Chapter 2

Asymmetric Functional Analysis

An asymmetric seminorm is a positive sublinear functional p on a real vector space X. If $p(x) = p(-x) = 0$ implies $x = 0$, then p is called an asymmetric norm. The conjugate asymmetric seminorm is given by $\bar{p}(x) = p(-x),\ x \in X$, and $p^s = p \vee \bar{p}$ is a seminorm, respectively a norm on X if p is an asymmetric norm. An important example is the asymmetric norm u on $\mathbb{R}$ given by $u(t) = t^+,\ t \in \mathbb{R}$, generating the upper topology on $\mathbb{R}$. In this case $\bar{u}(t) = t^-$ and $u^s(t) = |t|,\ t \in \mathbb{R}$. The dual $X_p^\flat$ of an asymmetric normed space (X,p) is formed by all upper semi-continuous linear functionals from (X,p) to $(\mathbb{R}, |\cdot|)$, or equivalently, by all continuous linear functionals from (X,p) to $(\mathbb{R},u)$. In contrast to the usual case, $X_p^\flat$ is not a linear space but merely a cone contained in the dual X^* of the normed space space (X,p^s). The aim of this chapter is to present some basic results on asymmetric normed spaces, their duals and on continuous linear operators acting between them. Applications are given to best approximation in asymmetric normed spaces. As important examples one considers asymmetric norms on normed lattices and spaces of semi-Lipschitz functions on quasi-metric spaces. Asymmetric locally convex spaces are considered as well.

2.1 Continuous linear operators between asymmetric normed spaces

The basic objects of functional analysis are normed spaces and locally convex spaces and spaces of continuous linear operators acting between them, with special emphasis on continuous linear functionals and the duals of these spaces. The situation is quite different in the asymmetric case, mainly due to the fact that the dual of an asymmetric normed space or of an asymmetric LCS X, meaning the set of all lower semi-continuous linear functionals on X, is not a linear space but merely a cone in the space of all continuous linear functionals on X. In spite of the existing differences, some results from the symmetric case have their counterparts in the asymmetric one, a study that was initiated in [90]. As an application one considers the important case of asymmetric norms on normed lattices.

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2.1.1 The asymmetric norm of a continuous linear operator

Let (X,p) and (Y,q) be two asymmetric seminormed spaces. Denote by $L_a(X,Y)$ the space of all linear operators from X to Y. A linear operator $A : X \to Y$ is called (p,q)-continuous if it is continuous with respect to the topologies τ_p on X and τ_q on Y. The set of all (p,q)-continuous linear operators from X to Y is denoted by $L_{p,q}(X,Y)$. For $\mu \in \{p,\bar{p},p^s\}$ and $\nu \in \{q,\bar{q},q^s\}$, the (μ,ν)-continuity and the set $L_{\mu,\nu}(X,Y)$ are defined similarly. The space of all continuous linear operators between the associated seminormed spaces (X,p^s) and (Y,q^s) is denoted by $L(X,Y)$.

In the case of linear functionals, i.e., when $Y=(\mathbb{R},u)$, we put $X^\flat_p = L_{p,u}(X,\mathbb{R})$ and $X^* = L((X,p^s),(\mathbb{R},|\cdot|))$. The meaning of $X^\flat_{\bar{p}}$ is clear. A linear operator $A:(X,p)\to(Y,q)$ is called (p,q)-*semi-Lipschitz* (or (p,q)-*bounded*) if there exists a number $\beta \geq 0$ such that

$$q(Ax) \leq \beta\, p(x)\ , \tag{2.1.1}$$

for all $x \in X$.

The characterizations of the continuity of linear mappings will be based on the following proposition.

Proposition 2.1.1. *Let X be a real vector space, $f,g : X \to \mathbb{R}$ sublinear functionals and $\alpha,\beta > 0$.*

Then the following conditions are equivalent:

$$\forall x \in X, \quad g(x) \leq \beta \ \Rightarrow\ f(x) \leq \alpha, \tag{2.1.2}$$

and

$$\forall x \in X, \quad u(f(x)) \leq \frac{\alpha}{\beta} u(g(x)). \tag{2.1.3}$$

If $g(x) \geq 0$ for all $x \in X$, then these two conditions are also equivalent to

$$\forall x \in X, \quad f(x) \leq \frac{\alpha}{\beta} g(x). \tag{2.1.4}$$

Proof. (2.1.2) $\Rightarrow$ (2.1.3) Let $x \in X$. If $g(x) \leq 0$, then $g(nx) = ng(x) \leq 0 < \alpha,\ n \in \mathbb{N},$ so that $nf(x) = f(nx) \leq \beta,\ n \in \mathbb{N},$ implying $f(x) \leq 0$ and

$$u(f(x)) = 0 = \frac{\alpha}{\beta} u(g(x))\ .$$

If $g(x) > 0$, then $g\left(\frac{\beta}{g(x)}x\right) = \beta$, so that

$$f\left(\frac{\beta}{g(x)}x\right) \leq \alpha \iff f(x) \leq \frac{\alpha}{\beta} g(x) \iff u(f(x)) \leq \frac{\alpha}{\beta} u(g(x))\ .$$

(2.1.3) $\Rightarrow$ (2.1.2) Let $x \in X$. If $g(x) \leq 0 < \beta$, then $u(g(x)) = 0$, so that

$$f(x) \leq u(f(x)) \leq \frac{\alpha}{\beta} u(g(x)) = \frac{\alpha}{\beta} g(x) \leq \alpha\ .$$

If $g(x) > 0$, then, by hypothesis,

$$f(x) \le u(f(x)) \le \frac{\alpha}{\beta} u(g(x)) = \frac{\alpha}{\beta} g(x) \le \alpha \, .$$

Since $g(x) \ge 0,\ x \in X,$ implies $u(g(x)) = g(x),\ x \in X,$ the equivalence (2.1.3) $\iff$ (2.1.4) is obvious. □

The following proposition contains some characterizations of continuity, similar to those known in the symmetric case.

Proposition 2.1.2. *For a linear operator A between two asymmetric seminormed spaces (X,p), (Y,q), the following are equivalent.*

1. *The operator A is continuous on X.*
2. *The operator A is continuous at $0 \in X$ (or at an arbitrary point $x_0 \in X$).*
3. *The operator A is (p,q)-semi-Lipschitz.*
4. *The operator A is $(\mathcal{U}_p,\mathcal{U}_q)$-quasi-uniformly continuous on X.*

Proof. The equivalence $1 \iff 2$ holds for any additive operator (see Proposition 1.1.42).

$2 \Rightarrow 3$ By the continuity of A at $0 \in X$ there exists $r > 0$ such that $A(B_p[0,r]) \subset B_q[0,1]$, meaning that

$$p(x) \le r \;\Rightarrow\; q(Ax) \le 1 \, ,$$

for every $x \in X$. Applying Proposition 2.1.1 to the sublinear functionals $f(x) = q(Ax)$ and $g(x) = p(x)$, it follows that

$$q(Ax) \le \frac{1}{r} p(x),$$

for all $x \in X$.

$3 \Rightarrow 4$ Suppose that (2.1.1) holds with some $\beta > 0$ and let $\varepsilon > 0$. For $V = \{(y_1,y_2) \in Y \times Y : q(y_2 - y_1) < \varepsilon\} \in \mathcal{U}_q$, let $U := \{(x_1,x_2) \in X \times X : p(x_2 - x_1) < \varepsilon/\beta\} \in \mathcal{U}_p$. Then for every $(x_1,x_2) \in U$, $q(Ax_2 - Ax_1) = q(A(x_2 - x_1)) \le \beta p(x_2 - x_1) < \varepsilon$, showing that $(Ax_1, Ax_2) \in V$.

The implication $4 \Rightarrow 1$ is a general result: any quasi-uniformly continuous mapping between two quasi-uniform spaces is continuous with respect to the topologies induced by the quasi-uniformities. □

Based on these properties one can introduce an asymmetric seminorm on the cone $L_{p,q}(X,Y)$ by

$$\|A|_{p,q} = \sup\{q(Ax) : x \in X,\ p(x) \le 1\} \, , \tag{2.1.5}$$

for every $A \in L_{p,q}(X,Y)$.

The seminorm $\|\cdot\| := \|\cdot|_{p^s,q^s}$ is the usual operator seminorm on the space $L(X,Y) = L_{p^s,q^s}(X,Y)$ given for $A \in L(X,Y)$ by

$$\|A\| = \sup\{q^s(Ax) : x \in X,\, p^s(x) \le 1\}\ . \tag{2.1.6}$$

We mention the following results, whose proofs are similar to those for normed spaces.

Proposition 2.1.3. *Let (X,p) and (Y,q) be asymmetric seminormed spaces and $A \in L_{p,q}(X,Y)$. Then the number $\|A|_{p,q}$ is the smallest semi-Lipschitz constant for A and $\|\cdot|_{p,q}$ is an asymmetric seminorm on the cone $L_{p,q}(X,Y)$, which is an asymmetric norm if q is an asymmetric norm.*

The asymmetric seminorm $\|A|_{p,q}$ can be calculated also by the formula

$$\|A|_{p,q} = \sup\{q(Ax)/p(x) : x \in X,\, p(x) > 0\}\ . \tag{2.1.7}$$

A *cone* (in fact, a *convex cone*) is a subset Z of a linear space X such that $w+z \in Z$ and $\lambda z \in Z$ for all $w,z \in Z$ and $\lambda \ge 0$. A cone is called also a *semilinear space*. In order to study spaces of linear operators between asymmetric normed spaces and the duals of such spaces, we shall consider asymmetric norms on cones.

Remark 2.1.4. In fact, one can define an abstract notion of cone as a set K with two operations, addition $+$ which is supposed to be commutative, associative and having a neutral element denoted by 0, and multiplication by nonnegative scalars (denoted by $\cdot$), satisfying the properties

$$\begin{aligned}&\text{(i) } (\lambda\mu)a = \lambda(\mu a), \quad \text{(ii) } \lambda(a+b) = \lambda a + \lambda b, \quad \text{(iii)} (\lambda+\mu)a = \lambda a + \mu a,\\ &\text{(iv) } 1\cdot a = 1, \quad \text{ and } \quad \text{(v) } 0 \cdot a = 0\ .\end{aligned} \tag{2.1.8}$$

The cone X is called *cancellative* if $a+c = b+c \Rightarrow a = b$ for all $a,b,c \in X$. A cone X is cancellative if and only if it can be embedded in a vector space.

The theory of locally convex cones, with applications to Korovkin type approximation theory for positive operators and to vector-measure theory, is developed in the books by Keimel and Roth [109] and Roth [212], respectively. A recent paper by Galanis [85] discusses Gâteaux and Hukuhara differentiability on topological cones (called by him topological semilinear spaces and meaning cones for which the operations of addition and multiplication by positive scalars are continuous).

The following proposition shows that continuous linear operators between two asymmetric normed spaces are continuous with respect to the associated norm topologies. Also the set of all these continuous linear operators is a cone (a cancellative one).

Proposition 2.1.5. *Let (X,p) and (Y,q) be asymmetric seminormed spaces. Any (p,q)-continuous linear operator $A : X \to Y$ is also (p^s,q^s)-continuous and the set $L_{p,q}(X,Y)$ is a convex cone in $L(X,Y)$. Also*

$$L_{p,q}(X,Y) = L_{\bar p,\bar q}(X,Y)\ , \tag{2.1.9}$$

and

$$\|A|_{p,q} = \|A|_{\bar{p},\bar{q}} \geq \|A|_{p^s,q^s}, \tag{2.1.10}$$

for every $A \in L_{p,q}(X,Y)$.

In particular, every (p,u)*-continuous linear functional is* $(p^s, |\cdot|)$*-continuous and* $X_p^\flat$ *is a cone in the space* X^* *of all continuous linear functionals on the normed space* (X, p^s).

Proof. Observe that

$$\forall x \in X,\ q(Ax) \leq \beta p(x) \iff \forall x \in X,\ \bar{q}(Ax) \leq \beta \bar{p}(x),$$

proving the equality (2.1.9). Taking into account Proposition 2.1.3 (the fact that the seminorm is the smallest semi-Lipschitz constant), this equivalence implies also the equality $\|A|_{p,q} = \|A|_{\bar{p},\bar{q}}$.

Also, $q(Ax) \leq \beta p(x) \leq \beta p^s(x)$ and $\bar{q}(Ax) \leq \beta \bar{p}(x) \leq \beta p^s(x)$, implies $q^s(Ax) \leq \beta p^s(x)$, proving the inclusion $L_{p,q}(X,Y) \subset L_{p^s,q^s}(X,Y)$ and the inequality $\|A|_{p^s,q^s} \leq \|A|_{p,q}$. □

The following example shows that $L_{p,q}(X,Y)$ is not a subspace of $L(X,Y)$.

Example 2.1.6. On the space $X = C_0[0;1]$ from Example 1.1.43 consider the functional $\varphi(f) = f(1)$, $f \in C_0[0;1]$. Then φ is continuous on (X,p), but $-\varphi$ is not continuous.

Another example is furnished by the functional $\mathrm{id} : (\mathbb{R}, u) \to (\mathbb{R}, u)$ which is (p,u)-continuous, but $-\mathrm{id}$ is not.

Indeed, $\varphi(f) = f(1) \leq \max f([0;1]) = p(f)$, $f \in C_0[0;1]$, so that φ is (p,u)-continuous. Taking $f_n(t) = 1 - nt$, $t \in [0;1]$, it follows that $p(f_n) = 1$ and $-\varphi(f_n) = -f_n(1) = n - 1$, so that $-\varphi$ is not bounded on the unit ball of (X,p), so it is not continuous.

2.1.2 Continuous linear functionals on an asymmetric seminormed space

In this subsection we consider an asymmetric seminormed space (X,p) with nontrivial seminorm p (that is $p \neq 0$) with conjugate seminorm $\bar{p}$ and the (symmetric) seminorm p^s. Note that in this case the fact that a linear functional $\varphi : (X,p) \to \mathbb{R}$ is (p,u)-semi-Lipschitz is equivalent to

$$\varphi(x) \leq \beta p(x)\ , \tag{2.1.11}$$

for all $x \in X$.

Note also that the continuity of a function f from an asymmetric normed space (X, p) to $(\mathbb{R}, u)$ is equivalent to its upper semicontinuity from (X, p) to $(\mathbb{R}, |\cdot|)$.

Denote by $X_p^\flat$ and $X_{\bar{p}}^\flat$ the cones of p-continuous, respectively $\bar{p}$-continuous, linear functionals on X and let $X^* = (X, p^s)^*$ be the dual of the seminormed space (X, p^s). When there is no danger of confusion the space $X_p^\flat$ will be denoted simply by $X^\flat$ and we shall call it *the asymmetric dual* of the space (X, p).

Let

$$B_p = \{x \in X : p(x) \le 1\} \quad \text{and} \quad B'_p = \{x \in X : p(x) < 1\}$$

be the closed, respectively open, unit ball of X.

The functionals given by

$$\|\varphi|_p = \sup \varphi(B_p),\ \varphi \in X_p^\flat, \quad \text{and} \quad \|\psi|_{\bar{p}} = \sup \psi(B_{\bar{p}}),\ \psi \in X_{\bar{p}}^\flat, \tag{2.1.12}$$

are asymmetric norms on $X_p^\flat$ and $X_{\bar{p}}^\flat$, respectively.

The functional defined by

$$\|x^*\| = \sup x^*(B_{p^s}) = \sup\{x^*(x) : x \in X,\ p^s(x) \le 1\},\ x^* \in X^*\ , \tag{2.1.13}$$

is a norm on the space $X^* = (X, p^s)^*$ and the space X^* is complete with respect to this norm, i.e., it is a Banach space.

We mention the following properties of continuous linear functionals.

Proposition 2.1.7. *Let (X, p) be a space with asymmetric seminorm.*

1. *The functionals given by* (2.1.12) *are asymmetric norms on $X_p^\flat$, and $X_{\bar{p}}^\flat$, respectively, satisfying*

$$\varphi = 0 \iff \|\varphi|_p = 0 \quad \textit{and} \quad \psi = 0 \iff \|\psi|_{\bar{p}} = 0\ . \tag{2.1.14}$$

Also, if $\varphi \ne 0$, then

$$\varphi(x_0) = \|\varphi|_p \ \Rightarrow\ p(x_0) = 1\ , \tag{2.1.15}$$

for any $x_0 \in B_p$.

2. *The norm $\|\varphi|_p$ of a functional $\varphi \in X_p^\flat$ is the smallest semi-Lipschitz constant for φ and it can be also calculated by the formula*

$$\|\varphi|_p = \sup\{\varphi(x)/p(x) : x \in X,\ p(x) > 0\}\ .$$

3. *The cones $X_p^\flat$ and $X_{\bar{p}}^\flat$ are contained in X^* and*

$$\|\varphi\| \le \|\varphi|_p \quad \textit{and} \quad \|\psi\| \le \|\psi|_{\bar{p}}\ ,$$

for every $\varphi \in X_p^\flat$ and every $\psi \in X_{\bar{p}}^\flat$.

4. *The following hold:*

$$\varphi \in X_p^\flat \iff -\varphi \in X_{\bar{p}}^\flat \quad \text{and} \quad \|\varphi|_p = \| -\varphi|_{\bar{p}} \, .$$

Consequently, $X_p = -X_{\bar{p}}$ *and the linear spans of* $X_p^\flat$ *and* $X_{\bar{p}}^\flat$ *agree, being given by*

$$\operatorname{sp}(X_p^\flat) = \operatorname{sp}(X_{\bar{p}}^\flat) = X_p^\flat + X_{\bar{p}}^\flat . \, .$$

Proof. All of the assertions from 1 and 2, excepting (2.1.14) and (2.1.15), are consequences of Propositions 2.1.2 and 2.1.3.

To prove (2.1.14), observe that if $\varphi \neq 0$, then there exists $x_0 \in X$ such that $\varphi(x_0) > 0$, implying

$$0 < \varphi(x_0) \le \|\varphi|_p \, p(x_0) \, .$$

If there exists $x_0 \in X$ with $p(x_0) < 1$ such that $\varphi(x_0) = \|\varphi|_p > 0$, then $x_1 = [p(x_0)]^{-1} x_0$ satisfies $p(x_1) = 1$ and $\varphi(x_1) = \varphi(x_0)/p(x_0) > \|\varphi|_p$, a contradiction, showing that (2.1.15) holds, too.

3. Let $\varphi \in X_p^\flat$. The inequalities

$$\varphi(x) \le \|\varphi|_p p(x) \le \|\varphi|_p p^s(x), \ x \in X \, ,$$

imply $\varphi \in X^*$ and $\|\varphi\| \le \|\varphi|_p$. The situation for the conjugate seminorm $\bar{p}$ is similar.

4. Let now $\varphi : X \to \mathbb{R}$ be a linear functional. The assertion will be a consequence of the following equalities:

$$\| -\varphi|_{\bar{p}} = \sup\{-\varphi(x) : \bar{p}(x) \le 1\} = \sup\{\varphi(-x) : p(-x) \le 1\} = \|\varphi|_p \, .$$

It follows that $X_p^\flat = -X_{\bar{p}}^\flat$, so that $\operatorname{sp}(X_p^\flat) = X_p^\flat - X_p^\flat = X_p^\flat + X_{\bar{p}}^\flat = \operatorname{sp}(X_{\bar{p}}^\flat)$. □

In the following proposition we collect some simple properties of the norm $\| \cdot |_p$ that we shall need in the proofs of the separation theorems.

Proposition 2.1.8. *If* φ *is a continuous linear functional on a space with asymmetric seminorm* (X, p), $p \neq 0$, *then the following assertions hold.*

1. *We have*

$$\begin{aligned} \|\varphi|_p &= \sup\{\varphi(x) : x \in X, \ p(x) < 1\} \\ &= \sup\{\varphi(x) : x \in X, \ p(x) = 1\} \end{aligned} \tag{2.1.16}$$

and

$$\begin{aligned} -\|\varphi|_{\bar{p}} &= \inf\{\varphi(x) : x \in X, \ p(x) < 1\} \\ &= \inf\{\varphi(x) : x \in X, \ p(x) = 1\} \, . \end{aligned} \tag{2.1.17}$$

2. *If* φ *is both* p*- and* $\bar{p}$*-continuous (i.e.,* $\varphi \in X_p^\flat \cap X_{\bar{p}}^\flat$*), then*

$$\varphi(B_p') = (-\|\varphi|_{\bar{p}}; \|\varphi|_p) \quad \text{and} \quad \varphi(B_{\bar{p}}') = (-\|\varphi|_p; \|\varphi|_{\bar{p}}) \, .$$

3. *If* $\varphi \in X_p^\flat \setminus X_{\bar p}^\flat$ *and* $\psi \in X_{\bar p}^\flat \setminus X_p^\flat$, *then*

$$\varphi(B_p') = (-\infty, \|\varphi|_p) \quad \textit{and} \quad \psi(B_{\bar p}') = (-\|\psi|_{\bar p}; \infty)\ .$$

Proof. 1. We can suppose $\varphi \neq 0$. Then $c := \sup\{\varphi(x) : p(x) < 1\} \leq \|\varphi|_p$. If $x \in X$ is such that $p(x) = 1$, then $p(n(n+1)^{-1}x) = n(n+1)^{-1}p(x) < 1$, so that $n(n+1)^{-1}\varphi(x) = \varphi(n(n+1)^{-1}x) \leq c$, for all $n \in \mathbb{N}$. Letting $n \to \infty$, it follows that $\varphi(x) \leq c$, implying $\|\varphi|_p = \sup \varphi(B_p) \leq c$, that is $\|\varphi|_p = c$.

By Proposition 2.1.7,

$$\|\varphi|_p = \sup\{\varphi(x/p(x)) : x \in X,\, p(x) > 0\} = \sup\{\varphi(y) : y \in X,\, p(y) = 1\}.$$

By (2.1.16) applied to $\bar p$, $\|\varphi|_{\bar p} = \sup \varphi(B_{\bar p}')$, so that

$$\inf\{\varphi(x):p(x)<1\}=\inf\{\varphi(-x):p(-x)<1\}=-\sup\{\varphi(x):\bar p(x)<1\}=-\|\varphi|_{\bar p}\,,$$

proving the first equality in (2.1.17).

The second equality is proved similarly.

2. By the first assertion of the proposition,

$$\sup \varphi(B_p') = \|\varphi|_p \quad \text{and} \quad \inf \varphi(B_p') = -\|\varphi|_{\bar p}\ .$$

By Proposition 2.1.7.1, $-\|\varphi|_{\bar p} < \varphi(x) < \|\varphi|_p$ for every $x \in B_p'$.

The convexity of B_p' and the linearity of φ imply that $\varphi(B_p')$ is convex, that is it is an interval in $\mathbb{R}$, so that

$$\varphi(B_p') = (\inf \varphi(B_p'); \sup \varphi(B_p')) = (-\|\varphi|_{\bar p}; \|\varphi|_p)\ .$$

3. If $\varphi \in X_p^\flat \setminus X_{\bar p}^\flat$, then

$$\infty = \sup\{\varphi(x) : p(-x) < 1\} = \sup\{-\varphi(x) : p(x) < 1\} = -\inf\{\varphi(x) : p(x) < 1\}\ ,$$

that is $\inf \varphi(B_p') = -\infty$, and $\sup \varphi(B_p') = \|\varphi|_p$.

Reasoning as above, it follows that $\varphi(B_p') = (-\infty; \|\varphi|_p)$.

For the second equality, note that $\sup \varphi(B_p') = \infty$ and, by (2.1.17), $\inf \varphi(B_p') = -\|\varphi|_{\bar p}$, implying $\varphi(B_{\bar p}') = (-\|\varphi|_{\bar p}; \infty)$. □

2.1.3 Continuous linear mappings between asymmetric locally convex spaces

Let (X,P), (Y,Q) be two asymmetric locally convex spaces with the topologies τ_P and τ_Q generated by the families P and Q of asymmetric seminorms on X and Y, respectively. In the following when we say that (X,P) is an asymmetric locally convex space, we understand that X is a real vector space, P is a family of asymmetric seminorms on X and τ_P is the asymmetric locally convex topology associated to P.

A linear mapping $A : X \to Y$ is called (P, Q)-*bounded* if for every $q \in Q$ there exist $F \in \mathcal{F}(P)$ (the family of all nonempty finite subsets of P) and $\beta \geq 0$ such that

$$\forall x \in X,\ q(Ax) \leq \beta \max\{p(x) : p \in F\}. \tag{2.1.18}$$

If the family P is directed, then the (P, Q)-boundedness of A is equivalent to the condition: for every $q \in Q$ there exist $p \in P$ and $\beta \geq 0$ such that

$$\forall x \in X,\ q(Ax) \leq \beta p(x). \tag{2.1.19}$$

The continuity of the mapping A from (X, τ_P) to (Y, τ_Q) is called (τ_P, τ_Q)-*continuity*. We shall use also the terms (P, Q)-continuity for this property, and (P, u)-continuity in the case of (τ_P, τ_u)-continuous linear functionals, where u is the quasi-metric on $\mathbb{R}$ given in Example 1.1.3.

Because both of the topologies τ_P and τ_Q are translation invariant, we have the following result.

Proposition 2.1.9. *Let (X, P) and (Y, Q) be asymmetric locally convex spaces and $A : X \to Y$ a linear mapping. The following conditions are equivalent.*

1. *The mapping A is (P, Q)-continuous on X.*
2. *The mapping A is continuous at $0 \in X$.*
3. *The mapping A is continuous at some point $x_0 \in X$.*

The following proposition emphasizes the equivalence of continuity and boundedness for linear mappings.

Proposition 2.1.10. *Let (X, P) and (Y, Q) be two asymmetric locally convex spaces and $A : X \to Y$ a linear mapping. The following assertions are equivalent.*

1. *The mapping A is (P, Q)-continuous on X.*
2. *The mapping A is continuous at $0 \in X$.*
3. *The mapping A is (P, Q)-bounded.*
4. *The mapping A is $(\mathcal{U}_P, \mathcal{U}_Q)$-quasi-uniformly continuous on X.*

Proof. The equivalence $1 \iff 2$ follows from the preceding proposition.

Suppose the families P and Q to be directed.

$2 \Rightarrow 3$. For $q \in Q$ consider the τ_Q-neighborhood $V = B_q(0, 1)$ of $A(0) = 0 \in Y$, and let U be a neighborhood of $0 \in X$ such that $A(U) \subset V$. If $p \in P$ and $r > 0$ are such that $B_p(0, r) \subset U$, then

$$\forall x \in X, \quad p(x) \leq r \ \Rightarrow\ q(Ax) \leq 1\ .$$

By Proposition 2.1.1 applied to f(x) = q(Ax) and $g(x) = p(x)$, this relation implies

$$\forall x \in X, \quad q(Ax) \leq \frac{1}{r} p(x)\ .$$

$3 \Rightarrow 4$. For $W \in \mathcal{U}_Q$ there exists $q \in Q$ and $\varepsilon > 0$ such that $W_{q,\varepsilon} = \{(y, y') \in Y \times Y : q(y' - y) < \varepsilon\} \subset W$. If $p \in P$ and $\beta > 0$ are such that (2.1.19) holds, then $(f(x), f(y)) \in W_{q,\varepsilon}$ for every $(x, y) \in U_{p,r}$, where $r := \varepsilon/\beta$.

The implication $4 \Rightarrow 1$ holds for arbitrary quasi-uniform spaces. □

In the case of linear functionals on an asymmetric locally convex space we have the following characterization of continuity, where u is as in Example 1.1.3.

Proposition 2.1.11. *Let (X, P) be an asymmetric locally convex space, where P is a directed family of asymmetric seminorms on X, and $\varphi : X \to \mathbb{R}$ a linear functional. The following assertions are equivalent.*

1. *The functional φ is (P, u)-continuous at $0 \in X$.*
2. *The functional φ is (P, u)-continuous on X.*
3. *The functional φ is upper semi-continuous from (X, τ_P) to $(\mathbb{R}, |\cdot|)$.*
4. *There exist $p \in P$ and $\beta \geq 0$ such that*
$$\forall x \in X, \quad \varphi(x) \leq \beta p(x). \tag{2.1.20}$$
5. *The functional φ is (P, u)-quasi-uniformly continuous on X.*

Using Proposition 2.1.1 and the inequality
$$f(y) - f(x) \leq f(y - x),\ x, y \in X, \tag{2.1.21}$$
valid for any sublinear functional on a vector space X, it is easy to check that these results hold for the slightly more general case of sublinear functionals.

Proposition 2.1.12. *Let (X, P) be an asymmetric locally convex space, where P is a directed family of asymmetric seminorms on X, and let $f : X \to \mathbb{R}$ be a sublinear functional. The following assertions are equivalent.*

1. *The functional f is (P, u)-continuous at $0 \in X$.*
2. *The functional f is (P, u)-continuous on X.*
3. *The functional f is upper semi-continuous from (X, τ_P) to $(\mathbb{R}, |\cdot|)$.*
4. *There exist $p \in P$ and $\beta > 0$ such that*
$$\forall x \in X, \quad f(x) \leq \beta p(x)\ .$$
5. *The functional f is (P, u)-quasi-uniformly continuous on X.*

Proof. $1 \Rightarrow 2$ Let $x \in X$. For $\varepsilon > 0$ there exist $p \in P$ and $r > 0$ such that
$$p(z) < r \ \Rightarrow\ f(z) < \varepsilon.$$

Then, for any $y \in X$ such that $p(y - x) < r$ we have
$$f(y) - f(x) \leq f(y - x) < \varepsilon,$$
proving the (P, u)-continuity of f at x.

The equivalence 2 $\iff$ 3 is true for an arbitrary function from X to $\mathbb{R}$.

2 $\Rightarrow$ 4. Since $(-\infty; 1]$ is a τ_u-neighborhood of $f(0) = 0 \in \mathbb{R}$, there exist $p \in P$ and $r > 0$ such that $f(B_p[0,r]) \subset (-\infty; 1]$, i.e.,

$$\forall x \in X, \quad p(x) \le r \;\Rightarrow\; f(x) \le 1 \; .$$

By Proposition 2.1.1 this implies

$$\forall x \in X, \quad f(x) \le \frac{1}{r} p(x) \; .$$

4 $\Rightarrow$ 5. For $\varepsilon > 0$ let $U_{u,\varepsilon} = \{(s,t) \in \mathbb{R}^2 : u(t-s) < \varepsilon\}$. If $p \in P$ and $\beta > 0$ are given by 4, let $V_{p,\varepsilon/\beta} = \{(x,y) \in X^2 : p(y-x) < \varepsilon/\beta\}$.

The inequalities

$$f(y) - f(x) \le f(y-x) \le \beta p(y-x) < \varepsilon$$

show that $f(V_{p,\varepsilon/\beta}) \subset U_{u,\varepsilon}$.

The implication 5 $\Rightarrow$ 1 is a general topological property. □

The above proposition has the following useful corollary.

Corollary 2.1.13. *Let (X,P) be an asymmetric LCS.*

1. *Let f, g be sublinear functionals defined on an asymmetric locally convex space (X,P). If $f \le g$ and g is (P,u)-continuous, then f is (P,u)-continuous too. In particular the result is true when f is linear.*
2. *Every asymmetric seminorm $p \in P$ is quasi-uniformly (P,u)-continuous.*

Proof. By Proposition 2.1.12, there exist $p \in P$ and $\beta \ge 0$ such that $\forall x \in X,\; g(x) \le \beta p(x)$. It follows that $\forall x \in X,\; f(x) \le g(x) \le \beta p(x)$, which, by the same proposition, implies the continuity of f.

The second assertion is a consequence of the first one and of the translation invariance of the topology. □

Remark 2.1.14. If the family P is not directed, then the family $\widetilde{P} = \{p_F : F \in \mathcal{F}(P)\}$, where $p_F(x) = \max\{p(x) : p \in F\}$, is a directed family of seminorms generating the same topology as P.

Consequently, the (P,u)-continuity of a functional φ (or f) is equivalent to the condition: there exist $F \in \mathcal{F}(P)$ and $\beta \ge 0$ such that

$$\forall x \in X, \quad \varphi(x) \le \beta p_F(x) = \beta \max\{p(x) : p \in F\}. \tag{2.1.22}$$

As in the case of asymmetric normed spaces, see Subsection 2.1.4, the set $L_{P,Q}(X,Y)$ of all (P,Q)-continuous mappings between two asymmetric LCS (X,P) and (Y,Q) is a convex cone in the space $L_a(X,Y)$ of all linear operators between X and Y. In fact it is contained in the linear space $L(X,Y) = L((X,P^s),(Y,Q^s))$

of all continuous linear operators between the locally convex spaces (X, P^s) and (Y, Q^s). Indeed, supposing P, Q directed, then for $A \in L_{P,Q}(X,Y)$ and $q \in Q$ there exist $p \in P$ and $\beta \geq 0$ such that

$$\forall x \in X, \quad q(Ax) \leq \beta p(x) \leq \beta p^s(x) .$$

Since

$$\bar{q}(Ax) = q(-Ax) = q(A(-x)) \leq \beta p(-x) \leq \beta p^s(x) ,$$

it follows that

$$\forall x \in X, \quad q^s(Ax) \leq \beta p^s(x) ,$$

showing that $A \in L(X,Y)$.

Here for $p \in P$, $\bar{p}(x) = p(-x)$, $p^s(x) = \max\{p(x), p(-x)\}$, $x \in X$, and $P^s = \{p^s : p \in P\}$ with similar definitions for the family Q.

For an asymmetric locally convex space (X, P) denote by $X^\flat = X^\flat_P$ the set of all linear (P, u)-continuous functionals. It follows that $X^\flat_P$ is a convex cone in $X^\#$ the algebraic dual space of X, i.e., the space of all linear functionals on X. In fact, by the above remark, it is contained in the dual space $X^* = L((X, P^s), (\mathbb{R}, |\cdot|))$.

Remark 2.1.15. It is easy to check that a linear functional $\varphi(t) = at$, $t \in \mathbb{R}$, is (τ_u, τ_u)-continuous if and only if $a \geq 0$. Indeed if $a \geq 0$, then $\varphi(t) = at \leq u(at) = au(t)$, $t \in \mathbb{R}$. If $a < 0$, then, reasoning as above, one concludes that φ fails to be continuous.

2.1.4 Completeness properties of the normed cone of continuous linear operators

Following [90], we can consider an extended asymmetric norm on the space $L(X,Y)$ of all linear continuous operators from (X, p^s) to (Y, q^s), defined by the same formula:

$$\|A|^*_{p,q} = \sup\{q(Ax) : x \in X,\, p(x) \leq 1\} = \sup\{q(Ax) : x \in B_p\} , \qquad (2.1.23)$$

for every $A \in L(X,Y)$. If $A \in L_{p,q}(X.Y)$, then $A \in L(X,Y)$, and so $-A \in L(X,Y)$, but, as the above examples show, it is possible that $\| -A|^*_{p,q} = \infty$, so that $\|\cdot|^*_{p,q}$ could be effectively an extended asymmetric norm.

With the asymmetric norm $\|\cdot|^*_{p,q}$ one associates a symmetric extended norm on $L(X,Y)$ defined by

$$\|A\|^*_{p,q} = \|A|^*_{p,q} \vee \| -A|^*_{p,q} . \qquad (2.1.24)$$

Since

$$\begin{aligned} \| -A|^*_{p,q} = \sup\{q(-Ax) : p(x) \leq 1\} &= \sup\{q(Ax) : p(-x) \leq 1\} \\ &= \sup\{q(Ax) : x \in B_{\bar{p}}\} , \end{aligned}$$

it follows that

$$\|A\|^*_{p,q} = \sup\{q(Ax) : x \in B_p \cup B_{\bar{p}}\} . \tag{2.1.25}$$

The norm $\|A\|^*_{p,q}$ can be calculated also by the formula

$$\|A\|^*_{p,q} = \sup\{q^s(Ax) : x \in B_p\} . \tag{2.1.26}$$

Indeed, denoting by λ the right side member of the above equality, we have $q(Ax) \le q^s(Ax) \le \lambda$ for every $x \in B_p$, so that $\|A|^*_{p,q} \le \lambda$. Similarly $q(-Ax) \le q^s(Ax) \le \lambda$ for every $x \in B_p$ implies $\| - A|^*_{p,q} \le \lambda$, so that $\|A\|^*_{p,q} \le \lambda$. Also, $q(Ax) \le \|A|^*_{p,q} \le \|A\|^*_{p,q}$ and $q(-Ax) \le \| - A|^*_{p,q} \le \| - A\|^*_{p,q} = \|A\|^*_{p,q}$ for every $x \in B_p$, implies $\lambda \le \|A\|^*_{p,q}$.

Recall that an asymmetric normed space (X, p) is called biBanach if the associated normed space (X, p^s) is a Banach space (i.e., a complete normed space).

Proposition 2.1.16 ([90]). *Let (X, p) and (Y, q) be asymmetric normed spaces.*

1. *The functional $\| \cdot |^*_{p,q}$ given by (2.1.23) is an extended asymmetric norm on the space $L(X, Y)$ and $\|A\| \le \|A\|^*_{p,q}$ for all $A \in L(X, Y)$. An operator $A \in L(X, Y)$ belongs to $L_{p,q}(X, Y)$ if and only if $\|A|^*_{p,q} < \infty$. Also $\|A|^*_{p,q} = \|A|_{p,q}$ for $A \in L_{p,q}(X, Y)$.*
2. *If the space (Y, q) is biBanach, then the space $(L(X, Y), \| \cdot \|^*)$ is complete.*
3. *The set $L_{p,q}(X, Y)$ is closed in $(L(X, Y), \|\cdot\|^*_{p,q})$, so it is complete with respect to the restriction of the extended norm $\| \cdot \|^*_{p,q}$ to $L_{p,q}(X, Y)$.*

Proof. We shall omit the subscripts p, q in what follows.

1. It is easy to check that $\| \cdot |^*$ is an extended asymmetric norm on $L(X, Y)$.

We can suppose $\|A\|^* < \infty$. Then

$$q(Ax) \le \|A|p(x) \le \|A\|^* p^s(x), \quad \text{and}$$
$$q(-Ax) \le \| - A|p(x) \le \|A\|^* p^s(x) ,$$

so that $q^s(Ax) \le \|A\|^* p^s(x)$, for all $x \in X$, implying $\|A\| \le \|A\|^*$.

2. Let (A_n) be a $\| \cdot \|^*$-Cauchy sequence in $L(X, Y)$, that is for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\|A_m - A_n\|^* \le \varepsilon \tag{2.1.27}$$

holds for all $m, n \ge n_0$. By the first point of the theorem, $\|A_m - A_n\| \le \|A_m - A_n\|^*$, so that (A_n) is a Cauchy sequence in the Banach space $(L(X, Y), \| \cdot \|)$, and so (A_n) has a $\| \cdot \|$-limit $A \in L(X, Y)$.

It remains to show that (A_n) converges to A with respect to the norm $\| \cdot \|^*$. The inequality $q^s(A_n x - Ax) \le \|A_n - A\| \, p^s(x)$ implies

$$\lim_{n\to\infty} q^s(A_n x - Ax) = 0 , \tag{2.1.28}$$

that is the sequence $(A_n x)$ converges to Ax in the normed space (Y, q^s), for every $x \in X$.

For $\varepsilon > 0$ let n_0 be such that (2.1.27) holds. Then, by (2.1.26), for every $x \in B_p$,

$$q^s(A_m x - A_n x) \le \|A_m - A_n\|^* \le \varepsilon \ .$$

By (2.1.28), the inequality $q^s(A_m x - A_n x) \le \varepsilon$ yields for $n \to \infty$,

$$q^s(A_m x - Ax) \le \varepsilon \ ,$$

for every $x \in B_p$ and $m \ge n_0$. Taking into account (2.1.26), it follows that $\|A_m - A\|^* \le \varepsilon$ for every $m \ge n_0$.

