Chapter 2

Pseudomonotone or weakly continuous mappings

The basic modern approach to boundary-value problems in differential equations of the type (0.1)–(0.2) is the so-called *energy-method* technique which took the name after a-priori estimates having sometimes physical analogies as bounds of an energy.[1] This technique originated from modern theory of linear partial differential equations where, however, other approaches are efficient, too. On the abstract level, this method relies on relative weak compactness of bounded sets in reflexive Banach spaces, and either pseudomonotonicity or weak continuity of differential operators which are understood as bounded from one Banach space to another (necessarily different) Banach space. On the concrete-problem level, the main tool is a weak formulation of boundary-value problems in question, Poincaré and Hölder inequalities, and fine issues from the theory of Sobolev spaces.

2.1 Abstract theory, basic definitions, Galerkin method

Throughout this chapter (and most of the others), V will be a separable reflexive Banach space and V^* its dual space, with $\|\cdot\|$ and $\|\cdot\|_*$ denoting briefly their norms instead of $\|\cdot\|_V$ and $\|\cdot\|_{V^*}$, respectively.

Definition 2.1 (Monotonicity modes)**.** Let A be a mapping $V \to V^*$.

(i) $A : V \to V^*$ is *monotone* iff $\forall u, v \in V :\ \langle A(u) - A(v), u - v\rangle \geq 0$.

(ii) If A is monotone and $u \neq v$ implies $\langle A(u) - A(v), u - v\rangle > 0$, then A is *strictly monotone.*

[1] Cf. Example 6.7 or e.g. also (11.120) or (12.11).

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(iii) Considering an increasing function $d : \mathbb{R}^+ \to \mathbb{R}$, we say that $A : V \to V^*$ is *d-monotone* with respect to a seminorm $|\cdot|$,

$$\big\langle A(u) - A(v), u - v\big\rangle \geq \Big(d\big(|u|\big) - d\big(|v|\big)\Big)\big(|u| - |v|\big). \tag{2.1}$$

If $|\cdot|$ is the norm $\|\cdot\|$ on V, we say simply that A is d-monotone. Moreover, A is called *uniformly monotone* if

$$\big\langle A(u) - A(v), u - v\big\rangle \geq \zeta\big(\|u - v\|\big)\|u - v\| \tag{2.2}$$

for some increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$. If $\zeta(r) = \delta r$ for some $\delta > 0$, then A is called *strongly monotone.*

(iv) The mapping $A : V \to V^*$ is called *pseudomonotone* iff

$$A \text{ is bounded, and} \tag{2.3a}$$

$$\left.\begin{array}{r} u_k \rightharpoonup u \\ \limsup\limits_{k\to\infty} \big\langle A(u_k), u_k{-}u\big\rangle \leq 0 \end{array}\right\} \Rightarrow \begin{array}{l} \forall v{\in}V : \\ \big\langle A(u), u{-}v\big\rangle \leq \liminf\limits_{k\to\infty} \big\langle A(u_k), u_k{-}v\big\rangle. \end{array} \tag{2.3b}$$

Remark 2.2. Let us emphasize that the monotonicity due to Definition 2.1(i) has no direct relation with monotonicity of mappings with respect to an ordering. E.g., if $V^* = V$, the composition of monotone operators has a good sense but need not be monotone. Definition 2.1(iv) represents a suitable extent[2] of generalization of the monotonicity concept from the viewpoint of quasilinear differential equations of the type (0.2).

Definition 2.3 (Continuity modes)**.**

(i) $A : V{\to}V^*$ is *hemicontinuous* iff $\forall u, v, w{\in}V$ the function $t \mapsto \langle A(u{+}tv), w\rangle$ is continuous, i.e. A is directionally weakly continuous.

(ii) If it holds only for $v = w$, i.e. $\forall u, v{\in}V :\ t \mapsto \langle A(u{+}tv), v\rangle$ is continuous, then A is called *radially continuous.*

(iii) $A : V{\to}V^*$ is *demicontinuous* iff $\forall w{\in}V$ the functional $u \mapsto \langle A(u), w\rangle$ is continuous; i.e. A is continuous as a mapping $(V, \text{norm}) \to (V^*, \text{weak})$.

(iv) $A : V{\to}V^*$ is *weakly continuous* iff $\forall w{\in}V$ the functional $u \mapsto \langle A(u), w\rangle$ is weakly continuous; i.e. A is continuous as a mapping $(V, \text{weak}) \to (V^*, \text{weak})$.

(v) $A : V{\to}V^*$ is *totally continuous* if it is continuous as a mapping $(V, \text{weak}) \to (V^*, \text{norm})$.

Lemma 2.4. *Any pseudomonotone mapping A is demicontinuous.*

Proof. Suppose $u_k \to u$. By (2.3a), the sequence $\{A(u_k)\}_{k\in\mathbb{N}}$ is bounded in a reflexive space V^*. Then, as V is assumed separable, by the Banach Theorem 1.7 after taking a subsequence (denoted, for simplicity, by the same indices) we have

[2] In the sense that the premise of (2.3b) still can be proved under reasonable assumptions and the conclusion of (2.3b) still suffices to prove convergence of various approximate solutions.

$A(u_k) \rightharpoonup f$ for some $f \in V^*$. Then $\lim_{k\to\infty}\langle A(u_k), u_k - u\rangle = \langle f, u-u\rangle = 0$ and therefore, by (2.3b),

$$\big\langle A(u), u-v\big\rangle \le \liminf_{k\to\infty}\big\langle A(u_k), u_k - v\big\rangle = \langle f, u-v\rangle \tag{2.4}$$

for any $v \in V$. From this we get $A(u) = f$. In particular, f is determined uniquely, and thus even the whole sequence (not only the selected subsequence) must converge. □

Definition 2.5 (Coercivity). $A : V \to V^*$ is *coercive* iff $\exists\zeta : \mathbb{R}^+ \to \mathbb{R}^+$: $\lim_{s\to+\infty}\zeta(s) = +\infty$ and $\langle A(u), u\rangle \ge \zeta(\|u\|)\|u\|$. In other words, A coercive means

$$\lim_{\|u\|\to\infty}\frac{\big\langle A(u), u\big\rangle}{\|u\|} = +\infty. \tag{2.5}$$

Theorem 2.6 (Brézis [64]). *Any A pseudomonotone and coercive is surjective; this means, for any $f \in V^*$, there is at least one solution to the equation*

$$A(u) = f. \tag{2.6}$$

Proof. Let us divide the proof into four particular steps.

