

# Contents

<b>Preface</b>	<b>vii</b>
<b>Introduction</b>	<b>xiii</b>
<b>Acknowledgments</b>	<b>xx</b>
<b>1 A Glance at the Classical Theory</b>	<b>1</b>
1.1 Isometries of hyperbolic $n$ -space. The conformal group . . . . .	2
1.1.1 Poincaré models for hyperbolic space . . . . .	2
1.1.2 Möbius groups in dimensions 2 and 3 . . . . .	6
1.1.3 Geometric classification of the elements in $\text{Iso}_+(\mathbb{H}_{\mathbb{R}}^n)$ . . . .	7
1.1.4 Isometric spheres . . . . .	10
1.2 Discrete subgroups . . . . .	12
1.2.1 Properly Discontinuous Actions . . . . .	12
1.2.2 The limit set and the discontinuity region. . . . .	15
1.2.3 Fundamental domains . . . . .	19
1.2.4 Fuchsian groups . . . . .	20
1.2.5 Kleinian and Schottky groups . . . . .	25
1.3 Rigidity and ergodicity . . . . .	28
1.3.1 Moore's Ergodicity Theorem . . . . .	28
1.3.2 Mostow's rigidity theorem . . . . .	29
1.3.3 On the Patterson-Sullivan measure . . . . .	31
1.3.4 Sullivan's theorem on nonexistence of invariant lines . . . .	38
<b>2 Complex Hyperbolic Geometry</b>	<b>41</b>
2.1 Some basic facts on Projective geometry . . . . .	42
2.2 Complex hyperbolic geometry. The ball model . . . . .	45
2.2.1 Totally geodesic subspaces . . . . .	48
2.2.2 Bisectors and spines . . . . .	49
2.3 The Siegel domain model . . . . .	51
2.3.1 Heisenberg geometry and horospherical coordinates . . . .	52
2.3.2 The geometry at infinity . . . . .	53

2.4	Isometries of the complex hyperbolic space . . . . .	54
2.4.1	Complex reflections . . . . .	55
2.4.2	Dynamical classification of the elements in $\mathrm{PU}(2, 1)$ . . . .	57
2.4.3	Traces and conjugacy classes in $\mathrm{SU}(2, 1)$ . . . . .	60
2.5	Complex hyperbolic Kleinian groups . . . . .	66
2.5.1	Constructions of complex hyperbolic lattices . . . . .	67
2.5.2	Other constructions of complex hyperbolic Kleinian groups . . .	69
2.6	The Chen-Greenberg limit set . . . . .	73
<b>3</b>	<b>Complex Kleinian Groups</b>	<b>77</b>
3.1	The limit set: an example . . . . .	77
3.2	Complex Kleinian groups: definition and examples . . . . .	80
3.3	Limit sets: definitions and some basic properties . . . . .	83
3.3.1	The Limit set of Kulkarni . . . . .	83
3.3.2	Elementary groups . . . . .	87
3.4	On the subgroups of the affine group . . . . .	88
3.4.1	Fundamental groups of Hopf manifolds . . . . .	88
3.4.2	Fundamental groups of complex tori . . . . .	88
3.4.3	The suspension or cone construction . . . . .	88
3.4.4	Example of elliptic affine surfaces . . . . .	90
3.4.5	Fundamental groups of Inoue surfaces . . . . .	90
3.4.6	A group induced by a hyperbolic toral automorphism . . . .	91
3.4.7	Crystallographic and complex affine reflection groups . . . .	92
<b>4</b>	<b>Geometry and Dynamics of Automorphisms of <math>\mathbb{P}_{\mathbb{C}}^2</math></b>	<b>93</b>
4.1	A qualitative view of the classification problem . . . . .	94
4.2	Classification of the elements in $\mathrm{PSL}(3, \mathbb{C})$ . . . . .	97
4.2.1	Elliptic Transformations in $\mathrm{PSL}(3, \mathbb{C})$ . . . . .	97
4.2.2	Parabolic Transformations in $\mathrm{PSL}(3, \mathbb{C})$ . . . . .	101
4.2.3	Loxodromic Transformations in $\mathrm{PSL}(3, \mathbb{C})$ . . . . .	105
4.3	The classification theorems . . . . .	112
<b>5</b>	<b>Kleinian Groups with a Control Group</b>	<b>119</b>
5.1	$\mathrm{PSL}(2, \mathbb{C})$ revisited: nondiscrete subgroups . . . . .	120
5.1.1	Main theorems for nondiscrete subgroups of $\mathrm{PSL}(2, \mathbb{C})$ . . .	120
5.2	Some basic examples . . . . .	122
5.3	Elementary groups . . . . .	123
5.3.1	Groups whose equicontinuity region is the whole sphere . .	123
5.3.2	Classification of elementary groups . . . . .	126
5.4	Consequences of the classification theorem . . . . .	126
5.5	Controllable and control groups: definitions . . . . .	129
5.5.1	Suspensions . . . . .	129
5.6	Controllable groups . . . . .	131
5.7	Groups with control . . . . .	133