3. To show that $L_{p,q}(X,Y)$ is closed in $(L(X,Y), \|\cdot\|^*)$ let (A_n) be a sequence in $L_{p,q}(X,Y)$ which is $\|\cdot\|^*$-convergent to some $A \in L(X,Y)$. For $\varepsilon = 1$ let $n_0 \in \mathbb{N}$ be such that $\|A_n - A\|^* \le 1$ for all $n \ge n_0$. Then $\|A\|^* \le \|A - A_{n_0}\|^* + \|A_{n_0}\|^* \le 1 + \|A_{n_0}\|^*$, which implies that both A and $-A$ belong to $L_{p,q}(X,Y)$. □

Remark 2.1.17. 1. If a sequence (A_n) in $L_{p,q}(X,Y)$ converges to $A \in L(X,Y)$ with respect to the conjugate norm $(\|\cdot|^*_{p,q})^{-1}$ of $\|\cdot|^*_{p,q}$, then $A \in L_{p,q}(X,Y)$.

2. As it is known, in the classical case, the completeness of $(L(X,Y), \|\cdot\|)$ implies the completeness of the normed space Y. We do not know if a similar result holds for the extended norm $\|\cdot\|^*$.

Let n_0 be such that $\|A - A_n|^* \le 1$ for all $n \ge n_0$. Then $\|A|^* \le \|A - A_{n_0}|^* + \|A_{n_0}|^* \le 1 + \|A_{n_0}|^* < \infty$, shows that $A \in L_{p,q}(X,Y)$, proving the validity of the assertion from 1.

2.1.5 The bicompletion of an asymmetric normed space

As it is well known, any normed space $(X, \|\cdot\|)$ has a completion, meaning that there exists a Banach space $(\widetilde{X}, \|\cdot\|\tilde{})$ such that $(X, \|\cdot\|)$ is isometrically isomorphic to a dense subspace of $(\widetilde{X}, \|\cdot\|\tilde{})$. The Banach space $(\widetilde{X}, \|\cdot\|\tilde{})$, called the *completion* of X is uniquely determined, in the sense that any Banach space Z such that X is isometrically isomorphic to a dense subspace of Z is isometrically isomorphic to $(\widetilde{X}, \|\cdot\|\tilde{})$.

A *bicompletion* of an asymmetric normed space (X,p) is a bicomplete asymmetric normed space (Y,q) such that X is isometrically isomorphic to a τ_{q^s}-dense subspace of Y. An *isometry* between two asymmetric normed spaces $(X,p), (Y,q)$ is a mapping $T : X \to Y$ such that

$$q(Tx - Ty) = p(x - y), \ \text{ for all } \quad x, y \in X \ . \tag{2.1.29}$$

If T is linear, then (2.1.29) is equivalent to

$$q(Tx) = p(x), \ \text{ for all } \quad x \in X \ . \tag{2.1.30}$$

Note that, as defined, the isometry T is in fact an isometry between the associated normed spaces $(X, p^s), (Y, q^s)$, because

$$q^s(Tx - Ty) = q(Tx - Ty) \vee q(Ty - Tx) = p(x - y) \vee p(y - x) = p^s(x - y) ,$$

for all $x, y \in X$. The construction of the bicompletion of an asymmetric normed space was done in [88] (see [170] for the case of normed cones), following ideas from the normed case, adapted to the asymmetric one. We only sketch the construction, referring for details to the mentioned papers.

Let (X, p) be an asymmetric normed space. In the set of all p^s-Cauchy sequences in X define an equivalence relation by

$$(x_n) \sim (y_n) \iff \lim_{n\to\infty} p^s(x_n - y_n) = 0 . \tag{2.1.31}$$

Lemma 2.1.18.

1. *The relation $\sim$ is an equivalence relation on the set of all p^s-Cauchy sequences in X.*
2. *If (x_n) is a p^s-Cauchy sequence then the sequence $(p(x_n))$ is convergent and $\lim_n p(x_n) = \lim_n p(y_n)$ whenever (y_n) is a p^s-Cauchy sequence equivalent to (x_n).*

Proof. The verification of 1 is routine.

2. The inequalities $p(x_n) - p(x_m) \le p(x_n - x_m) \le p^s(x_n - x_m)$, valid for all $m, n \in \mathbb{N}$, and the fact that (x_n) is p^s-Cauchy, imply that $(p(x_n))$ is a Cauchy sequence in $\mathbb{R}$, so it converges to some $\alpha \in \mathbb{R}$. If (y_n) is another p^s-Cauchy sequence equivalent to (x_n), then the inequalities

$$\begin{aligned} &p(x_n) - p(y_n) \le p(x_n - y_n) \le p^s(x_n - y_n), \\ &p(y_n) - p(x_n) \le p(y_n - x_n) \le p^s(y_n - x_n) = p^s(x_n - y_n) \end{aligned}$$

and the condition $\lim_n p^s(x_n - y_n) = 0$ imply $\lim_n p(x_n) = \lim_n p(y_n)$. $\square$

Denote by $\tilde{X}$ the linear space of all equivalence classes of p^s-Cauchy sequences with addition and multiplication by scalars defined, as usual, by $[(x_n)] + [(y_n)] = [(x_n + y_n)]$ and $\lambda[(x_n)] = [(\lambda x_n)]$, where $[(x_n)]$ denotes the equivalence class containing the p^s-Cauchy sequence (x_n). Based on Lemma 2.1.18, one can define on the space $\widetilde{X}$ an asymmetric norm $\tilde{p}$ by

$$\tilde{p}([(x_n)]) = \lim_n p(x_n) , \tag{2.1.32}$$

for any p^s-Cauchy sequence (x_n) in X.

As it is known, equipped with the norm

$$\widetilde{p^s}([(x_n)]) = \lim_n p^s(x_n) , \tag{2.1.33}$$

the space $(\tilde{X}, \widetilde{p^s})$ is a Banach space and the mapping $i : X \to \tilde{X}$ defined by

$$i(x) = [(x_n)] \quad \text{where} \quad x_n = x, \forall n \in \mathbb{N} , \tag{2.1.34}$$

is a linear isometry of X into $(\tilde{X}, \widetilde{p^s})$ such that $i(X)$ is $\widetilde{p^s}$-dense in $\tilde{X}$. Hence, the fact that (X, p) is biBanach, meaning that $(\tilde{X}, (\tilde{p})^s)$ is Banach, will follow once we prove the equality

$$(\tilde{p})^s = \widetilde{p^s} . \tag{2.1.35}$$

For a p^s-Cauchy sequence (x_n) in X, let $\alpha_n = p(x_n)$ and $\beta_n = p(-x_n)$, $n \in \mathbb{N}$. Then

$$\begin{aligned}\widetilde{p^s}([x_n]) &= \lim_n p^s(x_n) = \lim_n(\alpha_n \vee \beta_n) = (\lim_n \alpha_n) \vee (\lim_n \beta_n) \\ &= \tilde{p}([(x_n)]) \vee \tilde{p}(-[(x_n)]) = (\tilde{p})^s([(x_n)]) .\end{aligned}$$

Remark 2.1.19. The fact that $\alpha_n \to \alpha$ and $\beta_n \to \beta$ implies $\alpha_n \vee \beta_n \to \alpha \vee \beta$ follows from the relations

$$\alpha_n \vee \beta_n = \frac{\alpha_n + \beta_n + |\alpha_n - \beta_n|}{2} \to \frac{\alpha + \beta + |\alpha - \beta|}{2} = \alpha \vee \beta .$$

We summarize the results in the following theorem.

Theorem 2.1.20. *Let (X, p) be an asymmetric normed space, $\tilde{X}$ the space constructed above and $\tilde{p}$, $\widetilde{p^s}$ the norms on $\tilde{X}$ given by* (2.1.32) *and* (2.1.33)*, respectively.*

1. *The space $(\tilde{X}, \tilde{p})$ is biBanach, or, equivalently, $(\tilde{X}, (\tilde{p})^s)$ is a Banach space.*
2. *The mapping $i : X \to \tilde{X}$, defined by* (2.1.34)*, is a linear isometry of (X, p) into $(\tilde{X}, \tilde{p})$ and the space $i(X)$ is $\tilde{p}^s$-dense in $\tilde{X}$.*
3. *If (Y, q) is an asymmetric biBanach space such that (X, p) is isometrically isomorphic to a q^s-dense subspace of Y, then (Y, q) is isometrically isomorphic to $(\tilde{X}, \tilde{p})$.*

2.1.6 Asymmetric topologies on normed lattices

Alegre, Ferrer and Gregori, [77] and [6, 8], introduced an asymmetric norm on a normed lattice and studied the properties of the induced quasi-uniformity and topology, in connection with the usual properties of normed lattices.

An *ordered vector space* is real vector space X equipped with a partial order relation such that

$$x \le y \ \Rightarrow \ x + z \le y + z \quad \text{and} \quad tx \le ty , \tag{2.1.36}$$

for all $z \in X$ and $t \ge 0$. Denoting by X_+ the cone of positive elements, $X_+ = \{x \in X; x \ge 0\}$, it follows that

$$x \le y \iff y - x \in X_+ .$$

The ordered vector space $(X, \leq)$ is called a *vector lattice* if every pair x, y of elements in X admits a lowest upper bound $x \vee y$. Since

$$x \leq y \iff -y \leq -x ,$$

it follows that

$$x \wedge y = -\left((-x) \vee (-y)\right) ,$$

so that every pair of elements $x, y \in X$ has a greatest lower bound.

Put

$$x^+ = x \vee 0, \;\; x^- = (-x) \vee 0 = -(x \wedge 0) \quad \text{and} \quad |x| = x^+ + x^- .$$

It follows that $x = x^+ - x^-$.

A norm $\|\cdot\|$ on an ordered vector space $(X, \leq)$ is called a *lattice norm* if it satisfies one of the following equivalent conditions:

$$\begin{aligned} &\text{(i)} \quad |x| \leq |y| \;\; \Rightarrow \;\; \|x\| \leq \|y\|, \\ &\text{(ii)} \quad 1^\circ. \;\; \||x|\| = \|x\|, \;\; \text{and} \;\; 2^\circ. \;\; 0 \leq x \leq y \;\; \Rightarrow \;\; \|x\| \leq \|y\| , \end{aligned} \tag{2.1.37}$$

for all $x, y \in X$. An ordered vector space equipped with a lattice norm is called a *normed lattice* and is denoted by $(X, \|\cdot\|, \leq)$. If, in addition, $(X, \|\cdot\|)$ is a Banach space, then $(X, \|\cdot\|, \leq)$ is called a *Banach lattice*.

A normed lattice $(X, \|\cdot\|, \leq)$ is called an L-space, M-space or an E-space, provided that

$$\begin{aligned} &\text{(L)} && \|x + y\| = \|x\| + \|y\|, \\ &\text{(M)} && \|x \vee y\| = \|x\| \vee \|y\|, \\ &\text{(E)} && \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 , \end{aligned}$$

for all positive $x, y \in X$.

To an asymmetric norm p on a vector space X one can associate the following norms, defined for $x \in X$ by the equalities:

$$\begin{aligned} &p_L^s(x) = p_L^s(x) = p(x) + p(-x), \qquad p_M^s(x) = p^s(x) = p(x) \vee p(-x), \\ &p_E^s(x) = \left(p(x)^2 + p(-x)^2\right)^{1/2} . \end{aligned} \tag{2.1.38}$$

The norm p_M^s is the usual norm p^s we have associated to an asymmetric norm p and all the norms given in (2.1.38) are equivalent.

A subset Z of a normed lattice X is called *increasing* if for every $y, z \in X,\ z \in Z$ and $z \leq y$ implies $y \in Z$. It is called *decreasing* if for every $y, z \in X,\ z \in Z$ and $y \leq z$ implies $y \in Z$.

For a normed lattice $(X, \|\cdot\|, \leq)$ consider the functional

$$p(x) = \|x \vee 0\|, \quad x \in X . \tag{2.1.39}$$

Proposition 2.1.21. *The functional p given by* (2.1.39) *is an asymmetric norm on X with conjugate*

$$\bar{p}(x) = \|x^-\|, \quad x \in X \; . \tag{2.1.40}$$

The topology τ_p ($\tau_{\bar{p}}$) generated by p ($\bar{p}$) is T_0 but not T_1.

Proof. If $p(x) = p(-x) = 0$, then $\|x^+\| = 0$ implies $x^+ = 0$ and $\|(-x)^+\| = 0$ implies $x^- = (-x)^+ = 0$, so that $x = x^+ - x^- = 0$.

Also, $0 \le (x+y)^+ \le x^+ + y^+$ implies

$$p(x+y) = \|(x+y)^+\| \le \|x^+ + y^+\| \le \|x^+\| + \|y^+\| = p(x) + p(y) \; .$$

The positive homogeneity is obvious.

The conjugate $\bar{p}$ of p satisfies the equalities

$$\bar{p}(x) = p(-x) = \|(-x)^+\| = \|x^-\| \; .$$

Since p is an asymmetric norm the topology τ_p is T_0. The topology is T_1 if and only if $p(x) > 0$ for every $x \neq 0$ (see Proposition 1.1.8.3). But $p(x) = \|x^+\| = 0$ is equivalent to $x^+ = 0$, that is $x \le 0$. That is, excepting the trivial case $X_+ = X$, there are non-null elements $x \in X$ with $x \le 0$. $\square$

The remark concerning the separation properties is taken from [51], where some properties of convergent sequences were also proved. Recall that we denote by $L_\rho((x_n))$ the set of all ρ-limits of a sequence (x_n) in a quasi-semimetric space (X, ρ), see (1.1.8).

Proposition 2.1.22 ([51]). *Let $(X, \|\cdot\|)$ be a normed lattice, p the asymmetric norm given by* (2.1.39) *and (x_n) a sequence in X.*

1. *If (x_n) is p-convergent, then $L_\rho((x_n))$ is increasing.*
2. *The sequence (x_n) is p-convergent to every $z \in X$ such that $x_n \le z$ for all $n \in \mathbb{N}$.*

   Similar results hold for $\bar{p}$-convergence.

3. *If (x_n) is $\bar{p}$-convergent, then $L_{\bar{\rho}}((x_n))$ is decreasing.*
4. *The sequence (x_n) is $\bar{p}$-convergent to every $y \in X$ such that $y \le x_n$ for all $n \in \mathbb{N}$.*

The following proposition contains some properties of this asymmetric norm and of the corresponding topology and quasi-uniformity.

Proposition 2.1.23. *Let $(X, \|\cdot\|, \le)$ be a normed lattice and p the asymmetric norm given by* (2.1.39).

1. *The norms p^s_M, p^s_L and p^s_E are mutually equivalent norms on X which are also equivalent to the original norm. Further, if X is an M-space, an L-space, or an E-space, then p^s_M, p^s_L, respectively p^s_E agree with the original norm $\|\cdot\|$.*

2. *The quasi-uniformity* $\mathcal{U}_p$ determines *the normed lattice structure in the sense that* $\tau_{\|\cdot\|} = \tau(\mathcal{U}_p^s)$ *and* $\mathrm{Graph}(\le) = \bigcap \mathcal{U}_{\bar{p}}$, *where* $\mathrm{Graph}(\le) = \{(x,y) \in X \times X : x \le y\}$.
3. *A linear functional* $\varphi : X \to \mathbb{R}$ *is* (p,u)*-continuous if and only if it is* $(\|\cdot\|, |\cdot|)$*-continuous and positive, that is* $\varphi(x) \ge 0$ *whenever* $x \ge 0$.
4. *A subset* Y *of* X *is* p*-open* ($\bar{p}$*-open*) *if and only if it is* $\|\cdot\|$*-open and decreasing* (*resp. increasing*).

Proof. 1. Since the norms p_L^s, p_L^s, p_L^s are mutually equivalent, it remains to show their equivalence with $\|\cdot\|$. From (2.1.40)

$$\|x\| = \|x^+ - x^-\| \le \|x^+\| + \|x^-\| = p(x) + p(-x) = p_L^s(x) .$$

On the other side, by (2.1.37), $x^+ \le |x|$, $x^- \le |x|$ implies $\|x^+\| \le \|x\|$ and $\|x^-\| \le \|x\|$, so that

$$p_L^s(x) \le \|x^+\| + \|x^-\| \le 2\|x\| .$$

If X is an L-space, then

$$p_L^s(x) = \|x^+\| + \|x^-\| = \|x^+ + x^-\| = \||x|\| = \|x\| .$$

The case when X is an M-space follows similarly using the equality $|x| = x^+ \vee x^-$.

If X is an E-space, then

$$\begin{aligned} 2\left(p_E^s(x)\right)^2 &= 2\left(\|x^+\|^2 + \|x^-\|^2\right) \\ &= \|x^+ + x^-\|^2 + \|x^+ - x^-\|^2 = \||x|\|^2 + \|x\|^2 = 2\|x\|^2 . \end{aligned}$$

2. The quasi-uniformity $\mathcal{U}_p$ is generated by the set

$$U_\varepsilon = \{(x,y) \in X \times X : p(y-x) < \varepsilon\} ,$$

and the uniformity $\mathcal{U}_p^s$ by the sets

$$U_\varepsilon^s = \{(x,y) \in X \times X : p_M^s(y-x) < \varepsilon\} .$$

Since the norm p_M^s is equivalent to the norm $\|\cdot\|$, it follows that it generates the same topology as $\|\cdot\|$.

To prove the equality $\mathrm{Graph}(\le) = \bigcap \mathcal{U}_{\bar{p}}$ observe that

$$\begin{aligned} (x,y) \in \bigcap \mathcal{U}_{\bar{p}} &\iff \forall \varepsilon,\ \bar{p}(y-x) < \varepsilon \iff \bar{p}(y-x) = 0 \iff \|(x-y)^+\| = 0 \\ &\iff (x-y)^+ = 0 \iff x - y \le 0 \iff x \le y . \end{aligned}$$

3. Let $f : (X,p) \to (\mathbb{R}, u)$ be linear and continuous. Then f is $(p_M^s, |\cdot|)$-continuous, and since the norm p_M^s is equivalent to $\|\cdot\|$, f is $(\|\cdot\|, |\cdot|)$-continuous.

The (p,u)-continuity of f implies the existence of $\beta \geq 0$ such that

$$\forall z \in X, \quad f(z) \leq \beta p(z)\ .$$

If $f(x) < 0$ for some $x \geq 0$, then $y = x/f(x) \leq 0$, and so $p(y) = \|y^+\| = 0$, that leads to the contradiction

$$1 = f(y) \leq \beta p(y) = 0\ .$$

Conversely, suppose that $f : (X, \|\cdot\|) \to \mathbb{R}, |\cdot|)$ is linear, continuous and positive. Then there exists $\beta \geq 0$ such that

$$f(x) \leq \beta\|x\|,\ x \in X\ ,$$

implying

$$f(x) = f(x^+) - f(x^-) \leq f(x^+) \leq \beta\|x^+\| = \beta p(x)\ ,$$

for all $x \in X$, proving the (p,u)-continuity of the functional f.

4. Suppose that $G \subset X$ is p-open. For $x \in G$ there exists $r > 0$ such that $B_p(x,r) \subset G$ Then $B_{\|}\cdot\|(x,r) \subset B_p(x,r)$, showing that G is $\|\cdot\|$-open, and $y \leq x$ implies $(y-x)^+ = 0$ and $p(y-x) = \|(y-x)^+\| = 0 < r$.

Conversely, suppose that G is $\|\cdot\|$-open and decreasing and let $x \in G$. Then there exists $r > 0$ such that $B_{\|}\cdot\|(x,r) \subset G$. We shall show that $B_p(x,r) \subset G$. Indeed, let $y \in B_p(x,r)$. If $y \leq x$, then $y \in G$, because G is decreasing. If $y > x$, then $(y-x)^+ = y - x$, so that $\|y-x\| = \|(y-x)^+\| = p(y-x) < r$, showing that $y \in B_{\|}\cdot\|(x,r) \subset G$.

The case of $\bar{p}$-open sets can be treated similarly. □

Remark 2.1.24. Properties 1–3 are taken from [77] and 4 from [6].

In all examples given below of asymmetric normed lattices the order is the pointwise order

$$x \leq y \iff \forall k \in \mathbb{N},\ \ x_k \leq x_k,$$

if $x = (x_k)$ and $y = (y_k)$ are sequences (possibly finite) of real numbers, and

$$f \leq g \iff \forall t \in T,\ \ f(t) \leq g(t),$$

if f, g are real-valued functions defined on a set T.

Consider the Banach lattices ℓ^p, $1 \leq p < \infty$, of all real sequences $x = (x_k)$ such that

$$\|x\|_p := \big(\sum_k |x_k|^p\big)^{1,p}.$$

By ℓ^∞ one denotes the Banach lattice of all bounded sequences with the sup-norm $\|\cdot\|_\infty$ and by c_0 its subspace formed by all converging to 0 sequences.

If T is a Hausdorff compact topological space, then $C(T)$ denotes the Banach lattice of all real-valued continuous functions on T with the sup-norm $\|\cdot\|_\infty$.

In all of these cases we shall use the notation

$$\|x\|^+ = \|x^+\| \quad \text{and} \quad \|x\|^- = \|x^-\|$$

for the asymmetric norm and its conjugate in a normed lattice $(X, \|\cdot\|)$.

In $\mathbb{R}^m$ one considers the Euclidean norm $\|\cdot\|_2$, but the results hold for any other lattice norm on $\mathbb{R}^m$, because for two equivalent lattice norms $\|\cdot\|$, $|||\cdot|||$ on $\mathbb{R}^m$, the corresponding asymmetric norms $\|\cdot\|^+$, $|||\cdot|||^-$ are also equivalent ([51, Proposition 2.5].

We mention the following characterizations of convergence in some of these asymmetric normed lattices obtained in [51].

Proposition 2.1.25.

1. *If a sequence* $x_n = (x_{n,i})_{i\in\mathbb{N}}$, $n \in \mathbb{N}$, *in* $X = \ell^p$, $1 \le p \le \infty$, *or in* $X = c_0$, *is* $\|\cdot\|_p^+$*-convergent to* $x = (x_i)_{i\in\mathbb{N}} \in X$, *then*

$$\forall i \in \mathbb{N}, \quad x_{n,i} \xrightarrow{u} x_i \quad \text{as} \quad n \to \infty. \tag{2.1.41}$$

2. *If* $1 \le p < \infty$, *then* $x_n \xrightarrow{\|\cdot\|_p^+} x$ *if and only if* (2.1.41) *holds and*

$$\sup_{n\in\mathbb{N}} \sum_{i=m}^{\infty} (x_{n,i}^+)^p \to 0 \quad \text{as} \quad m \to \infty. \tag{2.1.42}$$

3. *If* $X = c_0$, *then* $x_n \xrightarrow{\|\cdot\|_\infty^+} x$ *if and only if* (2.1.41) *holds and*

$$\sup_{n\in\mathbb{N}} \sup_{i\ge m} x_{n,i}^+ \to 0 \quad \text{as} \quad m \to \infty. \tag{2.1.43}$$

The following completeness results for these concrete Banach lattices were obtained also in [51], as consequences of some more general results concerning completeness in asymmetric normed lattices.

Theorem 2.1.26. *The asymmetric normed lattices* $(\mathbb{R}^m, \|\cdot\|_2^{\pm})$, $(C(T), \|\cdot\|_\infty^{\pm})$, $(\ell^\infty, \|\cdot\|_\infty^{\pm})$, $(c_0, \|\cdot\|_\infty^{+})$ *and* $(\ell^p, \|\cdot\|_p^{+})$, $1 \le p < \infty$, *are all left K-sequentially complete.*

Now we shall present some Baire properties of the asymmetric topology. As it is remarked in [6, Proposition 1] the asymmetric topology τ_p of a normed lattice is never Baire.

Proposition 2.1.27. *Let* $(X, \|\cdot\|, \le)$ *be a normed lattice.*

1. *Every nonempty p-open subset of X is p-dense in X.*
2. *Any p-dense increasing subset of X is $\|\cdot\|$-dense in X. Similarly, a $\bar{p}$-dense decreasing subset of X is $\|\cdot\|$-dense in X.*
3. *The associated asymmetric normed space (X, p) is never Baire.*

Proof. 1. Let G be a nonempty p-open subset of X. For an arbitrary nonempty $U \in \tau_p$ let $x \in G$ and $y \in U$. Since both G and U are decreasing it follows that $z = x \wedge y \in G \cap U$, so that G is p-dense in X.

2. Suppose that $Y \subset X$ is p-dense and increasing. For $x \in X$ and $\varepsilon > 0$ there exists $y \in Y$ such that $p(y - x) < \varepsilon$. Then

$$y = x + (y - x) \leq x + (y - x)^+$$

implies $z := x + (y - x)^+ \in Y$. Since

$$\|z - x\| = \|(y - x)^+\| = p(y - x) < \varepsilon ,$$

it follows that Y is $\|\cdot\|$-dense in X.

The second case can be treated similarly.

3. For $x < 0$ in X consider the family $G_n = B_p(nx, 1),\ n \in \mathbb{N}$, of p-open sets and show that $\cap_n G_n = \emptyset$. Indeed, if $y \in \cap_n G_n = \emptyset$, then

$$\begin{aligned} n\|x\| = \|nx\| = \|-nx\| = \|(-nx)^+\| &= \|(y - nx - y)^+\| \\ \leq \|(y - nx)^+\| + \|(-y)^+\| &< 1 + \|(-y)^+\| , \end{aligned}$$

for all $n \in \mathbb{N}$, leading to the contradiction $\|(-y)^+\| = \infty$.

Consequently, $G_n,\ n \in \mathbb{N}$, is a family of p-open p-dense subsets of X whose intersection is not p-dense in X, showing that X is not a Baire space. □

For this reason the authors defined in *loc. cit.* another property: a normed lattice is called *quasi-Baire* if the intersection of any sequence of monotonic (all of the same kind) $\|\cdot\|$-dense sets is $\|\cdot\|$-dense. By a monotonic set one understands a set that is increasing or decreasing. Recall that a bitopological space (T, τ, ν) is called *pairwise Baire* provided the intersection of any sequence of τ-open ν-dense sets is ν-dense, and the intersection of any sequence of ν-open τ-dense sets is τ-dense (see Subsection 1.2.4).

Let $(X, \|\cdot\|, \leq)$ be a normed lattice and X^* the dual space of $(X, \|\cdot\|)$. A subset $F \subset X^*$ is called *order determining* if

$$x \leq y \iff \forall \varphi \in F,\ \varphi(x) \leq \varphi(y) .$$

The following proposition puts in evidence the relevance of this notion for the quasi-Baire property.

Proposition 2.1.28. *Let $(X, \|\cdot\|, \leq)$ be a normed lattice and F an order determining subset of X^*. If*

$$\sup_{x \in X} \inf_{\varphi \in F} \varphi(x) > 0 ,$$

then X is a quasi-Baire space.

In fact, the proof given in [6] shows that X does not contain decreasing proper dense subsets, so it is quasi-Baire in a trivial way.

Based on this proposition one can give the following example of a quasi-Baire lattice.

Example 2.1.29 ([6]). Let $\mathcal{A}$ be an algebra of subsets of a nonempty set T. Consider the linear space $\ell_0^\infty(T,\mathcal{A})$ generated by the set $\{\chi_A : A \in \mathcal{A}\}$ of characteristic functions of sets in $\mathcal{A}$, equipped with the norm $\|x\| = \sup\{|x(t)| : t \in T\}$ and the pointwise order.

Then $\ell_0^\infty(T,\mathcal{A})$ is a quasi-Baire space.

Let $X = \ell_0^\infty(T,\mathcal{A})$. It is well known that the dual of X is the space $\mathrm{ba}(T,\mathcal{A})$ of all finitely additive bounded measures on $\mathcal{A}$. It is obvious that the Dirac measures $\delta_t,\ t \in T$, defined by $\delta_t(x) = x(t),\ x \in X$, determines the order in X. Because $\delta_t(\chi_T) = 1$, for all $t \in T$, it follows that

$$\sup\{\inf_{t\in T} \delta_t(x) : x \in X\} \geq \inf_{t\in T} \delta_t(\chi_T) = 1 ,$$

so that, By Proposition 2.1.28, X is quasi-Baire.

Theorem 2.1.30 ([6]). *A normed lattice $(X, \|\cdot\|, \leq)$ is quasi-Baire if and only if the associated bitopological space $(X, \tau_p, \tau_{\bar{p}})$ is pairwise Baire.*

Proof. Suppose that X is quasi-Baire and let $G_n,\ n \in \mathbb{N}$, be a family of p-open $\bar{p}$-dense subsets of X. By Proposition 2.1.27.1, each G_n is decreasing, so that, by the second assertion of the same proposition, G_n is $\|\cdot\|$-dense in X. Since X is quasi-Baire, it follows that $\cap_n G_n$ is $\|\cdot\|$-dense in X, and so $\bar{p}$-dense too.

Similarly, if $G_n,\ n \in \mathbb{N}$, is a family of $\bar{p}$-open p-dense subsets of X, then $\cap_n G_n$ is p-dense in X.

Consequently, X is pairwise Baire.

Conversely, suppose that X is pairwise Baire and let $G_n,\ n \in \mathbb{N}$, be a family of decreasing $\|\cdot\|$-open and $\|\cdot\|$-dense subsets of X. By Proposition 2.1.23.4, each G_n is p-open and $\bar{p}$-dense in X, so that their intersection is $\bar{p}$-dense in X. By Proposition 2.1.27.2, $\cap_n G_n$ is $\|\cdot\|$-dense in X.

If the sets $G_n,\ n \in \mathbb{N}$, are increasing $\|\cdot\|$-open and $\|\cdot\|$-dense in X, one proceeds similarly. □

We shall present now an example given by Alegre [2] of an asymmetric dual of a normed lattice.

Consider the real Banach space ℓ^1 with the usual norm

$$\|x\|_1 = \sum_{i=1}^{\infty} |x_i|, \quad \text{for} \quad x = (x_i) \in \ell^1 ,$$

and the pointwise order

$$x \leq y \iff \forall i \in \mathbb{N}, \quad x_i \leq y_i .$$

In this case

$$x^+ = (x_i^+)_{i\in\mathbb{N}} \, .$$

The dual of ℓ^1 is the space ℓ^∞ of all bounded sequences of real numbers with the supremum norm

$$\|x\|_\infty = \sup\{|x_i| : i \in \mathbb{N}\}, \quad \text{for} \quad x = (x_i) \in \ell^\infty \, .$$

The mapping $y \mapsto f_y$, where for $y \in \ell^\infty$,

$$f_y(x) = \sum_{i=1}^{\infty} x_i y_i, \quad \text{for} \quad x = (x_i) \in \ell^1 \, ,$$

is an isometric isomorphism between $(\ell^1)^*$ and ℓ^∞.

Let

$$X = \{x = (x_n) \in \ell^1 : x_1 + x_2 = 0\} \quad \text{and} \quad H = \{y = (y_n) \in \ell^\infty : y_1 + y_2 = 0\} \, .$$

Consider on X the induced norm $\|\cdot\|_1$ and the associated asymmetric norm $p(x) = \|x^+\|_1$, $x \in X$, and consider H equipped with the norm $\|\cdot\|_\infty$. Let $X_p^\flat = (X,p)^\flat$ be the asymmetric dual of X and let $\mathrm{sp}(X_p^\flat) = X_p^\flat - X_p^\flat$ be the linear subspace of $X^* = (X, \|\cdot\|_1)^*$ generated by $X_p^\flat$ equipped with the norm

$$\|f\|_1^* = \sup\{|f(x)| : x \in X,\ \|x\|_1 \le 1\} \, .$$

Denote by (e_n) the canonical Schauder basis of ℓ^1, where for each $n \in \mathbb{N}$,

$$e_{n,i} = \delta_{n,i}, \quad i \in \mathbb{N} \, ,$$

($\delta_{n,i}$ is the Kronecker symbol).

Proposition 2.1.31. *The mapping Φ defined on $(\mathrm{sp}(X_p^\flat), \|\cdot\|_1^*)$ by*

$$\Phi(f) = \left(\frac{1}{2}(e_1 - e_2), \frac{1}{2}(e_2 - e_1), f(e_3), \dots\right), \quad f \in \mathrm{sp}(X_p^\flat) \, , \tag{2.1.44}$$

is an isometrical isomorphism between the spaces $(\mathrm{sp}(X_p^\flat), \|\cdot\|_1^*)$ and $(H, \|\cdot\|_\infty)$.

Proof. Let $f \in \mathrm{sp}(X_p^\flat) \subset X^*$. Then there exists $\beta \ge 0$ such that

$$|f(x)| \le \beta \|x\|_1, \quad x \in X \, ,$$

which shows that $\Phi(f) \in H$. It is obvious that $\Phi : X \to H$ is linear.

If $\Phi(f) = 0$, then $f(e_1 - e_2) = 0, f(e_3) = 0, \dots$. Since every $x \in X$ can be written as $x = x_1(e_1 - e_2) + x_3 e_3 + \cdots$ it follows that $f(x) = 0$, that is f is the null functional on X, showing that Φ is injective.

To show that Φ is surjective, for $y = (y_n) \in H$ put

$$\varphi(x) = 2x_1 y_1^+ + \sum_{i=3}^{\infty} x_i y_i^+ \quad \text{and} \quad \psi(x) = 2x_1 y_1^- + \sum_{i=3}^{\infty} x_i y_i^- \,.$$

Since $\varphi, \psi : X \to \mathbb{R}$ are positive and $(\|\cdot\|_1, |\cdot|)$-continuous, it follows that $\varphi, \psi \in X_p^\flat$, and $f = \varphi - \psi \in \mathrm{sp}(X_p^\flat)$. Since $\Phi(f) = y$, the mapping Φ is surjective, so bijective, and

$$\Phi^{-1}(y)(x) = 2x_1 y_1 + \sum_{i \geq 3} x_i y_i \,,$$

for any $x = (x_1, -x_1, x_3, \dots) \in X$.

The functional $\Phi^{-1}(y)$ acts on ℓ^1 by the rule

$$\Phi^{-1}(y)(z) = z_1 y_1 - z_2 y_2 + \sum_{i \geq 3} z_i y_i \,,$$

for any $z = (z_1, z_2, \dots) \in \ell^1$.

It remained to prove that Φ is an isometry.

For $z \in \ell^1$ with $\|z\|_1 \leq 1$, the element x given by

$$x = \left(\frac{1}{2}(z_1 - z_2), \frac{1}{2}(z_2 - z_1), z_3, \dots \right)$$

belongs to X, $\|x\|_1 \leq \|z\|_1 \leq 1$ and $\Phi^{-1}(y)(x) = \Phi^{-1}(y)(z)$.

It follows that

$$|\Phi^{-1}(y)(z)| = |\Phi^{-1}(y)(x)| \leq \|\Phi^{-1}(y)\|_1^* \,.$$

Since the dual of $(\ell^1, \|\cdot\|_1)$ is $(\ell^\infty, \|\cdot\|_\infty)$,

$$\sup\{|\Phi^{-1}(y)(z)| : z \in \ell^1, \ \|z\|_1 \leq 1\} = \|y\|_\infty \,,$$

so that $\|y\|_\infty \leq \|\Phi^{-1}(y)\|_1^*$.

On the other side, for every $x \in X$,

$$|\Phi^{-1}(y)(x)| = |2x_1 y_1 + \sum_{i \geq 3} x_i y_i| \leq \|y\|_\infty \, \|x\|_1 \,,$$

implying $\|\Phi^{-1}(y)\|_1^* \leq \|y\|_\infty$.

Consequently, $\|\Phi^{-1}(y)\|_1^* = \|y\|_\infty$. □

2.2 Hahn-Banach type theorems and the separation of convex sets

One of the fundamental principles of functional analysis is the Hahn-Banach extension theorem for a linear functional dominated by a sublinear functional. Based on this theorem one can prove extension results for lsc linear functionals on asymmetric normed spaces and on asymmetric LCS. Some separation results for convex subsets of asymmetric LCS, relying on the properties of the Minkowski gauge functional and on the extension results, are also proved. As an application, an asymmetric version of the Krein-Milman theorem is proved.

2.2.1 Hahn-Banach type theorems

Let X be a real vector space. Recall that a functional $p : X \to \mathbb{R}$ is called *sublinear* if

$$\text{(i) } p(\lambda x) = \lambda p(x) \quad \text{and} \quad \text{(ii)} \quad p(x + y) \leq p(x) + p(y) ,$$

for all $x, y \in X$ and $\lambda \geq 0$.

Notice that, as defined, a sublinear functional need not be positive. A positive sublinear functional is an asymmetric seminorm.

A sublinear functional is called a *seminorm* if instead of (i) it satisfies

$$p(\lambda x) = |\lambda| p(x) ,$$

for all $x \in X$ and $\lambda \in \mathbb{R}$. A seminorm is necessarily positive, that is $p(x) \geq 0$ for all $x \in X$. A seminorm is called a norm if

$$\text{(iii)} \quad p(x) = 0 \iff x = 0 .$$

A function $f : X \to \mathbb{R}$ is said to be *dominated* by a function $g : X \to \mathbb{R}$ if

$$f(x) \leq g(x) ,$$

for all $x \in X$. If the above inequality holds only for x in a subset Y of X, then we say that f is dominated by g on Y.

Theorem 2.2.1 (Hahn-Banach Extension Theorem)**.** *Let X be a real vector space and $p : X \to \mathbb{R}$ a sublinear functional. If Y is a subspace of X and $f : Y \to \mathbb{R}$ is a linear functional dominated by p on Y then there exists a linear functional $F : X \to \mathbb{R}$ dominated by p on X such that $F|_Y = f$.*

We present now the extension results in the asymmetric case.

Theorem 2.2.2. *Let (X, p) be a space with asymmetric seminorm.*

1. *If Y is a subspace of X and $\varphi_0 : Y \to \mathbb{R}$ is a continuous linear functional on the asymmetric seminormed space $(Y, p|_Y)$, then there exists a continuous linear functional $\varphi : X \to \mathbb{R}$ such that*

$$\varphi|_Y = \varphi_0 \quad \textit{and} \quad \|\varphi|_p = \|\varphi_0|_p \ .$$

2. *If x_0 is a point in X with $p(x_0) > 0$, then there exists a continuous linear functional $\varphi : X \to \mathbb{R}$ such that*

$$\|\varphi|_p = 1 \quad \textit{and} \quad \varphi(x_0) = p(x_0) \ . \tag{2.2.1}$$

Proof. 1. The inequality

$$\varphi_0(y) \le \|\varphi_0|p(y) \ ,$$

valid for all $y \in Y$, shows that the linear functional φ_0 is dominated by the sublinear functional $q(\cdot) = \|\varphi_0|p(\cdot)$. By the Hahn-Banach Theorem (Theorem 2.2.1), it has a linear extension $\varphi : X \to \mathbb{R}$ dominated by q, that is

$$\varphi(x) \le \|\varphi_0|p(x) \ ,$$

for all $x \in X$, implying $\|\varphi| \le \|\varphi_0|$.

The relations

$$\begin{aligned}\|\varphi| &= \sup\{\varphi(x) : x \in X,\ p(x) \le 1\} \ge \sup\{\varphi(y) : y \in X,\ p(y) \le 1\} \\ &= \sup\{\varphi(x) : x \in X,\ p(x) \le 1\} \ ,\end{aligned}$$

prove the reverse inequality, so that $\|\varphi| = \|\varphi_0|$.

2. Let $Y = \mathbb{R}x_0$ the one-dimensional subspace generated by x_0. Define $\varphi_0 : Y \to \mathbb{R}$ by $\varphi_0(tx_0) = tp(x_0),\ t \in \mathbb{R}$.

Since for $t \ge 0,\ \varphi_0(tx_0) = tp(x_0) = p(tx_0)$ and $\varphi_0(tx_0) = tp(x_0) \le 0 \le p(tx_0)$ for $t < 0$, it follows that the linear functional φ_0 is dominated by p on Y, and $\|\varphi_0| \le 1$. The equality $\varphi(x_0/p(x_0)) = 1$ implies $\|\varphi_0| = 1$. □

Remark 2.2.3. 1. The conditions (2.2.1) are equivalent to

$$\varphi(x_0) = p(x_0) \quad \text{and} \quad \forall x \in X,\ \ \varphi(x) \le p(x) \ . \tag{2.2.2}$$

2. Taking $\psi = (1/p(x_0))\,\varphi$ it follows that ψ satisfies the conditions

$$\psi(x_0) = 1 \quad \text{and} \quad \forall x \in X,\ \ \psi(x) \le \frac{1}{p(x_0)}\, p(x) \ ,$$

or, equivalently,

$$\|\psi|_p = \frac{1}{p(x_0)} \quad \text{and} \quad \psi(x_0) = 1 \ .$$

We agree to call a functional φ satisfying the conclusions of the first point of the theorem above a *norm preserving extension* of φ_0.