Step 1: (*Abstract Galerkin approximation.*) As V is supposed separable, we can take a sequence of finite-dimensional subspaces

$$\forall k \in \mathbb{N}: \quad V_k \subset V_{k+1} \subset V \quad \text{ and } \quad \bigcup_{k\in\mathbb{N}} V_k \text{ is dense in } V. \tag{2.7}$$

Then we define a *Galerkin approximation* $u_k \in V_k$ by the identity:

$$\forall v \in V_k: \qquad \big\langle A(u_k), v\big\rangle = \langle f, v\rangle. \tag{2.8}$$

Step 2: (*Existence of approximate solutions u_k.*) In other words, we seek $u_k \in V_k$ solving $I_k^*(A(u_k) - f) = 0$ where $I_k : V_k \to V$ is the canonical inclusion so that the adjoint operator $I_k^* : V^* \to V_k^*$ represents the restriction $I_k^* f = f|_{V_k}$. Besides, as V_k is finite-dimensional, we will identify $V_k \cong V_k^*$ by a linear homeomorphism $J_k : V_k \to V_k^*$ such that $\langle J_k u, u\rangle = \|u\|^2_{V_k}$, $\|J_k u\|_{V_k^*} = \|u\|_{V_k}$, and $\langle J_k u, J_k^{-1} f\rangle = \langle f, u\rangle$.[3]

As A is coercive, for ϱ sufficiently large we have

$$\|u\|_{V_k} = \varrho \quad \Longrightarrow \quad \big\langle A(u) - f, u\big\rangle \ge \big\langle A(u), u\big\rangle - \|f\|_*\|u\| > 0. \tag{2.9}$$

[3] If necessary, we can re-norm the finite-dimensional V_k to impose a Hilbert structure (i.e. V_k is then homeomorphic with a Euclidean space). Then J_k can be taken as in (3.1) below; note that, by Lemma 3.2, J_k is a homeomorphism. Also note that (2.5) restricted on V_k holds in the new, equivalent norm, as well; possibly, the function ζ in Definition 2.5 is changed by this renormalization.

Suppose, for a moment, that $I_k^*A(u) \neq I_k^*f$ for any $u \in V_k$ with $\|u\|_{V_k} \le \varrho$. Then the mapping

$$u \mapsto \varrho \frac{J_k^{-1} I_k^*\big(f - A(u)\big)}{\big\|I_k^*(f - A(u))\big\|_{V_k^*}} \tag{2.10}$$

maps the convex compact set $\{u \in V_k;\ \|u\| \le \varrho\}$ into itself because $\|J_k^{-1}\| = 1$; note that $\|J_k^{-1} f\|_{V_k} = \|f\|_{V_k^*}$. Also, by Lemma 2.4, the mapping $u \mapsto \langle A(u), v\rangle : V_k \to \mathbb{R}$ is continuous for any v so that also $u \mapsto I_k^*A(u) : V_k \to V_k^*$ is continuous. By the Brouwer fixed-point Theorem 1.10, the mapping (2.10) has a fixed point u, this means

$$u = \varrho \frac{J_k^{-1} I_k^*\big(f - A(u)\big)}{\big\|I_k^*(f - A(u))\big\|_{V_k^*}}. \tag{2.11}$$

As $\|J_k^{-1} f\|_{V_k} = \|f\|_{V_k^*}$, (2.11) implies $\|u\|_{V_k} = \varrho$. Testing (2.11) by $J_k u \|I_k^*(f - A(u))\|_{V_k^*}$, one gets

$$\begin{aligned} \varrho^2 \big\|I_k^*(f - A(u))\big\|_{V_k^*} &= \langle J_k u, u\rangle \big\|I_k^*(f - A(u))\big\|_{V_k^*} \\ &= \varrho\big\langle J_k u, J_k^{-1} I_k^*(f - A(u))\big\rangle = \varrho\big\langle I_k^*(f - A(u)), u\big\rangle \\ &= \varrho\big\langle f - A(u), I_k u\big\rangle = \varrho\big\langle f - A(u), u\big\rangle \end{aligned} \tag{2.12}$$

which yields $\langle A(u) - f, u\rangle = -\varrho\|I_k^*(A(u) - f)\|_{V_k^*} \le 0$, a contradiction with (2.9).

Step 3: (*An a-priori estimate.*) Moreover, putting $v := u_k$ into (2.8), we can estimate[4]

$$\zeta\big(\|u_k\|\big)\|u_k\| \le \big\langle A(u_k), u_k\big\rangle = \langle f, u_k\rangle \le \|f\|_*\|u_k\| \tag{2.13}$$

with a suitable increasing function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{\xi\to\infty} \zeta(\xi) = +\infty$, cf. the coercivity (2.5) of A. Then $\|u_k\| \le \zeta^{-1}(\|f\|_*) < +\infty$, so that u_k is bounded in V independently of k. This holds even for any solution to (2.8).

Step 4: (*Limit passage.*) Since $\{u_k\}_{k\in\mathbb{N}}$ is bounded and V is reflexive and separable, by the Banach Theorem 1.7 together with Proposition 1.3, there is a subsequence and $u \in V$ such that $u_k \rightharpoonup u$. From (2.8), we have also

$$\big\langle A(u_k), v_m - u_k\big\rangle = \langle f, v_m - u_k\rangle \tag{2.14}$$

for any $k \ge m$ and $v_m \in V_m \subset V_k$. By density of $\bigcup_{k\in\mathbb{N}} V_k$ in V, we can take $v_k \to u$. Then, by (2.14), one gets

$$\begin{aligned} \limsup_{k\to\infty} \langle A(u_k), u_k - u\rangle &= \limsup_{k\to\infty} \Big(\langle A(u_k), u_k - v_k\rangle + \langle A(u_k), v_k - u\rangle\Big) \\ &\le \lim_{k\to\infty} \Big(\langle f, u_k - v_k\rangle + \|A(u_k)\|_*\|v_k - u\|\Big) = 0. \end{aligned} \tag{2.15}$$

[4]Here we forget possible renormalization of the finite-dimensional subspaces V_k and come back to the original norm on V.

