Introduction

The main goal of this book is to present the basic results on asymmetric normed spaces. Since the basic topological tools come from quasi-metric spaces and quasi-uniform spaces, the first chapter contains a thorough presentation of some fundamental results from the theory of these spaces. The focus is on those which are most used in functional analysis – completeness, compactness and Baire category. For a good presentation of the general theory of quasi-uniform and quasi-metric spaces, a well-established and thickly developed branch of general topology, one can consult the classical monograph by Fletcher and Lindgren [80] and some subsequent survey papers by Künzi (see the bibliography at the end of the book). The survey paper [45] may be viewed as a skeleton of this book.

A quasi-metric is a function ρ on $X \times X$ satisfying all the axioms of a metric with the exception of the symmetry: it is possible that $\rho(y,x) \neq \rho(x,y)$ for some $x, y \in X$. In this case $\bar{\rho}(x,y) = \rho(y,x)$ is another quasi-metric on X, called the conjugate of ρ, and $\rho^s = \rho \vee \bar{\rho}$ is a metric on X. Asymmetric metric spaces are called quasi-metric spaces. The term quasi-metric was proposed as early as 1931 by Wilson [239], see also [1]. Quasi-metric spaces were considered also by Niemytzki [168] in connection with the axioms defining a metric space and metrizability. In [27], [186] they are called oriented metric spaces and in [187] spaces with weak metric.

This apparently innocent modification of the axioms of a metric space drastically changes the whole theory, mainly with respect to completeness, compactness and total boundedness. There are a lot of completeness notions in quasi-metric and quasi-uniform spaces, all agreeing with the usual notion of completeness in the case of metric or uniform spaces, each of them having its advantages and weaknesses.

Also, concerning compactness, the situation is totally different in quasi-metric spaces – for instance, sequential compactness does not agree with compactness, in contrast to the case of metric spaces. In spite of these peculiarities there are a lot of positive results relating compactness with various kinds of completeness and total boundedness. Baire category also needs a special treatment, including some bitopological results.

Quasi-uniform spaces form a natural extension of both quasi-metric spaces and uniform spaces. A quasi-uniformity is a family $\mathcal{U}$ of subsets of $X \times X$, called entourages, satisfying all the axioms of a uniformity excepting symmetry: one does not suppose that $\mathcal{U}$ has a base formed of symmetric entourages. Again, $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$ is another quasi-uniformity on X, called the conjugate of $\mathcal{U}$, and $\mathcal{U}^s = \mathcal{U} \vee \mathcal{U}^{-1}$ is a uniformity. The notions of completeness can be transposed from quasi-metric spaces to quasi-uniform spaces by replacing sequences with nets or filters. Again the focus is on the relations between compactness, completeness and total boundedness within this framework.

On a quasi-metric space (X, ρ) there are two natural topologies generated by the quasi-metric ρ and its conjugate $\bar{\rho}$, respectively by the quasi-uniformity $\mathcal{U}$ and its conjugate $\mathcal{U}^{-1}$, making quasi-metric and quasi-uniform spaces bitopological spaces. For this reason the first chapter of the book contains a quite detailed introduction to bitopological spaces, including Urysohn and Tietze type theorems for semi-continuous functions on bitopological spaces, compactness and Baire category.

Following the advice of Einar Hille [105] that "a functional analyst is an analyst, first and foremost, and not a degenerate species of a topologist", after this detour in topology we turn to functional analysis. Functional analysis in the asymmetric case, meaning the study of asymmetric normed spaces, asymmetric locally convex spaces and of operators acting between them, with emphasis on linear functionals and dual spaces, is treated in the second chapter.

An asymmetric norm is a positive definite sublinear functional p on a real vector space X. Since the possibility that $p(x) = p(-x)$ for some $x \in X$ is not excluded, $\bar{p}(x) = p(-x),\ x \in X$, is another asymmetric norm on X called the conjugate of p, and $p^s = p \vee \bar{p}$ is a norm on X. The topological notion are considered with respect to the attached metric $\rho_p(x, y) = p(y - x),\ x, y \in X$. Any asymmetric norm can be obtained as the Minkowski gauge functional of an absorbing convex subset of X. Asymmetric locally convex spaces are defined as vector spaces equipped with a topology generated by a family of asymmetric seminorms.

Of great importance is the asymmetric norm u on $\mathbb{R}$ given by $u(t) = t^+,\ t \in \mathbb{R}$, with conjugate $\bar{u}(t) = t^-$ and $u^s = |\cdot|$. The topology generated by u is called the upper topology of $\mathbb{R}$, while that generated by its conjugate $\bar{u}$, the lower topology. If (T, τ) is a topological space, then a real-valued function f on T is upper semi-continuous as a function from T to $(\mathbb{R}, |\cdot|)$ if and only if it is continuous from T to $(\mathbb{R}, u)$. Similarly, f is $(\tau, |\cdot|)$-lower semi-continuous if and only if it is $(\tau, \bar{u})$-continuous.