From the second point of the theorem one obtains as corollary a well-known and useful result in normed spaces.

Corollary 2.2.4. *Let (X,p) be an asymmetric seminormed space, $x_0 \in X$ and $X_p^\flat$ its dual. If $p(x_0) > 0$ then*

$$p(x_0) = \sup\{\varphi(x_0) : \varphi \in X_p^\flat,\ \|\varphi|_p \le 1\} .$$

Proof. Denote by s the supremum in the right-hand side of the above formula. Since $\varphi(x_0) \le \|\varphi|_p\, p(x_0) \le p(x_0)$ for every $\varphi \in X_p^\flat$ with $\|\varphi|_p \le 1$, it follows that $s \le p(x_0)$. Choosing $\varphi \in X_p^\flat$ as in Theorem 2.2.2.2, it follows that $p(x_0) = \varphi(x_0) \le s$. □

In the case of asymmetric LCS one obtains the existence of continuous linear extensions.

Proposition 2.2.5. *Let (X,P) be an asymmetric LCS and Y a subspace of X. Then every $\tau(P)$-continuous linear functional on Y has a $\tau(P)$-continuous linear extension to the whole space X.*

Proof. Supposing P directed, then for $\psi \in Y_P^\flat$ there exists $p \in P$ and $\beta \ge 0$, such that

$$\forall y \in Y,\ \ \psi(y) \le \beta p(y) .$$

By Theorem 2.2.1 applied to the sublinear functional $q(\cdot) = \beta p(\cdot)$, ψ has a linear extension $\varphi : X \to \mathbb{R}$ such that $\varphi(x) \le \beta p(x)$ for all $x \in X$. It follows that φ is a continuous linear extension of ψ. □

As in the symmetric case, one can prove the existence of some functionals on an asymmetric normed space related to distances to a subspace, see [40, 48, 49]. This result will be applied in Sect. 2.5 to best approximation problems in asymmetric normed spaces.

Let (X,p) be an asymmetric normed space, Y a nonempty subset of X and $x \in X$. Due to the asymmetry of the norm we have to consider two *distances* from x to Y:

$$\begin{aligned} &\text{(i)}\ \ d_p(x,Y) = \inf\{p(y-x) : y \in Y\}, \quad \text{and} \\ &\text{(ii)}\ \ d_p(Y,x) = \inf\{p(x-y) : y \in Y\} . \end{aligned} \tag{2.2.3}$$

Observe that $d_p(Y,x) = d_{\bar{p}}(x,Y)$, where $\bar{p}$ is the norm conjugate to p.

Theorem 2.2.6. *Let Y be a subspace of a space with asymmetric norm (X,p) and $x_0 \in X$. If $d := d_p(x_0,Y) > 0$, then there exists a p-bounded linear functional $\varphi : X \to \mathbb{R}$ such that*

$$\text{(i) } \varphi|_Y = 0, \quad \text{(ii) } \|\varphi|_p = 1, \quad \textit{and} \quad \text{(iii) } \varphi(-x_0) = d .$$

If $\overline{d} := d_p(Y, x_0) > 0$, *then there exists a* p*-bounded linear functional* $\psi : X \to \mathbb{R}$ *such that*

$$\text{(j) } \psi|_Y = 0, \quad \text{(jj) } \|\psi|_p = 1, \quad \textit{and} \quad \text{(jjj) } \psi(x_0) = \overline{d}\,.$$

Proof. Suppose first that $\overline{d} = d_p(Y, x_0) > 0$, so that $x_0 \notin Y$. Let $Z := Y \dot{+} \mathbb{R}x_0$ ($\dot{+}$ stands for the direct sum) and let $\psi_0 : Z \to \mathbb{R}$ be defined by

$$\psi_0(y + tx_0) = t, \; y \in Y, \; t \in \mathbb{R}\,.$$

Then ψ_0 is linear, $\psi_0(y) = 0$, $\forall y \in Y$, and $\psi_0(x_0) = 1$. For $t > 0$ we have

$$p(y + tx_0) = tp(x_0 + t^{-1}y) \geq t\overline{d} = \overline{d} \cdot \psi_0(y + tx_0)\,,$$

so that

$$\psi_0(y + tx_0) = t \leq \frac{1}{\overline{d}} p(y + tx_0)\,.$$

Since this inequality obviously holds for $t \leq 0$, it follows that $\|\psi_0| \leq 1/\overline{d}$. Let (y_n) be a sequence in Y such that $p(x_0 - y_n) \to \overline{d}$ for $n \to \infty$ and $p(x_0 - y_n) > 0$ for all $n \in \mathbb{N}$. Then

$$\|\psi_0| \geq \psi_0 \left(\frac{x_0 - y_n}{p(x_0 - y_n)} \right) = \frac{1}{p(x_0 - y_n)} \to \frac{1}{\overline{d}}\,,$$

implying $\|\psi_0| \geq 1/\overline{d}$. Therefore $\|\psi_0| = 1/\overline{d}$.

If $\psi_1 : X \to \mathbb{R}$ is a linear functional such that

$$\psi_1|_Z = \psi_0 \quad \text{and} \quad \|\psi_1| = \|\psi_0|\,,$$

then the linear functional $\psi = \overline{d} \cdot \varphi_1$ fulfills the conditions (j)–(jjj).

Suppose now that $d = d_p(x_0, Y) > 0$, and let $Z := Y \dot{+} \mathbb{R}x_0$. Define $\varphi_0 : Z \to \mathbb{R}$ by

$$\varphi_0(y + tx_0) = -t \iff \varphi_0(y - tx_0) = t \quad \text{for } y \in Y \text{ and } t \in \mathbb{R}\,.$$

Then φ_0 is linear and, for $t > 0$, we have

$$p(y - tx_0) = tp(\frac{1}{t}y - x_0) \geq td = \overline{d} \cdot \varphi_0(y - tx_0)\,,$$

so that

$$\varphi_0(y - tx_0) \leq \frac{1}{d} p(y - tx_0)\,,$$

for $t > 0$. Since this inequality is obviously true if $\varphi_0(y - tx_0) = t \leq 0$, it follows that φ_0 is bounded and $\|\varphi_0| \leq 1/d$. Reasoning as above, one obtains the existence of a functional φ satisfying the conditions (i)–(iii). □

Other extension results can be found in [8, 28, 90, 224]. By studying quasi-uniformities on real vector spaces, Alegre, Ferrer and Gregori [5] were able to prove a Hahn-Banach type extension theorem for pseudo-topological vector spaces. More general and sophisticated versions of the Hahn-Banach theorem are also known, see, for instance, [82], [122], [188], and the book [83]. For an extensive list of references see the survey [30].

2.2.2 The Minkowski gauge functional – definition and properties

A subset Y of a vector space X is called *absorbing* if

$$\forall x \in X,\ \exists t > 0, \quad \text{such that} \quad x \in tY\ .$$

If Y is absorbing, then the *Minkowski functional* (or the *gauge function*) p_Y of the set Y is defined by

$$p_Y(x) = \inf\{t > 0 : x \in tY\}\ .$$

It follows that p_Y is a positive and positively homogeneous functional, and

$$Y \subset \{x \in X : p_Y(x) \le 1\}\ .$$

If Y is convex and absorbing, then p_Y is a positive sublinear functional and

$$\{x \in X : p_Y(x) < 1\} \subset Y \subset \{x \in X : p_Y(x) \le 1\}\ . \tag{2.2.4}$$

Now suppose that (X, P) is an asymmetric LCS and look for conditions on the set Y ensuring the (P, u)-continuity of p_Y.

Proposition 2.2.7. *Let Y be a convex absorbing subset of an asymmetric locally convex space (X, P).*

1. *The Minkowski functional p_Y is (P, u)-continuous if and only if 0 is a τ_P-interior point of Y.*
2. *If p_Y is (P, u)-continuous, then*

$$\tau_P\text{-}\operatorname{int}(Y) = \{x \in X : p_Y(x) < 1\}. \tag{2.2.5}$$

Proof. Suppose the family P to be directed.

1. If 0 is a τ_P-interior point of Y, then there exist $p \in P$ and $r > 0$ such that

$$B_p(0, r) \subset Y \subset \{x \in X : p_Y(x) \le 1\}\ ,$$

i.e.,

$$\forall x \in X, \quad p(x) \le r\ \Rightarrow\ p_Y(x) \le 1\ .$$

By Proposition 2.1.1, we have

$$\forall x \in X, \quad p_Y(x) \le \frac{1}{r} p(x)\ ,$$

which, by Proposition 2.1.12, implies the (P, u)-continuity of p_Y.

Conversely, suppose that p_Y is (P, u)-continuous. Since the set $(-\infty; 1)$ is τ_u-open in $\mathbb{R}$, the set $\{x \in X : p_Y(x) < 1\} = p_Y^{-1}((-\infty; 1))$ is τ_P-open, contains 0, and is contained in Y, implying $0 \in \tau_P\text{-int}(Y)$.

2. Suppose that p_Y is (P,u)-continuous. By the proof of the first point, $\{x \in X : p_Y(x) < 1\} \subset \tau_P\text{-int}(Y)$, so it remains to prove the reverse inclusion.

If $x \in \tau_P\text{-int}(Y)$, then there exist $p_1 \in P$ and $r > 0$ such that $B_{p_1}(x,r) \subset Y$. By Proposition 2.1.12 there exist $p_2 \in P$ and $\beta > 0$ such that $\forall x \in X,\ p_Y(x) \leq \beta p_2(x)$. If $p \in P$ is such that $p \geq p_i,\ i = 1,2,$ then $B_p(x,r) \subset B_{p_1}(x,r) \subset Y$ and

$$\forall x \in X, \quad p_Y(x) \leq \beta p(x) \ .$$

If $p(x) = 0$, then, by the above inequality, $p_Y(x) = 0 < 1$. If $p(x) > 0$, put $x_\alpha = (1+\alpha)x$ for $\alpha > 0$. If $0 < \alpha < r/p(x),$ then $p(x_\alpha - x) = \alpha p(x) < r$ so that $x_\alpha \in Y$ and $p_Y(x_\alpha) \leq 1$. But then, for any such α we have

$$p_Y(x) = \frac{1}{1+\alpha} p_Y(x_\alpha) \leq \frac{1}{1+\alpha} < 1 \ . \qquad \square$$

2.2.3 The separation of convex sets

The separation theorems for convex sets are very efficient tools in the treatment of optimization problems in Banach or locally convex spaces. The so far developed machinery allows us to prove the asymmetric analogs of the Eidelheit and Tukey separation theorems (Theorems 2.2.26 and 2.2.28 in [149]). The presentation follows [41].

Theorem 2.2.8. *Let (X,P) be an asymmetric locally convex space and $Y_1,\ Y_2$ two disjoint nonempty convex subsets of X with Y_1 τ_P-open.*

Then there exists a τ_P-continuous linear functional $\varphi : X \to \mathbb{R}$ such that

$$\forall y_1 \in Y_1,\ \forall y_2 \in Y_2 \quad \varphi(y_1) < \varphi(y_2) \ .$$

Proof. Let $y_i^0 \in Y_i,\ i = 1,2,$ and let $x_0 = y_2^0 - y_1^0$. Since the set Y_1 is τ_P-open and the topology τ_P is translation invariant, the set

$$Y := x_0 + Y_1 - Y_2 = \cup\{x_0 - y_2 + Y_1 : y_2 \in Y_2\}$$

is τ_P-open too. It is obvious that Y is also convex.

We have $0 = x_0 + y_1^0 - y_2^0 \in Y$ and $x_0 \notin Y$. Indeed, if $x_0 = x_0 + y_1 - y_2$, for some $y_1 \in Y_1$ and $y_2 \in Y_2$, then the element $y = y_1 = y_2$ would belong to the empty set $Y_1 \cap Y_2$.

By Proposition 2.2.7, the Minkowski functional p_Y of the τ_P-open convex set Y is sublinear, (P,u)-continuous and

$$Y = \{x \in X : p_Y(x) < 1\} \ . \tag{2.2.6}$$

Since $x_0 \notin Y$, it follows that $p_Y(x_0) \geq 1$. By Proposition 2.2.2.2, there exists a p_Y-bounded linear functional $\psi : X \to \mathbb{R}$ such that $\psi(x_0) = p_Y(x_0)$ and $\psi(x) \leq p_Y(x),\ x \in X$. Taking $\varphi = (1/p_Y(x_0))\psi$ it follows that

$$\varphi(x_0) = 1 \quad \text{and} \quad \forall x \in X,\ \varphi(x) = \frac{1}{p_Y(x_0)}\psi(x) \leq \frac{1}{p_Y(x_0)} p_Y(x) \leq p_Y(x) \ .$$

By Proposition 2.1.11, the functional φ is (P,u)-continuous. Because Y is τ_P-open and $0 \in Y$, by Proposition 2.2.7 we have $Y = \{x \in X : p_Y(x) < 1\}$. From (2.2.6) and the fact that $\varphi(x_0) = 1$, one obtains

$$\forall y_1 \in Y_1,\ \forall y_2 \in Y_2,\ 1 + \varphi(y_1) - \varphi(y_2) = \varphi(x_0 + y_1 - y_2) \le p_Y(x_0 + y_1 - y_2) < 1\ ,$$

implying

$$\forall y_1 \in Y_1,\ \forall y_2 \in Y_2, \quad \varphi(y_1) < \varphi(y_2)\ . \qquad \square$$

We prove now the asymmetric analog of Tukey's separation theorem.

Theorem 2.2.9. *Let (X,P) be an asymmetric locally convex space and Y_1, Y_2 two nonempty disjoint convex subsets of X, with Y_1 τ_P -compact and Y_2 τ_P-closed.*

Then there exists a τ_P-continuous linear functional $\varphi : X \to \mathbb{R}$ such that

$$\sup \varphi(Y_1) < \inf \varphi(Y_2)\ . \tag{2.2.7}$$

Proof. Suppose that P is directed. For $p \in P$ denote by B'_p the open unit p-ball, $B'_p = \{x \in X : p(x) < 1\}$.

Since $Y_1 \cap Y_2 = \emptyset$ and Y_2 is τ_P-closed, for every $y \in Y_1$ there exist $p_y \in P$ and $r_y > 0$ such that

$$(y + 2r_y B'_{p_y}) \cap Y_2 = \emptyset. \tag{2.2.8}$$

The τ_P-open cover $\{y + r_y B'_{p_y} : y \in Y_1\}$ of the τ_P-compact set Y_1, contains a finite subcover $\{y_k + r_k B'_{p_k} : k = 1, 2, \dots, n\}$, where $p_k = p_{y_k}$ and $r_k = r_{y_k}$ for $k = 1, \dots, n$. Take $p \in P$ such that $p \ge p_k,\ k = 1, 2, \dots, n,$ put $r := \min\{r_k : k = 1, 2, \dots, n\}$ and show that

$$(Y_1 + rB'_p) \cap Y_2 = \emptyset. \tag{2.2.9}$$

Indeed, if $y' = y + ru \in Y_2$ for some $y \in Y_1$, $u \in B'_p$, then, choosing $k \in \{1, 2, \dots, n\}$ such that $y \in y_k + r_k B'_{p_k}$, we have

$$y' = y + ru \in y_k + rB'_p + r_k B'_{p_k} \subset y_k + r_k B'_{p_k} + r_k B'_{p_k} = y_k + 2r_k B'_{p_k}\ ,$$

in contradiction to (2.2.8).

The set $Z := Y_1 + rB'_p$ is convex, τ_P-open and disjoint from Y_2. By Theorem 2.2.8, there exists $\varphi \in X^\flat_P$ such that

$$\forall y \in Y_1,\ \forall u \in B'_p,\ \forall y' \in Y_2 \quad \varphi(y) + r\varphi(u) < \varphi(y'). \tag{2.2.10}$$

By Proposition 2.1.11, there exists $q_1 \in P$ and $\beta > 0$ such that $\forall x \in X,\ \varphi(x) \le \beta q_1(x)$. If $q \in P$ is such that $q \ge \max\{p, q_1\}$, then $\varphi(x) \le \beta q(x),\ x \in X,$ and $B'_q \subset B'_p$, so that

$$\forall y \in Y_1,\ \forall u \in B'_q,\ \forall y' \in Y_2 \quad \varphi(y) + r\varphi(u) < \varphi(y'). \tag{2.2.11}$$

By (2.2.10), $\varphi \neq 0,$ so that by Propositions 2.1.7 and 2.1.8, $\|\varphi|_q = \sup \varphi(B'_q) > 0.$ Passing in (2.2.11) to supremum with respect to $u \in B'_q$, we get

$$\forall y \in Y_1,\ \forall y' \in Y_2 \quad \varphi(y) + r\|\varphi|_q \leq \varphi(y')\ ,$$

implying

$$r\|\varphi|_q + \sup \varphi(Y_1) \leq \inf \varphi(Y_2)\ .$$

It follows that

$$\sup \varphi(Y_1) < \inf \varphi(Y_2)\ . \qquad \square$$

Remark 2.2.10. The inequality in Theorem 2.2.8 can not be reversed, in the sense that, under the same hypotheses on the sets Y_1 and Y_2, we can not find a (P,u)-continuous linear functional ψ on X such that

$$\forall y_2 \in Y_2,\ \forall y_1 \in Y_1 \quad \psi(y_2) < \psi(y_1)\ .$$

This is due, on one hand, to the fact that the functional $-\varphi$ need not be (P,u)-continuous, where φ is the linear functional given by Theorem 2.2.8. On the other hand, analyzing the proof of Theorem 2.2.8, it follows that we should work with the set $Y' := x_0 + Y_2 - Y_1$ which need not be τ_P-open, because the τ_P-openness of Y_1 does not imply the τ_P-openness of $-Y_1$. For instance, if (X,p) is an asymmetric LCS, then the fact that the set Y_1 is τ_p-open implies that $-Y_1$ is $\tau_{\bar{p}}$-open, so that, in this case, there exists a $\bar{p}$-bounded linear functional $\psi : X \to \mathbb{R}$ such that

$$\forall y_2 \in Y_2\ \forall y_1 \in Y_1, \quad \psi(y_2) < \psi(y_1)\ .$$

The same caution must be taken when applying Theorem 2.2.9.

2.2.4 Extreme points and the Krein-Milman theorem

Following the ideas from the symmetric case, one can prove a Krein-Milman type theorem for asymmetric LCS. The proof is based on Tukey's separation theorem and on the fact that the intersection of an arbitrary family of extremal subsets of a convex set Y is an extremal subset of Y, provided it is nonempty. The presentation follows [41]. The asymmetric normed case was treated in [40].

We start by recalling some notions and facts. A point e in a convex subset of a vector space X is called an *extreme point* of Y provided that $(1-t)x + ty = e$ for some $x, y \in Y$ and $0 < t < 1$, implies $x = y = e$. A nonempty convex subset Z of Y is called an *extremal subset* of Y if $(1-t)x + ty \in Z$, for some $x, y \in Y$ and some $0 < t < 1$ implies $x, y \in Z$ (in fact, $[x;y] \subset Z$, by the convexity of Z). An extremal subset is called also a *face* of Y. Obviously, a one-point set $Z = \{e\}$ is an extremal subset of Y if and only if e is an extreme point of Y. Also, if W is an extremal subset of the extremal subset Z of Y then W is an extremal subset of Y too. In particular, if e is an extreme point of an extremal subset Z of Y, then e is an extreme point of Y. The intersection of a family of extremal subsets of Y is an extremal subset of Y provided it is nonempty. We denote by $\operatorname{ext} Y$ the (possibly empty) set of extreme points of the convex set Y.

The following proposition is an immediate consequence of the definitions.

Proposition 2.2.11. *Let Y be a nonempty convex subset of a vector space X and f a linear functional on X.*

If the set $Z = \{z \in Y : f(z) = \sup f(Y)\}$ is nonempty, then it is an extremal subset of Y.

A similar assertion holds for the set $W = \{w \in Y : f(w) = \inf f(Y)\}$.

We can state and prove now the Krein-Milman theorem in the asymmetric case.

Theorem 2.2.12. *Let (X, P) be an asymmetric locally convex space such that the topology τ_P is Hausdorff.*

Then any nonempty convex τ_P-compact subset Y of X coincides with the τ_P-closed convex hull of the set of its extreme points

$$Y = \tau_P\text{-cl-co}(\text{ext Y}) .$$

Proof. All the topological notions will concern the τ_P-topology of X so that we shall omit sometimes "τ_P-" in the following. By Proposition 1.1.63, for every $x \in X,\ x \neq 0,$ there exists $p \in P$ such that $p(x) > 0$, so that, by Theorem 2.2.2 (see also Remark 2.2.3.2), there exists $\varphi \in X_P^\flat$ with $\varphi(x) = 1$.

Fact I. *Every nonempty convex compact subset Z of X has an extreme point.*

Let

$$\mathcal{F} := \{F : F \text{ is a closed extremal subset of } Z\} ,$$

and define the order in $\mathcal{F}$ by $F_1 \leq F_2 \iff F_1 \subset F_2$ and show that the set $\mathcal{F}$ is nonempty and downward inductively ordered. Because Y is τ_P-compact and the topology τ_P is Hausdorff, it follows that Y is convex and τ_P-closed, so that $Y \in \mathcal{F}$. Since a totally ordered subfamily $\mathcal{G}$ of $\mathcal{F}$ has the finite intersection property, by the compactness of the set Z the set $G = \cap\mathcal{G}$ is nonempty, closed and extremal. Therefore $G \in \mathcal{F}$ is a lower bound for $\mathcal{G}$. By Zorn's Lemma the ordered set $\mathcal{F}$ has a minimal element F_0. If we show that F_0 is a one-point set, $F_0 = \{x_0\}$, then x_0 will be an extreme point of Z.

Suppose that F_0 contains two distinct points x_1, x_2, and let $p \in P$ be such that $p(x_1 - x_2) > 0$. Let φ be a p-bounded linear functional such that $\varphi(x_1 - x_2) = p(x_1 - x_2) > 0$ (see Theorem 2.2.2.2). It follows that $\varphi \in X^\flat$, so that φ is upper semi-continuous as a mapping from (X, τ_P) to $(\mathbb{R}, |\cdot|)$. By the compactness of the set F_0 the set

$$F_1 = \{x \in F_0 : \varphi(x) = \sup \varphi(F_0)\} = \{x \in F_0 : \varphi(x) \geq \sup \varphi(F_0)\}$$

is nonempty and closed. By Proposition 2.2.11, F_1 is an extremal subset of F_0, and so an extremal subset of Z. Therefore, $F_1 \in \mathcal{F},\ F_1 \subset F_0,$ and $x_2 \in F_0 \setminus F_1$ in contradiction to the minimality of F_0.

Fact II. $Y = \tau_P\text{-cl co}(\text{ext}\, Y)$.

The inclusion $\text{ext}(Y) \subset Y$ implies $Y_1 := \tau_P\text{-cl co}(\text{ext}\, Y) \subset Y$. As a closed subset of a compact set, the set Y_1 is convex and compact. Supposing that there exists a point $y_0 \in Y \setminus Y_1,$ then, by Theorem 2.2.9, there exists $\varphi \in X^\flat$ such that

$$\sup \varphi(Y_1) < \varphi(y_0). \tag{2.2.12}$$

Using again the upper semi-continuity of φ as a mapping from (X, τ_P) to $(\mathbb{R}, |\cdot|)$, we see that the set

$$F = \{y \in Y : \varphi(y) = \sup \varphi(Y)\} = \{y \in Y : \varphi(y) \geq \sup \varphi(Y)\} \ ,$$

is nonempty, convex and compact, so that, by Fact I, it has an extreme point e_1. Since F is an extremal subset of Y, it follows that e_1 is an extreme point of Y, implying $e_1 \in Y_1$. Taking into account (2.2.12) we obtain the contradiction

$$\sup \varphi(Y) = \varphi(e_1) \leq \sup \varphi(Y_1) < \varphi(y_0) \leq \sup \varphi(Y) \ . \qquad \square$$

The following question remains open.

Problem. It is known that in locally convex spaces a kind of converse of the Krein-Milman theorem holds: If Y is convex compact and $Y = \overline{\text{co}}(Z)$ for some subset Z of Y, then $\text{ext}(Y) \subset \bar{Z}$. Is this result true in the asymmetric case too?

Now we shall present the existence of a norm preserving extension that preserves also the extremality of the original functional. In the case of normed spaces the result was obtained by Singer [221] and in the asymmetric case in [40]. The result will be applied in Section 2.5 to the characterization of best approximation elements in asymmetric normed spaces.

Let (X, p) be an asymmetric normed space. In the following theorem, the symbols $B_{Y_p^\flat}$ and $B_{X_p^\flat}$ stand for the closed unit balls of the dual spaces $Y_p^\flat$ and $X_p^\flat$,

$$B_{Y_p^\flat} = \{\varphi \in Y_p^\flat : \|\varphi|_p \leq 1\}$$

and

$$B_{X_p^\flat} = \{\psi \in X_p^\flat : \|\psi|_p \leq 1\} \ .$$

Theorem 2.2.13. *Let (X, p) be a space with asymmetric norm and Y a subspace of X.*

If φ_0 is an extreme point of the closed ball $\|\varphi_0| \cdot B_{Y_p^\flat}$ then there exists a norm preserving extension φ of φ_0 which is an extreme point of the ball $\|\varphi_0| \cdot B_{X_p^\flat}$ of $X_p^\flat$.

Proof. Denote by $B^\flat$ the closed unit ball of the space $X^\flat$. Because ψ is an extreme point of the ball $B^\flat$ if and only if $r \cdot \psi$ is an extreme point of the ball $rB^\flat$, it is sufficient to prove the theorem for $\|\varphi_0| = 1$. Let φ_0, $\|\varphi_0| = 1$, be an extreme point of the unit ball $B_{Y^\flat}$ of $Y^\flat$ and let

$$E(\varphi_0) = \{\psi \in X^\flat : \psi|_Y = \varphi_0 \text{ and } \|\psi| = 1\} .$$

Clearly, the set $E(\varphi_0)$ convex; by Theorem 2.2.2 it is also nonempty. Further, the set $E(\varphi_0)$ is an extremal subset of $B^\flat$ because $(1-\alpha)\varphi_1 + \alpha\varphi_2 \in E(\varphi_0)$, for some $\varphi_1, \varphi_2 \in B^\flat$ and some $0 < \alpha < 1$, implies $(1-\alpha)\varphi_1|_Y + \alpha\varphi_2|_Y = \varphi_0$, so that, by the extremality of φ_0, we have $\varphi_1|_Y = \varphi_2|_Y = \varphi_0$. Since $\|(1-\alpha)\varphi_1 + \alpha\varphi_2| = 1$ and $\varphi_k \in B^\flat$ it follows that $\|\varphi_k| = 1$, $k = 1, 2$, so that $\varphi_k \in E(\varphi_0)$, $k = 1, 2$.

We show now that the set $E(\varphi_0)$ is a w^*-closed subset of the closed ball B^* of $X^* = (X, p^s)^*$. Let $(\varphi_\gamma : \gamma \in \Gamma)$ be a net in $E(\varphi_0)$ that is w^*-convergent to an element $\varphi \in B^*$, i.e.,

$$\forall x \in X, \quad \varphi_\gamma(x) \to \varphi(x) \quad \text{in } (\mathbb{R}, |\cdot|) .$$

Since for every $x \in X$ and every $\gamma \in \Gamma$, we have $\varphi_\gamma(x) \le p(x)$, it follows that $\varphi(x) \le p(x)$, i.e., $\|\varphi| \le 1$. Also, for every $y \in Y$ and $\gamma \in \Gamma$, $\varphi_\gamma(y) = \varphi_0(y)$, so that $\varphi|_Y = \varphi_0$, and $\|\varphi| \ge \|\varphi_0| = 1$. It follows that $\varphi \in E(\varphi_0)$, showing that $E(\varphi_0)$ is a w^*-closed subset of the w^*-compact set B^*, so it is w^*-compact too.

By the Krein-Milman theorem the convex w^*-compact set $E(\varphi_0)$ agrees with the closed convex hull of its extreme points, so that it has extreme points. Taking an extreme point φ of the extremal subset $E(\varphi_0)$ of $B^\flat$, it follows that φ is an extreme point of the unit ball $B^\flat$ of $X^\flat$ and $\varphi|_Y = \varphi_0$, $\|\varphi| = 1 = \|\varphi_0|$. □

2.3 The fundamental principles

Together with the Hahn-Banach extension theorem, the Open Mapping Theorem and the Closed Graph Theorem are the cornerstones of the whole edifice of classical functional analysis. Although in the asymmetric case they do not hold in full generality, some positive results have been obtained, which will be presented in this section.

2.3.1 The Open Mapping and the Closed Graph Theorems

As it is known, the proofs of two fundamental principles of functional analysis – the Open Mapping Theorem and the Closed Graph Theorem for Banach spaces – rely on Baire's category theorem. Based on Theorem 1.2.44, C. Alegre [2] extended these principles to asymmetric normed spaces.

Theorem 2.3.1 (The Open Mapping Theorem, [2]). *Let (X, p) and (Y, q) be asymmetric normed spaces. Suppose that (X, p) is right-K-complete and Y is Hausdorff* *and a $(q,\bar{q})$-Baire space. If $A : X \to Y$ is linear, surjective and (p,q)-continuous, then for every p-open subset G of X, $A(G)$ is q-open in Y.*

Proof. Let B_p' be the open unit ball of (X,p). Since A is surjective, $A(B_p')$ is an absorbing convex subset of Y, and so will be the $\bar{q}$-closed convex set $\bar{q}\text{-}\operatorname{cl}(A(B_p'))$, implying $X = \cup_{n=1}^{\infty} n\ \bar{q}\text{-}\operatorname{cl}(A(B_p'))$. Since Y is a $(q,\bar{q})$-Baire space, there exists $n \in \mathbb{N}$ such that $q\text{-}\operatorname{int}\big(n\ \bar{q}\text{-}\operatorname{cl}(A(B_p'))\big) \neq \emptyset$ (see Theorem 1.2.49). It follows

$$q\text{-}\operatorname{int}\big(\bar{q}\text{-}\operatorname{cl}(A(B_p'))\big) \neq \emptyset ,$$

so that, by Proposition 1.1.67.4, there exists $\varepsilon > 0$ such that

$$B_q(0,\varepsilon) \subset \bar{q}\text{-}\operatorname{cl}(A(B_p')) . \tag{2.3.1}$$

We show that

$$B_q(0,\frac{\varepsilon}{2^2}) \subset A(B_p') . \tag{2.3.2}$$

Indeed, by (2.3.1), $y \in B_q(0,2^{-2}\varepsilon) \subset \bar{q}\text{-}\operatorname{cl}(A(2^{-2}B_p')$ implies $B_{\bar{q}}(y,2^{-3}\varepsilon) \cap A(2^{-2}B_p') \neq \emptyset$, so that there exists $x_1 \in 2^{-2}B_p'$ such that

$$\bar{q}(Ax_1 - y) < \frac{\varepsilon}{2^3} \iff q(y - Ax_1) < \frac{\varepsilon}{2^3} .$$

It follows that

$$y - Ax_1 \in B_q(0,\frac{\varepsilon}{2^3}) \subset \bar{q}\text{-}\operatorname{cl}(A(2^{-3}B_p') ,$$

so that $B_{\bar{q}}(y - Ax_1, 2^{-4}\varepsilon) \cap A(2^{-3}B_p') \neq \emptyset$, implying the existence of an element $x_2 \in 2^{-3}B_p'$ such that

$$\bar{q}(Ax_2 + Ax_1 - y) < \frac{\varepsilon}{2^4} \iff q(y - Ax_1 - Ax_2) < \frac{\varepsilon}{2^4} .$$

Continuing in this manner, one obtains the elements $x_k \in 2^{-(k+1)}B_p'$ such that

$$\bar{q}(Ax_n + \cdots + Ax_2 + Ax_1 - y) < \frac{\varepsilon}{2^{n+2}} , \tag{2.3.3}$$

for all $n \in \mathbb{N}$, implying $Ax_1 + \cdots + Ax_n \xrightarrow{\bar{q}} y$ as $n \to \infty$.

Since $p(x_k) < 2^{-k-1}$ the sequence $s_n = x_1 + \cdots + x_n,\ n \in \mathbb{N}$, is p-left-K-Cauchy (see Proposition 1.2.6), or, equivalently, $\bar{p}$-right-K-Cauchy. The right K-completeness of (X,p) implies the right K-completeness of $(X,\bar{p})$, so there exists $x \in X$ such that $\bar{p}(s_n - x) \to 0$ as $n \to \infty$. The (p,q)-continuity of the linear operator A implies its $(\bar{p},\bar{q})$-continuity, so that $Ax_1 + \cdots + Ax_n \xrightarrow{\bar{q}} Ax$ as $n \to \infty$. Since (Y,τ_q) is Hausdorff, $(Y,\tau_{\bar{q}})$ is also Hausdorff, so that $y = Ax$. From

$$\begin{aligned} p(x) \leq & p(x - s_n) + p(s_n) \leq \bar{p}(s_n - x) + p(x_1) + \cdots + p(x_n) \\ < & \bar{p}(s_n - x) + \frac{1}{2^2} + \cdots + \frac{1}{2^{n+1}} \underset{(n\to\infty)}{\longrightarrow} \frac{1}{2} , \end{aligned}$$

it follows that $x \in B_p'$.

Finally, we show that (2.3.2) implies that $A(G)$ is q-open for every p-open subset G of X.

If $y = Ax$ for some $x \in G$, then there exists $r > 0$ such that $x + rB'_p \subset G$. It follows that

$$y + \frac{r\varepsilon}{4} B'_q \subset Ax + rA(B'_p) = A(x + rB'_p) \subset A(G) ,$$

showing that the set $A(G)$ is q-open. □

Remark 2.3.2. Theorem 2.3.1 is proved in [2] under the hypotheses that the asymmetric normed space (X, p) is sequentially right K-complete and (Y, q) is sequentially right K-complete and Hausdorff. The proof is based on Lemma 3 asserting that in an asymmetric normed space (Y, q) which is of second Baire category in itself (and so, by Proposition 1.2.43, a Baire space), q-cl(A) and $\bar{q}$-cl(A) are 0-neighborhoods in (Y, q), for any absorbing and starshaped with respect to 0 subset A of X. While the assertion concerning q-cl(A) follows from the Baire property, it is not founded concerning $\bar{q}$-cl(A). So we reformulated the theorem by asking the space (Y, q) to be $(q, \bar{q})$-Baire.

A consequence of this deep result is the inverse mapping theorem which, in essence, is an equivalent form of the open mapping theorem.

Corollary 2.3.3. *Let (X, p) and (Y, q) be two asymmetric normed spaces. If (X, p) is right K-sequentially and (Y, q) is $(q, \bar{q})$-Baire and Hausdorff, then the inverse of any bijective continuous linear mapping $A : (X, \tau_p) \to (Y, \tau_q)$ is continuous.*

For two asymmetric normed spaces (X, p) and (Y, q) consider $X \times Y$ endowed with the asymmetric norm

$$r(x, y) = p(x) + q(y),\ (x, y) \in X \times Y . \tag{2.3.4}$$

The proof of the results from the following lemma are similar to those in the symmetric case.

Lemma 2.3.4. *Let (X, p), (Y, q) be asymmetric normed spaces.*

1. *The norm r defined by* (2.3.4) *generates the product topology $\tau_p \times \tau_q$ on $X \times Y$.*
2. *If (X, p) and (Y, q) are right* (*left*) *K-sequentially complete, then $(X \times Y, r)$ is right* (*left*) *K-sequentially complete.*
3. *A closed subset of a right* (*left*) *K-sequentially complete normed space is right* (*left*) *K-sequentially complete.*

As in the case of Banach spaces, the closed graph theorem can easily be derived from the open mapping theorem. The graph G_f of a mapping $f : X \to Y$ is the subset of $X \times Y$ given by $G_f = \{(x, y) \in X \times Y : y = f(x)\}$.

Proposition 2.3.5. *If (X, τ), (Y, ν) are topological spaces, with Y Hausdorff, and $f : X \to Y$ is continuous, then the graph G_f of f is closed in $X \times Y$ with respect to the product topology $\tau \times \nu$.*

Proof. Let (x_i, y_i), $i \in I$, be a net in $X \times Y$ converging to some $(x, y) \in X \times Y$ with respect to the product topology. This is equivalent to $x_i \to x$ in X and $y_i \to y$ in Y. Since $y_i = f(x_i)$, $i \in I$, the continuity of f and the uniqueness of the limit (Y is Hausdorff) implies $f(x) = y$, that is $(x, y) \in G_f$. □

If X and Y are topological spaces, then a mapping $f : X \to Y$ is said to have *closed graph* provided G_f is closed in $X \times Y$ with respect to the product topology.

Theorem 2.3.6 (The Closed Graph Theorem, [2]). *Let (X, p) and (Y, q) be asymmetric normed spaces. If (X, p) is right K-sequentially complete, $(p, \bar{p})$-Baire and Hausdorff and (Y, q) is right K-sequentially complete, then any linear mapping $A : X \to Y$ with closed graph is continuous.*

Proof. By Lemma 2.3.4, the product topology $\tau_p \times \tau_q$ is generated by the asymmetric norm $r(x, y) = p(x) + q(y)$ and the space $(X \times Y, r)$ is right K-sequentially complete.

By hypothesis, the graph G_A of A is a closed subspace of $(X \times Y, r)$ so it is also right K-sequentially complete with respect to r. The projections $P_1, P_2 : G_A \to X$ defined by $P_1(x, y) = x$ and $P_2(x, y)$, for $(x, y) \in G_A$, are linear and continuous mappings. The projection P_1 is also bijective, so that, by the inverse mapping theorem (Corollary 2.3.3), $P_1^{-1} : X \to G_A$ is also continuous. Observe that P_1^{-1} is given by $P_1^{-1}(x) = (x, Ax)$, $x \in X$, so that $A = P_2 \circ P_1^{-1}$ will be continuous. □

2.3.2 The Banach-Steinhaus principle

In this subsection we shall prove an asymmetric version of the Banach-Steinhaus uniform boundedness principle.

Theorem 2.3.7. *Let (X, p) be a right K-complete asymmetric normed space, (Y, q) an asymmetric normed space and $\mathcal{A} \subset L_{p,q}(X, Y)$. If the family $\mathcal{A}$ is both q- and $\bar{q}$-upper pointwise bounded, that is for every $x \in X$,*

$$\sup_{A\in\mathcal{A}} q(Ax) < \infty \quad \textit{and} \quad \sup_{A\in\mathcal{A}} \bar{q}(Ax) < \infty \,, \tag{2.3.5}$$

then

$$\sup_{A\in\mathcal{A}} \sup\{\bar{q}(Ax) : x \in B_p\} < \infty \quad \textit{and} \quad \sup_{A\in\mathcal{A}} \sup\{q(Ax) : x \in B_{\bar{p}}\} < \infty \,. \tag{2.3.6}$$

Proof. Since the case $X = \{0\}$ is trivial, we can suppose $X \neq \{0\}$.

Suppose that

$$\sup_{A\in\mathcal{A}} \sup\{\bar{q}(Ax) : x \in B_p\} = \infty \tag{2.3.7}$$

and that the family $\mathcal{A}$ is q-upper pointwise bounded.

Put

$$S_{\mathcal{A}} = \{x \in X : \sup_{A\in\mathcal{A}} \bar{q}(Ax) = \infty\}$$

and

$$X_n = \{x \in X : \exists A \in \mathcal{A},\ \bar{q}(Ax) > n\}, \quad n \in \mathbb{N} .$$

It is clear that

$$S_{\mathcal{A}} = \cap_{n=1}^{\infty} X_n \ . \tag{2.3.8}$$

Since (X, p) is right K-complete it follows that $(X, \bar{p})$ is also right K-complete so that, by Theorem 1.2.44 and Proposition 1.2.43, (X, τ_p) is a Baire space. If we show that each X_n is p-open and p-dense in (X, τ_p), then by (2.3.8), the set $S_{\mathcal{A}}$ is p-G_δ and p-dense in (X, τ_p), in contradiction to the $\bar{q}$-upper pointwise boundedness of the family $\mathcal{A}$ (which is equivalent to $S_{\mathcal{A}} = \emptyset$).