5.8	On the limit set . . . . .	134
5.8.1	The limit set for suspensions extended by a group . . . . .	134
5.8.2	A discontinuity region for some weakly semi-controllable groups . . . . .	134
<b>6</b>	<b>The Limit Set in Dimension 2</b>	<b>137</b>
6.1	Montel's theorem in higher dimensions . . . . .	138
6.2	Lines and the limit set . . . . .	139
6.3	The limit set is a union of complex projective lines . . . . .	140
<b>7</b>	<b>On the Dynamics of Discrete Subgroups of <math>\mathrm{PU}(n, 1)</math></b>	<b>145</b>
7.1	Discrete subgroups of $\mathrm{PU}(n, 1)$ revisited . . . . .	146
7.2	Some properties of the limit set . . . . .	148
7.3	Comparing the limit sets $\Lambda_{\mathrm{Kul}}(G)$ and $\Lambda_{CG}(G)$ . . . . .	150
7.4	Pseudo-projective maps and equicontinuity . . . . .	156
7.5	On the equicontinuity region . . . . .	160
7.6	Geometric Applications . . . . .	162
7.6.1	The Kobayashi Metric . . . . .	162
7.6.2	Complex Hyperbolic groups and $k$ -chains . . . . .	162
7.7	The two-dimensional case revisited . . . . .	164
<b>8</b>	<b>Projective Orbifolds and Dynamics in Dimension 2</b>	<b>167</b>
8.1	Geometric structures and the developing map . . . . .	168
8.1.1	Projective structures on manifolds . . . . .	169
8.1.2	The developing map and holonomy . . . . .	170
8.2	Real Projective Structures and Discrete Groups . . . . .	171
8.2.1	Projective structures on real surfaces . . . . .	172
8.2.2	On divisible convex sets in real projective space . . . . .	173
8.3	Projective structures on complex surfaces . . . . .	175
8.4	Orbifolds . . . . .	177
8.4.1	Basic notions on orbifolds . . . . .	177
8.4.2	Description of the compact $(\mathbb{P}_{\mathbb{C}}^2, \mathrm{PSL}_3(\mathbb{C}))$ -orbifolds . . . . .	181
8.5	Discrete groups and divisible sets in dimension 2 . . . . .	182
8.6	Elementary quasi-cocompact groups of $\mathrm{PSL}(3, \mathbb{C})$ . . . . .	184
8.7	Nonelementary affine groups . . . . .	187
8.8	Concluding remarks . . . . .	192
8.8.1	Summary of results for quasi-cocompact groups . . . . .	192
8.8.2	Comments and open questions . . . . .	193
<b>9</b>	<b>Complex Schottky Groups</b>	<b>195</b>
9.1	Examples of Schottky groups . . . . .	196
9.1.1	The Seade-Verjovsky complex Schottky groups . . . . .	196
9.1.2	Nori's construction of complex Schottky groups . . . . .	199
9.2	Schottky groups: definition and basic facts . . . . .	199

9.3	On the limit set and the discontinuity region . . . . .	203
9.3.1	Quotient spaces of the region of discontinuity . . . . .	205
9.3.2	Hausdorff dimension and moduli spaces . . . . .	209
9.4	Schottky groups do not exist in even dimensions . . . . .	216
9.4.1	On the dynamics of projective transformations . . . . .	216
9.4.2	Nonrealizability of Schottky groups in $\mathrm{PSL}(2n+1, \mathbb{C})$ . . .	220
9.5	Complex kissing-Schottky groups . . . . .	223
9.6	The “zoo” in dimension 2 . . . . .	225
9.7	Remarks on the uniformisation of projective 3-folds . . . . .	228
<b>10</b>	<b>Kleinian Groups and Twistor Theory</b>	<b>231</b>
10.1	The twistor fibration . . . . .	231
10.1.1	The twistor fibration in dimension 4 . . . . .	232
10.1.2	The twistor fibration in higher dimensions . . . . .	234
10.2	The Canonical Lifting . . . . .	235
10.2.1	Lifting $\mathrm{Conf}_+(\mathbb{S}^4)$ to $\mathrm{PSL}(4, \mathbb{C})$ . . . . .	235
10.2.2	The Canonical lifting in higher dimensions . . . . .	238
10.3	Complex Kleinian groups on $\mathbb{P}_{\mathbb{C}}^3$ . . . . .	239
10.4	Kleinian groups and twistor spaces in higher dimensions . . . . .	244
10.5	Patterson-Sullivan measures on twistor spaces . . . . .	247
10.6	Some remarks . . . . .	250
	<b>Bibliography</b>	<b>253</b>
	<b>Index</b>	<b>269</b>

Complex Kleinian Groups

Cano, A.; Navarrete, J.P.; José Antonio, S.K.

2013, XX, 272 p., Hardcover

ISBN: 978-3-0348-0480-6

A product of Birkhäuser Basel