The main differences with respect to the classical functional analysis (meaning analysis over the fields $\mathbb{R}$ or $\mathbb{C}$) come from the fact that the asymmetric norm p does not generate a vector topology on X: the addition is continuous with respect to the product topology on X, but the multiplication by scalars is continuous only when restricted to $(0; \infty) \times X$. Also, for each fixed x, the function $f : \mathbb{R} \to X$ given by $f(t) = tx,\ t \in \mathbb{R}$, is continuous. The dual space of an asymmetric normed space (X, p), denoted by $X_p^\flat$, formed by all linear and $|\cdot|$-upper semi-continuous functions, or, equivalently, linear continuous functionals from (X, p) to $(\mathbb{R}, u)$, is not a linear space but merely a cone contained in the dual space $X^* = (X, p^s)^*$ of the associated normed space (X, p^s). The situation is similar for the set of all continuous linear operators between two asymmetric normed spaces, as well as for asymmetric locally convex spaces.

In spite of these differences, many results from classical functional analysis have their counterparts in the asymmetric case, by taking care of the interplay between the asymmetric norm p and its conjugate $\bar{p}$. Among the positive results we mention: Hahn-Banach type theorems and separation results for convex sets, Krein-Milman type theorems, analogs of the fundamental principles – open mapping and closed graph theorems – an analog of the Schauder theorem on the compactness of a conjugate mapping. Applications are given to best approximation problems and, as relevant examples, one considers normed lattices equipped with asymmetric norms and spaces of semi-Lipschitz functions on quasi-metric spaces.

It is difficult to localize the first moment when asymmetric norms were used, but it goes back as early as 1968 in a paper by Duffin and Karlovitz (1968) [70], who proposed the term asymmetric norm. Krein and Nudelman (1973) [129] used also asymmetric norms in their study of some extremal problems related to the Markov moment problem. Remark that the relevance of sublinear functionals for some problems of convex analysis and of mathematical analysis was emphasized also by H. König in the 1970s. A systematic study of the properties of asymmetric normed spaces started with the papers of S. Romaguera, from the Polytechnic University of Valencia, and his collaborators from the same university and from other universities in Spain: Alegre, Ferrer, García-Raffi, Sánchez Pérez, Sánchez Álvarez, Sanchis, Valero (see the bibliography). Besides its intrinsic interest, their study was motivated also by applications in Computer Science, namely to the complexity analysis of programs, results obtained in cooperation with Professor Schellekens from the National University of Ireland.

Containing very recent results, some of them appearing for the first time in print, in the focus of current research, on quasi-metric, quasi-uniform, asymmetric normed and asymmetric locally convex spaces, the book can be used as a reference by researchers in this domain. Due to the detailed exposition of the subject, the book can be also used as an introductory text for newcomers.

Acknowledgement. The author expresses his gratitude to the staff of Birkhäuser-Springer, particularly to the Editors Anna Mätzener and Sylvia Lotrovsky, for the professional work and excellent cooperation during the publication process, and to Ute McCrory from Springer DE for support.

I want to mention also that this research was supported by
Grant CNCSIS 2261, ID 543.

Notation. We present here, for the convenience of the reader, some symbols that are used throughout the text, which could differ from the standard ones. Other notations are standard or explained in the text, some of them being included in the index at the end of the book.

- $\mathbb{N} = \{1, 2, \dots\}$ – the set of natural numbers (positive integers);
- $[a; b]$, $(a; b)$, $(a; b]$, $[a; b)$ – intervals;
- (a, b) – an ordered pair;
- $B_\rho[x, r] = \{y \in X : \rho(x, y) \leq r\}$ – a closed ball in a quasi-metric space (X, ρ);
- $B_\rho(x, r) = \{y \in X : \rho(x, y) < r\}$ – an open ball;
- $\rho_p(x, y) = p(y - x)$ – the quasi-metric associated to an asymmetric norm p;
- $B_p = \{x \in X : p(x) \leq 1\}$ – the closed unit ball of an asymmetric normed space (X, p);
- $B'_p = \{x \in X : p(x) < 1\}$ – the open unit ball;
- $S_p = \{x \in X : p(x) = 1\}$ – the unit sphere;
- u is the standard asymmetric norm $u(t) = t^+$ on $\mathbb{R}$.

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http://www.springer.com/978-3-0348-0477-6

Functional Analysis in Asymmetric Normed Spaces
Cobzas, S.
2013, X, 219 p. 1 illus. in color., Softcover
ISBN: 978-3-0348-0477-6
A product of Birkhäuser Basel