For $n \in \mathbb{N}$ and $A \in \mathcal{A}$ put

$$X_{n,A} = \{x \in X : \bar{q}(Ax) > n\} \ .$$

Claim I. The set $X_{n,A}$ is p-open.

Since the norm $\bar{q}$ is q-lsc (see Proposition 1.1.8.4) and A is (p, q)-continuous, it follows that $\bar{q} \circ A$ is p-lsc, implying that the set $X_{n,A}$ is p-open.

Since

$$X_n = \cup\{X_{n,A} : A \in \mathcal{A}\} \ ,$$

it follows that the set X_n is p-open too.

Claim II. The set X_n is p-dense in X.

Suppose, on the contrary, that some X_n is not p-dense in X. Then there exists $x_0 \in X$ and $r_0 > 0$ such that

$$B_p[x_0, r_0] \cap X_n = \emptyset \ ,$$

implying

$$\forall A \in \mathcal{A},\ \forall x \in B_p[x_0, r_0], \quad \bar{q}(Ax) \le n \ .$$

Let $M_{x_0} = \sup\{q(Ax_0) : A \in \mathcal{A}\}$. Since for every $u \in B_p,\ x := x_0 + r_0 u \in B_p[x_0, r_0]$, it follows that $u = r_0^{-1}(x - x_0)$ and

$$\bar{q}(Au) \le \frac{1}{r_0}\left(\bar{q}(Ax) + \bar{q}(-Ax_0)\right) \le \frac{1}{r_0}\left(n + q(Ax_0)\right) \le \frac{1}{r_0}\left(n + M_{x_0}\right) \ ,$$

for every $A \in \mathcal{A}$, in contradiction to (2.3.7).

Consequently, each X_n is p-dense in X.

Since $L_{p,q}(X, Y) = L_{\bar{p},\bar{q}}(X, Y)$, the proof of the second inequality in (2.3.6) proceeds similarly. □

The proof given above allows the following formulation of the principle of condensation of singularities.

Theorem 2.3.8. *Let (X, p) be a right K-complete asymmetric normed space, (Y, q) an asymmetric normed space and let $\mathcal{A} \subset L_{p,q}(X, Y)$.*

1. *If the family $\mathcal{A}$ is q-upper pointwise bounded and*

$$\sup_{A\in\mathcal{A}} \sup\{\bar{q}(Ax) : x \in B_p\} = \infty \,, \tag{2.3.9}$$

then the set

$$\{x \in X : \sup_{A\in\mathcal{A}} \bar{q}(Ax) = \infty\} \tag{2.3.10}$$

is p-G_δ and p-dense in X.

2. *Similarly, if the family $\mathcal{A}$ is $\bar{q}$-upper pointwise bounded and*

$$\sup_{A\in\mathcal{A}} \sup\{q(Ax) : x \in B_{\bar{p}}\} = \infty \,, \tag{2.3.11}$$

then the set

$$\{x \in X : \sup_{A\in\mathcal{A}} q(Ax) = \infty\} \tag{2.3.12}$$

is $\bar{p}$-G_δ and $\bar{p}$-dense in X.

This principle has the following curious consequence.

Corollary 2.3.9. *If (X,p) is a right K-complete asymmetric normed space and (Y,q) is an asymmetric normed space, then $L_{p,q}(X,Y) = L_{p,\bar{q}}(X,Y)$.*

Proof. Suppose that there exists $A \in L_{p,q}(X,Y) \setminus L_{p,\bar{q}}(X,Y)$. Then

$$\sup\{\bar{q}(Ax) : x \in B_p\} = \infty \,.$$

Applying Theorem 2.3.8 to the family $\mathcal{A} = \{A\}$, one obtains that $\emptyset = \{x \in X : \bar{q}(Ax) = \infty\}$ is p-dense in X, a contradiction. $\square$

2.3.3 Normed cones

We shall present some results on abstract normed cones as defined in Remark 2.1.4. As we did mention in Remark 2.1.4, the study of the duals of asymmetric normed spaces requires the consideration of normed cones.

A *linear mapping* between two cones X, Y is an additive and positively homogeneous mapping $A : X \to Y$.

An *asymmetric seminorm* on a cone X is a mapping $p : X \to \mathbb{R}^+$ such that

(i) $p(0) = 0$ and $(x, -x \in X \wedge p(x) = p(-x) = 0) \Rightarrow x = 0$
(ii) $p(\alpha x) = \alpha p(x)$;
(iii) $p(x+y) \le p(x) + p(y)$,

for all $x, y \in X$ and $\alpha \ge 0$. If

(iv) $p(x) = 0 \iff x = 0$,

then p is called an *asymmetric norm.*

Starting from an asymmetric seminorm p on a cone X one can define an extended quasi-semimetric e_p on X by the formula

$$e_p(x,y) = \begin{cases} \inf\{p(z) : z \in X,\, y = x+z\} & \text{if} \quad y \in x + X, \\ \infty & \text{otherwise} . \end{cases} \tag{2.3.13}$$

An extended quasi-semimetric $d : X \times X \to [0;\infty]$ on a cone X is called *subinvariant* provided that

$$\begin{aligned} &\text{(i)} \qquad d(x+z, y+z) \le d(x,y), \quad \textit{and} \\ &\text{(ii)} \qquad d(\alpha x, \alpha y) = \alpha d(x,y) \ , \end{aligned} \tag{2.3.14}$$

for all $x, y, z \in X$ and $\alpha \ge 0$.

For instance, $\mathbb{R}^+$ is a cancellative cone and $u(\alpha) = \alpha$ is an asymmetric norm on $\mathbb{R}^+$. The associated extended quasi-metric, given by $e_u(x,y) = y - x$ if $x \le y$ and $e_u(x,y) = \infty$, otherwise, induces the Sorgenfrey topology on $\mathbb{R}^+$ (see Example 1.1.6).

The topological notions for a normed cone (X,p) will be considered with respect to this extended quasi-semimetric. As before, one associates to e_p the conjugate quasi-semimetric $\bar{e}_p(x,y) = e_p(y,x)$ and the (symmetric) extended semimetric $e_p^s(x,y) = \max\{e_p(x,y), \bar{e}_p(x,y))$.

Some properties of this quasi-semimetric are collected in the following proposition.

Proposition 2.3.10. *Let (X,p) be an asymmetric normed cone.*

1. *The function e_p defined by* (2.3.13) *is a subinvariant extended quasi-semimetric on X.*
2. *The equality*
$$r \cdot B_{e_p}(x,\varepsilon) = rx + \{y \in X : p(y) \le \varepsilon r\} \ ,$$
holds for every $x \in X$ and $r, \varepsilon > 0$.
3. *The translations with respect to $+$ and $\cdot$ are $\tau(e_p)$-open, that is, if $Z \subset X$ is $\tau(e_p)$-open, then both $x + Z$ and $r \cdot Z$ are $\tau(e_p)$-open.*

Continuous linear mapping between normed cones have properties similar to those between asymmetric normed spaces, see Proposition 2.1.2.

Proposition 2.3.11 ([234])**.** *Let $(X,p), (Y,q)$ be asymmetric normed cones and $A : X \to Y$ a linear operator. The following are equivalent.*

1. *The operator A is continuous on X.*
2. *The operator A is continuous at $0 \in X$.*
3. *The operator A is upper bounded on every ball $B_{e_p}[0,r]$.*
4. *There exists $\beta \ge 0$ such that $q(Ax) \le \beta\, p(x)$, for all $x \in X$.*

Proof. The implication 1 $\Rightarrow$ 2 is trivial and the proofs of the implications 2 $\Rightarrow$ 3, and 3 $\Rightarrow$ 4 are similar to those from Proposition 2.1.2.

4 $\Rightarrow$ 1. For $x \in X$ and $\varepsilon > 0$ let $r > 0$ be such that $\beta r < \varepsilon$. If $e_p(x, x') < r$, then there exists $z \in X$ such that $x' = x + z$ and $p(z), r$. It follows that $Ax' = Ax + Az$ and $q(Az) \leq \beta p(z) < \beta r < \varepsilon$, proving the continuity of A at x. $\square$

Based on this proposition one can introduce an asymmetric norm on the space $L_{p,q}(X, Y)$ of continuous linear operators between two asymmetric normed cones (X, p) and (Y, q), by

$$\|A|_{p,q} = \sup\{q(Ax) : p(x) \leq 1\} . \tag{2.3.15}$$

It follows that

$$q(Ax) \leq \|A|_{p,q}\, p(x) , \tag{2.3.16}$$

for all $x \in X$ and that $L_{p,q}(X, Y)$ is an asymmetric normed cone with respect to (2.3.15), see [234].

We shall present now some closed graph and open mapping results for normed cones proved by Valero [235]. A uniform boundedness principle for locally convex cones was proved by Roth [211] (see also Roth [212] and the paper [184]).

An asymmetric normed cone is called *bicomplete* if it is complete with respect to the extended metric e_p^s. As it was shown by examples in [235], the Closed Graph and the Open Mapping Theorems do not hold for bicomplete asymmetric normed cones, some supplementary hypotheses being necessary.

A mapping f between two topological spaces (S, ν) and (T, τ) is called *almost continuous* at $s \in S$ if for every open subset V of T such that $f(s) \in V$, the set $\mathrm{cl}_\nu(f^{-1}(V))$ is a ν-neighborhood of s. A subset A of a bitopological space (T, τ_1, τ_2) is called (τ_1, τ_2)-*preopen* if $A \subset \tau_1\text{-}\operatorname{int}(\tau_2\text{-}\operatorname{cl} A)$. A mapping f from a topological space (S, ν) to a bitopological space (T, τ_1, τ_2) is called *almost open* if $f(U)$ is (τ_1, τ_2)-preopen for every ν-open subset U of S.

The closed graph theorem proved by Valero [235] is the following.

Theorem 2.3.12. *Let (X, p) and (Y, q) be two asymmetric normed cones such that the cone Y is right K-sequentially complete with respect to the conjugate extended quasi-metric $\bar{e}_q$. If $A : X \to Y$ is a linear mapping with closed graph in $(X \times Y, \bar{e}_p \times \bar{e}_q)$ which is $(\bar{e}_p, e_q)$-almost continuous at 0, then A is continuous.*

An open mapping theorem holds in similar conditions.

Theorem 2.3.13. *Let (X, p) and (Y, q) be two asymmetric normed cones such that the cone Y is right K-sequentially complete with respect to the conjugate extended quasi-metric $\bar{e}_q$. If $A : X \to Y$ is a linear mapping with closed graph in $(X \times Y, \bar{e}_p \times \bar{e}_q)$ which is almost open as a mapping from $(X, \tau(e_p))$ to $(Y, \tau(e_q), \tau_(\bar{e}_q))$, then A is continuous.*

There are also other results on normed cones: the paper [94] discusses the metrizability of the unit ball of the dual of a normed cone, Oltra and Valero [170] study the isometries and bicompletions of normed cones and Valero [234] defines and studies the properties of quotient normed cones (the study of quotient spaces of asymmetric normed spaces is done in [3]). Other properties are investigated in a series of papers by Romaguera, Sánchez Pérez and Valero: in [203] one considers generalized monotone normed cones, quasi-normed monoids are discussed in [201], the dominated extension of functionals, of V-convex functions and duality on cancellative and noncancellative normed cones are treated in [200] and [202], respectively.

2.4 Weak topologies

The aim of this section is to present some basic results on weak topologies on asymmetric normed spaces and on asymmetric LCS. As it is well known, the weak topologies play a crucial role in functional analysis. Their asymmetric counterparts were studied in the fundamental paper [90], where an Alaoglu-Bourbaki type theorem on the weak*-compactness of the closed unit ball of the dual space is proved. The locally convex variant on the weak-compactness of the polar of a 0-neighborhood in an asymmetric LCS was proved in [41]. With an appropriate definition of the bidual, reflexivity is studied and an analogue of Goldstine's theorem on the weak density of a Banach space in its bidual is proved.

2.4.1 The $w^\flat$-topology of the dual space $X_p^\flat$

This is the analog of the w^*-topology on the dual of a normed space, which we shall present following [90].

Let (X,p) be a space with asymmetric norm and $X_p^\flat$ its asymmetric dual. The $w^\flat$-topology on $X_p^\flat$ is the topology admitting as a neighborhood basis of a point $\varphi \in X_p^\flat$ the sets

$$V_{x_1,\dots,x_n;\varepsilon}(\varphi) = \{\psi \in X_p^\flat : \psi(x_k) - \varphi(x_k) < \varepsilon,\ k = 1, 2, \dots, n\}\ , \tag{2.4.1}$$

for all $\varepsilon > 0$, and all $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$.

The topology $w^\flat$ is derived from a quasi-uniformity $\mathcal{W}_p^\flat$ on $X_p^\flat$ with a basis formed of the sets

$$V_{x_1,\dots,x_n;\,\varepsilon} = \{(\varphi_1, \varphi_2) \in X_p^\flat \times X_p^\flat : \varphi_2(x_i) - \varphi_1(x_i) \le \varepsilon,\ i = 1, \dots, n\}\ , \tag{2.4.2}$$

for $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$ and $\varepsilon > 0$. Note that, for fixed $\varphi_1 = \varphi$, one obtains the neighborhoods from (2.4.1).

By the definition of the topology $w^\flat$, the $w^\flat$-convergence of a net (φ_γ) in $X_p^\flat$ to $\varphi \in X_p^\flat$ is equivalent to

$$\forall x \in X, \quad \varphi_\gamma(x) \to \varphi(x) \quad \text{in} \quad (\mathbb{R}, u)\ .$$

The following proposition shows that, in fact, a stronger result holds.

Proposition 2.4.1. *Let (X,p) be an asymmetric normed space with dual $X_p^\flat$.*

The $w^\flat$-topology is the restriction to $X_p^\flat$ of the w^*-topology of the space $X^* = (X,p^s)^*$. Consequently, it is Hausdorff and the $w^\flat$-convergence of a net (φ_γ) in $X_p^\flat$ to $\varphi \in X_p^\flat$ is equivalent to

$$\forall x \in X, \quad \varphi_\gamma(x) \to \varphi(x) \quad \textit{in} \quad (\mathbb{R}, |\cdot|)\,.$$

Proof. The first assertion is a direct consequence of the definition of the topology $w^\flat$. The second assertion follows from the remarks that

$$V_{\varepsilon,x} \cap V_{\varepsilon,-x} = \{\psi \in X_p^\flat : |\psi(x) - \varphi(x)| < \varepsilon\}$$

is a $w^\flat$-neighborhood of $0 \in X$ and $X_p^\flat \subset X^*$. □

A deep result in Banach space theory is the Alaoglu-Bourbaki Theorem:

THEOREM. (Alaoglu-Bourbaki) *The closed unit ball B_{X^*} of the dual of a normed space X is w^*-compact.*

The analog of this theorem for asymmetric normed spaces was proved in [90, Theorem 4]. We include a slightly simpler proof of this result.

Proposition 2.4.2. *The closed unit ball $B_p^\flat = B_{X_p^\flat}$ of the space $X_p^\flat$ is a w^*-closed subset of the closed unit ball B^* of the space $X^* = (X,p^s)^*$.*

Proof. Let (φ_γ) be a net in $B_p^\flat$ that is w^*-convergent to an element $\varphi \in X^*$, i.e., for every $x \in X$ the net $(\varphi_\gamma(x))$ converges to $\varphi(x)$ in $(\mathbb{R}, |\cdot|)$. Because $\forall x \in X,\ \varphi_\gamma(x) \le p(x)$, it follows that $\varphi(x) \le p(x)$ for all $x \in X$, showing that $\varphi \in B_p^\flat$. □

Theorem 2.4.3. *The closed unit ball $B_p^\flat$ of the dual $X_p^\flat$ of an asymmetric normed space (X,p) is $w^\flat$-compact.*

Proof. By the Alaoglu-Bourbaki theorem the ball B^* is w^*-compact, so that, as a w^*-closed subset of B^*, the ball $B_p^\flat$ will be w^*-compact too. Since the $w^\flat$-topology is the restriction of w^* to $X_p^\flat$, it follows that the set $B_p^\flat$ is also $w^\flat$-compact. □

Remark 2.4.4. In [41, Proposition 2.11] the Alaoglu-Bourbaki theorem was extended to asymmetric locally convex spaces: the polar of any neighborhood of 0 is a $w^\flat$-compact convex subset of the asymmetric dual cone $X_P^\flat$, see Theorem 2.4.30.

2.4.2 Compact subsets of asymmetric normed spaces

In this subsection we shall present, following [86] and [4], some results on compactness specific to asymmetric normed spaces. The proof will be given in the next subsection within the more general context of asymmetric LCS.

Let (X, p) be an asymmetric normed space. For $x \in X$ put

$$\theta(x) = \{y \in X : p(y - x) = 0\} . \tag{2.4.3}$$

It is clear that $\theta(x) = x + \theta(0)$ and $Y + \theta(0) = \cup\{\theta(y) : y \in Y\}$. Also $\theta(x) = \tau(\bar{p})\text{-cl}(\{x\})$, as can be seen from the equivalences

$$\begin{aligned} y \in \theta(x) &\iff p(y - x) = 0 \iff \bar{p}(x - y) = 0 \\ &\iff \forall \varepsilon > 0,\ \bar{p}(x - y) < \varepsilon \iff \forall \varepsilon > 0,\ x \in B_{\bar{p}}(y, \varepsilon) \\ &\iff y \in \tau(\bar{p})\text{- cl}(\{x\}) . \end{aligned}$$

The following properties hold.

Proposition 2.4.5 ([86]). *Let (X, p) be an asymmetric normed normed space, $x \in X$ and $\varepsilon > 0$. Then $B_p(x, \varepsilon) = \theta(0) + B_p(x, \varepsilon) + \theta(0)$. Also, if Y is a $\tau(p)$-open subset of X, then $Y = Y + \theta(0)$.*

As it is shown by García-Raffi [86] the sets $\theta(x)$ are involved in the study of compactness in asymmetric normed spaces.

Proposition 2.4.6. *Let (X, p) be an asymmetric normed space and $K \subset X$.*

1. *The set K is τ_p-compact if and only if $K + \theta(0)$ is τ_p-compact.*
   If $K + \theta(0)$ is τ_p-compact, $K_0 \subset K + \theta(0)$ and $K_0 + \theta(0) = K + \theta(0)$, then K_0 is τ_p-compact.
2. *A finite sum and a finite union of p-precompact sets is p-precompact.*
3. *The convex hull of a p-precompact set is p-precompact.*
4. *The set K is p-precompact if and only if the $\tau(\bar{p})$-closure of K is p-precompact.*
5. *If $K_0 \subset K \subset K_0 + \theta(0)$ and K_0-is $\tau(p^s)$-compact, then K is p-compact.*

García-Raffi [86] obtained characterizations of finite-dimensional normed spaces similar to those known for normed spaces. In the following proposition all topological notions refer to $\tau(p)$.

Theorem 2.4.7. *Let (X, p) be an asymmetric normed space.*

1. *If X is finite dimensional, of dimension $n \geq 1$, and T_1, then X is topologically isomorphic to the Euclidean space $\mathbb{R}^n$.*
2. *If (X, p) is T_1, then X is finite dimensional if and only if its closed unit ball B_p is $\tau(p)$-compact.*

3. *Suppose that X is finite dimensional. Then X is T_1 if and only if every $\tau(p)$-compact subset of X is $\tau(p)$-closed.*

As it is shown in [4, Example 12], the property 5 from Proposition 2.4.6 does not characterize the p-compactness. This paper contains also some further results on the relations between the $\tau(p^s)$-compactness of K_0 and the p-compactness of K, involving a notion of boundedness called right-boundedness. The unit closed ball B_p of (X, p) is called *right-bounded* if there exists $r > 0$ such that

$$r\, B_p \subset B_{p^s} + \theta(0)\ .$$

Note that the inclusion $B_{p^s} + \theta(0) \subset B_p$ is always true.

Theorem 2.4.8. *Let (X, p) be an asymmetric normed normed space.*

1. ([86]) *If X is finite dimensional and the unit closed ball B_p is right-bounded, then B_p is $\tau(p)$-compact.*
2. ([4]) *Suppose that (X, p) is biBanach with B_p right-bounded with $r = 1$. If $K \subset X$ is p-precompact, then there exists a p^s-compact subset K_0 of K such that $K \subset K_0 + \theta(0)$.*
3. ([4]) *If K_0 is p^s-precompact and $K \subset K_0 + \theta(0)$, then K is outside p-precompact.*

2.4.3 Compact sets in LCS

In this subsection we shall present, following [44], some properties of compact sets in LCS.

A subset Y of a quasi-uniform space $(X, \mathcal{U})$ is called *precompact* if for every $U \in \mathcal{U}$ there exists a finite subset Z of Y such that $Y \subset U[Z]$. The set Y is called *totally bounded* if for every U there exists a finite family $A_1, \dots, A_n$ of subsets of Y such that $A_i \times A_i \subset U$, $i = 1, \dots, n$, and $Y \subset \cup_{i=1}^n A_i$. Note that the total boundedness with respect to $\mathcal{U}$ is equivalent to the total boundedness with respect to the associated uniformity $\mathcal{U}_s$.

If in the above definition of precompactness one asks that the finite set Z be contained in X, then one obtains the notions of *outside precompactness* considered in [4]. Obviously, the precompactness implies the outside precompactness, but the reverse implication is not true, even in asymmetric normed spaces, see Example 1.2.23. In uniform spaces the total boundedness, the precompactness and the outside precompactness agree, and a set is compact if and only if it is totally bounded and complete.

If p is an asymmetric seminorm on a vector space X, we say that a subset Y of X is *p-precompact* if for every $\varepsilon > 0$ there exists a finite subset Z of Y such that

$$Y \subset \bigcup_{z \in Z} B_p(z, \varepsilon). \tag{2.4.4}$$

If for every $\varepsilon > 0$ there exists a finite subset Z of X such that (2.4.4) holds, then the set Y is called *outside p-precompact*. One obtains an equivalent notion if one asks that Y is covered by the family $B_p[z, \varepsilon]$, $z \in Z$, of closed balls. The set Z is called also a (p, ε)-net for Y (in both cases).

A subset of an asymmetric LCS (X, P) is called *precompact* if it is precompact with respect to the quasi-uniformity $\mathcal{U}_P$. The following proposition contains a useful characterization of precompactness in asymmetric LCS in terms of seminorms. The proof follows immediately from the definition of the quasi-uniformity $\mathcal{U}_P$ (the fact that $U_{p,\varepsilon}(x) = B_p(x, \varepsilon)$).

Proposition 2.4.9. *A subset Y of an asymmetric LCS (X, P) is (outside) precompact if and only if it is (outside) p-precompact for every $p \in P$.*

Based on this proposition, the proof of Proposition 1.2.18.3 can be adapted to obtain the following relation between precompactness and outside precompactness.

Proposition 2.4.10. *Let (X, P) be an asymmetric LCS. A subset Y of X is P-precompact if and only if for every $p \in P$ and every $\varepsilon > 0$ there exists a finite subset $\{x_1, \dots, x_n\}$ of X such that $Y \subset \cup_{i=1}^n B_p(x_i, \varepsilon)$ and $Y \cap B_{\bar{p}}(x_i, \varepsilon) \neq \emptyset$ for all $i \in \{1, \dots, n\}$.*

As a consequence of Proposition 1.1.59, one obtains the following relations between various notions of compactness and precompactness. A subset Y of an asymmetric LCS (X, P) is called *P-bounded* provided $\sup\{p(y) : y \in Y\} < \infty$ for every $p \in P$. This is equivalent to the fact that it is absorbed by every $\tau(P)$-neighborhood of 0, that is for every $\tau(P)$-neighborhood V of 0 there exists $\lambda > 0$ such that $\lambda Y \subset V$, or, in other words, Y is topologically bounded.

Proposition 2.4.11. *Let (X, P) be an asymmetric LCS and Y a subset of X.*

1. *If the set Y is P^s-precompact, then it is P-precompact and $\bar{P}$-precompact. The same is true for the outside precompactness.*
2. *If the set Y is $\tau(P^s)$-compact, then it is $\tau(P)$-compact and $\tau(\bar{P})$-compact.*
3. *The outside P-precompact subsets of X are P-bounded. In particular, the P-precompact subsets of X are P-bounded as well.*
4. *A subset of X is P-precompact if and only if its $\tau(\bar{P})$-closure is P-precompact. The same is true for outside P-precompactness.*

Proof. 1. For $\varepsilon > 0$ and $p \in P$ there exists a finite subset $\{y_1, \dots, y_n\}$ of Y such that $Y \subset \cup_{i=1}^n B_{p^s}(y_i, \varepsilon)$. Since $B_{p^s}(y_i, \varepsilon) \subset B_p(y_i, \varepsilon)$, $i = 1, \dots, n$, it follows that $Y \subset \cup_{i=1}^n B_p(y_i, \varepsilon)$, so that Y is P-precompact. Similarly, $B_{p^s}(y_i, \varepsilon) \subset B_{\bar{p}}(y_i, \varepsilon)$, $i = 1, \dots, n$, implies that Y is $\bar{P}$-precompact. The case of outside precompactness can be treated exactly in the same way.

2. This follows from the fact that a compact subset of a topological space remains compact for every coarser topology.

3. For $p \in P$ there exists a finite subset $\{x_1, \ldots, x_n\}$ of X such that $Y \subset \{x_1, \ldots, x_n\} + B_p(0,1)$, implying $p(y) \le \max\{p(x_i) : 1 \le i \le n\} + 1$ for every $y \in Y$.

4. We give a proof different from that in [4]. Suppose first that Y is P-precompact and show that $Z = \tau(\bar{P})\text{-cl}\, Y$ is also P-precompact, which is equivalent to the fact that Z is p-precompact for every $p \in P$.

Let $p \in P$ and $\varepsilon > 0$. Since Y is p-precompact there exists $y_1, \ldots, y_n \in Y$ such that

$$Y \subset \cup_{i=1}^n B_p[y_i, \varepsilon]. \tag{2.4.5}$$

By Proposition 1.1.59.4, the ball $B_p[0, \varepsilon]$ is $\tau(\bar{P})$-closed, so that the set

$$\cup_{i=1}^n B_p[y_i, \varepsilon] = \{y_1, \ldots, y_n\} + B_p[0, \varepsilon]$$

is also $\tau(\bar{P})$-closed. But then, the inclusion (2.4.5) implies

$$\tau(\bar{P})\text{-}\operatorname{cl} Y \subset \cup_{i=1}^n B_p[y_i, \varepsilon]\ .$$

Conversely, suppose that $Z = \tau(\bar{P})\text{-cl}\, Y$ is P-precompact and prove that Y is P-precompact.

For $p \in P$ and $\varepsilon > 0$ there exist $z_1, \ldots, z_n \in Z$ such that

$$Z \subset \cup_{i=1}^n B_p(z_i, \varepsilon). \tag{2.4.6}$$

For every $i \in \{1, \ldots, n\}$ there exists $y_i \in Y \cap B_{\bar{p}}(z_i, \varepsilon)$, that is an $y_i \in Y$ such that $\bar{p}(y_i - z_i) < \varepsilon$, or, equivalently, $p(z_i - y_i) < \varepsilon$.

Let $y \in Y \subset Z$. By (2.4.6) there exists $j \in \{1, \ldots, n\}$ such that $y \in B_p(z_j, \varepsilon)$. But then

$$p(y - y_j) \le p(y - z_j) + p(z_j - y_j) < 2\varepsilon\ ,$$

showing that $Y \subset \cup_{i=1}^n B_p(y_i, 2\varepsilon)$.

In the case of outside precompactness, a subset of an outside precompact set is also outside precompact, so the outside precompactness of the $\tau(\bar{P})$-closure of Y implies the outside precompactness of Y. The reverse implication can be proved exactly as in the case of the precompactness. □

Remark 2.4.12. In the case of asymmetric normed spaces, the result from the assertion 4 of the above proposition was proved by García-Raffi [86, Prop. 9].

The following property is a consequence of Propositions 1.1.64.1 and 1.1.67.1, but for the sake of convenience, we give the proof.

Lemma 2.4.13. *Let (X, P) be an asymmetric LCS, $Q \subset P$ and $D \subset \mathbb{R}$ such that the family $\{B_q(0,r) : q \in Q,\, r \in D\}$ is a basis of $\tau(P)$-neighborhoods of 0. Then*

$$\tau(P)\text{-}\operatorname{cl} Y = \bigcap \{Y + B_{\bar{q}}(0,r) : q \in Q,\, r \in D\}, \tag{2.4.7}$$

for every subset Y of X.

Proof of Lemma 2.4.13. Let $x \in \tau(P)\text{-}\operatorname{cl} Y$, $q \in Q$ and $r \in D$. Then $x + B_q(0,r)$ is a $\tau(P)$-neighborhood of x, so that $Y \cap (x + B_q(0,r)) \neq \emptyset$, implying $x + u = y$, for some $u \in B_q(0,r)$ and $y \in Y$. But

$$u \in B_q(0,r) \iff q(u) \le r \iff \bar{q}(-u) \le r \iff -u \in B_{\bar{q}}(0,r) ,$$

so that $x = y - u \in Y + B_{\bar{q}}(0,r)$.

Conversely, suppose that x belongs to the intersection from the right-hand side of the equality (2.4.7). For a $\tau(P)$-neighborhood V_0 of 0, let $q \in Q$ and $r \in D$ be such that $B_q(0,r) \subset V_0$. By hypothesis, $x = y + v$ for some $y \in Y$ and $v \in B_{\bar{q}}(0,r)$, which, as above, implies that

$$y = x - v \in x + B_q(0,r) \subset x + V_0 .$$

Consequently, $(x + V_0) \cap Y \neq \emptyset$, showing that $x \in \tau(P)\text{-}\operatorname{cl} Y$. □

Proposition 2.4.14. *Let (X,P) be an asymmetric LCS whose topology $\tau(P)$ is T_1. Then X is finite dimensional if and only if there exists an outside P-precompact $\tau(P)$-neighborhood of 0.*

Proof. Necessity. If $\dim X = m$, then, by Proposition 1.1.68, it is isomorphic, algebraically and topologically, to the Euclidean space $\mathbb{R}^m$. Let $\Phi : \mathbb{R}^m \to X$ be an isomorphism. Since the closed unit ball $B_{\mathbb{R}^m}$ is a compact neighborhood of $0 \in \mathbb{R}^m$, its image by Φ will be a $\tau(P)$-compact neighborhood of $0 \in X$ which will be P-precompact and so outside P-precompact.

Sufficiency. Let $U = B_{p_0}(0, r_0)$ be an outside P-precompact $\tau(P)$-neighborhood of 0. Then there exists a finite subset $\{x_1, \dots, x_n\}$ of X such that

$$U \subset \{x_1, \dots, x_n\} + \frac{1}{2}U$$

implying

$$U \subset Z + \frac{1}{2}U, \tag{2.4.8}$$

where $Z = \operatorname{sp}\{x_1, \dots, x_n\}$ is the linear space generated by $\{x_1, \dots, x_n\}$. By (2.4.8)

$$\frac{1}{2}U \subset \frac{1}{2}Z + \frac{1}{2^2}U = Z + \frac{1}{2^2}U ,$$

so that

$$U \subset Z + \frac{1}{2}U \subset Z + Z + \frac{1}{2^2}U = Z + \frac{1}{2^2}U .$$

Repeating the argument, one obtains

$$U \subset Z + \frac{1}{2^n}U, \tag{2.4.9}$$

for all $n \in \mathbb{N}$.

We show that $\{\frac{1}{2^n}U : n \in \mathbb{N}\}$ is a basis of $\tau(P)$-neighborhoods of 0. For a $\tau(P)$-neighborhood V of 0, there exists $p \in P$ and $r > 0$ such that $B_p(0,r) \subset V$. Since a P-precompact set is topologically bounded (with respect to $\tau(P)$), there exists $\lambda > 0$ such that $\lambda U = \lambda B_{p_0}(0,r_0) \subset B_p(0,r)$. If $n \in \mathbb{N}$ is such that $2^{-n} < \lambda$, then

$$\frac{1}{2^n}U = \frac{1}{2^n}B_{p_0}(0,r_0) \subset \lambda B_{p_0}(0,r_0) \subset B_p(0,r) \subset V\ .$$

It is easy to check that $\{\frac{1}{2^n}B_{\bar{p}_0}(0,r_0) : n \in \mathbb{N}\}$ is a basis of $\tau(\bar{P})$-neighborhoods of 0, so that, by Lemma 2.4.13 and by (2.4.9),

$$U \subset \bigcap\{Z + \frac{1}{2^n}U : n \in \mathbb{N}\} = \tau(\bar{P})\text{-}\operatorname{cl} Z. \tag{2.4.10}$$

If we show that $\tau(\bar{P})$-$\operatorname{cl} Z = Z$, then by (2.4.10), for every $x \in X \setminus \{0\}$ there exists $\lambda > 0$ such that $\lambda x \in U \subset Z$, showing that $X = Z$ is finite dimensional.

Let $x \in \tau(\bar{P})$-$\operatorname{cl} Z \setminus Z$. Suppose that $\dim Z = m$ and let $e_1, \ldots, e_m$ be an algebraic basis of Z. The space $W = \operatorname{sp}(Z \cup \{x\})$ has dimension $m+1$ and $e_1, \ldots, e_m, x$ is an algebraic basis of W. Since $\{\{\frac{1}{2^n}B_{\bar{p}_0}(0,r_0) : n \in \mathbb{N}\}$ is a basis of $\tau(\bar{P})$-neighborhoods of 0, it follows that the topology $\tau(\bar{P})$ is generated by $\bar{p}_0$, so we can work with sequences. Suppose that $z_k = \alpha_{1,k}\, e_1 + \cdots + \alpha_{m,k}\, e_m + 0 \cdot x$, $k \in \mathbb{N}$, is a sequence in Z which converges to $x = 0 \cdot e_1 + \cdots + 0 \cdot e_m + 1 \cdot x$. Since the topology $\tau(P)$ is T_1, Proposition 1.1.63.2, implies that the topology $\tau(\bar{P})$ is also T_1. By Lemma 1.1.70, $\lim_k \alpha_{i,k} = 0$, $i = 1, \ldots, m$, and $0 = \lim_k \alpha_{m+1,k} = \alpha_{m+1} = 1$, a contradiction. Consequently, $\tau(\bar{P})$-$\operatorname{cl} Z = Z$, and Proposition 2.4.14 is completely proved. □

The following proposition is the analog of a known result in normed spaces. In the case of asymmetric normed spaces it was proved in [4, Proposition 8].

Proposition 2.4.15. *If Y is a precompact subset of an asymmetric LCS (X,P), then the convex hull $\operatorname{co} Y$ of Y is also precompact.*

Proof. By Proposition 2.4.9 it is sufficient to show that $\operatorname{co} Y$ is p-precompact for every $p \in P$.

Let $p \in P$ and $\varepsilon > 0$. By the precompactness of Y there exists a finite subset $\{y_1^0, \ldots, y_n^0\}$ of Y such that

$$Y \subset \cup_{i=1}^n B_p(y_i^0, \varepsilon). \tag{2.4.11}$$

Let $\Delta_n = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}_+^n : \sum_{i=1}^n \lambda_i = 1\}$ be the standard simplex in $\mathbb{R}_+^n$. The mapping $\Phi : \mathbb{R}_+^n \times X^n \to X$ given by $\Phi((\alpha_1, \ldots, \alpha_n), (x_1, \ldots, x_n)) = \sum_{i=1}^n \alpha_i x_i$ is continuous and $W = \operatorname{co}\{y_1^0, \ldots, y_n^0\}$ is the image by this mapping of the compact subset $\Delta_n \times \{y_1^0, \ldots, y_n^0\}$ of $\mathbb{R}_+^n \times X^n$, so it is compact and consequently P-precompact.

Therefore, there exists a subset $\{w_1^0, \ldots, w_m^0\} \subset W$ such that

$$W \subset \cup_{i=1}^m B_p(w_i^0, \varepsilon). \tag{2.4.12}$$

We show that

$$Y \subset \cup_{i=1}^{m} B_p(w_i^0, 2\varepsilon). \tag{2.4.13}$$

Let $x \in \operatorname{co} Y$, $x = \sum_{i=1}^{l} \alpha_i y_i$ for some $\alpha_i \geq 0$, $y_i \in Y$, $i = 1, \ldots, l$, $\sum_{i=1}^{l} \alpha_i = 1$. By (2.4.11), for every $i \in \{1, \ldots, l\}$ there exists $j(i) \in \{1, \ldots, n\}$ such that $p(y_i - y_{j(i)}^0) \leq \varepsilon$. Putting $w := \sum_{i=1}^{l} \alpha_i y_{j(i)}^0$, it follows that

$$p(x - w) = p(\sum_{i=1}^{l} \alpha_i (y_i - y_{j(i)}^0)) \leq \sum_{i=1}^{l} \alpha_i p(y_i - y_{j(i)}^0) \leq \varepsilon \ .$$

Since $w \in W$, the equality (2.4.12) implies the existence of $i_0 \in \{1, \ldots, m\}$ such that $p(w - w_{i_0}^0) \leq \varepsilon$. But then

$$p(x - w_{i_0}^0) \leq p(x - w) + p(w - w_{i_0}^0) \leq 2\varepsilon \ ,$$

showing that (2.4.13) holds. □

Some relations between precompactness and compactness in asymmetric normed spaces were studied in [86] and [4]. These results were extended in [44] to asymmetric LCS.

Let (X, P) be an asymmetric LCS. For $p \in P$ let

$$\begin{aligned} \theta_{0,p} &= \{z \in X : p(z) = 0\}, \quad \text{and} \\ \theta_0 &= \cap_{p \in P} \theta_{0,p} \ . \end{aligned}$$

Let also

$$\theta_{x,p} = \{z \in X : p(z - x) = 0\} = x + \theta_{0,p} \ .$$

It is immediate that θ_x agrees with the $\tau(\bar{P})$-closure of the set $\{x\}$. Indeed

$$\begin{aligned} y \in \tau(\bar{P})\text{-}\operatorname{cl}\{x\} &\iff \forall p \in P,\ \forall \varepsilon > 0,\ \bar{p}(x - y) < \varepsilon \\ &\iff \forall p \in P,\ \bar{p}(x - y) = 0 \\ &\iff \forall p \in P,\ p(y - x) = 0 \\ &\iff y \in \theta_x \ . \end{aligned}$$

As it was shown in [86]

$$B_p(x, \varepsilon) = B_p(x, \varepsilon) + \theta_{0,p} \ .$$

Based on this equality one obtains immediately that

$$Y = Y + \theta_0 \ ,$$

for every $\tau(P)$-open subset Y of X. Indeed, $0 \in \theta_0$ implies $Y \subset Y + \theta_0$. Conversely, let $x = y + z$ for some $y \in Y$ and $z \in \theta_0$. Since Y is $\tau(P)$-open there exist $p \in P$ and $\varepsilon > 0$ such that $B_p(y, \varepsilon) \subset Y$, implying $x = y + z \in B_p'(y, \varepsilon) + \theta_0 \subset B_p'(y, \varepsilon) + \theta_{0,p} = B_p'(y, \varepsilon) \subset Y$.

As a consequence of this last equality, one obtains the analog of Proposition 6 from [86].

Proposition 2.4.16. *A subset K of an asymmetric LCS is $\tau(P)$-compact if and only if $K+\theta_0$ is $\tau(P)$-compact.*

Also, if K is $\tau(P)$-compact, then every subset Z of $K+\theta_0$ is $\tau(P)$-compact.

Remark 2.4.17. In the case of an asymmetric normed space (X,p), Alegre et al. [4] give characterizations of τ_p-compact subsets of X. Among other results, they prove, under some supplementary hypotheses, that a subset K of X is τ_p-compact if and only if there exists a τ_{p^s}-compact subset K_0 of X such that $K_0 \subset K \subset K+\theta_0$ ([4, Theorem 20]). It is possible that similar characterizations hold in the locally convex case, a topic for further investigation.

2.4.4 The conjugate operator, precompact operators and a Schauder type theorem

A Schauder type theorem on the compactness of the conjugate of a compact linear operator on an asymmetric normed space was proved in [43]. We shall briefly present this result, referring for the details to the mentioned paper.