Note that the sequence $\{u_k\}_{k\in\mathbb{N}}$ has been proved bounded so $\{\|A(u_k)\|_*\}_{k\in\mathbb{N}}$ is bounded by (2.3a) and that, in fact, even an equality holds in (2.15) and "limsup" is a limit. By pseudomonotonicity (2.3b) of A, we get

$$\forall v\in V: \qquad \liminf_{k\to\infty} \big\langle A(u_k), u_k - v\big\rangle \ge \big\langle A(u), u-v\big\rangle. \tag{2.16}$$

On the other hand, from (2.14) we also have

$$\forall v\in \bigcup_{m\in\mathbb{N}} V_m: \qquad \liminf_{k\to\infty} \big\langle A(u_k), u_k - v\big\rangle = \lim_{k\to\infty} \langle f, u_k - v\rangle = \langle f, u-v\rangle. \tag{2.17}$$

Combining (2.16) and (2.17), one gets $\langle A(u), u-v\rangle \le \langle f, u-v\rangle$ for any v ranging over a dense subset of V, namely $\bigcup_{m\in\mathbb{N}} V_m$, which shows that $A(u) = f$. $\square$

Remark 2.7 (Nonconstructivity)**.** Let us emphasize three aspects of high nonconstructivity of the above proof:
✓ usage of Brouwer's fixed-point theorem,
✓ a contradiction argument, and
✓ a selection of a convergent subsequence by a compactness argument.

Remark 2.8 (Necessity of approximation)**.** The approximation (Step 1) is necessary in the above proof, otherwise one would have to think about usage of Schauder's type fixed point Theorem 1.9 instead of the Brouwer one. This would need additional assumptions about weak continuity of A and the Hilbert structure of V, cf. Exercise 2.56, which is not fitted with general quasilinear differential equations, cf. Sect. 2.5 where omitting the approximation would also hurt for not allowing for a weaker concept of A as $V \to Z^*$ with $Z \subsetneqq V$.

2.2 Some facts about pseudomonotone mappings

Brézis' Theorem 2.6 showed the importance of the class of pseudomonotone mappings. It is therefore worth knowing some more specific cases leading to such mappings.

Lemma 2.9 (Brézis [64])**.** *Radially continuous monotone mappings satisfy* (2.3b)*. In particular, bounded radially continuous monotone mappings are pseudomonotone.*

Proof. Take $u_k \rightharpoonup u$ and assume $\limsup_{k\to\infty}\langle A(u_k), u_k - u\rangle \le 0$. Since A is monotone, $\langle A(u_k), u_k - u\rangle \ge \langle A(u), u_k - u\rangle \to 0$ so that $\liminf_{k\to\infty}\langle A(u_k), u_k - u\rangle \ge 0$ and therefore altogether

$$\lim_{k\to\infty} \langle A(u_k), u_k - u\rangle = 0. \tag{2.18}$$

We take $u_\varepsilon = (1-\varepsilon)u + \varepsilon v$, $\varepsilon > 0$, and write the monotonicity condition of A between u_k and u_ε:

$$0 \le \langle A(u_k) - A(u_\varepsilon), u_k - u_\varepsilon\rangle = \langle A(u_k) - A(u_\varepsilon), \varepsilon(u-v) + u_k - u\rangle \tag{2.19}$$

and, by a simple algebraic manipulation, we obtain

$$\varepsilon\langle A(u_k), u-v\rangle \geq \langle A(u_\varepsilon), u_k-u\rangle - \langle A(u_k), u_k-u\rangle + \varepsilon\langle A(u_\varepsilon), u-v\rangle. \tag{2.20}$$

Therefore, fixing $\varepsilon > 0$ and passing with k to infinity, by (2.18) we get

$$\varepsilon \liminf_{k\to\infty}\langle A(u_k), u-v\rangle \geq \varepsilon\langle A(u_\varepsilon), u-v\rangle. \tag{2.21}$$

Then divide it by ε, which gives $\liminf_{k\to\infty}\langle A(u_k), u-v\rangle \geq \langle A(u_\varepsilon), u-v\rangle = \langle A(u+\varepsilon(v-u)), u-v\rangle$. Passing with $\varepsilon \to 0$ and using the radial continuity of A, we eventually get

$$\liminf_{k\to\infty}\langle A(u_k), u-v\rangle \geq \lim_{\varepsilon\searrow 0}\langle A(u+\varepsilon(v-u)), u-v\rangle = \langle A(u), u-v\rangle. \tag{2.22}$$

The pseudomonotonicity of A then follows by using (2.22) with (2.18):

$$\liminf_{k\to\infty}\langle A(u_k), u_k-v\rangle = \lim_{k\to\infty}\langle A(u_k), u_k-u\rangle + \liminf_{k\to\infty}\langle A(u_k), u-v\rangle \geq \langle A(u), u-v\rangle.$$

$\square$

Lemma 2.10. *Any bounded demicontinuous mapping $A : V \to V^*$ satisfying*

$$\Big(u_k \rightharpoonup u \quad \& \quad \limsup_{k\to\infty}\big\langle A(u_k)-A(u), u_k-u\big\rangle \leq 0\Big) \quad \Rightarrow \quad u_k \to u \tag{2.23}$$

is pseudomonotone.

Proof. The premise of (2.3b), i.e. $u_k \rightharpoonup u$ and $\limsup_{k\to\infty}\langle A(u_k), u_k-u\rangle \leq 0$, yields

$$\limsup_{k\to\infty}\big\langle A(u_k)-A(u), u_k-u\big\rangle = \limsup_{k\to\infty}\big\langle A(u_k), u_k-u\big\rangle - \lim_{k\to\infty}\big\langle A(u), u_k-u\big\rangle \leq 0,$$

so that by (2.23) we have $u_k \to u$, and by demicontinuity of A also $A(u_k) \rightharpoonup A(u)$, and eventually $\lim_{k\to\infty}\langle A(u_k), u_k-v\rangle = \langle A(u), u-v\rangle$ for any $v \in V$. $\square$

Lemma 2.11.

(i) *The sum of any pseudomonotone mappings remains pseudomonotone, i.e. A_1 and A_2 pseudomonotone implies $u \mapsto A_1(u) + A_2(u)$ pseudomonotone.*

(ii) *A shift of a pseudomonotone mapping remains pseudomonotone, i.e. A pseudomonotone implies $u \mapsto A(u+w)$ pseudomonotone for any $w \in V$.*

Proof. The boundedness (2.3a) of $A_1 + A_2$ and $A(\cdot + w)$ is obvious hence we need to show only (2.3b).

