but, strictly speaking, this knowledge is not critical for the understanding of the main body of the book.

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Whitney-Besicovitch Coverings, and Singular Integrals

Kislyakov, S.; Kruglyak, N.

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