For a continuous linear operator $A:(X,p)\to(Y,q)$ between two asymmetric normed spaces, one defines the *conjugate operator* $A^\flat : Y_q^\flat \to X_p^\flat$ by the formula

$$A^\flat\psi = \psi\circ A, \quad \psi\in Y_q^\flat\ . \tag{2.4.14}$$

Concerning the continuity we have.

Proposition 2.4.18. *Let $(X,p),(Y,q)$ be asymmetric normed spaces and $A:X\to Y$ a continuous linear operator.*

1. *The operator $A^\flat : (Y_q^\flat, \|\cdot|_q) \to (X_p^\flat, \|\cdot|_p)$ is additive, positively homogeneous and continuous. So it is also quasi-uniformly continuous with respect to the quasi-uniformities $\mathcal{U}_q^\flat$ and $\mathcal{U}_p^\flat$ on $Y_q^\flat$ and $X_p^\flat$, respectively.*
2. *The operator $A^\flat$ is also quasi-uniformly continuous with respect to the $w^\flat$-quasi-uniformities $\mathcal{W}_q^\flat$ on $Y_q^\flat$ and $\mathcal{W}_p^\flat$ on $X_p^\flat$.*

Proof. 1. It is obvious that $A^\flat$ is properly defined, additive and positively homogeneous.

For every $\psi\in Y_q^\flat$,

$$\|A^\flat\psi|_q = \|\psi\circ A^\flat|_q \le \|\psi|_q\, \|A|_{p,q}\ ,$$

implying the continuity of $A^\flat$, which, in its turn, by the linearity of A, implies the quasi-uniform continuity with respect to the quasi-uniformities $\mathcal{U}_q^\flat$ and $\mathcal{U}_p^\flat$.

2. For $x_1,\dots,x_n\in X$ and $\varepsilon>0$ let

$$V=\{(\varphi_1,\varphi_2)\in X_p^\flat\times X_p^\flat : \varphi_2(x_i)-\varphi_1(x_i)\le\varepsilon,\ i=1,\dots,n\}$$

be a $w^\flat$-entourage in $X_p^\flat$. Then

$$U = \{(\psi_1, \psi_2) \in Y_q^\flat \times Y_q^\flat : \psi_2(Ax_i) - \psi_1(Ax_i) \le \varepsilon,\ i = 1, \dots, n\}\ ,$$

is a $w^\flat$-entourage in $Y_q^\flat$ and $(A^\flat\psi_1, A^\flat\psi_2) \in V$ for every $(\psi_1, \psi_2) \in U$, proving the quasi-uniform continuity of $A^\flat$ with respect to the $w^\flat$-quasi-uniformities on $Y_q^\flat$ and $X_p^\flat$. $\square$

A linear operator $A : (X, p) \to (Y, q)$ between two asymmetric normed spaces is called (p, q)-*precompact* if the image $A(B_p)$ of the closed unit ball B_p of X by the operator A is a q-precompact subset of Y. We shall denote by $(X, Y)_{p,q}^k$ the set of all (p, q)-precompact operators from X to Y. A subset Y of a quasi-uniform space $(X, \mathcal{U})$ is called $\mathcal{U}$-*precompact* if for every $\varepsilon > 0$ there exists a finite subset Z of Y such that $Y \subset U[Z]$. If there exists a set $Z \subset X$ such that $Y \subset U[Z]$, then Y is called *outside* $\mathcal{U}$-*precompact*. It is clear that a subset Y of an asymmetric normed space (X, p) is (outside) p-precompact if and only if it is (outside) $\mathcal{U}_p$-precompact.

For $\mu \in \{p, \bar{p}, p^s\}$ and $\nu \in \{q, \bar{q}, q_s\}$ denote by $(X, Y)_{\mu,\nu}^\flat$ the cone of all continuous linear operators from (X, μ) to (Y, ν). The space $(X, Y)_s^* := (X, Y)_{p^s,q^s}^\flat$ is the space of all continuous linear operators between the associated normed spaces (X, p^s) and (Y, p^s), which was denoted also by $L(X, Y)$.

On the space $(X, Y)_s^*$ we shall consider several quasi-uniformities. For $\mu \in \{p, \bar{p}, p^s\}$ and $\nu \in \{q, \bar{q}, q^s\}$ let $\mathcal{U}_{\mu,\nu}$ be the quasi-uniformity generated by the basis

$$U_{\mu,\nu;\,\varepsilon} = \{(A, B); A, B \in (X, Y)_s^*,\ \nu(Bx - Ax) \le \varepsilon,\ \forall x \in B_\mu, \},\ \ \varepsilon > 0\ , \quad (2.4.15)$$

where $B_\mu = \{x \in X : \mu(x) \le 1\}$ denotes the unit ball of (X, μ). The induced quasi-uniformity on the semilinear subspace $(X, Y)_{\mu,\nu}^\flat$ of $(X, Y)_s^*$ is denoted also by $\mathcal{U}_{\mu,\nu}$ and the corresponding topologies by $\tau(\mu, \nu)$. The uniformity $\mathcal{U}_{p^s,q^s}$ and the topology $\tau(p^s, q^s)$ are those corresponding to the norm (2.1.6) on the space $(X, Y)_s^*$, while the quasi-uniformity $\mathcal{U}_{p,q}$ corresponds to the extended asymmetric norm $\| \cdot |_{p,q}^*$ given by (2.1.23). In the case of the dual space $X_\mu^\flat$ we shall use the notation $\mathcal{U}_\mu^\flat$ for the quasi-uniformity $\mathcal{U}_{\mu,u}$.

Notice that, for $\mu = p^s$ and $\nu = q^s$, the space $(X, Y)_{p^s,q^s}^\flat$ agrees with $(X, Y)_s^*$, the (p^s, q^s)-compact operators are the usual linear compact operators between the normed spaces (X, p^s) and (Y, q^s), so the following proposition extends some well-known results for compact operators on normed spaces. For $\mu \in \{p, \bar{p}, p^s\}$ and $\nu \in \{q, \bar{q}, q_s\}$ one denotes by $(X, Y)_{\mu,\nu}^k$ the set of all (μ, ν)-precompact linear operators from (X, μ) to (Y, ν).

Proposition 2.4.19. *Let $(X, p), (Y, q)$ be asymmetric normed spaces. The following assertions hold.*

1. $(X, Y)_{\mu,\nu}^k$ *is a subcone of the cone* $(X, Y)_{\mu,\nu}^\flat$ *of all continuous linear operators from X to Y.*
2. $(X, Y)_{p,q}^k$ *is* $\tau(p, q^s)$*-closed in* $(X, Y)_{p,q}^\flat$.

Proof. 1. We give the proof in the case $\mu = p$ and $\nu = q$. The other cases can be treated similarly.

If $A : X \to Y$ is (p, q)-precompact, then there exists $x_1, \ldots, x_n \in B_p$ such that

$$\forall x \in B_p,\ \exists i \in \{1, \ldots, n\}, \quad q(Ax - Ax_i) \leq 1. \tag{2.4.16}$$

If for $x \in B_p$, $i \in \{1, \ldots, n\}$ is chosen according to (2.4.16), then

$$q(Ax) \leq q(Ax - Ax_i) + q(Ax_i) \leq 1 + \max\{q(Ax_j) : 1 \leq j \leq n\},$$

showing that the operator A is (p, q)-bounded.

Suppose that $A_1, A_2 : X \to Y$ are (p, q)-precompact and let $\varepsilon > 0$. By the (p, q)-precompactness of the operators A_1, A_2, there exist $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ in B_p such that

$$\forall x \in B_p,\ \exists i \in \{1, \ldots, m\},\ \exists j \in \{1, \ldots, n\},$$
$$q(A_1x - A_1x_i) \leq \varepsilon \text{ and } q(A_2x - A_2y_j) \leq \varepsilon.$$

It follows that for every $x \in B_p$ there exists a pair (i, j) with $1 \leq i \leq m$ and $1 \leq j \leq n$ such that

$$q(A_1x + A_2x - A_1x_i - A_2y_j) \leq q(A_1x - A_1x_i) + q(A_2x - A_2y_j) \leq 2\varepsilon,$$

showing that $\{A_1x_i + A_2y_j : 1 \leq i \leq m,\ 1 \leq j \leq n\}$ is a finite 2ε-net for $(A_1 + A_2)(B_p)$.

The proof of the precompactness of αA, for $\alpha > 0$ and A precompact, is immediate and we omit it.

2. *The $\tau(p, q^s)$-closedness of $(X, Y)^k_{p.q}$.*

Let (A_n) be a sequence in $(X, Y)^k_{p,q}$ which is $\tau(p, q^s)$-convergent to $A \in (X, Y)^\flat_{p,q}$.

For $\varepsilon > 0$ choose $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0,\ \forall x \in B_p,\ \ q(A_nx - Ax) \leq \varepsilon \text{ and } q(Ax - A_nx) \leq \varepsilon\,. \tag{2.4.17}$$

Let $x_1, \ldots, x_m \in B_p$ be such that the points $A_{n_0}x_i$, $1 \leq i \leq m$, form an ε-net for $A_{n_0}(B_p)$. Then for every $x \in B_p$ there exists $i \in \{1, \ldots, m\}$ such that

$$q(A_{n_0}x - A_{n_0}x_i) \leq \varepsilon,$$

so that, by (2.4.17),

$$q(Ax - Ax_i) \leq q(Ax - A_{n_0}x) + q(A_{n_0}x - A_{n_0}x_i) + q(A_{n_0}x_i - Ax_i) \leq 3\varepsilon.$$

Consequently, Ax_i, $1 \leq i \leq m$, is a 3ε-net for $A(B_p)$, showing that $A \in (X, Y)^k_{p,q}$. $\square$

Now we are prepared to state and prove the analog of the Schauder theorem.

Theorem 2.4.20. *Let $(X,p),(Y,q)$ be asymmetric normed spaces. If the linear operator $A : X \to Y$ is (p,q^s)-precompact, then $A^\flat(B_q^\flat)$ is precompact with respect to the quasi-uniformity $\mathcal{U}_p^\flat$ on $X_p^\flat$.*

Proof. For $\varepsilon > 0$ let

$$U_\varepsilon = \{(\varphi_1,\varphi_2) \in X_p^\flat \times X_p^\flat : \varphi_2(x) - \varphi_1(x) \le \varepsilon,\ \forall x \in B_p\}\ ,$$

be an entourage in $X_p^\flat$ for the quasi-uniformity $\mathcal{U}_p^\flat$.

Since A is (p,q^s)-precompact, there exist $x_1,\dots,x_n \in B_p$ such that

$$\forall x \in B_p\,,\ \exists j \in \{1,\dots,n\}, \quad q(Ax - Ax_j) \le \varepsilon \ \text{ and } \ q(Ax_j - Ax) \le \varepsilon. \tag{2.4.18}$$

By the Alaoglu-Bourbaki theorem, Theorem 2.4.3, the set $B_q^\flat$ is $w^\flat$-compact, so by the $(w^\flat,w^\flat)$-continuity of the operator $A^\flat$ (Proposition 2.4.18), the set $A^\flat(B_q^\flat)$ is $w^\flat$-compact in $X_p^\flat$. Consequently, the $w^\flat$-open cover of $A^\flat(B_q^\flat)$,

$$V_\psi = \{\varphi \in X_p^\flat : \varphi(x_i) - A^\flat\psi(x_i) < \varepsilon,\ i = 1,\dots,n\},\ \psi \in B_q^\flat\ ,$$

contains a finite subcover, i.e., there exist $m \in \mathbb{N}$ and $\psi_k \in B_q^\flat$, $1 \le k \le m$, such that

$$A^\flat(B_q^\flat) \subset \bigcup\{V_{\psi_k} : 1 \le k \le m\}\ . \tag{2.4.19}$$

Now let $\psi \in B_q^\flat$. By (2.4.19) there exists $k \in \{1,\dots,m\}$ such that

$$A^\flat\psi(x_i) - A^\flat\psi_k(x_i) < \varepsilon,\ i = 1,\dots,n\ .$$

If $x \in B_p$, then, by (2.4.18), there exists $j \in \{1,\dots,n\}$, such that

$$q(Ax - Ax_j) \le \varepsilon \ \text{ and } \ q(Ax_j - Ax) \le \varepsilon\ .$$

It follows that

$$\begin{aligned}
&\psi(Ax) - \psi_k(Ax)\\
&= \psi(Ax - Ax_j) + \psi(Ax_j) - \psi_k(Ax_j) + \psi_k(Ax_j - Ax)\\
&\le q(Ax - Ax_j) + \varepsilon + q(Ax_j - Ax) \le 3\varepsilon\ .
\end{aligned}$$

Consequently,

$$\forall x \in B_p\,, \quad (A^\flat\psi - A^\flat\psi_k)(x) \le 3\varepsilon\ ,$$

proving that

$$A^\flat(B_q^\flat) \subset U_{3\varepsilon}[\{A^\flat\psi_1,\dots,A^\flat\psi_m\}]\ .$$

□

Remark 2.4.21. As a measure of precaution, we have restricted our study to precompact linear operators A on an asymmetric normed space (X,p) with values in another asymmetric normed space (Y,q), meaning that the image $A(B_p)$ of the unit ball of X by A is a q-precompact subset of Y. A compact linear operator should be defined by the condition that $A(B_p)$ is a relatively compact subset of Y, that is the τ_q-closure of $A(B_p)$ is τ_q-compact subset of Y, in concordance to the definition of compact linear operators between normed spaces.

As can be seen from Section 1.2, the relations between precompactness, total boundedness and completeness are considerably more complicated in the asymmetric case than in the symmetric one. The compactness properties of the set $A(B_p)$ need a study of the completeness of the space $(X,Y)^\flat_{\mu,\nu}$ with respect to various quasi-uniformities and various notions of completeness, which could be a topic of further investigation.

2.4.5 The bidual space, reflexivity and Goldstine theorem

The bidual space was introduced in [90], including the definition of the canonical embedding of an asymmetric normed space in its bidual and the definition of a reflexive asymmetric normed space. Further properties were proved in [93].

Let (X,p) be an asymmetric normed space. Denote by $X^\flat_p$ the cone of all continuous linear functionals $f:(X,p)\to(\mathbb{R},u)$ and by X^* the dual space of the associated normed space (X,p^s). On X^* one considers the extended asymmetric norm and the extended norm given by

$$\|f|^* = \sup f(B_p) \quad \text{and} \quad \|f\|^* = \max\{\|f|^*, \|-f|^*\}\ ,$$

respectively. The restriction of $\|\cdot|^*$ to $X^\flat_p$ is denoted by $\|\cdot|$. As we have seen, $\operatorname{sp} X^\flat_p = X^\flat_p - X^\flat_p \subset X^*$ (Proposition 2.1.7) and a linear functional $f\in X^*$ belongs to $X^\flat_p$ if and only if $\|f|^*<\infty$ (Proposition 2.1.16).

Let

$$\begin{aligned} X^{**}_e &= \{\varphi : (X^*, \|\cdot\|^*) \to (\mathbb{R}, |\cdot|) : \varphi \text{ is linear and continuous}\} \\ &= (X^*, \|\cdot\|^*)^*\ , \end{aligned} \tag{2.4.20}$$

and

$$X^{*\flat}_e = \{\varphi : (X^*, \|\cdot|^*) \to (\mathbb{R}, u) : \varphi \text{ is linear and continuous}\} = (X^*, \|\cdot|^*)^\flat\ . \tag{2.4.21}$$

(Here the subscript "e" comes from "extended" and the superscript $*$ means that we consider linear continuous functionals, while the superscript $\flat$ means that the functionals are linear and upper semi-continuous, or, equivalently, continuous to $(\mathbb{R},u)$.)

The set X^{**}_e is a linear space, $X^{*\flat}_e$ is a cone in X^{**}_e, and

$$\operatorname{sp}(X^{*\flat}_e) = X^{*\flat}_e - X^{*\flat}_e \subset X^{**}_e\ .$$

For $\varphi \in X_e^{*\flat}$ let

$$\|\varphi|^{*\flat} = \sup\{\varphi(f) : f \in B_p^\flat\} , \tag{2.4.22}$$

where $B_p^\flat = \{f \in X_p^\flat : \|f| \le 1\}$.

Then $\|\cdot|^{*\flat}$ is an asymmetric norm and the asymmetric normed cone $(X_e^{*\flat}, \|\cdot|^{*\flat})$ is called the *bidual space* of (X, p).

For $x \in X$ define $\Lambda(x) : X^* \to \mathbb{R}$ by

$$\Lambda(x)(f) = f(x),\ f \in X^* . \tag{2.4.23}$$

Proposition 2.4.22. *Let (X, p) be an asymmetric normed space and let Λ be the mapping defined by* (2.4.23). *Then*

$$\Lambda(x) \in X_e^{*\flat} \quad \textit{and} \quad \|\Lambda(x)|^{*\flat} = p(x) ,$$

for every $x \in X$. Moreover, the mapping Λ is linear, so that it defines a linear isometric embedding of (X, p) into the semilinear space $(X_e^{*\flat}, \|\cdot|^{*\flat})$.

Proof. It is obvious that $\Lambda(x)$ is a linear functional on X^*. The inequality

$$\Lambda(x)(f) = f(x) \le \|f|^* p(x) ,$$

valid for every $f \in X^*$, implies $\|\Lambda(x)|^{*\flat} \le p(x)$. Since, by Theorem 2.2.2.2, there exists $f_0 \in B_p^\flat$ such that $f_0(x) = p(x)$, it follows that $\|\Lambda(x)|^{*\flat} = p(x)$.

The linearity of the mapping $\Lambda : X \to X_e^{*\flat}$ is easily verified. □

The space X^* induces a topology on $\mathrm{sp}(X_e^{*\flat})$ denoted by $\sigma(X_e^{*\flat}, X^*)$ having as basis of neighborhoods of 0 the sets

$$V_{f_1,\dots,f_n;\varepsilon} = \{\varphi \in \mathrm{sp}(X_e^{*\flat}) : |\varphi(f_k)| < \varepsilon,\ k = 1, \dots, n\} , \tag{2.4.24}$$

for $\varepsilon > 0$, $f_1, \dots, f_n \in X^*$, $n \in \mathbb{N}$. The neighborhoods of an arbitrary point $\varphi \in \mathrm{sp}(X_e^{*\flat})$ are obtained by translating the neighborhoods of 0.

It is obvious that the topology $\sigma(X_e^{*\flat}, X^*)$ is a locally convex topology on $\mathrm{sp}(X_e^{*\flat})$ generated by the family $p_f,\ f \in X_e^{*\flat}$, of seminorms, where for $f \in X^*$ the seminorm $p_f : \mathrm{sp}(X_e^{*\flat}) \to \mathbb{R}$ is given by

$$p_f(\varphi) = |\varphi(f)|, \quad \varphi \in \mathrm{sp}(X_e^{*\flat}) . \tag{2.4.25}$$

The following proposition says that, in essence, the spaces X^* and $\mathrm{sp}\, X_e^{*\flat}$ form a dual pair.

Proposition 2.4.23 ([93])**.** *Let (X, p) be an asymmetric normed space. Then the following hold.*

1. *For each $f \in X^*$ the linear functional $e_f : \mathrm{sp}\, X_e^{*\flat} \to \mathbb{R}$ defined by $e_f(\varphi) = \varphi(f)$, $\varphi \in X_e^{*\flat}$, is continuous from $(\mathrm{sp}\, X_e^{*\flat}, \sigma(X_e^{*\flat}, X^*))$ to $(\mathbb{R}, |\cdot|)$.*
2. *If $\Psi : \mathrm{sp} X_e^{*\flat} \to \mathbb{R}$ is a linear functional, continuous from $(\mathrm{sp} X_e^{*\flat}, \sigma(X_e^{*\flat}, X^*))$ to $(\mathbb{R}, |\cdot|)$, then there exists $f \in X^*$ such that $\Psi = e_f$.*

Proof. The first assertion is almost trivial, because for $f \in X^*$ and $\varepsilon > 0$ the set

$$(e_f)^{-1}((-\varepsilon;\varepsilon)) = \{\varphi \in \mathrm{sp}(X_e^{*\flat}) : |\varphi(f)| < \varepsilon\} = V_{f,\varepsilon}$$

is a $\sigma(X_e^{*\flat}, X^*)$-neighborhood of 0.

To prove the converse assertion 2, let $\Psi : \mathrm{sp}\, X_e^{*\flat} \to \mathbb{R}$ be a linear functional, continuous from $(\mathrm{sp}\, X_e^{*\flat}, \sigma(X_e^{*\flat}, X^*))$ to $(\mathbb{R}, |\cdot|)$. By the characterization of the continuity of linear functionals on locally convex spaces [238, Korollar VIII.2.4], there exist $f_1, \dots, f_n \in X^*$ and $\beta > 0$ such that

$$\begin{aligned} |\Psi(\varphi)| &\le \beta \max\{p_{f_k}(\varphi) : 1 \le k \le n\} \\ &= \beta \max\{|e_{f_k}(\varphi)| : 1 \le k \le n\} \,. \end{aligned}$$

By [238, Lemma VIII.3.3] this implies that Ψ is a linear combination of e_{f_k}, $k = 1, \dots, n$, $\Psi = \sum_{k=1}^n \alpha_k e_{f_k}$. But then $\Psi = e_f$, where $f = \sum_{k=1}^n \alpha_k f_k$. □

The remarkable theorem of Goldstine on the weak density of B_X in $B_{X^{**}}$ was extended in [93] to asymmetric normed spaces.

Theorem 2.4.24 (Goldstine Theorem)**.** *Let (X, p) be an asymmetric normed space. The image by Λ of the unit ball B_p is $\sigma(X_e^{*\flat}, X^*)$-dense in the unit ball $B_{X_e^{*\flat}}$ of $X_e^{*\flat}$.*

Proof. Denote by C the $\sigma(X_e^{*\flat}, X^*)$-closure of $\Lambda(B_p)$ in $Z = X_e^{*\flat} - X_e^{*\flat}$ and let $\varphi \in C$.

Claim I. $\varphi \in X_e^{*\flat}$.

We have to show that φ is usc from $(X^*, \|\cdot|_p^*)$ to $(\mathbb{R}, |\cdot|)$. Suppose that (f_n) is a sequence in X^* which is $\|\cdot|_p^*$-convergent to $f \in X^*$. Then, given $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\forall n \ge n_\varepsilon, \quad \|f_n - f|_p^* < \varepsilon \,. \tag{2.4.26}$$

Since φ belongs to the $\sigma(X_e^{*\flat}, X^*)$-closure of $\Lambda(B_p)$ in Z, for every $n \in \mathbb{N}$ there exists $x_n \in B_p$ such that

$$|\varphi(f_n - f) - (f_n - f)(x_n)| = |\varphi(f_n - f) - \Lambda(x_n)(f_n - f)| < \varepsilon \,. \tag{2.4.27}$$

But then, by (2.4.27) and (2.4.26),

$$\varphi(f_n) - \varphi(f) < (f_n - f)(x_n) + \varepsilon \le \|f_n - f|_p^* + \varepsilon < 2\varepsilon \,,$$

for all $n \ge n_\varepsilon$, proving the required upper semi-continuity of the functional φ.

Claim II. $\|\varphi|^{*\flat} \le 1$.

Let $\varepsilon > 0$ be given. By the definition of C, for every $f \in B_p^\flat$ there exists $x \in B_p$ such that

$$|\varphi(f) - f(x)| = |\varphi(f) - \Lambda(x)(f)| < \varepsilon \,,$$

implying

$$\varphi(f) < f(x) + \varepsilon \le \|f|_p + \varepsilon \le 1 + \varepsilon \ .$$

Since $\varepsilon >$ was arbitrarily taken, it follows that $\varphi(f) \le 1$ for every $f \in B_p^\flat$, so that $\|\varphi|^{*\flat} \le 1$.

By Claim I and Claim II, $C \subset B_{X_e^{*\flat}}$.

Claim III. $B_{X_e^{*\flat}} \subset C$.

Suppose that there exists $\varphi_0 \in C \setminus B_{X_e^{*\flat}}$. Applying Tukey's separation theorem ([149, Theorem 2.28]) in the locally convex space $(Z, \sigma(X_e^{*\flat}, X^*)$ in follows the existence of a $\sigma(X_e^{*\flat}, X^*)$-continuous linear functional $\Psi : Z \to \mathbb{R}$ such that

$$\sup\{\Psi(\varphi) : \varphi \in C\} < \Psi(\varphi_0) \ . \tag{2.4.28}$$

By Proposition 2.4.23.2 there exists $f_0 \in X^*$ such that $\Psi(\varphi) = \varphi(f_0),\ \varphi \in Z$.

It follows that

$$\varphi_0(f_0) = \Psi(\varphi_0) > \sup\{\varphi)(f_0) : \varphi \in C\} \le \sup\{\Lambda(x)(f_0) : x \in B_p\} = \|f_0|_p^* \ ,$$

implying $f_0 \in X_p^\flat$. But then

$$\|f_0|_p^* < \varphi_0(f_0) \le \|\varphi_0|^{*\flat} \|f_0|_p^* \le \|f_0|_p^* \ ,$$

a contradiction which shows that $B_{X_e^{*\flat}} \subset C$. □

Based on Proposition 2.4.22 one can define the reflexivity of an asymmetric normed space: an asymmetric normed space (X, p) is called *reflexive* if $\Lambda(X) = X_e^{*\flat}$. In spite of the fact that this definition imposes a strong condition on the cone $X_e^{*\flat}$, namely to be a linear space, there are many interesting examples justifying this definition, see [90].

Example 2.4.25. If the normed space (X, p^s) is reflexive, then the asymmetric normed space (X, p) is reflexive in the above sense. In particular any finite-dimensional asymmetric normed space is reflexive.

This follows from the inclusions

$$\Lambda(X) \subset X_e^{*\flat} \subset X_e^{*\flat} - X_e^{*\flat} \subset X^{**} = \Lambda(X) \ .$$

The paper [93] contains a characterization of reflexivity of an asymmetric normed space (X, p) in terms of the completeness of the unit ball B_p with respect to the weak uniformity induced by the space X^* on X that will be presented in what follows.

The locally convex topology $w_e^\flat = \sigma(X_e^{*\flat}, X^*)$ on the space $Z = X_e^{*\flat} - X_e^{*\flat}$ is generated by the uniformity $\mathcal{U}_\flat$ formed of the sets

$$\{(\varphi, \psi) \in X_e^{*\flat} \times X_e^{*\flat} : \psi - \varphi \in V\}, \quad V \in \mathcal{N}(0) \ ,$$

where $\mathcal{N}(0)$ denotes the family of all $w_e^\flat$-neighborhoods of $0 \in Z$.

The uniformity $\mathcal{U}_\flat$ is generated by the sets

$$U_{F;\varepsilon} = \{(\varphi, \psi) \in X_e^{*\flat} \times X_e^{*\flat} : \forall f \in F,\ |\varphi(f) - \psi(f)| < \varepsilon\}\ , \tag{2.4.29}$$

for $F \subset X^*$ nonempty finite and $\varepsilon > 0$.

Proposition 2.4.26. *The uniformity* $\mathcal{U}_\flat$ *given by* (2.4.29) *is complete on* $B_{X_e^{*\flat}}$.

Proof. Consider $Z \subset \mathbb{R}^{X^*}$ and let τ denote the product topology on $(\mathbb{R}, |\cdot|)^{X^*}$. Let $(\varphi_i : i \in I)$ be a net in Z and $\varphi \in Z$. The equivalences

$$\begin{aligned} \varphi_i \xrightarrow{w_e^\flat} \varphi &\iff \forall f \in X^*, \quad \varphi_i(f) \xrightarrow{|\cdot|} \varphi(f) \\ &\iff \varphi_i \xrightarrow{\tau} \varphi\ , \end{aligned}$$

show that $w_e^\flat$ is the restriction of the product topology to Z. The product uniformity of $(\mathbb{R}, |\cdot|)^{X^*}$ is complete since each factor $(\mathbb{R}, |\cdot|)$ is complete. Consequently, the uniformity $\mathcal{U}_\flat$ will be complete on $B_{X_e^{*\flat}}$ provided that $B_{X_e^{*\flat}} = \{\varphi \in Z : \|\varphi|^{*\flat} \le 1\}$ is τ-closed in $(\mathbb{R}, |\cdot|)^{X^*}$.

Let $(\varphi_i : i \in I)$ be a net in $B_{X_e^{*\flat}}$ which is τ-convergent to some $\varphi \in (\mathbb{R}, |\cdot|)^{X^*}$, meaning that

$$\forall f \in X^*, \quad \varphi_i(f) \xrightarrow{|\cdot|} \varphi(f)\ . \tag{2.4.30}$$

The linearity of φ follows from (2.4.30), the linearity of each φ_i, and the fact that the addition and multiplication are continuous operations in $(\mathbb{R}, |\cdot|)$.

Since each φ_i belongs to $B_{X_e^{*\flat}}$,

$$\forall f \in B_p,\ \forall i \in I, \quad \varphi_i(f) \le 1\ .$$

Passing to the limit with respect to $i \in I$, one obtains

$$\forall f \in B_p,\ \varphi(f) \le 1\ ,$$

showing that $\|\varphi|^{*\flat} = \sup \varphi(B_p) \le 1$, that is $\varphi \in B_{X_e^{*\flat}}$. □

Denote by w^s the topology on (X, p) induced by the dual $X^* = (X, p^s)^*$. Again this topology is generated by a uniformity $\mathcal{W}_s$ formed of the sets

$$\{(x, y) \in X \times X : x - y \in V\}, \quad V \in \mathcal{V}_s(0)\ ,$$

where $\mathcal{V}_s(0)$ denotes the family of all w^s-neighborhoods of $0 \in X$.

The uniformity $\mathcal{W}_s$ is generated by the sets

$$W_{F;\varepsilon} = \{(x, y) \in X \times X : \forall f \in F,\ |f(x) - f(y)| < \varepsilon\}, \tag{2.4.31}$$

for $F \subset X^*$ nonempty finite and $\varepsilon > 0$.

Using this uniformity one can give a characterization of the reflexivity of an asymmetric normed space.

Theorem 2.4.27. *An asymmetric normed space (X,p) is reflexive if and only if the uniformity $\mathcal{W}_s$ is complete on the unit ball B_p of (X,p).*

Proof. Suppose that B_p is complete with respect to the uniformity $\mathcal{W}_s$ and let $\varphi \in B_{X_e^{*\flat}}$. By Goldstine's theorem, Theorem 2.4.24, $\Lambda(B_p)$ is $w_e^\flat$-dense in $B_{X_e^{*\flat}}$, so that for every $F \subset X^*$ nonempty finite and $\varepsilon > 0$ there exists $x_{F,\varepsilon} \in X$ such that

$$\forall f \in F, \quad |f(x_{F,\varepsilon}) - \varphi(f)| = |\Lambda(x_{F,\varepsilon})(f) - \varphi(f)| < \varepsilon \; . \tag{2.4.32}$$

The set $\mathcal{F}(X^*) \times (0;\infty)$ of all these pairs is directed with respect to the order

$$(F_1,\varepsilon_1) \le (F_2,\varepsilon_2) \iff F_1 \subset F_2 \quad \text{and} \quad \varepsilon_2 \le \varepsilon_1 \; ,$$

and (2.4.32) shows that the net $(x_{F,\varepsilon} : (F,\varepsilon) \in \mathcal{F}(X^*) \times (0;\infty))$ is $w_e^\flat$-convergent to φ.

Let us show that the net $(x_{F,\varepsilon})$ is Cauchy with respect to the uniformity $\mathcal{W}_s$. Let $F_0 \subset X^*$ finite nonempty and $\varepsilon_0 > 0$ be given. Then for every $F_1, F_2 \supset F_0$ and $\varepsilon_1, \varepsilon_2 \le \varepsilon_0$, by the choice of $x_{F,\varepsilon}$ (see (2.4.32)) we have

$$\begin{aligned} |f(x_{F_1,\varepsilon_1}) - f(x_{F_2,\varepsilon_2})| \le& |f(x_{F_1,\varepsilon_1}) - \varphi(f)| + |f(x_{F_2,\varepsilon_2}) - \varphi(f)| \\ <& \varepsilon_1 + \varepsilon_2 \le 2\varepsilon_0 \; , \end{aligned}$$

for all $f \in F_0$, showing that $x_{F_1,\varepsilon_1} - x_{F_2,\varepsilon_2} \in W_{F;\varepsilon}$. Consequently the net $(x_{F,\varepsilon})$ is $\mathcal{W}_s$-Cauchy, so that, by hypothesis, it is w^s-convergent to some $x \in B_p$. Since the topology w^s is Hausdorff it follows that $\varphi = \Lambda(x)$.

If $\varphi \in X_e^{*\flat} \setminus B_{X_e^{*\flat}}$, then $\psi = \varphi/\|\varphi|^* \in B_{X_e^{*\flat}}$, so that, by the first part of the proof, there exists $y \in B_p$ such that $\psi = \Lambda(y)$, implying $\varphi = \|\varphi|^*\psi$.

We have shown that $\Lambda(X) = X_e^{*\flat}$, i.e., the space $X_e^{*\flat}$ is reflexive.

Conversely, suppose that $\Lambda(X) = X_e^{*\flat}$. Then the uniform spaces $(X,\mathcal{W}_s)$ and $(\Lambda(X),\mathcal{U}_\flat)$ can be identified. By Proposition 2.4.26, $B_{\Lambda(X)}$ is $\mathcal{U}_\flat$-complete, implying that B_X is $\mathcal{W}_s$-complete. □

Remark 2.4.28. It is known that, by the Alaoglu-Bourbaki theorem, the closed unit ball $B_{X^{**}}$ of the bidual X^{**} of a a normed space X is $\sigma(X^{**},X^*)$-compact, a result that is no longer true in the asymmetric case, as it is shown by the example of the space $(\mathbb{R},u)$.

Indeed, $\mathrm{id} \in \mathbb{R}_u^\flat = (\mathbb{R},u)^\flat$, $(\mathbb{R},|\cdot|)^* = \mathbb{R}^*$, the sequence $t_n = -n$, $n \in \mathbb{N}$, is in the unit ball B_u of $(\mathbb{R},u)$, but $|\,\mathrm{id}(-n) - \mathrm{id}(-m)| = |n-m| \ge 1$, showing that B_u is not $\sigma(\mathbb{R}_u^\flat,\mathbb{R}^*)$-compact.

2.4.6 Weak topologies on asymmetric LCS

We shall present, following [41], some properties of weak topologies on asymmetric LCS.

The topology $w^\flat$
We consider first the analog of the weak*-topology (w^*-topology) on the dual of a locally convex space. In the case of an asymmetric normed space (X, p) it was considered in [90], see Subsection 2.4.1.

Let (X, P) be an asymmetric locally convex space and $X^\flat = X^\flat_P$ the asymmetric dual cone. A $w^\flat$-neighborhood of an element $\varphi \in X^\flat$ is a subset W of $X^\flat$ for which there exist $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$ such that

$$V_{x_1,\ldots,x_n;\varepsilon}(\varphi) := \{\psi \in X^\flat : \psi(x_i) - \varphi(x_i) < \varepsilon,\ i = 1, \ldots, n\} \subset W .$$

The $w^\flat$-convergence of a net $\{\varphi_i, i \in I\}$ to $\varphi \in X^\flat$ is equivalent to the fact that for every $x \in X$ the net $\{(\varphi_i - \varphi)(x), i \in I\}$ converges to 0 in $(\mathbb{R}, u)$, that is

$$\forall x \in X,\ \forall \varepsilon > 0,\ \exists i_0 \in I \quad \text{such that} \quad \forall i \geq i_0, \quad (\varphi_i - \varphi)(x) < \varepsilon .$$

Since $X^\flat \subset X^*$ and

$$V_{x;\varepsilon} \cap V_{-x;\varepsilon}(\varphi) = \{\psi \in X^\flat : |(\psi - \varphi)(x)| < \varepsilon\} ,$$

it follows that the $w^\flat$-topology on $X^\flat$ is induced by the w^*-topology of the space X^*.

Asymmetric polars
Let (X, P) be an asymmetric locally convex space, (X, P^s) the associated locally convex space, $X^\flat$ the asymmetric dual of (X, P) and $X^* = (X, P^s)^*$ the conjugate space of (X, P^s).

The *polar* of a nonempty subset Y of (X, P^s) is defined by

$$Y^\circ = \{x^* \in X^* : \forall y \in Y,\ x^*(y) \leq 1\} .$$

Define the corresponding set in the case of the asymmetric dual $X^\flat$ by

$$Y^\alpha = Y^\circ \cap X^\flat = \{\varphi \in X^\flat : \forall y \in Y,\ \varphi(y) \leq 1\} ,$$

and call it the *asymmetric polar* of the set Y.

As it is well known, the set Y° is a convex w^*-closed subset of X^* (see, e.g., [238, p. 341]). Since the $w^\flat$-topology on $X^\flat \subset X^*$ is induced by the w^*-topology on X^*, we have the following result.

Proposition 2.4.29. *The asymmetric polar Y^α of a nonempty subset Y of an asymmetric locally convex space (X, P), is a convex $w^\flat$-closed subset of $X^\flat$.*

In the following proposition we prove the asymmetric analog of the Alaoglu-Bourbaki theorem, see, e.g., [238, Satz VIII.3.11].

Theorem 2.4.30. *The asymmetric polar of a neighborhood of the origin of an asymmetric locally convex space (X, P) is a convex $w^\flat$-compact subset of the asymmetric dual $X^\flat$.*

Proof. Suppose that P is directed. If V is a τ_P-neighborhood of $0 \in X$, then there exist $p \in P$ and $r > 0$ such that $B_p(0, r) \subset V$. Because $p^s(x) \le r$ implies $p(x) \le p^s(x) \le r$, it follows that $B_{p^s}(0, r) \subset B_p(0, r) \subset V$, so that V is a neighborhood of 0 in the locally convex space (X, P). By the Alaoglu-Bourbaki theorem it follows that V° is a convex w^*-compact subset of the dual X^*. Since $w^\flat$-compactness of V^α is equivalent to its w^*-compactness in X^*, it is sufficient to show that the set V^α is w^*-closed in X^*.

Let $\{\varphi_i : i \in I\}$ be a net in V^α that is w^*-convergent to $f \in X^*$. This means that for every $x \in X$ the net $\{\varphi_i(x) : i \in I\}$ converges to $f(x)$ in $(\mathbb{R}, |\cdot|)$. Since for every $v \in V$, $\varphi_i(v) \le 1$, for all $i \in I$, it follows that $f(v) \le 1$ for all $v \in V$. Because f is linear it is sufficient to prove its (P, u)-continuity at $0 \in X$. Consider for some $\varepsilon > 0$ the τ_u-neighborhood $(-\infty\, ; \varepsilon)$ of $f(0) = 0 \in \mathbb{R}$. Then $U = \frac{\varepsilon}{2}V$ is a τ_P-neighborhood of $0 \in X$, and for $v \in V$ and $u = \frac{\varepsilon}{2}v \in U$ we have

$$f(u) = \frac{\varepsilon}{2} f(v) \le \frac{\varepsilon}{2} < \varepsilon\,,$$

i.e., $f(U) \subset (-\infty\, ; \varepsilon)$, proving the (P, u)-continuity of f at 0.

It follows that $f \in V^\alpha$, so that V^α is w^*-closed in X^*. □

The topology w^α

The weak topology of a locally convex space (X, Q) is defined by the locally convex basis $\mathcal{W}$ formed by the sets of the form

$$V_{x_1^*,\dots,x_n^*;\varepsilon} = \{x \in X : |x_i^*(x)| < \varepsilon,\ 1 \le i \le n\}, \tag{2.4.33}$$

for $\in \mathbb{N}$, $x_1^*, \dots, x_n^* \in X^*$ and $\varepsilon > 0$. Obviously, we can suppose $x_i^* \ne 0$, $i = 1, \dots, n$.

The duality theory for locally convex spaces is based on the following key lemma of algebraic nature.

Lemma 2.4.31. ([238, Lemma VIII.3.3]) *Let X be a vector space and $f, f_1, \dots, f_n : X \to \mathbb{R}$ linear functionals. The following assertions are equivalent.*

1. $f \in \operatorname{sp}\{f_1, \dots, f_n\}$.
2. *There exists $L \ge 0$ such that*

$$\forall x \in X, \quad f(x) \le L \max\{f_1(x), \dots, f_n(x)\}\,.$$

3. $\bigcap_{i=1}^n \ker f_i \subset \ker f$.