To prove (i), let A_1, A_2 be pseudomonotone, $u_k \rightharpoonup u$ and $\limsup_{k\to\infty}\langle [A_1 + A_2](u_k), u_k - u\rangle \leq 0$. Let us verify that

$$\limsup_{k\to\infty}\big\langle A_1(u_k), u_k-u\big\rangle \leq 0 \quad \text{and} \quad \limsup_{k\to\infty}\big\langle A_2(u_k), u_k-u\big\rangle \leq 0. \tag{2.24}$$

Suppose, for a moment, that $\limsup_{k\to\infty}\langle A_2(u_k),u_k-u\rangle=\varepsilon>0$. Taking a subsequence, we can suppose that $\lim_{k\to\infty}\langle A_2(u_k),u_k-u\rangle=\varepsilon>0$ and therefore

$$\limsup_{k\to\infty}\big\langle A_1(u_k),u_k-u\big\rangle\le-\varepsilon<0. \tag{2.25}$$

As A_1 is pseudomonotone, we get $\liminf_{k\to\infty}\langle A_1(u_k),u_k-v\rangle\ge\langle A_1(u),u-v\rangle$ for any $v\in V$. In particular, for $v=u$ we get $\liminf_{k\to\infty}\langle A_1(u_k),u_k-u\rangle\ge0$, which contradicts (2.25). Thus (2.24) holds. By the pseudomonotonicity both for A_1 and for A_2, we get

$$\begin{aligned}\liminf_{k\to\infty}\big\langle[A_1{+}A_2](u_k),u_k{-}v\big\rangle&\ge\liminf_{k\to\infty}\big\langle A_1(u_k),u_k{-}v\big\rangle+\liminf_{k\to\infty}\big\langle A_2(u_k),u_k{-}v\big\rangle\\&\ge\big\langle A_1(u),u{-}v\big\rangle+\big\langle A_2(u),u{-}v\big\rangle\ge\big\langle[A_1{+}A_2](u),u{-}v\big\rangle.\end{aligned}$$

As to (ii), let $u_k\rightharpoonup u$ and $\limsup_{k\to0}\langle A(u_k{+}w),u_k{-}u\rangle\le0$. Then obviously $u_k{+}w\rightharpoonup u{+}w$ and $\limsup_{k\to0}\langle A(u_k{+}w),(u_k{+}w)-(u{+}w)\rangle\le0$. If A is pseudomonotone, then $\liminf_{k\to0}\langle A(u_k{+}w),u_k{-}v\rangle=\liminf_{k\to0}\langle A(u_k{+}w),(u_k{+}w)-(v{+}w)\rangle\ge\langle A(u{+}w),(u{+}w)-(v{+}w)\rangle=\langle A(u{+}w),u{-}v\rangle$, hence $A(\cdot+w)$ is pseudomonotone. □

Corollary 2.12. *A perturbation of a pseudomonotone mapping by a totally continuous mapping is pseudomonotone.*

Proof. Realize that any totally continuous mapping is pseudomonotone; indeed, it is bounded (which can be easily proved by contradiction) and, if $u_k\rightharpoonup u$, then $A(u_k)\to A(u)$ and thus $\lim_{k\to\infty}\langle A(u_k),u_k-v\rangle=\langle A(u),u-v\rangle$ so that (2.3b) is trivial. □

2.3 Equations with monotone mappings

Monotone mappings (with boundedness and radial continuity properties) are a special class of pseudomonotone mappings, cf. Lemma 2.9, and, as such, they allow special treatment with a bit stronger results than a general "pseudomonotone theory" can yield, cf. Theorem 2.14 vs. Proposition 2.17.

Lemma 2.13 (Minty's trick [286]). *Let $A:V\to V^*$ be radially continuous and let $\langle f{-}A(v),u{-}v\rangle\ge0$ for any $v{\in}V$. Then $f=A(u)$.*

Proof. Replace v with $u+\varepsilon w$ with $w\in V$ arbitrary. This gives

$$\big\langle f-A(u{+}\varepsilon w),-\varepsilon w\big\rangle\ge0. \tag{2.26}$$

Divide it by $\varepsilon>0$ and pass to the limit with ε by using radial continuity of A:

$$0\ge\big\langle f-A(u{+}\varepsilon w),w\big\rangle\to\big\langle f-A(u),w\big\rangle. \tag{2.27}$$

As w is arbitrary, one gets $A(u)=f$. □

Theorem 2.14. *Let A be bounded,*[5] *radially continuous, monotone, coercive. Then:*

(i) *A is surjective; this means, for any $f \in V^*$, there is u solving* (2.6). *Moreover, the set of solutions to* (2.6) *is closed and convex.*

(ii) *If, in addition, A is strictly monotone, then $A^{-1} : V^* \to V$ does exist, is strictly monotone, bounded, and demicontinuous. If A is also d-monotone and V uniformly convex, then $A^{-1} : V^* \to V$ is continuous.*

(iii) *If, in addition, A is uniformly (resp. strongly) monotone, then $A^{-1} : V^* \to V$ is uniformly (resp. Lipschitz) continuous.*

Proof. By Lemma 2.9, A is pseudomonotone. As A is supposed also coercive, the surjectivity of A follows from Theorem 2.6. By Lemma 2.4, A is demicontinuous, hence the set of solutions to (2.6) is closed in the norm topology of V. Hence, to prove convexity of this set, it suffices to show that $u = \frac{1}{2}u_1 + \frac{1}{2}u_2$ solves (2.6) provided u_1 and u_2 do so, cf. Proposition 1.6. Thus we have

$$\begin{aligned}\big\langle f - A(v), u - v\big\rangle &= \frac{1}{2}\big\langle f - A(v), u_1 - v\big\rangle + \frac{1}{2}\big\langle f - A(v), u_2 - v\big\rangle \\ &= \frac{1}{2}\big\langle A(u_1) - A(v), u_1 - v\big\rangle + \frac{1}{2}\big\langle A(u_2) - A(v), u_2 - v\big\rangle \geq 0 \qquad (2.28)\end{aligned}$$

because of $A(u_1) = f = A(u_2)$ and of monotonicity of A. Then, by Lemma 2.13, one gets $A(u) = f$.

Let us go on to (ii). If A is strictly monotone, we have $\langle A(u_1) - A(u_2), u_1 - u_2\rangle = \langle f - f, u_1 - u_2\rangle = 0$ which is possible only if $u_1 = u_2$. In other words, the equation (2.6) has a unique solution so that the inverse A^{-1} does exist.

The mapping A^{-1} is strictly monotone: For $f_1, f_2 \in V^*$, $f_1 \neq f_2$, put $u_i = A^{-1}(f_i)$. Then also $u_1 \neq u_2$. As A is strictly monotone, one has

$$\big\langle f_1 - f_2, A^{-1}(f_1) - A^{-1}(f_2)\big\rangle = \big\langle A(u_1) - A(u_2), u_1 - u_2\big\rangle > 0. \qquad (2.29)$$

The mapping A^{-1} is bounded: by the coercivity of A, there is $\zeta : \mathbb{R}^+ \to \mathbb{R}$ such that $\lim_{\xi\to\infty} \zeta(\xi) = +\infty$ and $\langle A(u), u\rangle \geq \|u\|\zeta(\|u\|)$. Therefore

$$\zeta\big(\|u\|\big)\|u\| \leq \big\langle A(u), u\big\rangle = \big\langle f, u\big\rangle \leq \|f\|_*\|u\| \qquad (2.30)$$

so that $\zeta(\|A^{-1}(f)\|) = \zeta(\|u\|) \leq \|f\|_*$. Thus A^{-1} maps bounded sets in V^* into bounded sets in V.

