Preface

The purpose of this monograph is to lay down the foundations of the theory of complex Kleinian groups, a concept and a name introduced by José Seade and Alberto Verjovsky in the late 1990s, though their origin traces back to classical work by Henri Poincaré, Emile Picard, Georges Giraud and many others. This brings together several important areas of mathematics, as for instance classical Kleinian group actions, complex hyperbolic geometry, chrystallographic groups and the uniformization problem for complex manifolds. Each of these is in itself a fascinating area of mathematics, with a vast literature, both classical and modern. In fact, real and complex hyperbolic geometry are indeed at the very roots of the theory of complex Kleinian groups and therefore we have devoted the first two chapters of this work to giving a fast overview of these rich areas of mathematics.

A classical Kleinian group is a discrete group of conformal automorphisms of the Riemann sphere $\mathbb{S}^2$, acting on the sphere with a nonempty region of discontinuity. Since the Riemann sphere is biholomorphic to the complex projective line $\mathbb{P}^1_{\mathbb{C}}$, and the orientation preserving conformal automorphisms of $\mathbb{S}^2$ are exactly the elements in $\mathrm{PSL}(2,\mathbb{C})$, one has that in this dimension, the classical Kleinian groups can be regarded too as being groups of holomorphic automorphisms of $\mathbb{P}^1_{\mathbb{C}}$.

When going into higher dimensions, there is a dichotomy: Should we look at conformal automorphisms of the n-sphere $\mathbb{S}^n$?, or should we look at holomorphic automorphisms of the complex projective space $\mathbb{P}^n_{\mathbb{C}}$? These two theories are different because in higher dimensions, neither are conformal maps always holomorphic, nor are holomorphic maps necessarily conformal. In the first case we are talking about groups of isometries of real hyperbolic spaces, an area of mathematics where there is a rich body of knowledge thanks to the contributions of people like Ahlfors, Thurston, Margulis, Sullivan, Mostow, Kapovich, McMullen and many others. In the second case we are talking about an area of mathematics that still is in its childhood, and its study is the theme of this work. Complex Kleinian groups are discrete subgroups of $\mathrm{PSL}(n+1,\mathbb{C})$, the group of holomorphic automorphisms of $\mathbb{P}^n_{\mathbb{C}}$, having a nonempty invariant set where the action is properly discontinuous.

The group $\mathrm{PU}(n,1)$ of holomorphic isometries of complex hyperbolic n-space consists of the elements in $\mathrm{PSL}(n+1,\mathbb{C})$ that preserve a ball, while the affine group $\mathrm{Aff}\,(\mathbb{C}^n) \cong \mathrm{GL}\,(n,\mathbb{C}) \ltimes \mathbb{C}^n$ consists of the elements in $\mathrm{PSL}(n+1,\mathbb{C})$ that leave invariant a given projective hyperplane. These are two very important subgroups of $\mathrm{PSL}(n+1,\mathbb{C})$, but there are others, as for instance all the Lorentz groups $\mathrm{PU}(p,q)$ with $p+q=n+1$, the groups coming via twistor theory, Schottky type groups, and many others. Thus we see that the study of complex Kleinian groups is at the very heart of complex geometry.

An important difference with the classical case springs from the notion of the limit set. We know that if we consider a Kleinian group G acting on the n-sphere $\mathbb{S}^n$, then its limit set Λ is the set of accumulation points of the orbits. Its complement Ω is the region of discontinuity; this is the maximal region where the action is properly discontinuous, and it is also the equicontinuity set of the family of transformations defined by the group action. The same definitions and properties apply to discrete subgroups of $\mathrm{PU}(n,1)$, essentially because when one looks at the action of isometry groups in real or complex hyperbolic space, one has the convergence property, in Misha Kapovich's language. Yet, even in the case of subgroups of $\mathrm{PU}(n,1)$, when we look at the action on the whole space $\mathbb{P}^n_{\mathbb{C}}$ and not only on the unit ball, there is not a well-defined notion of the limit set. There are actually several possible definitions of the limit set, each with its own properties and characteristics, and their study is one of the main features of this monograph.

Another important point to notice is that, in complex dimension 1, we have Sullivan's dictionary (highly enriched by McMullen and others) between the theory of Kleinian groups and the study of iterates of holomorphic functions of the Riemann sphere. There is currently much interesting work being done on iteration theory in several complex variables, and it is to be expected that there should be plenty of analogies (or perhaps a dictionary, to some extent) between iteration theory of endomorphisms of $\mathbb{P}^n_{\mathbb{C}}$ and the theory of complex Kleinian groups. Recent work by W. Barrera, A. Cano, J.- P. Navarrete and others, points in this direction, but there is still a lot to be understood.

We finish this preface by saying that the theory of complex Kleinian groups is a rich area of mathematics that is waiting to be explored. We believe that anyone who absorbs the material in this book, will get plenty of ideas and insights about interesting questions and further lines of research.

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