In our case this lemma takes the following form.

Lemma 2.4.32. *Let $f, f_1, \ldots, f_n$ be real linear functionals on a vector space X, with $f_1, \ldots, f_n$ linearly independent. Then the following assertions are equivalent.*

1. $\forall x \in X, \quad [f_i(x) \le 0,\ i = 1, \ldots, n \Rightarrow f(x) \le 0.]$
2. $\exists L \ge 0$ *such that* $\forall x \in X,\ f(x) \le L \max\{f_i(x) : 1 \le i \le n\}$.
3. $\exists a_1, \ldots, a_n \ge 0,$ *such that* $f = \sum_{i=1}^n a_i f_i$.

Proof. Since the implications 2)$\Rightarrow$ 1) and 3)$\Rightarrow$ 2) are obvious, it is sufficient to prove 1) $\Rightarrow$ 3).

If $f_i(x) = 0$ for $i = 1, \ldots, n$, then $f_i(-x) = -f_i(x) = 0,\ i = 1, \ldots, n$, so that $f(x) \le 0$ and $-f(x) = f(-x) \le 0$, implying $f(x) = 0$. Therefore the condition 3) from Lemma 2.4.31 is fulfilled, so that there exist $a_1, \ldots, a_n \in \mathbb{R}$ such that $f = \sum_{i=1}^n a_i f_i$. It remains to show that $a_j \ge 0$ for $j = 1, \ldots, n$. Because $f_1, \ldots, f_n$ are linearly independent, there exist the elements $x_j \in X$ such that $f_i(x_j) = -\delta_{ij} \le 0,\ i, j = 1, 2, \ldots, n,$ where δ_{ij} is the Kronecker symbol. It follows that $f(x_j) \le 0$ and

$$-a_j = \sum_{i=1}^n a_i f_i(x_j) = f(x_j) \le 0\,,$$

for $j = 1, \ldots, n$. □

Let (X, P) be an asymmetric locally convex space and $X^\flat = (X, P)^\flat$ its asymmetric dual cone.

Define the asymmetric weak topology w^α on an asymmetric locally convex space (X, P) as the asymmetric locally convex topology generated by the asymmetric locally convex base $\mathcal{W}_\alpha$ formed of the sets

$$V_{\varphi_1, \ldots, \varphi_n; \varepsilon} = \{x \in X : \varphi_i(x) < \varepsilon,\ 1 \le i \le n\}, \tag{2.4.34}$$

for $n \in \mathbb{N}$, $\varphi_1, \ldots, \varphi_n \in X^\flat$ and $\varepsilon > 0$. The neighborhoods of an arbitrary point $x \in X$ are subsets of X containing a set of the form $x + V_{\varphi_1, \ldots, \varphi_n; \varepsilon} = \{x' \in X : \varphi_i(x' - x) < \varepsilon,\ 1 \le i \le n\}$.

The sets

$$V^-_{\varphi_1, \ldots, \varphi_n; \varepsilon} = \{x \in X : \varphi_i(x) \le \varepsilon,\ 1 \le i \le n\}$$

generate the same topology.

In the following proposition we collect some properties of the topology w^α.

Proposition 2.4.33. *Let (X, P) be an asymmetric locally convex space and $X^\flat = (X, P)^\flat$ its asymmetric dual cone.*

1. *The topology τ_P is finer than w^α.*
2. *For $\varphi \in X^\flat$ and $\varepsilon > 0$ the set $\{x \in X : \varphi(x) < \varepsilon\}$ is w^α-open and $\{x \in X : \varphi(x) \ge \varepsilon\}$ is w^α-closed.*

3. *A net* $\{x_i : i \in I\}$ *in* X *is* w^α*-convergent to* $x \in X$ *if and only if for every* $\varphi \in X^\flat$ *the net* $\{\varphi(x_i\}$ *converges to* $\varphi(x)$ *in* $(\mathbb{R}, u)$*. This means the following:*

$$\forall \varphi \in X^\flat,\ \forall \varepsilon > 0,\ \exists i_0 \ \text{ such that } \forall i \geq i_0,\ \varphi(x_i - x) < \varepsilon\ .$$

4. *The asymmetric dual* $(X, w^\alpha)^\flat$ *of the asymmetric locally convex space* (X, w^α) *agrees with* $X^\flat$*.*

Proof. Suppose P is directed.

1. Let $V = V_{\varphi_1,\ldots,\varphi_n;\varepsilon}$ be an element of the locally convex basis (2.4.34). Because φ_i are (P, u)-continuous there exist $p_i \in P$ and $L_i \geq 0$ such that

$$\forall x \in X, \quad \varphi_i(x) \leq L_i p_i(x), \text{ for } i = 1, \ldots, n\ .$$

The multiball $U = \{x \in X : p_i(x) < \varepsilon/(L+1),\ 1 \leq i \leq n\}$, where $L = \max L_i$, is contained in V, showing that V is a τ_P-neighborhood of $0 \in X$.

2. If $V = \{x \in X : \varphi(x) < \varepsilon\}$ and $x_0 \in V$, then the w^α-neighborhood $\{x \in X : \varphi(x - x_0) < \varepsilon - \varphi(x_0)\}$ of x_0 is contained in V because

$$\varphi(x - x_0) < \varepsilon - \varphi(x_0) \ \Rightarrow\ \varphi(x) = \varphi(x - x_0) + \varphi(x_0) < \varepsilon\ .$$

The assertion 3 follows from definitions.

4. Because τ_P is finer than w^α, the identity map Id: $(X, \tau_P) \to (X, w^\alpha)$ is continuous, implying the (P, u)-continuity of $\varphi \circ \mathrm{Id}$ for any $\varphi \in (X, w^\alpha)^\flat$, i.e., $(X, w^\alpha)^\flat \subset (X, P)^\flat$.

Conversely, if φ is a (P, u)-continuous linear functional, then the set $V = \{x \in X : \varphi(x) < \varepsilon\}$ is a w^α-neighborhood of $0 \in X$ and $\varphi(V) \subset (-\infty; \varepsilon)$ for every $\varepsilon > 0$, proving the (w^α, τ_u)-continuity of φ at 0, and by the linearity of φ, on the whole of X. □

As in the symmetric case the closed convex sets are the same for the topologies τ_P and w^α.

Proposition 2.4.34. *Let* (X, P) *be an asymmetric locally convex space and* Y *a convex subset of* X*.*

Then Y *is* w^α*-closed if and only if it is* τ_P*-closed.*

Proof. Because τ_P is finer than w^α, it follows that any (not necessarily convex) w^α-closed subset of X is also τ_P-closed.

Suppose now that the convex set Y is τ_P but not w^α-closed. If x_0 is a point in w^α-cl$Y \setminus Y$, then, applying Theorem 2.2.9 to the sets $\{x_0\}$ and Y we get a functional $\varphi \in X^\flat$ such that

$$\varphi(x_0) < \inf \varphi(Y)\ .$$

If $m := \inf \varphi(Y)$, then $V = \{x \in X : \varphi(x - x_0) < 2^{-1}(m - \varphi(x_0))\}$ is a w^α-neighborhood of x_0. Because

$$\varphi(x) = \varphi(x - x_0) + \varphi(x_0) < \frac{m + \varphi(x_0)}{2} < m ,$$

for every $x \in V$, it follows that $V \cap Y = \emptyset$, in contradiction to $x_0 \in w^\alpha$-cl Y. $\square$

The proposition has the following corollary.

Corollary 2.4.35. *Let (X, P) be an asymmetric locally convex space. Then for every subset Z of X the following equality holds:*

$$w^\alpha\text{-cl co}(Y) = \tau_P\text{-cl co}(Y) .$$

Proof. By the definition of the closed convex hull and the preceding proposition we have the equalities

$$\begin{aligned} w^\alpha\text{-cl co}(Y) &= \bigcap\{Y : Y \subset X,\ Y \text{ convex and } w^\alpha\text{-closed}\} \\ &= \bigcap\{Y : Y \subset X,\ Y \text{ convex and } \tau_P\text{-closed}\} \\ &= \tau_P\text{-clco}(Y) . \end{aligned}$$

$\square$

Remark 2.4.36. We can define the asymmetric polar of a subset W of the dual $X^\flat$ of an asymmetric locally convex space (X, P) by

$$W_\alpha = \{x \in X : \forall \varphi \in W,\ \varphi(x) \le 1\} .$$

Since, for $\varphi \in X^\flat$, a set of the form $\{x \in X : \varphi(x) \le 1\}$ is not necessarily τ_P-closed, the set W_α need not be τ_P-closed. Therefore an asymmetric analog of the bipolar theorem (see [238, Satz WIII.3.9]), asserting that

$$(A^\circ)_\circ = \text{cl-co}(A \cup \{0\}) ,$$

for any subset A of a locally convex space (X, Q), does not hold in the asymmetric case.

2.4.7 Asymmetric moduli of rotundity and smoothness

A convex body in a normed space $(X, \|\cdot\|)$ is a bounded closed convex set with nonempty interior. A *rooted convex body* in a normed space X is a pair (K, z) where K is a convex body and z is a fixed point in the interior K, called a *root* for K. If (K, z) is a rooted convex body, then $0 \in \text{int}(K - z) = \text{int}(K) - z$, so that the Minkowski functional p_{K-z} is well defined and it is an asymmetric norm on X satisfying $p_{K-z}(x) > 0$ for $x \neq 0$. The topology defined by p_{K-z} is equivalent to the norm-topology $\tau_{\|\cdot\|}$ of X. This follows from the facts that there exists $0 < r < R$

such that $B_{\|\cdot\|}(0,r) \subset K - z \subset B_{\|\cdot\|}(0,R)$ and $K - z = \{x \in X : p_{K-z}(x) \le 1\}$ is the closed unit ball of the asymmetric normed space (X, p_{K-z}).

Starting from this definition one can introduce geometric properties of convex bodies, as rotundity, local uniform rotundity, smoothness, uniform smoothness, expressed in terms of Minkowski functionals, by analogy with those from Banach space geometry (see, for instance, the book by Megginson [149]). This was done in a series of papers (see [114, 115, 116]) by V. Klee, E. Maluta, C. Zanco and L. Vesely, mainly in connection with the existence of good tilings of normed spaces. A *tiling* of a normed linear space X is a covering $\mathcal{T}$ of X such that each *tile* (member of $\mathcal{T}$) is a convex body and no point of X is interior to more than one tile. In contrast to the genuine theory of tilings in finite-dimensional spaces and an extensive theory for the plane, few significant examples were known and not even a rudimentary theory of tilings in infinite-dimensional spaces was built, see [113]. In the above-mentioned papers substantial progress was made in the study of tilings in infinite-dimensional Banach spaces. Zanco and Zucchi [240] defined and studied some properties of the asymmetric moduli of smoothness, quantitative expressions (in terms of Minkowski functionals) of the rotundity and smoothness properties of convex bodies.

Some uniform geometric properties for families of convex bodies were defined in [115]. We shall restrict these properties to a single convex body, a situation that fits better our needs. A convex body K in a normed space $(X, \|\cdot\|)$ is called *smooth* if for any point $y \in \partial K$ (the boundary of K) there exists exactly one hyperplane $H_y = \{x \in X : x^*(x) = c\}$, for some $x^* \in X^*$ and $c \in \mathbb{R}$, supporting K at y. That is $x^*(y) = c$ and $x^*(x) \le c$ for all $x \in K$. The convex body K is called *rotund* if its boundary ∂K does not contain nontrivial line segments, or, equivalently, if $d(\frac{1}{2}(x+y), \partial K) > 0$ (the distance with respect to $\|\cdot\|$) for every pair of distinct elements $x, y \in \partial K$. The convex body K is called *uniformly rotund* if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $d(\frac{1}{2}(x+y), \partial K) \ge \delta(\varepsilon)$ for every $x, y \in K$ with $\|x - y\| \ge \varepsilon$. It is obvious that one obtains the same notion if we require that $x, y \in \partial K$.

A rooted convex body (K, z) is smooth if and only if the Minkowski functional $p = p_{K-z}$ is Gâteaux differentiable on $X \setminus \{z\}$. This means that for any point $x \in X \setminus \{z\}$ there exists a continuous linear functional $p'(x; \cdot) \in X^*$ such that, for every $h \in X$,

$$\lim_{t \to 0} \frac{p(x+th) - p(x)}{t} = p'(x; h) \ . \tag{2.4.35}$$

With a rooted smooth convex body (K, z) one can associate two *duality mappings* from ∂K to X^* defined for $y \in \partial K$ as follows:

- $J_{(K,z)}(y) =$ the unique functional $x^* \in X^*$ such that $\|x^*\| = 1$ and $H_y - z = \{x \in X : x^*(x) = c\}$, for some $c \in \mathbb{R}$, and
- $J^1_{(K,z)}(y) =$ the unique functional $x^* \in X^*$ such that $H_y - z = \{x \in X : x^*(x) = 1\}$.

If $K = B_X$ (the closed unit ball of $(X, \|\cdot\|)$) and $z = 0$, then $J = J^1$ is the ordinary duality mapping, a very important notion in the geometry of Banach spaces and its applications to nonlinear operator theory, see, for instance, the book by Ciorănescu [36]. Also, it is clear that $J_{K,z_1} = J_{K,z_2}$ for every $z_1, z_2 \in \operatorname{int} K$, so that the duality mapping J_{K,z_1} does not depend on the root z, consequently it can be denoted simply by J_K.

A normed space $(X, \|\cdot\|)$ is called *uniformly smooth* if the norm is uniformly Fréchet differentiable on the unit sphere S_X. This means that, putting $f(x) = \|x\|$, $x \in X$, for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for every $x \in S_X$ and every $h \in X$ with $\|h\| \le \delta$

$$|f(x+h) - f(x) - f'(x)h| \le \varepsilon \|h\| \,,$$

where $f'(x) \in X^*$ denotes the Fréchet derivative of f at x. The modulus of smoothness of the space X is defined for $\tau \ge 0$ by

$$\rho_X(\tau) = \sup \left\{ \frac{1}{2} \left(\|x+y\| + \|x-y\| - 2 \right) : x \in S_X ,\ \|y\| = \tau \right\} \,. \tag{2.4.36}$$

The uniform smoothness of X can be characterized by the condition

$$\lim_{\tau \searrow 0} \frac{\rho_X(\tau)}{\tau} = 0 \,,$$

see, e.g., [149]. Another characterization can be done in terms of the duality mapping J, namely, the space X is uniformly smooth if and only if the duality mapping $J : S_X \to S_{X^*}$ is norm-to-norm continuous (see [36]).

Starting from this property one can define the *uniform smoothness* of a rooted convex body (K, z) by asking that the duality mapping $J_{K,z} : \partial K \to S_{X^*}$ is norm-to-norm continuous (see [115]). The norm-to-norm continuity of the duality mapping $J_{K,z}$ is equivalent to the norm-to-norm continuity of the duality mapping $J^1_{K,z}$, so that one obtains the same notion of uniform smoothness by working with the duality mapping $J^1_{K,z}$.

A normed space $(X, \|\cdot\|)$ is called *uniformly rotund* (*uniformly convex* by some authors) if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|x + y\| \ge 2(1 - \delta)$ for every $x, y \in S_X$ with $\|x - y\| \le \varepsilon$.

This property admits also quantitative characterizations in terms of some moduli. We mention two of them.

Clarkson's modulus of uniform rotundity $\delta_X : [0; 2] \to [0; 1]$ defined by

$$\begin{aligned} \delta_X(\varepsilon) &= \inf \left\{ 1 - \frac{1}{2}\|x+y\| : x, y \in S_X ,\ \|x-y\| \ge \varepsilon \right\} \\ &= \inf \left\{ 1 - \frac{1}{2}\|x+y\| : x, y \in S_X ,\ \|x-y\| = \varepsilon \right\} \,. \end{aligned} \tag{2.4.37}$$

Gurarii's modulus of uniform rotundity $\gamma_X : [0;2] \to [0;1]$ defined by

$$\gamma_X(\varepsilon) = \inf\{\max_{0\le t\le 1}(1 - \|tx + (1-t)y\|) : x, y \in S_X,\ \|x-y\| = \varepsilon\}\ . \tag{2.4.38}$$

One has $\delta_X \le \gamma_X$, and a 2-dimensional example given in [240] shows that the inequality can be strict. Also it is unknown whether the condition $\|x-y\| = \varepsilon$ in the definition of Gurarii's modulus can be replaced by the condition $\|x-y\| \ge \varepsilon$, as in the definition of Clarkson's modulus. The uniform rotundity of the space X can be characterized in the following way:

$$X \text{ is uniformly rotund } \iff \forall\varepsilon \in (0;2],\ \delta_X(\varepsilon) > 0 \iff \forall\varepsilon \in (0;2],\ \gamma_X(\varepsilon) > 0\ .$$

The *characteristic of convexity* corresponding to the moduli δ_X and γ_X are $\varepsilon_X^0 = \sup\{\varepsilon \in [0;2] : \delta_X(\varepsilon) = 0\} = \{\varepsilon \in [0;2] : \gamma_X(\varepsilon) = 0\}$. In terms of this characteristic of convexity the uniform rotundity of X is characterized by the condition $\varepsilon_X^0 = 0$ and the rotundity by the condition $\delta_X(2) = \gamma_X(2) = 1$. Other geometric properties of the normed space X, as, for instance, the uniform non-squareness can also be expressed in terms of these moduli.

The analogs of these moduli (Gurarii's variant in the case of uniform rotundity) and their relevance for the smoothness and rotundity properties of rooted convex bodies were given in the paper by Zanco and Zucchi [240].

The modulus of smoothness of a rooted convex body (K,z), $\rho_{(K,z)} : [0;\infty) \to [0;1)$ can be defined by replacing the norm in (2.4.36) by the Minkowski functional:

$$\begin{aligned}\rho_{(K,z)}(\tau) = \sup\Big\{\frac{1}{2}(p_{K-z}(x+y) + p_{K-z}(x-y)) - 1 :\\ x \in \partial(K-z),\ y \in X, p_{K-z}(y) = \tau\Big\}\ .\end{aligned} \tag{2.4.39}$$

For the sake of simplicity suppose $z = 0$ and put $\rho_K = \rho_{(K,0)}$. The modulus of smoothness is a continuous convex function such that $\rho_K(\tau) \le \lambda_K\tau$, where

$$\lambda_K = \sup\{p_K(-y) : y \in K\} = \sup\{p_{K\cap K}(y) : y \in K\}\ ,$$

could be taken as a possible measure of eccentricity of K with respect to 0. By analogy with the normed case, the asymmetric norm p_K is called K-uniformly Fréchet differentiable if (2.4.35) holds uniformly with respect to $h \in K$ and $x \in \partial K$. One shows ([240, Th. 4.2]) that the following are equivalent:

- the rooted convex body (K,z) is uniformly smooth;
- the Minkowski functional p_{K-z} is K-uniformly Fréchet differentiable on ∂K;
- $\lim_{\tau\searrow 0}\big(\rho_{(K,z)}(\tau)/\tau\big) = 0$ for every $z \in \operatorname{int} K$.

The definition of the modulus of uniform rotundity is more involved and needs to consider the existence of diametral points in K. The *Minkowski diameter* (M-diameter) of a rooted convex body (K,z) is defined by

$$\operatorname{diam}_M(K,z) = \sup\{p_{K-z}(x-y) : x, y \in K\}\ . \tag{2.4.40}$$

A pair of points $x, y \in K$ is called *M-diametral* if

$$\operatorname{diam}_M(K, z) = \max\{p_{K-z}(x-y), p_{K-z}(y-x)\}.$$

The problem of the existence of diametral points of convex bodies with respect to the norm was studied by Garkavi [98] in connection with some minimax and maximin problems. The center of the largest ball (whose radius is denoted by r_K) contained in a convex body K is called an H-center for K, while the center of the smallest ball containing K is called a *Chebyshev center* of K. Both centers could not exist, and if they exist they need not be unique. If x_0 is an H-center for K, then the set $A_K(x_0) = \{y \in K : \|x^0 - y\| = r_K\}$ is called the critical set of the H-center x_0. Garkavi, *loc. cit.*, proved that a Banach space X is reflexive if and only if every convex body in X has an H-center. Also, the space X is finite dimensional if and only if for any convex body K in X and any H-center x_0 of K the set $A_K(x_0)$ of critical points is nonempty.

By a compactness argument it follows that if X is finite dimensional, then every rooted convex body admits M-diametral points. In the case of M-diametral points the authors show in [240] that, for every $p \in [1;\infty)$, the space ℓ_p contains a rooted convex body $(K, 0)$ whose M-diameter is not attained. The general problem of the validity of Garkavi's results for M-diameters remains open, with the conjecture that the existence of M-diametral points for every rooted convex body is equivalent to the finite dimensionality of X. For a rooted convex body (K, z) in a normed space let

$$\Delta_{(K,z)}(\varepsilon) = \inf\Big\{\max_{t\in[0;1]}(1 - p_{K-z}(tx + (1-t)y)) : x, y \in K - z,\, p_{K-z}(x-y) \geq \varepsilon\Big\}. \tag{2.4.41}$$

If (K, z) has diametral points, then $\Delta_{(K,z)}$ is defined on $[0; \operatorname{diam}_M(K, z)]$ with values in $[0; 1]$ and, if (K, z) does not have diametral points, then $\Delta_{(K,z)}$ is defined on $[0; \operatorname{diam}_M(K, z))$ with values in $[0; 1)$.

Based on this notion, one defines the modulus of uniform rotundity $\gamma_{(K,z)}$ of a rooted convex body (K, z) by the conditions:

- $\gamma_{(K,z)}(\varepsilon) = \Delta_{(K,z)}(\varepsilon), \quad$ for $0 \leq \varepsilon < \operatorname{diam}_M(K, z)$,
- $\gamma_{(K,z)}(\operatorname{diam}_M(K, z)) = \Delta_{(K,z)}(\operatorname{diam}_M(K, z)), \quad$ if $\dim X = 2$,

and

- $\gamma_{(K,z)}(\operatorname{diam}_M(K, z)) = \inf\big\{\gamma_{(Y\cap(K-z),0)}(\operatorname{diam}_M(Y \cap (K - z), 0)) :$ Y is a 2-dimensional subspace of $X\big\}$, in general.

The last formula is justified by the fact that a similar result holds for both moduli of uniform rotundity δ_X and γ_X.

One shows that the function $\gamma_{(K,z)}$ is continuous on some interval $[0; \beta)$, where the number $\beta > 0$ depends on the geometric properties of the convex body K, expressed in terms of the so-called directional M-diameters of K. Also the convex body K is rotund if and only if $\inf\{\gamma_{(K,z)}(\operatorname{diam}_M(K, z)) : z \in \operatorname{int} K\} = 1$, and is uniformly rotund if and only if $\varepsilon^0_{(K,z)} = 0$ for every $z \in \operatorname{int} K$, where $\varepsilon^0_{(K,z)} = \sup\{\varepsilon \in [0; \operatorname{diam}_M(K,z)) : \gamma_{(K,z)}(\varepsilon) = 0\}$ is the characteristic of convexity corresponding to the asymmetric modulus $\gamma_{(K,z)}$.

If a Banach space X contains a rooted convex body $(K,0)$ with $\varepsilon^0_{(K,0)} < 2$, then X is superreflexive. Recall that a Banach space X is called superreflexive if it admits an equivalent uniformly rotund renorming (it is known that any uniformly rotund Banach space is reflexive).

If $0 \in \operatorname{int} K$, then one defines the polar set of K by $K^\pi = \{x^* \in X^* : \forall y \in K,\ x^*(y) \leq 1\}$. If B_X is the unit ball of a normed space $(X, \|\cdot\|)$, then B_X^π is the unit ball of the dual space. Based on Corollary 2.2.4 from the next section, it follows that this relation is also true in the asymmetric case: $B_{X^\flat} = B_p^\pi$. The well-known duality relation between uniform smoothness (US) and uniform rotundity (UR) holds in this case too:

- the convex body K is UR $\iff$ the polar set K^π is US.

Remark 2.4.37. The results presented above stand in a normed space. It would be of interest to study these properties in an asymmetric normed space, and to see their significance for the properties of the corresponding asymmetric normed space. For instance, is there any connection between the asymmetric uniform rotundity and the reflexivity of the asymmetric normed space (as defined in Subsection 2.4.5), like in the case of Banach spaces?

The approximation properties of subsets of a Banach space heavily depend on the geometric properties of the underlying space, see, for instance, the survey [42]. It would be interesting to see to what extent can these properties be extended to asymmetric normed spaces?

2.5 Applications to best approximation

The aim of this section is to study best approximation in asymmetric normed spaces. Due to the asymmetry of the norm, two kind of distances from a point to a set have to be considered, exemplified on Ascoli's formula for the distance to a closed hyperplane. Some characterization and duality results for best approximation by elements in closed convex sets and by elements in sets with convex bounded complement are proved.

As it is known, the natural framework for treating the problem of best approximation is that of normed spaces, see the books by Singer [222, 223], so that it is very natural to consider the corresponding problem in asymmetric normed spaces. Some problems of best approximation with respect to an asymmetric norm, including approximation in spaces of continuous or integrable functions, were considered by Duffin and Karlovitz [70]. Dunham [71] treated the problem of best approximation by elements of a finite-dimensional subspace of an asymmetric normed normed space and proved existence results, uniqueness results (guaranteed by the rotundity of the asymmetric norm), and found some conditions ensuring the continuity of the metric projection. Pfankuche-Winkler [176] considered the best approximation problem in some asymmetric normed spaces of Orlicz type. De Blasi and Myjak [59] proved some generic existence results for the problem of best approximation with respect to an asymmetric norm in a Banach space. Similar problems were considered by Li and Ni [146] and Ni [167].

As it is well known, any closed convex subset of a Hilbert space is Chebyshev. A famous problem in best approximation theory is that of the convexity of Chebyshev sets: must any Chebyshev subset of a Hilbert space be convex? There are a lot of results in this direction as presented, for instance, in the survey paper by Balaganskii and Vlasov [20], but the general problem is still unsolved. In some of the papers dealing with this problem one works with asymmetric norms as, for instance, in Alimov [12].

2.5.1 Characterizations of nearest points in convex sets and duality

As in the normed case, linear functionals are useful in characterizing the nearest points and for the duality results in best approximation in asymmetric normed spaces. In the following we shall present some results obtained in the papers [40, 48, 49]. Let (X,p) be an asymmetric normed space, Y a nonempty subset of X and $x\in X$. Recall that, due to the asymmetry of the norm, we have to consider two *distances* from x to Y:

$$\text{(i)}\ \ d_p(x,Y)=\inf\{p(y-x):y\in Y\}\quad\text{and}\quad\text{(ii)}\ \ d_p(Y,x)=\inf\{p(x-y):y\in Y\}\,. \tag{2.5.1}$$

Observe that $d_p(Y,x)=d_{\bar p}(x,Y)$, where $\bar p$ is the norm conjugate to p. Let also

$$P_Y(x)=\{y\in Y:p(y-x)=d_p(x,Y)\}$$

and

$$\bar P_Y(x)=\{y\in Y:p(x-y)=d_p(Y,x)\}\,,$$

denote the metric projections on Y. An element y in $P_Y(x)$ is called a *p-nearest point* to x in Y, while an element $\bar y$ in $\bar P_Y(x)$ is called a *$\bar p$-nearest point* to x in Y.

The set Y is called:

- *p-proximinal* if $P_Y(x)\neq\emptyset$ for every $x\in X$,
- *p-semi-Chebyshev* if $\#P_Y(x)\le 1$ for every $x\in X$ (i.e., every $x\in X$ has at most one p-nearest point in Y),
- *p-Chebyshev* if $\#P_Y(x)=1$ for every $x\in X$ (i.e., every $x\in X$ has exactly one p-nearest point in Y).

The corresponding notions for the conjugate norm $\bar p$ are defined similarly.

A consequence of Theorem 2.2.6 is the following characterization of nearest points.

Theorem 2.5.1. *Let (X,p) be a space with asymmetric norm, Y a subspace of X and x_0 a point in X.*

1. *Let $d = d_p(x_0, Y) > 0$. An element $y_0 \in Y$ is a p-nearest point to x_0 in Y if and only if there exists a p-bounded linear functional $\varphi : X \to \mathbb{R}$ such that*
$$\text{(i) } \varphi|_Y = 0, \quad \text{(ii) } \|\varphi|_p = 1, \quad \text{(iii) } \varphi(-x_0) = p(y_0 - x_0)\ .$$
2. *Let $\bar{d} = d_{\bar{p}}(x_0, Y) > 0$. An element $y_1 \in Y$ is a $\bar{p}$-nearest point to x_0 in Y if and only if there exists a p-bounded linear functional $\psi : X \to \mathbb{R}$ such that*
$$\text{(j) } \psi|_Y = 0, \quad \text{(jj) } \|\psi|_p = 1, \quad \text{(jjj) } \psi(x_0) = p(x_0 - y_1)\ .$$

Proof. 1. Suppose that $y_0 \in Y$ is such that $p(y_0 - x_0) = d = d_p(x_0, Y) > 0$. By Theorem 2.2.6, there exists $\varphi \in X_p^\flat$, $\|\varphi|_p = 1$, such that $\varphi|_Y = 0$ and $\varphi(-x_0) = d = p(y_0 - x_0)$.

Conversely, if for $y_0 \in Y$ there exists $\varphi \in X^\flat$ satisfying the conditions (i)–(iii), then for every $y \in Y$,

$$p(y - x_0) \geq \varphi(y - x_0) = \varphi(y_0 - x_0) = p(y_0 - x_0)\ ,$$

implying $p(y_0 - x_0) = d_p(x_0, Y)$.

The second assertion can be proved in a similar way. □

For a nonempty subset Y of an asymmetric normed space (X,p) put

$$Y^\perp = Y_p^\perp = \{\varphi \in X_p^\flat : \varphi|_Y = 0\}\ .$$

A consequence of Eidelheit's Separation Theorem (Theorem 2.2.8) is the following duality result for best approximation by elements of convex sets in asymmetric normed spaces. These extend results obtained in the case of normed spaces by Nikolski [169], Garkavi [96, 97], Singer [221] (see also Singer's book [222, Appendix I] and [99]). Some duality results in the asymmetric case were proved also by Babenko [19]. The case of so-called p-convex sets was considered in [47] and [37, 38].

Theorem 2.5.2. *For a nonempty convex subset Y of a space (X,p) with asymmetric norm and $x_0 \in X$, the following duality relations hold:*

$$d_p(x_0, Y) = \sup_{\|\varphi|_p \leq 1} \inf_{y \in Y} \varphi(y - x_0) \tag{2.5.2}$$

and

$$d_p(Y, x_0) = \sup_{\|\varphi|_p \leq 1} \inf_{y \in Y} \varphi(x_0 - y). \tag{2.5.3}$$

If $d_p(x_0, Y) > 0$, then there exists $\varphi_0 \in X_p^\flat$, $\|\varphi_0|_p = 1$, such that $d_p(x_0, Y) = \inf\{\varphi_0(y - x_0) : y \in Y\}$, i.e., the supremum in the right-hand side of the relation (2.5.2) is attained.

A similar result holds for the second duality relation.

Proof. We shall prove first the relation (2.5.2) and obtain (2.5.3) as an immediate consequence. Let $i = d_p(x_0, Y)$ and denote by s the quantity in the right-hand side of the relation (2.5.2).

For any $\varphi \in X_p^\flat$ with $\|\varphi|_p \le 1$ we have

$$\forall y \in Y,\ \varphi(y - x_0) \le p(y - x_0)\ ,$$

implying

$$\forall \varphi \in X_p^\flat \text{ with } \|\varphi|_p \le 1, \quad \inf\{\varphi(y - x_0) : y \in Y\} \le i\ ,$$

so that $s \le i$. Taking $\varphi = 0$ in the definition of s it follows that $s \ge 0$, so that $s = i = 0$ if $i = 0$.

Suppose now $i > 0$ and let

$$Z := \{x \in X : p(x - x_0) < i\}\ .$$

It follows that Z is nonempty, convex, τ_p-open and $Z \cap Y = \emptyset$. By the first separation theorem (Theorem 2.2.8), there exists $\psi \in X_p^\flat$ such that

$$\forall z \in Z\ \forall y \in Y, \quad \psi(z) < \psi(y)\ .$$

Putting $\varphi = (1/\|\psi|_p)\psi$ we have $\|\varphi|_p = 1$ and

$$\forall z \in Z\ \forall y \in Y, \quad \varphi(z - x_0) < \varphi(y - x_0). \tag{2.5.4}$$

Since

$$\sup\{\varphi(z - x_0) : z \in Z\} = \sup\{\varphi(w) : p(w) < i\} = i\|\varphi|_p = i\ ,$$

the inequality (2.5.4) yields

$$i = \sup\{\varphi(z - x_0) : z \in Z\} \le \inf\{\varphi(y - x_0) : y \in Y\} \le s\ ,$$

so that $s = i$.

Let us show now that the relation (2.5.3) follows from (2.5.2). Since the relation (2.5.3) holds for $\bar{p}$ too, by Proposition 2.1.7.5 we can write

$$\begin{aligned}\inf_{y \in Y} p(x_0 - y) &= \sup\{\inf \psi(Y - x_0) : \psi \in X_{\bar{p}}^\flat,\ \|\psi|_{\bar{p}} \le 1\} \\ &= \sup\{\inf(-\psi)(x_0 - Y) : -\psi \in X_p^\flat,\ \|-\psi|_p \le 1\} \\ &= \sup\{\inf \varphi(x_0 - Y) : \varphi \in X_p^\flat,\ \|\varphi|_p \le 1\}\ ,\end{aligned}$$

showing that the relation (2.5.3) holds too.

Finally, suppose $i > 0$ and let (φ_n) be a sequence in the unit ball $B_{X_p^\flat}$ of $X_p^\flat$ such that $\lim_n \inf\{\varphi_n(y - x_0) : y \in Y\} = s$ or, equivalently,

$$\lim_n (\inf\ \varphi_n(Y) - \varphi_n(x_0)) = \ s. \tag{2.5.5}$$

Since the ball $B_{X_p^\flat}$ is a w^*-compact subset of B_{X^*} (see Proposition 2.4.2), it follows that the sequence (φ_n) contains a subnet $(\psi_j : j \in J)$ which is w^*-convergent to an element $\varphi_0 \in B_{X_p^\flat}$. It follows that

$$\forall z \in X, \quad \lim_j \psi_j(z) = \varphi_0(z) \ .$$

Then, by (2.5.5), $\lim_j \inf \psi_j(Y) = \varphi_0(x_0) + s$. Since, for every $y \in Y$, $\psi_j(y) \geq \inf \psi_j(Y)$, by passing to the limit for $j \in J$ we get

$$\forall y \in Y, \quad \varphi_0(y) \geq \varphi_0(x_0) + s \ ,$$

so that

$$s \leq \inf \varphi_0(Y) - \varphi_0(x_0) = \inf \varphi_0(Y - x_0) \ .$$

The definition of s implies $s = \inf \varphi_0(Y - x_0)$.

It remains to show that $\|\varphi_0|_p = 1$. If $\|\varphi_0|_p < 1$, then $\lambda = 1/\|\varphi_0|_p > 1$ and the functional $\psi_0 = \lambda\varphi_0$ satisfies $\|\psi_0|_p = 1$ and

$$s \geq \inf \psi_0(Y - x_0) = \lambda \inf \varphi_0(Y - x_0) = \lambda s \ ,$$

a contradiction, since $s = i > 0$. Observe that $s > 0$ implies $\varphi_0 \neq 0$, so that $\|\varphi_0|_p > 0$ and λ is properly defined. □

Based on this duality result one obtains the following characterization of nearest points.

Theorem 2.5.3. *Let (X, p) be a space with asymmetric norm, Y a nonempty subset of X, $x \in X$, and $y_0 \in Y$.*

If there exists a functional $\varphi_0 \in X_p^\flat$ such that

$$\text{(i) } \|\varphi_0|_p = 1, \quad \text{(ii) } \varphi_0(y_0 - x) = p(y_0 - x), \quad \text{(iii) } \varphi_0(y_0) = \inf \varphi_0(Y) \ , \quad (2.5.6)$$

then y_0 is a p-nearest point to x in Y.

Similarly, if for $z_0 \in Y$, there exists a functional $\psi_0 \in X_p^\flat$ such that

$$\text{(j) } \|\psi_0|_p = 1, \quad \text{(jj) } \psi_0(x - z_0) = p(x - z_0), \quad \text{(jjj) } \psi_0(z_0) = \sup \psi_0(Y) \ , \quad (2.5.7)$$

then z_0 is a $\bar{p}$-nearest point to x in Y.

Conversely, if Y is convex, $d_p(x, Y) > 0$, an y_0 is a p-nearest point to x in Y, then there exists a functional $\varphi_0 \in X_p^\flat$ satisfying the conditions (i)–(iii) *from above.*

Similarly, if $z_0 \in Y$ is a $\bar{p}$-nearest point to x in Y with $p(x - z_0) > 0$, then there exists a functional $\psi_0 \in X_p^\flat$ satisfying the conditions (j)–(jjj).

Proof. Suppose that $\varphi_0 \in X_p^\flat$ satisfies the conditions (i)–(iii). Then for every $y \in Y$,

$$p(y_0-x) = \varphi_0(y_0-x) = \varphi_0(y_0)-\varphi_0(x) = \inf \varphi_0(Y)-\varphi_0(x) \le \varphi_0(y-x) \le p(y-x) ,$$

showing that

$$p(y_0 - x) = \inf\{p(y - x) : y \in Y\} = d_p(x, Y) .$$

If $z_0 \in Y$ and $\psi_0 \in X_p^\flat$ are such that the conditions (j)–(jjj) hold, then

$$p(x-z_0) = \psi_0(x-z_0) = \psi(x)-\sup \psi_0(Y) = \inf \psi_0(x-Y) \le \psi_0(x-y) \le p(x-y) ,$$

for all $y \in Y$.

Suppose now that Y is convex and y_0 is a p-nearest point to x in Y, $p(y_0-x) = d_p(x, Y) > 0$. Let $\varphi_0 \in X_p^\flat$ be the functional whose existence is stated in the second part of Theorem 2.5.3, i.e., $\|\varphi_0|_p = 1$ and $\inf \varphi_0(Y - x) = d_p(x, Y) = p(y_0 - x)$.

Then

$$p(y_0 - x) = d_p(x, Y) = \inf \varphi_0(Y - x) \le \varphi(y_0 - x) \le p(y_0 - x) ,$$

implying $\varphi_0(y_0) = \inf \varphi_0(Y)$ and $\varphi_0(y_0 - x) = p(y_0 - x)$.

If $z_0 \in Y$ is such that $p(x - z_0) = d_p(Y, x) > 0$, then, by the second part of Theorem 2.5.3, there exists $\psi_0 \in X_p^\flat$ such that $\|\psi_0|_p = 1$, and $\inf \psi_0(x - Y) = \sup\{\inf \psi(x - Y) : \psi \in X_p^\flat,\ \|\psi|_p \le 1\} = d_p(Y, x)$. Then, by the duality relation (2.5.7), we have

$$p(x_0 - z_0) = d_p(Y, x) = \inf \psi_0(x - Y) \le \psi_0(x - z_0) \le p(x - z_0)$$

implying $\psi_0(x - z_0) = p(x - z_0)$ and $\psi_0(x) - \psi(z_0) = \inf \psi_0(x - Y) = \psi_0(x) - \sup \psi_0(Y)$, so that $\psi_0(z_0) = \sup \psi_0(Y)$. □

When Y is a subspace of a space with asymmetric norm (X, p), one obtains the following characterization of nearest points. Denote by $Y^\perp = \{\varphi \in X_p^\flat : \varphi|_Y = 0\}$ *the annihilator* of Y in $X_p^\flat$.

Corollary 2.5.4. *Let Y be a subspace of (X, p), $x \in X$ and $y_0 \in Y$. If there exists $\varphi_0 \in Y^\perp$ such that*

$$(\mathrm{i}')\ \|\varphi_0|_p = 1 \quad \textit{and} \quad (\mathrm{ii}')\ \varphi_0(y_0 - x) = p(y_0 - x) ,$$

then $p(y_0 - x) = d_p(x, Y)$, i.e., y_0 is a p-nearest point to x in Y.