The mapping A^{-1} is demicontinuous, i.e. (norm,weak)-continuous: take $f_k \to f$ in V^*. As A^{-1} was shown to be bounded, $\{A^{-1}(f_k)\}_{k\in\mathbb{N}}$ is bounded and (possibly up to a subsequence) $u_k = A^{-1}(f_k) \rightharpoonup u$ in V by Banach's Theorem 1.7. It remains to show $A(u) = f$. By the monotonicity of A, for any $v \in V$:

$$0 \leq \big\langle A(u_k) - A(v), u_k - v\big\rangle = \big\langle f_k - A(v), u_k - v\big\rangle. \qquad (2.31)$$

[5] If proved directly, i.e. without passing through pseudomonotone mappings, the boundedness assumption can be omitted; cf. Theorem 2.18 below.

Therefore, by (norm×weak)-continuity of the duality pairing, passing to the limit with $k \to \infty$ yields

$$0 \le \lim_{k\to\infty} \big\langle f_k - A(v), u_k - v\big\rangle = \big\langle f - A(v), u - v\big\rangle. \tag{2.32}$$

Then we apply again the Minty-trick Lemma 2.13, which gives $A(u) = f$. Thus even the whole sequence $\{u_k\}_{k\in\mathbb{N}}$ converges weakly.

If A is d-monotone, we can refine (2.31) used for $v := u$ as follows:

$$\begin{aligned}\big(d(\|u_k\|) - d(\|u\|)\big)(\|u_k\| - \|u\|) &\le \big\langle A(u_k) - A(u), u_k - u\big\rangle \\ &= \big\langle f_k - A(u), u_k - u\big\rangle \to \big\langle f - A(u), u - u\big\rangle = 0,\end{aligned} \tag{2.33}$$

which gives $\|u_k\| \to \|u\|$ because $d : \mathbb{R} \to \mathbb{R}$ is increasing. Hence $u_k \to u$ by Theorem 1.2. In other words, A^{-1} is continuous.

The point (iii): By (2.2) one has for any $A(u_1) = f_1$ and $A(u_2) = f_2$ the estimate

$$\begin{aligned}\zeta\big(\|u_1 - u_2\|\big)\|u_1 - u_2\| &\le \big\langle A(u_1) - A(u_2), u_1 - u_2\big\rangle \\ &= \langle f_1 - f_2, u_1 - u_2\rangle \le \|f_1 - f_2\|_*\|u_1 - u_2\|\end{aligned} \tag{2.34}$$

so that $\zeta(\|u_1 - u_2\|) \le \|f_1 - f_2\|_*$. By the assumed properties of ζ, the inverse mapping A^{-1} is uniformly continuous. The case of strong monotonicity is obvious. $\square$

Lemma 2.15. *Any monotone mapping $A : V \to V^*$ is locally bounded in the sense:*

$$\forall u\in V\ \exists\varepsilon > 0\ \exists M\in\mathbb{R}^+\ \forall v\in V : \quad \|v-u\| \le \varepsilon \quad \Rightarrow \quad \|A(v)\|_* \le M. \tag{2.35}$$

Proof. Suppose the contrary, i.e. (2.35) does not hold at some $u \in V$. Without loss of generality, assume $u = 0$. This means that there is a sequence $\{v_k\}$, $v_k \to 0$, such that $\|A(v_k)\|_* \to \infty$. Putting $c_k := 1 + \|A(v_k)\|_*\|v_k\|$, we can estimate by monotonicity of A that

$$\begin{aligned}\Big\langle \frac{A(v_k)}{c_k}, v\Big\rangle &\le \frac{\big\langle A(v_k), v_k\big\rangle + \big\langle A(v), v - v_k\big\rangle}{c_k} \\ &\le 1 + \big\|A(v)\big\|_*\big(\|v\| + \|v_k\|\big) \to 1 + \|A(v)\|_*\|v\|.\end{aligned} \tag{2.36}$$

Replacing v by $-v$, we can conclude that $\limsup_{k\to\infty} |\langle c_k^{-1}A(v_k), v\rangle| < +\infty$ for any $v \in V$. By Banach-Steinhaus' Theorem 1.1, $c_k^{-1}\|A(v_k)\|_* \le M$. This means $\|A(v_k)\|_* \le Mc_k = M(1+\|A(v_k)\|_*\|v_k\|)$, and then also $\|A(v_k)\|_* \le M/(1-M\|v_k\|) \to M$, which contradicts the fact that $\|A(v_k)\|_* \to \infty$. $\square$

Lemma 2.16. *Radially continuous monotone mappings are also demicontinuous.*

Proof. Take a sequence $\{u_k\}_{k\in\mathbb{N}}$ convergent to some $u \in V$. By Lemma 2.15, $\{A(u_k)\}_{k\in\mathbb{N}}$ is bounded in V^* and, by Banach Theorem 1.7, we can select a subsequence $\{A(u_{k_l})\}_{l\in\mathbb{N}}$ converging weakly to some $f \in V^*$. Then, by the monotonicity of A, $0 \le \lim_{l\to\infty}\langle A(u_{k_l}) - A(v), u_{k_l} - v\rangle = \langle f - A(v), u - v\rangle$. As v is arbitrary and we assume radial continuity of A, the Minty-trick Lemma 2.13 yields $f = A(u)$. As f is thus determined uniquely, even the whole sequence $\{A(u_k)\}_{k\in\mathbb{N}}$ must converge to it weakly. □

Proposition 2.17. *Let $A = A_1 + A_2 : V \to V^*$ be coercive, and A_1 be radially continuous and monotone and A_2 be totally continuous. Then A is surjective.*

Proof. As in the proof of Brézis' Theorem 2.6, consider $u_k \in V_k$ the Galerkin approximations (2.8), i.e. here

$$\big\langle A_1(u_k) + A_2(u_k), v\big\rangle = \langle f, v\rangle \qquad \forall v\in V_k, \tag{2.37}$$

and the a-priori estimate (2.13), and choose a weakly convergent subsequence $\{u_{k_i}\}_{i\in\mathbb{N}}$ with a limit $u \in V$. Use monotonicity of A_1 to write

$$0 \le \big\langle A_1(v_l) - A_1(u_{k_i}), v_l - u_{k_i}\big\rangle = \big\langle A_1(v_l), v_l - u_{k_i}\big\rangle + \big\langle A_2(u_{k_i}) - f, v_l - u_{k_i}\big\rangle \tag{2.38}$$

for any $v_l \in V_l$ with $l \le k$. Passing to the limit with $i \to \infty$, it gives

$$0 \le \big\langle A_1(v_l), v_l - u\big\rangle + \big\langle A_2(u) - f, v_l - u\big\rangle. \tag{2.39}$$

Then, by density of $\bigcup_{k\in\mathbb{N}} V_k$ in V, consider $v_l \to v$ for $v\in V$ arbitrary, use demicontinuity of A_1 (cf. Lemma 2.16), and pass to the limit with $l \to \infty$ to get:

$$0 \le \big\langle A_1(v), v - u\big\rangle + \big\langle A_2(u) - f, v - u\big\rangle. \tag{2.40}$$

Finally, replace v by $u + \varepsilon w$ with $w \in V$ arbitrary and use Minty's trick as in (2.26)–(2.27) to show that $A_1(u) + A_2(u) = f$. □

In principle, if A_1 is also bounded, one could use Lemma 2.9 and Corollary 2.12 to see that A from Proposition 2.17 is surjective; realize that A_2, being totally continuous, is certainly bounded. The above direct proof allowed us to avoid the boundedness assumption of A_1. In particular, for $A_2 = 0$, one thus obtains the celebrated assertion:

Theorem 2.18 (Browder [73] and Minty [286])**.** *Any monotone, radially continuous, and coercive $A : V \to V^*$ is surjective.*

As a very special case, one gets another celebrated result:

Theorem 2.19 (Lax and Milgram [253][6])**.** *Let V be a Hilbert space, $A : V \to V^*$ be a linear continuous operator which is positive definite in the sense $\langle Av, v\rangle \ge \varepsilon\|v\|^2$ for some $\varepsilon > 0$. Then A has a bounded inverse.*

[6] A usual formulation uses a bounded, positive definite, bilinear form $a : V \times V \to \mathbb{R}$. This form then determines $A : V \to V^*$ through the identity $\langle Au, v\rangle = a(u, v)$.

Sometimes, the following modification of Proposition 2.17 can be advantageously applied, obtaining also the strong convergence of Galerkin's approximate solutions.

Proposition 2.20. *Let $A = A_1 + A_2 : V \to V^*$ be coercive, and A_1 be monotone radially continuous and satisfy (2.23), and A_2 be demicontinuous and compact.*[7] *Then A is surjective.*

Proof. We have the Galerkin identity (2.37) and a subsequence $u_k \rightharpoonup u$, and write

$$\begin{aligned}\big\langle A_1(u_k)-A_1(u), u_k-u\big\rangle &= \big\langle A_1(u_k)-A_1(u), v_k-u\big\rangle + \big\langle A_1(u_k)-A_1(u), u_k-v_k\big\rangle \\ &= \big\langle A_1(u_k)-A_1(u), v_k-u\big\rangle + \big\langle f-A_2(u_k)-A_1(u), u_k-v_k\big\rangle =: I_k^{(1)} + I_k^{(2)}. \quad (2.41)\end{aligned}$$

As A_1 is monotone, for any $\varepsilon > 0$, then

$$\begin{aligned}\big\|A_1(u_k)\big\|_* &= \frac{1}{\varepsilon} \sup_{\|v\|\le\varepsilon} \big\langle A_1(u_k), v\big\rangle \le \frac{1}{\varepsilon} \sup_{\|v\|\le\varepsilon} \Big(\big\langle A_1(u_k), v\big\rangle \\ &+ \big\langle A_1(u_k)-A_1(v), u_k-v\big\rangle\Big) = \frac{1}{\varepsilon} \sup_{\|v\|\le\varepsilon} \Big(\big\langle A_1(u_k), u_k\big\rangle + \big\langle A_1(v), v-u_k\big\rangle\Big). \quad (2.42)\end{aligned}$$

Now we use that $\{\langle A_1(u_k), u_k\rangle\}_{k\in\mathbb{N}}$ is bounded because $\langle A_1(u_k), u_k\rangle = \langle f - A_2(u_k), u_k\rangle$ and the compact mapping A_2 is certainly bounded, and also $\{\langle A_1(v), v - u_k\rangle; \|v\| \le \varepsilon\}$ is bounded if $\varepsilon > 0$ is small enough because A_1 is locally bounded around the origin due to Lemma 2.15. Thus (2.42) shows that $\|A_1(u_k) - A_1(u)\|_*$ is bounded, and, choosing $v_k \to u$ in V, we obtain $\lim_{k\to\infty} I_k^{(1)} = 0$ in (2.41).

Taking a subsequence such that also $A_2(u_k)$ converges to some $\chi \in V^*$ (as we can because A_2 is compact), we get $I_k^{(2)} = \langle f - A_2(u_k) - A_1(u), u_k - v_k\rangle \to \langle f - \chi - A_1(u), u - u\rangle = 0$. As this limit is determined uniquely, even the whole sequence $\{I_k^{(2)}\}_{k\in\mathbb{N}}$ converges to 0.

Using (2.41), by (2.23) we get $u_k \to u$. By Lemma 2.16, $A_1(u_k) \rightharpoonup A_1(u)$. By demicontinuity of A_2, also $A_2(u_k) \rightharpoonup A_2(u)$. It allows us to pass to the limit in (2.37), obtaining $\langle A_1(u) + A_2(u) - f, v\rangle = 0$ for any $v \in \bigcup_{k\in\mathbb{N}} V_k$, hence $A(u) = f$. □

Remark 2.21 (d-monotone A on a uniformly convex V)**.** Any d-monotone $A : V \to V^*$ satisfies (2.23) if V is uniformly convex. Indeed, the premise of (2.23) with $\langle A(u_k) - A(u), u_k - u\rangle \ge (d(\|u_k\|) - d(\|u\|))(\|u_k\| - \|u\|)$, cf. (2.1), yields $\|u_k\| \to \|u\|$. Then, by uniform convexity of V and Theorem 1.2, we get immediately $u_k \to u$.

The nonconstructivity of Brézis' Theorem 2.6 pointed out in Remark 2.7 can be avoided in special situations by using Banach's fixed-point Theorem 1.12 for the iterative process

$$u_k = T_\varepsilon(u_{k-1}) := u_{k-1} - \varepsilon J^{-1}\big(A(u_{k-1}) - f\big)\ , \quad k \in \mathbb{N}, \quad u_0 \in V, \qquad (2.43)$$

[7] In fact, any demicontinuous and compact A_2 is automatically continuous.

if V is a Hilbert space and the linear operator $J : V \to V^*$ is defined by $\langle Ju, v\rangle := (u, v)$ with $(\cdot,\cdot)$ denoting here the inner product in V, cf. Remark 3.10. For weakening of the assumptions by further (constructive) approximation see Example 2.95.