Conversely, if $y_0 \in Y$ is such that $p(y_0 - x) = d_p(x, Y) > 0$, then there exists a functional $\varphi_0 \in Y^\perp$ which satisfies the conditions (i′) *and* (ii′).

Similarly, in order that $z_0 \in Y$ be a $\bar{p}$-nearest point to $x \in X$ it is sufficient and, if $d_p(Y, x) > 0$ also necessary, to exist a functional $\psi_0 \in Y^\perp$ such that

$$(j')\ \|\psi_0|_p = 1 \quad \textit{and} \quad (jj')\ \psi_0(x - z_0) = p(x - z_0) .$$

Proof. If $\varphi_0 \in Y^\perp$ satisfies the conditions (i$'$) and (ii$'$), then $\varphi_0(y_0) = 0 = \inf \varphi_0(Y)$, so that, by Theorem 2.5.3, it is a p-nearest point to x.

Conversely, if $p(y_0 - x) = d_p(x, Y) > 0$, then, by the necessity part of the same theorem, there exists $\varphi_0 \in X_p^\flat$ satisfying the conditions (i)–(iii). By (iii) we have

$$\forall y' \in Y,\ \varphi_0(y' - y_0) \le 0 \iff \forall y \in Y,\ \varphi_0(y) \le 0 \iff \forall y \in Y,\ \varphi_0(y) = 0\ ,$$

showing that $\varphi_0 \in Y^\perp$.

The case of a $\bar{p}$-nearest point z_0 is treated similarly. □

Based on Theorem 2.2.13 one can prove the following characterization theorem in terms of the extreme points of the unit ball of the dual space $X_p^\flat$. In the case of a normed space X the result was obtained by Singer [221] when Y is a subspace of X and by Garkavi [97] for convex sets. The asymmetric case was treated in [40].

Theorem 2.5.5. *Let (X, p) be a space with asymmetric norm, Y a nonempty subset of X, $x \in X$ and $y_0 \in Y$.*

If for every $y \in Y$ there is a functional $\varphi = \varphi_y$ in the unit ball $B_p^\flat$ of $X_p^\flat$ such that

$$\text{(i) } \varphi(y_0 - x) = p(y_0 - x) \quad \textit{and} \quad \text{(ii) } \varphi(y_0 - y) \le 0\ ,$$

then y_0 is a p-nearest point to x in Y.

Conversely, if Y is convex and $y_0 \in Y$ is such that

$$p(y_0 - x) = d_p(x, Y) > 0\ ,$$

then for every $y \in Y$ there exists an extreme point $\varphi = \varphi_y$ of the unit ball $B_p^\flat$ of $X_p^\flat$, satisfying the conditions (i) *and* (ii) *from above.*

Similarly, if $z_0 \in Y$ is such that for every $y \in Y$ there exists a functional $\psi = \psi_y \in B_p^\flat$ such that

$$\text{(j) } \psi(x - z_0) = p(x - z_0) \quad \textit{and} \quad \text{(jj) } \psi(y - z_0) \le 0\ ,$$

then z_0 is a $\bar{p}$-nearest point to x in Y.

Conversely, if $z_0 \in Y$ is such that $p(x - z_0) = d_p(Y, x) > 0$, then for every $y \in Y$ there exists an extreme point $\psi = \psi_y$ of the unit ball $B_p^\flat$ of $X_p^\flat$ satisfying the conditions (j) *and* (jj)*.*

Proof. Suppose that $y_0 \in Y$ is such that for every $y \in Y$ there is a functional $\varphi = \varphi_y$ in $X_p^\flat$ satisfying the conditions (i) and (ii). Then, for every $y \in Y$,

$$p(y_0 - x) = \varphi(y_0 - x) = \varphi(y_0 - y) + \varphi(y - x) \le \varphi(y - x) \le p(y - x)\ ,$$

showing that $p(y_0 - x) = d_p(x, Y)$.

Suppose now that Y is convex and that $y_0 \in Y$ is a p-nearest point to x in Y such that $d_p(x,Y) = p(y_0 - x) > 0$. The equalities

$$p(y_0 - x) = \inf\{p(y-x) : y \in Y\} = \inf\{p(w) : w \in Y - x\}$$

show that $y_0 - x$ is a p-nearest point to 0 in $Y - x$. For $y \in Y \setminus \{y_0\}$ let $Z := \mathrm{sp}\{y_0 - x, y - x\}$ – the space generated by $y_0 - x$ and $y - x$, and let $W := Z \cap (Y - x)$. Since $y_0 - x$ is a p-nearest point to 0 in W, by Theorem 2.5.3, there exists $\psi_0 \in Z_p^\flat,\ \|\psi_0|_p = 1,$ such that

$$\psi_0(y_0 - x) = p(y_0 - x) \quad \text{and} \quad \psi_0(y_0 - x) = \inf \psi_0(W)\ .$$

It follows that

$$\psi_0(y_0 - x) \le \psi_0(y - x) \iff \psi_0(y_0 - y) \le 0\ .$$

The set $B_{Z_p^\flat}$ is a w^*-compact convex subset of the two-dimensional space Z^*, so that, by the Carathéodory and Krein-Milman theorems,

$$\psi_0 = \sum_{i=1}^{r} \alpha_i \psi_i \tag{2.5.8}$$

where $1 \le r \le 3,\ \alpha_i > 0,\ \sum_i \alpha_i = 1,$ and ψ_i are extreme points of the set $B_{Z_p^\flat}$. The equality $\psi_0(y_0 - x) = p(y_0 - x)$ and (2.5.8) imply $\psi_i(y_0 - x) = p(y_0 - x),\ i = 1, \ldots, r$. Also, since $\psi_0(y_0 - y) \le 0$, at least one of the ψ_i, say ψ_1, must satisfy $\psi_1(y_0 - y) \le 0$.

By Theorem 2.2.13, ψ_1 has a norm preserving extension $\varphi \in X_p^\flat$ that is an extreme point of the unit ball $B_p^\flat$. The functional φ satisfies all the requirements of the theorem.

The case of $\bar{p}$-nearest points can be reduced to that of p-nearest points, by working in the space $(X, \bar{p})$, taking into account the equality $d_p(Y,x) = d_{\bar{p}}(x,Y)$, the fact that $\psi \in X_{\bar{p}}^\flat$ if and only if $-\psi \in X_p^\flat$ and that $\|\psi|_{\bar{p}} = \| - \psi|_p$ (see Proposition 2.1.7.5). Also, as it is easily seen, ψ is an extreme point of the unit ball $B_{\bar{p}}^\flat$ of $X_{\bar{p}}^\flat$ if and only if $-\psi$ is an extreme point of the ball $B_p^\flat$. □

2.5.2 The distance to a hyperplane

As it was shown in [48], the well-known formula for the distance to a closed hyperplane in a normed space (the so-called Arzelá formula) has an analog in spaces with asymmetric norm. Remark that in this case we have to work with both of the distances d_p and $d_{\bar{p}}$ given by (2.2.3).

Proposition 2.5.6. *Let (X,p) be a space with asymmetric norm, $\varphi \in X_p^\flat,\ \varphi \neq 0,\ c \in \mathbb{R}$,*

$$H = \{x \in X : \varphi(x) = c\}$$

the hyperplane corresponding to φ *and* c*, and*

$$H^{<} = \{x \in X : \varphi(x) < c\} \quad \text{and} \quad H^{>} = \{x \in X : \varphi(x) > c\} \ ,$$

the open half-spaces determined by H*.*

1. *We have*
$$d_{\bar{p}}(x_0, H) = \frac{\varphi(x_0) - c}{\|\varphi|_p} \tag{2.5.9}$$
for every $x_0 \in H^{>}$*, and*
$$d_p(x_0, H) = \frac{c - \varphi(x_0)}{\|\varphi|_p} \tag{2.5.10}$$
for every $x_0 \in H^{<}$*.*
2. *If there exists an element* $z_0 \in X$ *with* $p(z_0) = 1$ *such that* $\varphi(z_0) = \|\varphi|_p$ *then every element in* $H^{>}$ *has a* $\bar{p}$*-nearest point in* H*, and every element in* $H^{<}$ *has a* p*-nearest point in* H*.*
3. *If there is an element* $x_0 \in H^{>}$ *having a* $\bar{p}$*-nearest point in* H*, or there is an element* $x_0' \in H^{<}$ *having a* p*-nearest point in* H*, then there exists an element* $z_0 \in X,\ p(z_0) = 1,$ *such that* $\varphi(z_0) = \|\varphi|_p$*. It follows that, in this case, every element in* $H^{>}$ *has a* $\bar{p}$*-nearest point in* H*, and every element in* $H^{<}$ *has a* p*-nearest point in* H*.*

Proof. 1. Let $x_0 \in H^{>}$. Then, for every $h \in H,\ \varphi(h) = c$, so that

$$\varphi(x_0) - c = \varphi(x_0 - h) \le \|\varphi|p(x_0 - h) \ ,$$

implying

$$d_{\bar{p}}(x_0, H) \ge \frac{\varphi(x_0) - c}{\|\varphi|} \ .$$

By Proposition 2.1.8.1, there exists a sequence (z_n) in X with $p(z_n) = 1$, such that $\varphi(z_n) \to \|\varphi|$ and $\varphi(z_n) > 0$ for all $n \in \mathbb{N}$. Then

$$h_n := x_0 - \frac{\varphi(x_0) - c}{\varphi(z_n)} z_n$$

belongs to H and

$$d_{\bar{p}}(x_0, H) \le p(x_0 - h_n) = \frac{\varphi(x_0) - c}{\varphi(z_n)} \to \frac{\varphi(x_0) - c}{\|\varphi|} \ .$$

It follows that $d_{\bar{p}}(x_0, H) \ge (\varphi(x_0) - c)/\|\varphi|$, so that formula (2.5.9) holds.

To prove (2.5.10), observe that for $h \in H$,

$$c - \varphi(x_0') = \varphi(h - x_0') \le \|\varphi|p(h - x_0') \ ,$$

implying

$$d_p(x'_0, H) \geq \frac{c - \varphi(x'_0)}{\|\varphi|} \, .$$

If the sequence (z_n) is as above then

$$h'_n := \frac{c - \varphi(x'_0)}{\varphi(z_n)} z_n + x'_0$$

belongs to H and

$$d_p(x'_0, H) \leq p(h'_n - x'_0) = \frac{c - \varphi(x'_0)}{\varphi(z_n)} \to \frac{c - \varphi(x'_0)}{\|\varphi|} \, ,$$

so that $d_p(x'_0, H) \geq (c - \varphi(x'_0))/\|\varphi|$, and formula (2.5.10) holds too.

2. Let $z_0 \in X$ be such that $p(z_0) = 1$ and $\varphi(z_0) = \|\varphi|$. Then, for $x_0 \in H^>$ and $x'_0 \in H^<$, the elements

$$h_0 := x_0 - \frac{\varphi(x_0) - c}{\varphi(z_0)} z_0 \quad \text{and} \quad h'_0 := \frac{c - \varphi(x'_0)}{\varphi(z_0)} z_0 + x'_0$$

belong to H,

$$p(x_0 - h_0) = \frac{\varphi(x_0) - c}{\|\varphi|} = d_{\bar{p}}(x_0, H) \quad \text{and} \quad p(h'_n - x'_0) = \frac{c - \varphi(x'_0)}{\|\varphi|} = d_p(x'_0, H) \, .$$

If an element $x_0 \in H^>$ has a $\bar{p}$-nearest point $h_0 \in H$ then

$$p(x_0 - h_0) = d_{\bar{p}}(x_0, H) = \frac{\varphi(x_0) - c}{\|\varphi|} = \frac{\varphi(x_0 - h_0)}{\|\varphi|} \, .$$

It follows that $z_0 = (x_0 - h_0)/p(x_0 - h_0)$ satisfies the conditions $p(z_0) = 1$ and $\varphi(z_0) = \|\varphi|$. If an element $x'_0 \in H^<$ has a p-nearest point h'_0 in H, then $z'_0 = (h'_0 - x'_0)/p(h'_0 - x'_0)$ satisfies $p(z'_0) = 1$ and $\varphi(z'_0) = \|\varphi|$. □

Remark that, according to the assertions 2 and 3 of the above proposition, the hyperplane H generated by a functional $\varphi \in X_p^\flat$ has some proximinality properties if and only if the functional φ attains its norm on the unit ball of X, a situation similar to that in normed spaces.

2.5.3 Best approximation by elements of sets with convex complement

Best approximation by elements of sets with convex complement was considered by Klee [112], in connection with the still unsolved problem of convexity of Chebyshev sets in Hilbert space (see the survey [20]). Klee conjectured that if a Hilbert space contains a non convex Chebyshev set, then it contains a Chebyshev set whose complement is convex and bounded. The conjecture was solved affirmatively by Asplund [16] who proposed the term *Klee cavern* to designate a set whose complement is convex and bounded. This term was used by Franchetti and Singer [81] who proved duality and characterization results for best approximation by elements of caverns as well as some existence results. In [47] some of these results were extended to sets with p-convex complement. In the paper [40] it was shown that the duality result proved by Franchetti and Singer, *loc cit.*, holds in spaces with asymmetric norm too. The proof is based on the formula for the distance to a hyperplane, Proposition 2.5.6.

We call a subset Y of (X, p) *upper p-bounded* if there exists $r > 0$ and $x \in X$ such that $Y \subset B_p[x, r]$ or, equivalently, if $\sup p(Y) < \infty$.

The duality result is the following.

Theorem 2.5.7. *Let (X, p) be a space with asymmetric norm, Z a τ_p-open, upper p-bounded convex subset of X and $Y = X \setminus Z$.*

Then for every $x \in Z$ the following duality relation holds:

$$d_p(x, Y) = \inf\{\sup \varphi(Y) - \varphi(x) : \varphi \in X_p^\flat,\ \|\varphi|_p = 1\}. \tag{2.5.11}$$

Proof. Let $d = d_p(x, Y)$ and denote by λ the quantity in the right-hand side of the relation (2.5.11).

For $\varphi \in X_p^\flat$, $\|\varphi|_p = 1$, let $c = \sup \varphi(Z)$. Because Z is upper p-bounded and $\varphi \le p$, it follows that c is finite. Also $\varphi(z) < 1$ for every $z \in Z$. Indeed, as the set Z is τ_p-open, for every $z \in Z$ there exists $r > 0$ such that $B_p(z, r) \subset Z$. Choosing $u \in X$ such that $\varphi(u) = 1$, it follows that $p(u) \ge \varphi(u) = 1$ and $z + (r/p(u))u \in B_p(z, r) \subset Z$, so that

$$c \ge \varphi\left(z + \frac{r}{p(u)}u\right) = \varphi(z) + \frac{r}{p(u)} > \varphi(z)\ .$$

Therefore, $\varphi(y) = c$ implies $y \notin Z \iff y \in Y$, i.e., the hyperplane $H = \{x' \in X : \varphi(x') = c\}$ is contained in Y.

Taking into account this fact and the formula (2.5.10) we get

$$d = d_p(x, Y) \le d_p(x, H) = \frac{c - \varphi(x)}{\|\varphi|_p} = c - \varphi(x) = \sup \varphi(Y) - \varphi(x)\ .$$

Since $\varphi \in X_p^\flat$, $\|\varphi|_p = 1$, was arbitrarily chosen it follows that $d \le \lambda$.

To prove the reverse inequality, observe that $y \in Y$ implies $Z \cap \{y\} = \emptyset$, so that, by the first separation theorem (Theorem 2.2.8), there exists $\varphi \in X_p^\flat$ such that

$$\forall z \in Z, \quad \varphi(z) < \varphi(y)\ .$$

Dividing (if necessary) this inequality by $\|\varphi|_p$, we can suppose $\|\varphi|_p = 1$, so that

$$\lambda \le \sup \varphi(Z) - \varphi(x) \le \varphi(y - x) \le p(y - x)\ ,$$

for any $y \in Y$, implying $\lambda \le d$. $\square$

2.5.4 Optimal points

García Raffi and Sánchez Pérez [95] propose a finer approach to the best approximation problem in asymmetric normed spaces.

Let (X, p) be an asymmetric normed space and $p^s(x) = \max\{p(x), p(-x)\}$ the associated norm on X. A norm p_0^s on X that is equivalent to p^s is called a *p-associated norm* on X. For $Y \subset X$ nonempty and $x \in X$ a point $y_0 \in P_Y(x)$ is called *p_0^s-optimal distance point* provided that

$$p_0^s(y_0 - x) \leq p_0^s(y - x) \ , \tag{2.5.12}$$

for all $y \in P_Y(x)$. A p^s-optimal distance point is called simply an *optimal distance point*. The set of all p_0^s-optimal distance points to x in $P_Y(x)$ is denoted by $O_{Y,p_0^s}(x)$ and that of optimal distance points by $O_Y(x)$.

As it is remarked in [95], the size of the set $P_Y(x)$ of nearest points to x in Y depends on the set $\theta(x)$ given by (2.4.3).

Proposition 2.5.8. *Let (X, p) be an asymmetric normed space, $Y \subset X$, $x, y \in X$. If $P_Y(x) \neq \emptyset$, then*

1. $y \in P_Y(x) \ \Rightarrow \ \theta(y) \cap Y \subset P_Y(x)$.
2. $\theta(y) \cap Y \neq \emptyset \ \Rightarrow \ p(y - x) \geq d_p(x, Y)$.
3. $P_Y(x) = \bigcup\{\theta(y) \cap Y : y \in B_p[x, d]\}$, *where* $d = d(x, Y)$.

The paper contains also the following existence results for optimal distance points.

Theorem 2.5.9. *Let (X, p) be an asymmetric normed space, p_0^s a p-associated norm on X, Y a nonempty convex subset of X and $x \in X$ such that $P_Y(x) \neq \emptyset$.*

1. *If the normed space (X, p_0^s) is strictly convex, then there is at most one p_0^s-optimal distance point in $P_Y(x)$.*
2. *In any of the following cases there is at least one p_0^s-optimal distance point in $P_Y(x)$:*

   (i) *the set $P_Y(x)$ is locally compact (in particular, if Y is contained in a finite-dimensional subspace of X), or*

   (ii) *(X, p^s) is a reflexive Banach space and Y is p^s-closed.*

2.5.5 Sign-sensitive approximation in spaces of continuous or integrable functions

As we did mention, asymmetric normed spaces and the notation $\| \cdot |$ for an asymmetric norm were introduced and studied by Krein and Nudelman [129, Ch. IX, §5] in their book on the moment problem. As an example they considered some spaces of continuous functions with an asymmetric norm given by a weight function. Consider a pair $\varphi = (\varphi_+, \varphi_-)$ of continuous strictly positive functions on an interval $[a; b]$ and denote by $B(\varphi)$ the space of all continuous functions on $[a; b]$ equipped with the norm

$$\|f| = \max_{a \le t \le b} \left\{ \frac{f^+(t)}{\varphi_+(t)} + \frac{f^-(t)}{\varphi_-(t)} \right\} , \tag{2.5.13}$$

for $f \in C[a; b]$, where $f^+(t) = \max\{f(t), 0\}$ and $f^-(t) = \max\{-f(t), 0\}$. (It follows that $f = f^+ - f^-$ and $|f| = f^+ + f^-$.)

The asymmetric norm (2.5.13) is topologically equivalent to the usual sup-norm on $C[a; b]$ (and coincides with it for $\varphi_+ = \varphi_- \equiv 1$), so that the dual of $B(\varphi)$ agrees with the dual of $C[a; b]$, that is with the space of all functions with bounded variation on $[a; b]$ with the total variation norm. The authors give in [129] the expression of the asymmetric norm of continuous linear functionals in terms of (φ_+, φ_-) and study some extremal problems in this space.

Later Dolzhenko and Sevastyanov [67, 69] (see also the survey [68]) considered some best approximation problems in spaces of continuous functions on an interval $\Delta = [a; b]$ and studied the existence and uniqueness (finite-dimensional Chebyshev subspaces) and gave characterizations of the best approximation (alternance and Kolmogorov type criteria). They consider a pair $w = (w_+, w_-)$ of nonnegative functions and the asymmetric norm

$$\|f|_w = \sup\{|w_+(t)f^+(t) - w_-(t)f^-(t)| : t \in \Delta\} . \tag{2.5.14}$$

Asymmetric norms on spaces of integrable functions are defined analogously:

$$\|f|_{p,w}^p = \int_a^b |w_+(t)f^+(t) - w_-(t)f^-(t)|^p dt , \tag{2.5.15}$$

for $1 \le p < \infty$.

The study of sign-sensitive approximation is considerably more complicated than the usual approximation (in the sup-norm or in L^p-norms) and requires a fine analysis, based on the properties of weight functions. This analysis is done in several papers, mainly by Russian and Ukrainian authors. Among these we mention Dolzhenko and Sevastyanov with the papers quoted above, Sevastyanov [217], Babenko [17, 18], Kozko [125, 126, 127, 128], Ramazanov [180, 181, 182, 183], Simonov [219], Simonov and Simonova [220], and the references quoted in these papers.

2.6 Spaces of semi-Lipschitz functions

One of the most important classes of asymmetric normed spaces is that of semi-Lipschitz functions on a quasi-metric space. This section is concerned with their study, with emphasis on various completeness results and applications to best approximation in quasi-metric spaces.

The properties of the spaces of semi-Lipschitz functions were studied by Romaguera and Sanchis [204, 206] and Romaguera, Sánchez-Álvarez and Sanchis [198]. The paper by Mustăţa [160] is concerned with the behavior of the extreme points of the unit ball in spaces of semi-Lipschitz functions.

A good presentation of properties of spaces of Lipschitz functions on metric spaces is given in the book by Weaver [236].

2.6.1 Semi-Lipschitz functions – definition and the extension property

Let (X, ρ) be a metric space and $(Y, \|\cdot\|)$ a normed space. A function $f : X \to Y$ is called *Lipschitz* if there exists $L \geq 0$ such that

$$\|f(x) - f(y)\| \leq L\rho(x, y) \tag{2.6.1}$$

for all $x, y \in X$. A number $L \geq 0$ satisfying (2.6.1) is called a *Lipschitz constant* for f. The space of all Lipschitz functions from X to Y is denoted by $\mathrm{Lip}_{\rho,\|\cdot\|}(X, Y)$, respectively by $\mathrm{Lip}_\rho(X)$ when $Y = (\mathbb{R}, |\cdot|)$.

The formula

$$\|f\|_{\rho,\|\cdot\|} = \sup\left\{\frac{\|f(x) - f(y)\|}{\rho(x, y)} : x, y \in X,\ \rho(x, y) > 0\right\} \tag{2.6.2}$$

defines a norm on the space $\mathrm{Lip}_{\rho,\|\cdot\|}(X, Y)$, that is $\left(\mathrm{Lip}_{\rho,\|\cdot\|}(X, Y), \|\cdot\|_{\rho,\|\cdot\|}\right)$ is a normed space which is complete, provided Y is a Banach space, see [236]. The number $\|f\|_{\rho,\|\cdot\|}$ is the smallest Lipschitz constant for f.

Suppose now that (X, ρ) is a quasi-metric space, (Y, q) an asymmetric normed space. A function $f : X \to \mathbb{R}$ is called *semi-Lipschitz* provided there exists a number $L \geq 0$ such that

$$q(f(x) - f(y)) \leq L\,\rho(x, y)\ , \tag{2.6.3}$$

for all $x, y \in X$. A number $L \geq 0$ for which (2.6.3) holds is called a *semi-Lipschitz constant* for f and we say that f is L-semi-Lipschitz. We denote by $\mathrm{SLip}(X, Y)$ ($\mathrm{SLip}_{\rho,d}(X, Y)$ if more precision is needed) the set of all semi-Lipschitz functions from X to Y.

In particular, if Y is the space $(\mathbb{R}, u)$, $u(\alpha) = \alpha^+$, (see Example 1.1.3), the condition (2.6.3)is equivalent to

$$f(x) - f(y) \leq L\,\rho(x, y)\ , \tag{2.6.4}$$

for all $x, y \in X$. In this case one uses the notation $\mathrm{SLip}_\rho(X) = \mathrm{SLip}_\rho(X, \mathbb{R})$.

A function $f : X \to Y$ is called $\le_{\rho,q}$-*monotone* if $q(f(x)-f(y)) = 0$ whenever $\rho(x, y) = 0$. In particular, a function $f : X \to \mathbb{R}$ is $\le_{\rho,u}$-monotone, called $\le_\rho$-*monotone*, if and only if $f(x) \le f(y)$ whenever $\rho(x, y) = 0$.

Obviously, a semi-Lipschitz function is $\le_{\rho,q}$-monotone. Since the topology τ_ρ is T_1 if and only if $\rho(x, y) = 0 \iff x = y$, (Proposition 1.1.8) it follows that any function on a T_1 quasi-metric space is $\le_{\rho,q}$-monotone.

Remark 2.6.1. It is clear that for $\alpha, \beta \in \mathbb{R}$,

$$\alpha \le \beta \iff \alpha - \beta \le 0 \iff u(\alpha - \beta) = (\alpha - \beta)^+ = 0.$$

If p is an asymmetric seminorm on a vector space X, then

$$x \le_p y \iff p(x - y) = 0$$

defines an order relation on X. Similarly, in a quasi-semimetric space (X, ρ)

$$x \le_\rho y \iff \rho(x, y) = 0$$

also defines an order relation.

Taking into account these order relations, the $\le_{\rho,q}$-monotonicity can be expressed by the condition

$$x \le_\rho y \implies f(x) \le_q f(y),$$

justifying the term monotonicity.

Suppose now that (X, ρ) is a quasi-metric space and (Y, q) an asymmetric normed space. For an arbitrary function $f : X \to Y$ put

$$\|f|_{\rho,q} = \sup\left\{\frac{q(f(x) - f(y))}{\rho(x, y)} : x, y \in X,\ \rho(x, y) > 0\right\}\ , \tag{2.6.5}$$

and $\|f|_\rho = \|f|_{\rho,u}$ when Y is $(\mathbb{R}, u)$.

Proposition 2.6.2. *Let (X, ρ) be a quasi-metric space and (Y, q) an asymmetric normed space.*

1. *The set $\mathrm{SLip}_{\rho,q}(X, Y)$ is a cone in the linear space $\mathrm{Lip}_{\rho^s,q^s}(X, Y)$ of all Lipschitz functions from the metric space (X, ρ^s) to the normed space (Y, q^s) and $\|f\|_{\rho^s,q^s} \le \|f|_{\rho,q}$ for all $f \in \mathrm{SLip}_{\rho,q}(X, Y)$.*
2. *If f is semi-Lipschitz, then $\|f|_{\rho,q}$ is the smallest semi-Lipschitz constant for f.*
3. *A function $f : X \to Y$ is semi-Lipschitz if and only if f is $\le_{\rho,q}$-monotone and $\|f|_{\rho,q} < \infty$.*

Proof. 1. It is clear that $f+g,\ \alpha f \in \mathrm{SLip}_{\rho,q}(X,Y)$ for all $f,g \in \mathrm{SLip}_{\rho,q}(X,Y)$ and $\alpha \geq 0$.

If $f \in \mathrm{SLip}_{\rho,q}(X,Y)$, then

$$q(f(x)-f(y)) \leq L\rho(x,y) \leq L\rho^s(x,y),\ \ x,y \in X\ ,$$

implying

$$q^s(f(x)-f(y)) \leq L\rho^s(x,y)\ ,$$

for all $x,y \in X$, so that $f \in \mathrm{Lip}_{\rho^s,q^s}(X,Y)$ and $\|f\|_{\rho^s,q^s} \leq \|f|_{\rho,q}$.

2. The inequality $q(f(x)-f(y))/\rho(x,y) \leq \|f|_{\rho,q}$ implies $f(x)-f(y) \leq \|f|_\rho\, \rho(x,y)$ for all $x,y \in X$ with $\rho(x,y) > 0$. Since a semi-Lipschitz function is $\leq_{\rho,q}$-monotone, $\rho(x,y) = 0$ implies $q(f(x)-f(y)) = 0 = \|f|_{\rho,q}\, \rho(x,y)$. Consequently, $\|f|_{\rho,q}$ is a semi-Lipschitz constant for f.

Suppose that L is a semi-Lipschitz constant for f. Then

$$q(f(x)-f(y))/\rho(x,y) \leq L,$$

whenever $\rho(x,y) > 0$, so that $\|f|_{\rho,q} \leq L$, showing that $\|f|_{\rho,q}$ is the smallest semi-Lipschitz constant for f.

The above reasonings show also the validity of 3. □

The following example shows that the inequality from Proposition 2.6.2.1 can be strict.

Example 2.6.3. On a three point set set $X = \{x_1, x_2, x_3\}$ consider the quasi-metric $\rho(x_1,x_2) = 1$, $\rho(x_2,x_1) = 2, \rho(x_1,x_3) = \rho(x_3,x_1) = 2$, $\rho(x_2,x_3) = \rho(x_3,x_2) = 2$, and the function $f : X \to \mathbb{R}$ given by $f(x_1) = 1$, $f(x_2) = f(x_3) = 2$. Then $\rho^s(x_i,x_j) = 2$ for $i \neq j$ and

$$\begin{aligned}\|f|_{\rho,|\cdot|} &= \max\left\{\frac{|f(x_i)-f(x_j)|}{\rho(x_i,x_j)} : 1 \leq i,j \leq 3,\ i \neq j\}\right\} = 1\\ &> \frac{1}{2} = \max\left\{\frac{|f(x_i)-f(x_j)|}{\rho^s(x_i,x_j)} : 1 \leq i,j \leq 3,\ i \neq j\}\right\} = \|f|_{\rho^s,|\cdot|}.\end{aligned}$$

The following proposition puts in evidence some useful semi-Lipschitz functions.

Proposition 2.6.4. *Let (X,ρ) be a quasi-metric space, $y \in X$ and $Y \subset X$ nonempty.*

1. *The functions $\rho(\cdot,y) : X \to \mathbb{R}$ and $d(\cdot,Y) : X \to \mathbb{R}$ are semi-Lipschitz with semi-Lipschitz constant 1.*
2. *For fixed $a \in X$, the functions $f(x) = \rho(a,x_0) - \rho(a,x)$ and $g(x) = \rho(x,a) - \rho(x_0,a)$ belong to $\mathrm{SLip}_{\rho,0}(X)$ and $\|f|_\rho,\ \|g|_\rho \leq 1$.*

Proof. 1. The inequality

$$\rho(x,y) \le \rho(x,x') + \rho(x',y) , \tag{2.6.6}$$

valid for $x, x' \in X$, shows that the function $\rho(\cdot, y)$ is semi-Lipschitz. Since the inequality (2.6.6) holds for all $y \in Y$ and fixed x, x', passing to infimum with respect to $y \in Y$ one obtains $d(x,Y) \le \rho(x,x') + d(x',Y)$, which means that the function $d(\cdot, Y)$ is semi-Lipschitz, too.

The assertions from 2 follow from 1. □

An important result in the study of Lipschitz functions on metric spaces is the extension of Lipschitz functions, usually known as Kirszbraun's extension theorem, see, for instance, the book [237].

In the case of semi-Lipschitz functions a similar result was proved by Mustăţa [158] (see also [157, 164]). The extension problem for semi-Lipschitz functions on quasi-metric spaces was considered also by Matoušková [148]. The paper [87] discusses the existence of an extension of an asymmetric norm defined on a cone K to an asymmetric norm defined on the generated linear space $X = K - K$.

Proposition 2.6.5. *Let (X, ρ) be a quasi-metric space, Y a nonempty subset of X and $f : Y \to \mathbb{R}$ an L-semi-Lipschitz function.*

1. *The functions F, G defined for $x \in X$ by*

$$F(x) = \inf\{f(y) + L\rho(x,y) : y \in Y\} \tag{2.6.7}$$

and

$$G(x) = \sup\{f(y') - L\rho(y',x) : y' \in Y\} \tag{2.6.8}$$

are L-semi-Lipschitz extensions of f.

2. *Any other L-semi-Lipschitz extension H of f satisfies the inequalities*

$$G \le H \le F . \tag{2.6.9}$$

Proof. 1. Let $y, y' \in Y$ and $x \in X$. The inequalities $f(y') - f(y) \le L\,\rho(y',y) \le L\,\rho(y',x) + L\,\rho(x,y)$ imply

$$f(y') - L\,\rho(y',x) \le f(y) + L\,\rho(x,y) .$$

Passing to supremum with respect to $y' \in Y$ and to infimum with respect to $y \in Y$, it follows that G, F are well defined and $G \le F$.

Let $x \in Y$. Then $f(x) \le f(y) + L\,\rho(x,y)$ for every $y \in Y$, implies $f(x) \le F(x)$. Similarly, $f(y') - L\,\rho(y',x) \le f(x)$ implies $G(x) \le f(x)$. Taking $y = x$ in (2.6.7) and $y' = x$ in (2.6.8), it follows that $F(x) \le f(x)$ and $G(x) \ge f(x)$, so that $G(x) = f(x) = F(x)$ for every $x \in Y$.

To conclude, we have to show that the functions F, G are semi-Lipschitz. Let $x, x' \in X$. The inequalities

$$F(x) \leq f(y) + L\,\rho(x,y) \leq f(y) + L\,\rho(x,x') + L\,\rho(x',y)\ ,$$

valid for all $y \in Y$, yield $F(x) \leq F(x') + L\,\rho(x,x')$, showing that F is semi-Lipschitz.

Similar reasonings show that G is semi-Lipschitz too.

2. Let H be an L-semi-Lipschitz extension of f and $x \in X$. Since

$$H(x) \leq H(y) + L\,\rho(x,y) = f(y) + L\,\rho(x,y)\ ,$$

for every $y \in Y$, passing to infimum with respect to $y \in Y$ one obtains $H(x) \leq F(x)$.

Similarly $f(y') - H(x) = H(y') - H(x) \leq L\,\rho(y',x,)$ implies

$$f(y') - L\,\rho(y',x) \leq H(x)\ ,$$

for every $y' \in Y$. Passing to supremum with respect to $y' \in Y$ one obtains $G(x) \leq H(x)$. $\square$

The following corollary shows the existence of norm-preserving extensions of semi-Lipschitz functions.

Corollary 2.6.6. *Let X, Y and f be as in Proposition* 2.6.5 *and*

$$\|f|_\rho = \sup\{u(f(y) - f(y'))/\rho(y,y') : y, y' \in Y,\ \rho(y,y') > 0\}\ ,$$

and let F, G be given by (2.6.7) *and* (2.6.8) *for $L = \|f|_\rho$. Then*

1. *The functions F and G are semi-Lipschitz norm-preserving extensions of f, that is*
$$(i)\ F|_Y = G|_Y = f \quad \text{and} \quad (ii)\ \|F|_\rho = \|G|_\rho = \|f|_\rho\ .$$
2. *Any other semi-Lipschitz norm-preserving extension H of f satisfies the inequalities*
$$G \leq H \leq F\ .$$

2.6.2 Properties of the cone of semi-Lipschitz functions – linearity

In the case of real-valued Lipschitz functions on a metric space (X, ρ) the function $\|\cdot\| : \mathrm{Lip}(X, \mathbb{R}) \to \mathbb{R}$ given by

$$\|f\|_\rho = \sup\Big\{\frac{|f(x) - f(y)|}{\rho(x,y)} : x, y \in X,\ \rho(x,y) > 0\Big\}\ ,$$

is only a seminorm on $\mathrm{Lip}(X, \mathbb{R})$ because $\|f\|_\rho = 0$ for any constant function. To obtain a norm one can proceed in two ways:

- one takes the quotient space of $\mathrm{Lip}(X,\mathbb{R})$ with respect to the subspace of constant functions, or
- one fixes a point $x_0 \in X$ and one considers the subspace

$$\mathrm{Lip}_0(X,\mathbb{R}) = \{f \in \mathrm{Lip}(X,\mathbb{R}) : f(x_0) = 0\}$$

with the induced norm.

We shall follow the second way. A metric space with an a priori fixed point x_0 is called a *pointed metric space*. If, in addition, X is a vector space then one takes usually $x_0 = 0$, the null element of X.

For a pointed quasi-metric space (X,ρ,x_0) and an asymmetric normed space (Y,q) let

$$\mathrm{SLip}^0_{\rho,q}(X,Y) = \left\{f \in \mathrm{SLip}_{\rho,q}(X,Y) : f(x_0) = 0\right\} . \tag{2.6.10}$$

In the case $Y = (\mathbb{R},u)$ one uses the notation $\mathrm{SLip}^0_\rho(X)$.

The following proposition contains some simple remarks concerning the relations between ρ- and $\bar\rho$-semi-Lipschitz functions, obtained in [215] and in [206] in the case $Y = \mathbb{R}$.

Proposition 2.6.7. *Let (X,ρ) be a quasi-metric space, (Y,q) an asymmetric normed space and $f : X \to Y$ a function.*

1. *The function f belongs to $\mathrm{SLip}_{\rho,q}(X,Y)$ (resp. to $\mathrm{SLip}^0_{\rho,q}(X,Y)$) if and only if $-f$ belongs to $\mathrm{SLip}_{\bar\rho,q}(X,Y)$ (resp. to $\mathrm{SLip}^0_{\bar\rho,q}(X,Y)$). Also the following equality holds,*

$$\|f|_{\rho,q} = \|-f|_{\bar\rho,q} .$$

The correspondence $f \mapsto -f$ is an isometric isomorphism between the normed cones

$$\left(\mathrm{SLip}_{\rho,q}(X,Y), \|\cdot|_{\rho,q}\right) \quad \textit{and} \quad \left(\mathrm{SLip}_{\bar\rho,q}(X,Y), \|\cdot|_{\bar\rho,q}\right) ,$$

respectively between

$$\left(\mathrm{SLip}^0_{\rho,q}(X,Y), \|\cdot|_{\rho,q}\right) \quad \textit{and} \quad \left(\mathrm{SLip}^0_{\bar\rho,q}(X,Y), \|\cdot|_{\bar\rho,q}\right) .$$

2. *The sets $\mathrm{SLip}_{\rho,q}(X,Y) \cap \mathrm{SLip}_{\bar\rho,q}(X,Y)$ and $\mathrm{SLip}^0_{\rho,q}(X,Y) \cap \mathrm{SLip}^0_{\bar\rho,q}(X,Y)$ are linear spaces.*

A problem discussed in [206] and [215] is: under what conditions is the cone $\mathrm{SLip}_{\rho,0}(X)$ a linear space?

We mention also the following result from [215] and [206] (the case $Y = \mathbb{R}$).

Theorem 2.6.8. *Let (X,ρ) be a quasi-metric space and (Y,q) an asymmetric normed space. The following are equivalent.*

1. $\mathrm{SLip}^0_{\rho,q}(X,Y) = \mathrm{SLip}^0_{\bar\rho,q}(X,Y)$.
2. $\mathrm{SLip}^0_{\rho,q}(X,Y)$ *is a linear space.*

3. $\mathrm{SLip}^0_{\bar\rho,q}(X,Y)$ *is a linear space.*
4. $\mathrm{SLip}^0_{\bar\rho,q}(X,Y) \subset \mathrm{SLip}^0_{\rho,q}(X,Y)$.
5. $\mathrm{SLip}^0_{\rho,q}(X,Y) \subset \mathrm{SLip}^0_{\bar\rho,q}(X,Y)$.

Similar equivalence results hold for the spaces $\mathrm{SLip}_{\rho,q}(X,Y)$ *and* $\mathrm{SLip}_{\bar\rho,q}(X,Y)$.

Proof. Observe that $\mathrm{SLip}^0_{\rho,q}(X,Y)$ is a linear space if and only if $\big(\mathrm{SLip}^0_{\rho,q}(X,Y),+\big)$ is a group and similarly for $\mathrm{SLip}^0_{\bar\rho,q}(X,Y)$.

$1 \Rightarrow 2$. By Proposition 2.6.2, $\mathrm{SLip}^0_{\rho,q}(X,Y) = \mathrm{SLip}^0_{\rho,q}(X,Y) \cap \mathrm{SLip}^0_{\bar\rho,q}(X,Y)$ is a linear space.