Proposition 2.22 (BANACH FIXED-POINT TECHNIQUE)**.** *Let V be a Hilbert space, $A : V \to V^*$ be strongly monotone, i.e. $\zeta(r) = \delta r$ from (2.2) with $\delta > 0$, and also A be Lipschitz continuous, i.e. $\|A(u) - A(v)\|_* \le \ell\|u - v\|$. Then the nonlinear mapping T_ε defined by (2.43) is contractive for any $\varepsilon > 0$ satisfying*

$$\varepsilon < 2\delta/\ell^2 \tag{2.44}$$

and the fixed point of T_ε, i.e. $T_\varepsilon(u) = u$, does exist and obviously solves $A(u) = f$.

Proof. It holds that[8] $\langle f, J^{-1}f\rangle = \|f\|_*^2$, so that one has

$$\begin{aligned}\|T_\varepsilon(u) - T_\varepsilon(v)\|^2 &= \big\langle J(u-v) - \varepsilon(A(u) - A(v)), u - v - \varepsilon J^{-1}(A(u) - A(v))\big\rangle\\ &= \|u-v\|^2 - 2\varepsilon\big(u-v, J^{-1}(A(u) - A(v))\big) + \varepsilon^2\|J^{-1}A(u) - J^{-1}A(v)\|^2\\ &= \|u-v\|^2 - 2\varepsilon\langle A(u) - A(v), u-v\rangle + \varepsilon^2\|A(u) - A(v)\|_*^2\\ &\le \|u-v\|^2 - 2\varepsilon\delta\|u-v\|^2 + \varepsilon^2\ell^2\|u-v\|^2.\end{aligned}$$

The condition (2.44) just guarantees the Lipschitz constant $\sqrt{1 - 2\varepsilon\delta + \varepsilon^2\ell^2}$ of T_ε to be less than 1. $\square$

2.4 Quasilinear elliptic equations

We will illustrate the above abstract theory on boundary-value problems for the quasilinear 2nd-order partial differential equation

$$-\mathrm{div}\big(a(x, u, \nabla u)\big) + c(x, u, \nabla u) = g \tag{2.45}$$

considered on a bounded connected Lipschitz domain $\Omega \subset \mathbb{R}^n$. Here $a : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $c : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$; for more qualification see (2.54) and (2.55a,c) below. Recall that $\nabla u := \big(\frac{\partial}{\partial x_1}u, \dots, \frac{\partial}{\partial x_n}u\big)$ denotes the gradient of u. More in detail, (2.45) means

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i\big(x, u(x), \nabla u(x)\big) + c\big(x, u(x), \nabla u(x)\big) = g(x) \tag{2.46}$$

for $x \in \Omega$ but we will rather use the abbreviated form (2.45) in what follows. For some systems of the 2nd-order equations see Sect. 6.1 below while higher-order equations will be briefly mentioned in Sect. 2.4.4. Besides, we will confine ourselves to data with polynomial-growth; $p \in (1, +\infty)$ will denote the growth of the leading

[8]Realize that, for $v = J^{-1}f$, one has $\langle f, J^{-1}f\rangle = \langle f, v\rangle = \langle Jv, v\rangle = (v, v) = \|v\|^2 = \|f\|_*^2$.

nonlinearity $a(x, u, \cdot)$ which essentially determines the setting and the other data qualification. Also, $a(x, u, \cdot)$ will be assumed to behave monotonically, cf. (2.65), which is related to the adjective *elliptic*. For the linear case $a(x, r, s) = \mathbb{A}s$, the monotonicity (2.65) and coercivity (2.92a) below implies the matrix $\mathbb{A}$ is positive definite, which is what is conventionally called "elliptic", contrary to $\mathbb{A}$ indefinite (resp. semidefinite) which is addressed as *hyperbolic* (resp. *parabolic*).

Convention 2.23 (Omitting x-variable)**.** For brevity, we will often write $a(x, u, \nabla u)$ instead of $a(x, u(x), \nabla u(x))$ (as we already did in (2.45)) or sometimes even $a(u, \nabla u)$ if the dependence on x is automatic; hence, in fact, $\mathscr{N}_a(u, \nabla u) = a(u, \nabla u)$. Thus, e.g. $\int_\Omega c(u, \nabla u)v \,\mathrm{d}x$ will mean $\int_\Omega c(x, u(x), \nabla u(x))v(x) \,\mathrm{d}x$.

2.4.1 Boundary-value problems for 2nd-order equations

The equation (2.45) may admit very many solutions, which indicates some missing requirements. This is usually overcome by a boundary condition to be prescribed for the solution on the boundary $\Gamma := \partial\Omega$ of the domain Ω.

One option is to prescribe simply the trace $u|_\Gamma$ of u on the boundary, i.e.

$$u|_\Gamma = u_{\mathrm{D}} \quad \text{on } \Gamma \tag{2.47}$$

with u_{D} a fixed function on Γ. This condition is referred to as a *Dirichlet boundary condition*.

Having in mind the equation (2.45), the alternative natural possibility is to prescribe a local equation for the "boundary flux" $\nu \cdot a$, i.e.

$$\nu \cdot a(x, u, \nabla u) + b(x, u) = h \quad \text{on } \Gamma \tag{2.48}$$

where $\nu = (\nu_1, \ldots, \nu_n)$ denotes the unit outward normal to Γ and $h : \Gamma \to \mathbb{R}$ and $b : \Gamma \times \mathbb{R} \to \mathbb{R}$ are given functions qualified later. More in detail, (2.48) means $\sum_{i=1}^n \nu_i(x)\, a_i(x, u(x), \nabla u(x)) + b(x, u(x)) = h(x)$ for $x \in \Gamma$. This condition is referred to as a (nonlinear) *Newton boundary condition* or sometimes also a *Robin condition*. If $b = 0$, it is called a *Neumann boundary condition*.

One can still think about a combination of (2.47) and (2.48) on various parts of Γ. For this, let us divide (up to a zero-measure set) the boundary Γ on two disjoint open parts Γ_{D} and Γ_{N} such that $\mathrm{meas}_{n-1}\big(\Gamma \setminus (\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}})\big) = 0$, and then consider so-called *mixed boundary conditions*

$$u|_\Gamma = u_{\mathrm{D}} \qquad \text{on } \Gamma_{\mathrm{D}}, \tag{2.49a}$$

$$\nu \cdot a(x, u, \nabla u) + b(x, u) = h \qquad \text{on } \Gamma_{\mathrm{N}}. \tag{2.49b}$$

As either Γ_{D} or Γ_{N} may be empty, (2.49) covers also (2.48) and (2.47), respectively.