$2 \Rightarrow 3$. Since $f \in \mathrm{SLip}^0_{\bar\rho,q}(X,Y) \iff -f\,\mathrm{SLip}^0_{\rho,q}(X,Y)$ and $\mathrm{SLip}^0_{\rho,q}(X,Y)$ is a linear space, it follows that $f = -(-f) \in \mathrm{SLip}^0_{\rho,q}(X,Y) \iff -f\,\mathrm{SLip}^0_{\bar\rho,q}(X,Y)$.

$3 \Rightarrow 4$. Follows from $f \in \mathrm{SLip}^0_{\bar\rho,q}(X,Y) \Rightarrow -f \in \mathrm{SLip}^0_{\bar\rho,q}(X,Y)$ and $-f \in \mathrm{SLip}^0_{\bar\rho,q}(X,Y) \iff f \in \mathrm{SLip}^0_{\rho,q}(X,Y)$.

The proof of $4 \Rightarrow 5$ is similar to that of the above implication, and $5 \Rightarrow 4$ follows from the equality $\bar{\bar\rho} = \rho$.

The implication $5 \Rightarrow 1$ follows from the equivalence $4 \iff 5$. $\square$

Remark 2.6.9. The equality $\mathrm{SLip}^{x_0}_{\rho,q}(X,Y) = \mathrm{SLip}^{x_0}_{\bar\rho,q}(X,Y)$ does not depend on the point x_0, in the sense that if $\mathrm{SLip}^{x_0}_{\rho,q}(X,Y) = \mathrm{SLip}^{x_0}_{\bar\rho,q}(X,Y)$ for some $x_0 \in X$, then $\mathrm{SLip}^{x_1}_{\rho,q}(X,Y) = \mathrm{SLip}^{x_1}_{\bar\rho,q}(X,Y)$ for any other point $x_1 \in X$.

Indeed, the correspondence $f \mapsto f - f(x_0)$ applies $\mathrm{SLip}^{x_1}_{\rho,q}(X,Y)$ onto $\mathrm{SLip}^{x_0}_{\bar\rho,q}(X,Y)$ and preserves the Lipschitz constants.

In the case $Y = \mathbb{R}$ one obtains the following condition on the metric ρ.

Proposition 2.6.10 ([206])**.** *Let (X,ρ,x_0) be a pointed quasi-metric space. If*

$$\mathrm{SLip}^0_\rho(X) = \mathrm{SLip}^0_{\bar\rho}(X),$$

then the topology τ_ρ is T_1 and there exist $\beta,\beta' > 0$ such that

$$\rho(x,x_0) \le \beta\rho(x_0,x) \quad \text{and} \quad \rho(x_0,x) \le \beta'\rho(x,x_0)\ , \tag{2.6.11}$$

for all $x \in X$.

Proof. Let $a,b \in X$ be such that $\rho(a,b) = 0$. By Proposition 2.6.4, the function $f(x) = \rho(x,a) - \rho(x_0,a)$, $x \in X$, is ρ-semi-Lipschitz, and so belongs to $\mathrm{SLip}^0_{\rho,q}(X) = \mathrm{SLip}^0_{\bar\rho}(X)$, so that there exists $\lambda > 0$ such that $\rho(x,a) - \rho(y,a) = f(x) - f(y) \le \lambda\rho(y,x)$, for all $x,y \in X$. Taking $x = b$ and $y = a$ one obtains $\rho(b,a) \le \lambda\rho(a,b) = 0$. Since ρ is a quasi-metric, $\rho(b,a) = \rho(a,b) = 0$ implies $a = b$. By Proposition 1.1.8.3 the topology τ_ρ is T_1 (and $\tau_{\bar\rho}$ as well).

The function $g(x) = \rho(x,x_0)$, $x \in X$, belongs to $\mathrm{SLip}^0_\rho(X) = \mathrm{SLip}^0_{\bar\rho}(X)$, so there exists $\beta > 0$ such that $\rho(x,x_0) = f(x) - f(x_0) \le \beta\rho(x_0,x)$. The existence of $\beta' > 0$ satisfying the second inequality is proved similarly. $\square$

The above results have the following consequences.

Corollary 2.6.11. *Let (X, ρ, x_0) be a pointed quasi-metric space.*

1. *If* $\mathrm{SLip}^0_\rho(X) = \mathrm{SLip}^0_{\bar\rho}(X)$, *then* $\tau_\rho = \tau_{\bar\rho}$ *and hence the topology* τ_ρ *is metrizable.*
2. $\big(\mathrm{SLip}^0_\rho(X), +, \|\cdot|_\rho\big)$ *is a topological group if and only if* $\mathrm{SLip}^0_\rho(X) = \mathrm{SLip}^0_{\bar\rho}(X)$ *and* $\tau_\rho = \tau_{\bar\rho}$.

We mention also the following result.

Proposition 2.6.12. *Let (X, ρ, x_0) be a pointed quasi-metric space. The following assertions are equivalent.*

1. $\mathrm{SLip}^0_\rho(X)$ *is a vector space and* $\|\cdot|_\rho$ *is a complete norm on it, that is* $\big(\mathrm{SLip}^0_\rho(X), \|\cdot|_\rho\big)$ *is a Banach space.*
2. $\mathrm{SLip}^0_\rho(X) = \mathrm{SLip}^0_{\bar\rho}(X)$ *and* $\|\cdot|_\rho = \|\cdot|_{\bar\rho}$.
3. (X, ρ) *is a metric space.*

Proof. $1 \Rightarrow 2$. By Proposition 2.6.8, $\mathrm{SLip}^0_\rho(X) = \mathrm{SLip}^0_{\bar\rho}(X)$. Since $\|\cdot|_\rho$ is a norm on $\mathrm{SLip}^0_\rho(X)$ it follows $\|f|_{\bar\rho} = \|-f|_\rho = \|f|_\rho$.

$2 \Rightarrow 3$. If $\mathrm{SLip}^0_\rho(X) = \mathrm{SLip}^0_{\bar\rho}(X)$, then, by Proposition 2.6.10, the topology τ_ρ is T_1. Consequently, if ρ is not a metric, then there exists a pair $a, b \in X$ such that $\rho(a,b) > \rho(b,a) > 0$. The function $f(x) = \rho(x,b) - \rho(x_0,b)$, $x \in X$, is in $\mathrm{SLip}^0_\rho(X)$ and $\|f|_\rho \le 1$ (see Proposition 2.6.4). By hypothesis $f \in \mathrm{SLip}^0_{\bar\rho}(X)$. But

$$\|f|_{\bar\rho} = \sup\{u(f(x) - f(y))/\rho(y,x) : \rho(y,x) > 0\} \ge \frac{u(f(a) - f(b))}{\rho(b,a)} = \frac{\rho(a,b)}{\rho(b,a)} > 1\ ,$$

in contradiction to the hypothesis $\|f|_{\bar\rho} = \|f|_\rho$.

$3 \Rightarrow 1$. If (X, ρ) is a metric space, then $\mathrm{SLip}^0_\rho(X) = \mathrm{SLip}^0_{\bar\rho}(X) = \mathrm{Lip}_0(X)$ – the space of Lipschitz functions vanishing at x_0. It is well known that $\mathrm{Lip}_0(X)$ is a Banach space with respect to the Lipschitz norm $\|\cdot\|_\rho = \|\cdot|_\rho = \|\cdot|_{\bar\rho}$ (see [236]). $\square$

Remark 2.6.13. Concerning the validity of Proposition 2.6.12 in the case of spaces of semi-Lipschitz functions with values in an asymmetric normed space (Y, q), Sánchez-Álvarez [215] has shown that $2 \iff 3$ and $3 \Rightarrow 1$, but the implication $1 \Rightarrow 3$ does not hold in general.

2.6.3 Completeness properties of the spaces of semi-Lipschitz functions

In order to treat some completeness questions for spaces of semi-Lipschitz functions, one defines an extended quasi-metric on $\mathrm{SLip}_\rho(X)$ by the formula

$$\delta_\rho(f,g) = \sup\left\{\frac{u((g-f)(x) - (g-f)(y))}{\rho(x,y)} : x, y \in X,\ \rho(x,y) > 0\right\}\ . \quad (2.6.12)$$

For $f, g \in \mathrm{SLip}_\rho(X)$ put also

$$\bar{\delta}_\rho(f,g) = \delta_\rho(g,f) \quad \text{and} \quad \delta_\rho^s(f,g) = \delta_\rho(f,g) \vee \bar{\delta}_\rho(f,g) .$$

Because $[\varphi(x) \vee 0] \vee [(-\varphi(x)) \vee 0] = |\varphi(x)|$,

$$\begin{aligned} \delta_\rho^s(f,g) &= \delta_\rho(f,g) \vee \delta_\rho(g,f) \\ &= \sup\left\{ \frac{|(g-f)(x) - (g-f)(y)|}{\rho(x,y)} : x, y \in X,\ \rho(x,y) > 0 \right\} . \end{aligned}$$

In fact δ_ρ can be considered as an extended quasi-metric on the linear space $\mathrm{SLip}_\rho(X) - \mathrm{SLip}_\rho(X)$ generated by $\mathrm{SLip}_\rho(X)$ in $\mathrm{Lip}_\rho(X)$.

Remark 2.6.14. If the topology τ_ρ is T_1 (equivalently, $\rho(x,y) > 0$ whenever $x \neq y$), then

$$\bar{\delta}_\rho(f,g) = \delta_{\bar{\rho}}(f,g) .$$

Indeed

$$\begin{aligned} \delta_{\bar{\rho}}(f,g) &= \sup\left\{ \frac{u\left((g-f)(x) - (g-f)(y)\right)}{\rho(y,x)} : x, y \in X,\ x \neq y \right\} \\ &= \sup\left\{ \frac{u\left((f-g)(x) - (f-g)(y)\right)}{\rho(x,y)} : x, y \in X,\ x \neq y \right\} \\ &= \delta_\rho(g,f) = \bar{\delta}_\rho(f,g) . \end{aligned}$$

The following example, given by Romaguera and Sanchis [204], shows that δ_ρ could be effectively an extended quasi-metric.

Example 2.6.15. For $x, y \in \mathbb{R}$ let $\rho(x,y) = x - y$ if $x \geq y$ and $\rho(x,y) = 1$ if $x < y$, i.e., $(\mathbb{R}, \rho)$ is the Sorgenfrey line (see Example 1.1.6). The identity mapping $\mathrm{id} : \mathbb{R} \to \mathbb{R}$ is semi-Lipschitz with $\| \mathrm{id} |_\rho = 1$, so that $\delta_\rho(0, \mathrm{id}) = 1$, but $\delta_\rho(\mathrm{id}, 0) = \infty$ because $\sup\{((y-x) \vee 0)/\rho(x,y) : x \neq y\} = \infty$.

Theorem 2.6.16. *Let (X, ρ) be a quasi-metric space.*

1. ([204]) *The space $\mathrm{SLip}_\rho^0(X)$ is bicomplete with respect to the extended quasi-metric δ_ρ, that is complete with respect to the extended metric $\delta_\rho^s = \delta_\rho \vee \bar{\delta}_\rho$.*
2. ([206]) *The extended quasi-metric space $(\mathrm{SLip}_\rho^0(X), \bar{\delta}_\rho)$ is right K-complete.*

Proof. 1. Let (f_n) be a δ_ρ^s-Cauchy sequence in $\mathrm{SLip}_\rho^0(X)$. Then

$$\begin{aligned} \delta_\rho^s(f_n, f_{n+k}) &= \delta_\rho(f_n, f_{n+k}) \vee \delta_\rho(f_{n+k}, f_n) \\ &= \sup\left\{ \frac{|(f_{n+k} - f_n)(u) - (f_{n+k} - f_n)(v)|}{\rho(u,v)} : u, v \in X,\ \rho(u,v) > 0 \right\} , \end{aligned}$$

so that for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\frac{|(f_{n+k} - f_n)(u) - (f_{n+k} - f_n)(v)|}{\rho(u, v)} \le \varepsilon \ , \tag{2.6.13}$$

for all $n \ge n_\varepsilon,\ k \in \mathbb{N}$ and all $u, v \in X$ with $\rho(u, v) > 0$.

Claim I. For every $x \in X,\ (f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$.

Let $x \ne x_0$ be fixed and $\varepsilon' > 0$. If $\rho(x, x_0) > 0$ let $n_0 \in \mathbb{N}$ be such that (2.6.13) holds for $\varepsilon := \varepsilon'/\rho(x, x_0)$. Taking $u = x$ and $v = x_0$ in (2.6.13) it follows that

$$|(f_{n+k} - f_n)(x)| \le \varepsilon\rho(x, x_0) = \varepsilon' \ ,$$

for all $n \ge n_\varepsilon$ and all $k \in \mathbb{N}$. If $\rho(x_0, x) > 0$, then use (2.6.13) with $\varepsilon := \varepsilon'/\rho(x_0, x),\ u = x_0$ and $v = x$ to obtain

$$|(f_{n+k} - f_n)(x)| \le \varepsilon\rho(x_0, x) = \varepsilon' \ ,$$

for all $n \ge n_\varepsilon$ and all $k \in \mathbb{N}$.

Consequently $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$ for every $x \in X$, so that we can define a function $f : X \to \mathbb{R}$ by $f(x) = \lim_n f_n(x),\ x \in X$.

To end the proof we have to show that $f \in \mathrm{SLip}^0_\rho(X)$ and that $f_n \xrightarrow{\delta^s_\rho} f$.

Claim II. $f \in \mathrm{SLip}^0_\rho(X)$.

Let $m \in \mathbb{N}$ be such that (2.6.13) holds for $\varepsilon = 1$. Then for every $x, y \in X$ with $\rho(x, y) > 0$ and $k \in \mathbb{N}$,

$$|(f_{m+k} - f_m)(x) - (f_{m+k} - f_m(y)| \le \rho(x, y) \ ,$$

yielding for $k \to \infty$,

$$|(f - f_m)(x) - (f - f_m)(y)| \le \rho(x, y) \ .$$

But then

$$\begin{aligned} f(x) - f(y) &= (f - f_m)(x) - (f - f_m)(y) + f_m(x) - f_m(y) \\ &\le (1 + \|f_m|_\rho)\ \rho(x, y) \ , \end{aligned}$$

for all $x, y \in X$ with $\rho(x, y) > 0$. Since the pointwise limit of a sequence of $\le_\rho$-monotone functions is $\le_\rho$-monotone, it follows that $f \in \mathrm{SLip}^0_\rho(X)$.

Claim III. $f_n \xrightarrow{\delta^s_\rho} f$.

For $\varepsilon > 0$ let $n_\varepsilon \in \mathbb{N}$ be chosen according to (2.6.13). Then for every $x, y \in X$ with $\rho(x, y) > 0$

$$\forall n \ge n_\varepsilon,\ \ \forall k \in \mathbb{N}, \quad |(f_{n+k} - f_n)(x) - (f_{n+k} - f_n)(y)| \le \rho(x, y) \ ,$$

which yields for $k \to \infty$,

$$\forall n \geq n_\varepsilon, \quad |(f - f_n)(x) - (f - f_n)(y)| \leq \varepsilon\rho(x, y) .$$

It follows that

$$\forall n \geq n_\varepsilon, \quad \delta^s_\rho(f, f_n) \leq \varepsilon ,$$

proving Claim III.

2. Let (f_n) be a right $\bar{\delta}_\rho$-K-Cauchy sequence in $\mathrm{SLip}^0_\rho(X)$. Then for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\forall n \geq n_\varepsilon, \ \forall k \in \mathbb{N}, \quad \bar{\delta}_\rho(f_{n+k}, f_n) \leq \varepsilon ,$$

that is

$$\frac{u\left((f_{n+k} - f_n)(u) - (f_{n+k} - f_n)(v)\right)}{\rho(u, v)} \leq \varepsilon , \tag{2.6.14}$$

for all $n \geq n_\varepsilon$, all $k \in \mathbb{N}$ and all $u, v \in X$ with $\rho(u, v) > 0$.

Claim I. For every $x \in X$, $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$.

Let $x \in X \setminus \{x_0\}$ and $\varepsilon' > 0$.

Case 1. $\rho(x, x_0) > 0$ and $\rho(x_0, x) > 0$. Take n_0 such that (2.6.14) holds for $\varepsilon := \varepsilon'/\rho^s(x, x_0)$. Taking first $u = x, \ v = x_0$ and then $u = x_0, \ v = x$, one obtains

$$\begin{aligned} &(f_{n+k} - f_n)(x) \leq \varepsilon\rho(x, x_0) \leq \varepsilon\rho^s(x, x_0) = \varepsilon', \quad \text{respectively} \\ &(f_{n+k} - f_n)(x) \leq \varepsilon\rho(x_0, x) \leq \varepsilon\rho^s(x, x_0) = \varepsilon' . \end{aligned}$$

Consequently

$$\forall n \geq n_0, \quad |f_{n+k}(x) - f_n)(x)| \leq \varepsilon' ,$$

showing that the sequence $(f_n(x)$ is $|\cdot|$-Cauchy.

Case 2. $\rho(x, x_0) > 0$ and $\rho(x_0, x) = 0$. Reasoning like above, given $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that

$$\forall n \geq m_0, \ \forall k \in \mathbb{N}, \quad f_{n+k}(x) - f_n(x) \leq \varepsilon . \tag{2.6.15}$$

Since the functions f_n are $\leq_\rho$-monotone, $0 = f_n(x_0) \leq f_n(x)$ for all $n \in \mathbb{N}$.

Applying (2.6.14) for $\varepsilon = 1/\rho(x, x_0)$ and $u = x, \ v = x_0$, it follows that there exists $m_1 \in \mathbb{N}$ such that $f_{m_1+k}(x) - f_{m_1}(x) \leq 1$, implying $0 \leq f_{m_1+k}(x) \leq f_{m_1}(x) + 1$ for all $k \in \mathbb{N}$. Consequently, the sequence $(f_n(x))$ is bounded, so it contains a subsequence $f_{n_i}(x))_{i\in\mathbb{N}}$ converging to some $f(x) \in \mathbb{R}$.

Let $i_0 \in \mathbb{N}$ such that $n_{i_0} > m_0$ and

$$\forall i \geq i_0, \quad |f_{n_i}(x) - f(x)| \leq \varepsilon . \tag{2.6.16}$$

For $n > n_{i_0}$ and $k \in \mathbb{N}$ let $n_j > n + k$. Then, combining (2.6.15) and (2.6.16), one obtains

$$\begin{aligned} & f_n(x) - f_{n+k}(x) \\ & \quad = f_n(x) - f_{n_{i_0}}(x) + f_{n_{i_0}}(x) - f(x) + f(x) - f_{n_j}(x) + f_{n_j}(x) - f_{n+k}(x) \\ & \quad \leq 4\varepsilon \ . \end{aligned} \tag{2.6.17}$$

The inequalities (2.6.15) and (2.6.17) show that $(f_n(x))$ is a $|\cdot|$-Cauchy sequence.

Case 3. $\rho(x, x_0) = 0$ and $\rho(x_0, x) > 0$. In this case $f_n(x) \leq f_n(x_0) = 0$ and a reasoning analogous to that made in Case 2 shows that the sequence $(f_n(x))$ is Cauchy in this case too.

Consequently we can define a function $f : X \to \mathbb{R}$ by $f(x) = \lim_n f_n(x)$, $x \in X$.

To end the proof we have to show that $f \in \mathrm{SLip}_\rho^0(X)$ and that $f_n \xrightarrow{\bar{\delta}_\rho} f$.

Claim II. $f \in \mathrm{SLip}_\rho^0(X)$.

Applying (2.6.14) for $\varepsilon = 1$, it follows that there exists $m_1 \in \mathbb{N}$ such that for all $x, y \in X$ with $\rho(x, y) > 0$,

$$\forall k \in \mathbb{N}, \quad \frac{(f_{m_1+k} - f_{m_1})(x) - (f_{m_1+k} - f_{m_1})(y)}{\rho(x, y)} \leq 1 \ ,$$

yielding, for $k \to \infty$ and after some calculation,

$$\frac{f(x) - f(y)}{\rho(x, y)} \leq \frac{f_{m_1})(x) - f_{m_1})(y)}{\rho(x, y)} + 1 \leq \|f_{m_1}|_\rho + 1 \ .$$

As a pointwise limit of a sequence of $\leq_\rho$-monotone functions, the function f is $\leq_\rho$-monotone, so it belongs to $\mathrm{SLip}_\rho^0(X)$.

Claim III. $f_n \xrightarrow{\bar{\delta}_\rho} f$.

For $\varepsilon > 0$ let $n_\varepsilon \in \mathbb{N}$ such that (2.6.14) holds. Considering $n \geq n_\varepsilon$ fixed and letting $k \to \infty$, one obtains

$$\frac{u\left((f - f_n)(u) - (f - f_n)(v)\right)}{\rho(u, v)} \leq \varepsilon \ ,$$

for all $u, v \in X$ with $\rho(u, v) > 0$ and all $n \geq n_\varepsilon$. It follows that

$$\bar{\delta}(f, f_n) \leq \varepsilon \ ,$$

for all $n \geq n_\varepsilon$, proving Claim III. $\square$

Remark 2.6.17. Sánchez-Álvarez [215] studied the completeness of the space $\mathrm{SLip}^0_{\rho,q}(X,Y)$, for (X,ρ) a quasi-metric space and (Y,q) an asymmetric normed space, with respect to the extended quasi-metric

$$\delta_{\rho,q}(f,g) = \sup\big\{\frac{q((g-f)(x)-(g-f)(y))}{\rho(x,y)} : x,y\in X,\ \rho(x,y)>0\big\}\,. \tag{2.6.18}$$

He proved that if the asymmetric normed space (Y,q) is bicomplete, then $\mathrm{SLip}^0_{\rho,q}(X,Y)$ is complete with respect to the extended metric

$$\delta^s_{\rho,q}(f,g) = \delta_{\rho,q}(f,g)\vee\delta_{\rho,q}(g,f)$$

([215, Theorem 3.5]).

Assertion 2 from Theorem 2.6.16 was extended to the case when (Y,q) is a finite-dimensional bicomplete asymmetric normed space (Theorem 4.1 in the same paper).

We mention also the following result from [215].

Proposition 2.6.18. *Let (X,ρ) be a T_1 quasi-metric space and (Y,q) a bicomplete asymmetric normed space. Then the linear space $\mathrm{SLip}^0_{\rho,q}(X,Y)\cap\mathrm{SLip}^0_{\bar\rho,q}(X,Y)$ is complete with respect to the norm*

$$\|f\|_{\rho,q} = \sup\left\{\frac{q^s(f(x)-f(y))}{\rho(x,y)} : x,y\in X,\ \ \rho(x,y)>0\right\}\,.$$

Proof. Observe that

$$\begin{aligned}
\|f\|_{\rho,q} &= \sup\left\{\frac{q(f(x)-f(y))\vee q(f(y)-f(x))}{\rho(x,y)} : x,y\in X,\ \rho(x,y)>0\right\}\\
&= \sup\left\{\frac{q(f(x)-f(y))}{\rho(x,y)} : x,y\in X,\ \ \rho(x,y)>0\right\}\\
&\qquad\vee\sup\left\{\frac{q(f(y)-f(x))}{\rho(x,y)} : x,y\in X,\ \ \rho(x,y)>0\right\}\\
&= \|f|_{\rho,q}\vee\|-f|_{\rho,q} = \|f|^s_{\rho,q}\,,
\end{aligned}$$

so that $\|\cdot|^s_{\rho,q}$ is the norm associated to the asymmetric norm $\|\cdot|_{\rho,q}$ given by (2.6.5).

The proof of the completeness follows the line of the proof of the assertion 1 in Theorem 2.6.16. □

Remark 2.6.19. If the metric space (X,ρ) is T_1, then

$$\mathrm{SLip}^0_{\rho,q}(X,Y)\cap\mathrm{SLip}^0_{\bar\rho,q}(X,Y) = \mathrm{Lip}^0\left((X,\rho^s),(Y,q^s)\right)\,,$$

and the norm $\|\cdot\|_{\rho,q}$ agrees with the Lipschitz norm $\|\cdot\|_{\rho^s,q^s}$ given by (2.6.2).

Indeed,

$$\begin{aligned}\|f\|_{\rho,q} &= \sup\left\{\frac{q^s(f(x)-f(y))}{\rho(x,y)} : x,y\in X,\ x\neq y\right\}\\ &= \sup\left\{\frac{q^s(f(x)-f(y))}{\rho^s(x,y)} : x,y\in X,\ x\neq y\right\} = \|f\|_{\rho^s,q^s}\ .\end{aligned}$$

Doitchinov [62, 63, 64] defined and studied a notion of completeness for quasi-metric spaces with the aim to obtain a satisfactory theory of completion (see [62] for the quasi-metric case and [66] for quasi-uniform spaces). A sequence (x_n) in a quasi-metric space (X,ρ) is called *D-Cauchy* if there exists another sequence (y_n) such that $\lim_{m,n}\rho(y_m,x_n)=0$. The quasi-metric space (X,ρ) is called D-complete if every D-Cauchy sequence converges. A quasi-metric space (X,ρ) is called *balanced* if for all sequences $(x_n),(y_n)$ such that $\lim_{m,n}\rho(y_m,x_n)=0$ and for every $x,y\in X$ and $r_1, r_2\geq 0$, $\rho(x,x_n)\leq r_1$ and $\rho(y_n,y)\leq r_2$ for all $n\in\mathbb{N}$, implies $\rho(x,y)\leq r_1+r_2$. The concept of balancedness, meaning a kind of symmetry of a quasi-metric space, was also introduced by Doitchinov in [63], to develop a satisfactory theory of completion. He proved that a balanced quasi-metric generates a Hausdorff and completely regular topology, see Proposition 1.1.11 and Subsection 1.2.6.

The following completeness result was proved in [198].

Theorem 2.6.20. *Let (X,ρ) be a T_1 quasi-metric space and (Y,q) an asymmetric normed space. If the asymmetric normed space (Y,q) is biBanach, then the space $\mathrm{SLip}^0_{\rho,q}(X,Y)$ is balanced and D-complete with respect to the extended quasi-metric $\delta_{\rho,q}$ defined by* (2.6.18).

Proof. The metric $\boldsymbol{\delta_{\rho,q}}$ is balanced on $\mathrm{SLip}^0_{\rho,q}(X,Y)$.

Let $(f_m),(g_n)$ be two sequences in $\mathrm{SLip}^0_{\rho,q}(X,Y)$ and $f,g\in\mathrm{SLip}^0_{\rho,q}(X,Y)$ such that

$$\begin{aligned}&\text{(i)} && \lim_{m,n\to\infty}\delta_{\rho,q}(f_m,g_n)=0,\\ &\text{(ii)} && \forall m\in\mathbb{N},\ \ \delta_{\rho,q}(f_m,f)\leq r_1, && (2.6.19)\\ &\text{(iii)} && \forall n\in\mathbb{N},\ \ \delta_{\rho,q}(g,g_n)\leq r_2\ ,\end{aligned}$$

for some some numbers $r_1,r_2>0$.

We have to show that $\delta_{\rho,q}(g,f)\leq r_1+r_2$.

Since the metric ρ is T_1, $\rho(x,y)>0 \iff x\neq y$, so that the extended quasi-metric (2.6.18) is given by

$$\delta_{\rho,q}(h_1,h_2)=\sup_{x\neq y}\frac{q((h_2-h_1)(x)-(h_2-h_1)(y))}{\rho(x,y)}\ ,$$

for $h_1,h_2\in\mathrm{SLip}^0_{\rho,q}(X,Y)$.

Let $x \neq y$ be a fixed pair of distinct elements in X. Then, by condition (i) in (2.6.19), for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\frac{q((g_n - f_m)(u) - (g_n - f_m)(v))}{\rho(u,v)} \leq \varepsilon \ , \tag{2.6.20}$$

for all $m, n \geq n_\varepsilon$ and all $u, v \in X$ with $u \neq v$.

It follows that

$$\begin{aligned} &q((f-g)(x) - (f-g)(y)) \qquad (2.6.21)\\ &\quad \leq q((f - f_{n_\varepsilon})(x) - (f - f_{n_\varepsilon})(y)) + q((f_{n_\varepsilon} - g_{n_\varepsilon})(x) - (f_{n_\varepsilon} - g_{n_\varepsilon})(y)) \\ &\qquad + q((g_{n_\varepsilon} - g)(x) - (g_{n_\varepsilon} - g)(y) \leq r_1\rho(x,y) + \varepsilon\rho(y,x) + r_2\rho(x,y) \ . \end{aligned}$$

The inequality for the middle term in the second row follows from (2.6.20) and the equality

$$q((f_{n_\varepsilon} - g_{n_\varepsilon})(x) - (f_{n_\varepsilon} - g_{n_\varepsilon})(y)) = q((g_{n_\varepsilon} - f_{n_\varepsilon})(y) - (g_{n_\varepsilon} - f_{n_\varepsilon})(x)) \ .$$

Since $\varepsilon > 0$ is arbitrary in (2.6.21), it follows that

$$q((f-g)(x) - (f-g)(y)) \leq q((f - f_{n_\varepsilon})(x) - (f - f_{n_\varepsilon})(y)) \leq r_1\rho(x,y) + r_2\rho(x,y) \ ,$$

for all $x, y \in X$ with $x \neq y$, that is $\delta_{\rho,q}(g,f) \leq r_1 + r_2$.

The metric $\boldsymbol{\delta_{\rho,q}}$ is $\boldsymbol{D}$-complete on $\mathbf{SLip}^{\mathbf{0}}_{\boldsymbol{\rho,q}}\boldsymbol{(X,Y)}$.

Suppose that (f_n) is a D-Cauchy sequence in $\mathrm{SLip}^0_{\rho,q}(X,Y)$ and let (g_m) be a sequence in $\mathrm{SLip}^0_{\rho,q}(X,Y)$ such that

$$\lim_{m,n\to\infty} \delta_{\rho,q}(g_m, f_n) = 0 \ , \tag{2.6.22}$$

that is (g_n) is a cosequence for (f_n).

Claim I. *For every $x \in X$, $(f_n(x))$ is a Cauchy sequence in (Y, q^s).*

Let $x \in X,\ x \neq x_0$. For $\varepsilon > 0$ let $n_\varepsilon \in \mathbb{N}$ be chosen according to (2.6.20). Taking in (2.6.20) first $u = x,\ v = x_0$ and then $u = x_0,\ v = x$, one obtains

$$\begin{aligned} &q((f_n - g_m)(x) \leq \varepsilon\rho(x,x_0) \leq \varepsilon\rho^s(x,x_0), \quad \text{respectively} \\ &q((g_m - f_n)(x) \leq \varepsilon\rho(x_0,x) \leq \varepsilon\rho^s(x,x_0) \ , \end{aligned}$$

implying

$$q^s((f_n - g_m)(x) \leq \varepsilon\rho^s(x,x_0) \ , \tag{2.6.23}$$

for all $m, n \geq n_\varepsilon$. But then,

$$q^s((f_n - f_m)(x) \leq q^s((f_n - g_{n_\varepsilon})(x) + q^s((g_{n_\varepsilon} - f_m)(x) \leq 2\varepsilon\rho^s(x_0,x) \ ,$$

for all $m, n \geq n_\varepsilon$, proving that $(f_n(x))$ is a Cauchy sequence in the Banach space (Y, q^s), so it converges to some $y \in Y$.

It follows that we can define a function $f : X \to Y$ by $f(x) = \lim_n f_n(x),\ x \in X$. To end the proof we have to show that $f \in \mathrm{SLip}^0_{\rho,q}(X,Y)$ and $f_n \to f$ with respect to the extended quasi-metric $\delta_{\rho,q}$.

Observe that, by (2.6.23), the sequence $(g_m(x))$ also converges to $f(x)$ in the norm q^s.

Claim II. $f \in \mathrm{SLip}^0_{\rho,q}(X,Y)$.

Let $n_1 \in \mathbb{N}$ be such that (2.6.20) holds for $\varepsilon = 1$. Then, taking into account (2.6.19).(ii), one gets

$$\begin{aligned} q(f(u)-f(v)) &\leq q((f-f_{n_1})(u)-(f-f_{n_1})(v)) \\ &\qquad + q((f_{n_1}-g_{n_1})(u)-(f_{n_1}-g_{n_1})(v)) + q((g_{n_1})(u)-(g_{n_1})(v)) \\ &\leq (r_1 + 1 + \|g_{n_1}|_{\rho,q})\rho(u,v)\ , \end{aligned}$$

for all $u, v \in X$, proving that $f \in \mathrm{SLip}^0_{\rho,q}(X,Y)$.

Claim III. *$f_n \to f$ with respect to the extended quasi-metric $\delta_{\rho,q}$.*

Given $\varepsilon > 0$ let n_ε be chosen according to (2.6.20) and let $x \neq y$ in X, arbitrary, but fixed for the moment.

Let $n \geq n_\varepsilon$. Since $\lim_m q^s(g_m(u) - f(u)) = 0$ for every $u \in X$, there exists $m \geq n_\varepsilon$ such that

$$q^s(g_m(x)f(x)) \leq \varepsilon\rho(x,y) \quad \text{and} \quad q^s(g_m(y)-f(y)) \leq \varepsilon\rho(x,y)\ . \tag{2.6.24}$$

But then

$$\begin{aligned} &q((f_n-f)(x)-(f_n-f)(y)) \\ &\quad \leq q((f_n-g_m)(y)-(f_n-g_m)(y)) + q((g_m-f)(x)-(g_m-f)(y)) \\ &\quad \leq 3\varepsilon\rho(x,y)\ , \end{aligned}$$

because, by (2.6.24),

$$\begin{aligned} q((g_m-f)(x)-(g_m-f)(y)) &\leq q^s((g_m-f)(x)-(g_m-f)(y)) \\ \leq q^s((g_m-f)(x))(x)) &+ q^s((g_m-f)(x))(y)) \leq 2\varepsilon\rho(x,y)\ . \end{aligned}$$

Since the points $x \neq y$ were arbitrarily chosen in X, it follows that $\delta_{\rho,q}(f, f_n) \leq 3\varepsilon$ for all $n \geq n_\varepsilon$, proving the convergence of the sequence (f_n) to f with respect to the extended quasi-metric $\delta_{\rho,q}$. □

Let (X, p) be a T_1 asymmetric normed space and $X^\flat_p$ its dual. Then $X^\flat_p$ is contained in the cone $\mathrm{SLip}^0_{\bar{\rho}}(X)$, where

$$\bar{\rho}(x,y) = p(x-y) = \bar{p}(y-x) = \rho_{\bar{p}}(x,y),\ x,y \in X\ .$$

Indeed

$$\varphi \in X_\rho^\flat \;\Rightarrow\; \forall x, y \in X,\;\; \varphi(x-y) \le \|\varphi|_p\, p(x-y) = \|\varphi|_p\, \bar\rho(x,y) \;\Rightarrow \varphi \in \mathrm{SLip}_{\bar\rho}^0(X)\,.$$

It follows also that $\|\varphi|_{\bar\rho} \le \|\varphi|_p$. In fact we have equality:

$$\|\varphi|_{\bar\rho} = \sup\Big\{\frac{u(\varphi(x-y))}{p(x-y)} : x \neq y\Big\} = \sup\{u(\varphi(z)) : p(z) \le 1\} = \|\varphi|_p\,.$$

It follows that the restriction $\delta_p^\flat$ of the extended quasi-metric $\delta_{\bar\rho}$ to $X_p^\flat$ is given by

$$\begin{aligned}\delta_p^\flat(\varphi,\psi) &= \sup\{\frac{u((\psi-\varphi)(x-y))}{p(x-y)} : x \neq y\}\\ &= \sup\{u((\psi-\varphi)(z)) : X,\, p(z) \le 1\} = \|\psi-\varphi|_p\,,\end{aligned}$$

that is it agrees with the extended quasi-metric associated to the extended asymmetric norm $\|\cdot|_{p,u}$, see (2.1.23) and Proposition 2.1.3.

Theorem 2.6.21 ([198])**.** *If (X,p) is a T_1 asymmetric normed space, then the dual space $X_p^\flat$ is balanced and D-complete with respect to the extended quasi-metric $\delta_p^\flat$.*

Proof. Since $X_p^\flat \subset \mathrm{SLip}_{\bar\rho}^0(X)$ and the balancedness is a hereditary property it follows that $X_p^\flat$ is balanced with respect to the extended metric $\delta_p^\flat$.

If (φ_n) is a D-Cauchy sequence in $(X_p^\flat, \delta_p^\flat)$, then it is D-Cauchy in $(\mathrm{SLip}_\rho^0(X), \delta_{\bar\rho})$, so that, by Theorem 2.6.20, it converges to some $\varphi \in \mathrm{SLip}_{\bar\rho}^0(X)$. By the proof of the same theorem, for every $x \in X$ the sequence $(\varphi_n(x))$ converges in $(\mathbb{R}, |\cdot|)$ to $\varphi(x)$, implying the linearity of the limit φ. Since φ is semi-Lipschitz, there exists $\beta > 0$ such that

$$\varphi(x) = \varphi(x) - \varphi(0) \le u\left(\varphi(x) - \varphi(0)\right) \le \beta\, \bar\rho(x,0) = \beta\, p(x)\,,$$

showing that $\varphi \in X_p^\flat$. □

2.6.4 Applications to best approximation in quasi-metric spaces

Semi-Lipschitz functions can be used to study some best approximation problems in quasi-metric spaces. The notions can transposed from asymmetric normed spaces (see (2.2.3)) to quasi-metric spaces. Let (X,ρ) be a quasi-metric space. For $Y \subset X$ and $x \in X$ put

$$\begin{aligned} d(x,Y) &= \inf\{\rho(x,y) : y \in Y\} \quad \text{and}\\ d(Y,x) &= \inf\{\rho(y,x) : y \in Y\}\,,\end{aligned}$$

see (1.1.22). Consider also the set

$$Y^\perp = \{f \in \mathrm{SLip}_\rho^0(X) : f|_Y = 0\}\,.$$

The following result is the semi-Lipschitz analog of Theorem 2.5.1.

Proposition 2.6.22. *Let (X,ρ) be a quasi-metric space, $Y\subset X$ nonempty, $x_0\in X$ such that $d(x_0,Y)>0$ and $y_0\in Y$. Then y_0 is a nearest point to x_0 in Y if and only if there exists $f\in Y^\perp$ such that*

$$\text{(i)}\quad \|f|_\rho = 1 \quad \textit{and} \quad \text{(ii)}\quad \rho(x_0,y_0)=f(x_0)-f(y_0)\,.$$

In the case of metric spaces and Lipschitz functions similar results were obtained by Mustăţa [154, 155], who proved also many results on the characterization of the approximation properties in a quasi-metric space in terms of the semi-Lipschitz functions defined on it. Other results on best approximation and extensions were obtained in [160, 163].

Let $(X,\|\cdot\|)$ be a normed space and X^* its dual. For a subspace Y of X put

$$Y^\perp=\{x^*\in X^*: x^*|_Y=0\}$$

and denote by $E_Y(y^*)$ the set of all norm-preserving extensions of a continuous linear functional y^* on Y, that is,

$$E_Y(y^*)=\{x^*\in X^*: x^*|_Y=y^* \text{ and } \|x^*\|=\|y^*\|\}\,.$$

Phelps [177] proved the following remarkable result relating the approximation properties of the space $Y^\perp$ and the extension properties of the space Y.

Theorem 2.6.23 (Phelps [177]). *Let Y be a closed subspace of a normed space X. Then $Y^\perp$ is a proximinal subspace of X^* and for every $x^*\in X^*$ the following equality holds:*

$$P_{Y^\perp}(x^*)=E_Y(x^*|_Y)\,.$$

Consequently, $Y^\perp$ is a Chebyshev subspace of X^* if and only if every $y^*\in Y^*$ has a unique norm-preserving extension to the whole of X.

Extensions of this result to spaces of Lipschitz functions on metric spaces and to spaces of semi-Lipschitz functions on quasi-metric spaces were given by Mustăţa [156, 157, 159, 161, 162, 165] (see also the paper [39] containing a survey of various situations where a Phelps type result can occur). An iterative approximation method to find the global minimum of a semi-Lipschitz function is proposed in [166]. Romaguera and Sanchis give in [205] characterizations of preferences on separable quasi-metric spaces admitting semi-Lipschitz utility functions, with applications to theoretical computer science.

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