Completing the equation (2.45) with the boundary conditions (2.47) (resp. (2.48) or (2.49)), we will speak about a Dirichlet (resp. Newton or mixed) *boundary-value problem*. One can have an idea to seek a so-called *classical solution* u of it, i.e. such $u \in C^2(\bar{\Omega})$ satisfying the involved equalities everywhere on

Ω and Γ. This requires, however, very strong data qualifications both for a, b, and c and for Ω itself. Therefore, modern theories rely on a natural generalization of the notion of the solution. In this context, ultimate requirements on every sensible definition are[9]:

1. *Consistency*: Any classical solution to the boundary-value problem in question is the generalized solution.
2. *Selectivity*: If all data are smooth and if the generalized solution belongs to $C^2(\bar{\Omega})$, then it is the classical solution. Moreover, speaking a bit vaguely, in qualified cases the generalized solution is unique.

2.4.2 Weak formulation

Here, the generalized solution will arise from a so-called weak formulation of the boundary-value problem, which is the most frequently used concept and which just fits to the pseudomonotonicity approach. Later, we will present some other concepts, too.

For the full generality, we will treat the mixed boundary conditions (2.49). The *weak formulation* of (2.45) with (2.49) arises as follows:

Step 1: Multiply the differential equation, i.e. here (2.45), by a test function v.

Step 2: Integrate it over Ω.

Step 3: Use Green's formula (1.54), here with $z = a(x, u, \nabla u)$.

Step 4: Substitute the Newton boundary condition, i.e. here (2.49b), into the boundary integral, i.e. here $\int_{\Gamma_{\mathrm{N}}} v(z \cdot \nu)\,\mathrm{d}S = \int_{\Gamma_{\mathrm{N}}} (\nu \cdot a(x, u, \nabla u))v\,\mathrm{d}S$ in (1.54), while by considering $v|_{\Gamma_{\mathrm{D}}} = 0$, the integral over Γ_{D} simply vanishes.

This procedure looks here as

$$\begin{aligned}
&\int_\Omega \Big(-\operatorname{div}\big(a(x,u,\nabla u)\big) + c(x,u,\nabla u)\Big) v\,\mathrm{d}x \\
&\overset{\substack{\text{Green's}\\ \text{formula}}}{=} \int_\Omega a(x,u,\nabla u)\cdot\nabla v + c(x,u,\nabla u)v\,\mathrm{d}x - \int_\Gamma \big(\nu\cdot a(x,u,\nabla u)\big)v\,\mathrm{d}S \\
&\overset{\substack{\text{boundary}\\ \text{conditions}}}{=} \int_\Omega a(x,u,\nabla u)\cdot\nabla v + c(x,u,\nabla u)v\,\mathrm{d}x + \int_\Gamma \big(b(x,u) - h(x)\big)v\,\mathrm{d}S. \qquad (2.50)
\end{aligned}$$

Realizing still that the left-hand side in (2.50) is just $\int_\Omega gv\,\mathrm{d}x$, we come to the integral identity

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla v + c(x,u,\nabla u)v\,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} b(x,u)v\,\mathrm{d}S = \int_\Omega gv\,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} hv\,\mathrm{d}S. \qquad (2.51)$$

[9] See [360, Remark 5.3.8] or [370] for some examples of unsuitable concepts of so-called "measure-valued" solutions, cf. also DiPerna [124] or Illner and Wick [210].

As declared, we confine ourselves to a p-polynomial growth, cf. (2.55a) below, and then it is natural to seek the weak solution in the Sobolev space $W^{1,p}(\Omega)$. It leads to the following definition:

Definition 2.24. We call $u \in W^{1,p}(\Omega)$ a *weak solution* to the mixed boundary-value problem (2.45) and (2.49) if $u|_{\Gamma_D} = u_D$ and if the integral identity (2.51) holds for any $v \in W^{1,p}(\Omega)$ with $v|_{\Gamma_D} = 0$.

The above 4-step procedure to derive (2.51) guarantees automatically its consistency. On the other hand, its selectivity is related to the important fact that the space V of test-functions v's, i.e.

$$V = \big\{v\in W^{1,p}(\Omega);\ v|_{\Gamma_D} = 0\big\}, \tag{2.52}$$

is sufficiently rich, the restriction of v on Γ_D being compensated by direct involvement of the boundary condition (2.49a) in Definition 2.24:

Proposition 2.25 (Selectivity of the weak-solution definition)**.** *Let $a \in C^1(\bar\Omega \times \mathbb{R} \times \mathbb{R}^n; \mathbb{R}^n)$, $c \in C^0(\bar\Omega \times \mathbb{R} \times \mathbb{R}^n)$, and $b \in C^0(\bar\Gamma_N \times \mathbb{R})$, $g \in C(\bar\Omega)$, and $h \in C(\Gamma_N)$. Then any weak solution $u \in C^2(\bar\Omega)$ is the classical solution.*

Proof. Put $v \in V$ into (2.51) and use Green's formula (1.54). One gets

$$\int_\Omega \Big(\operatorname{div} a(x,u,\nabla u) - c(x,u,\nabla u) + g\Big) v \, dx \\ + \int_{\Gamma_N} \Big(h - b(x,u) - \nu{\cdot}a(x,u,\nabla u)\Big) v \, dS = 0. \tag{2.53}$$

Considering $v|_\Gamma = 0$, the boundary integral in (2.53) vanishes. As v is otherwise arbitrary, one deduces that (2.45) holds a.e., and hence even everywhere in Ω due to the assumed smoothness of a and c.[10] Hence, the first integral in (2.53) vanishes. Then, putting a general $v\in V$ into (2.53) shows the latter boundary condition in (2.49) valid,[11] while the former one is directly involved in Definition 2.24. □

The important issue now is to set up basic data qualification to give a sense to all integrals in (2.51). Recall that we keep the permanent assumption Ω to be a bounded Lipschitz domain (so that, in particular, ν is defined a.e. on Γ) and Γ_D and Γ_N are open in Γ (hence, in particular, measurable). To ensure measurability of integrands on the left-hand side of (2.51) we must assume:

$$a_i, c : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}\ ,\quad b : \Gamma \times \mathbb{R} \to \mathbb{R} \quad \text{are Carathéodory functions,} \tag{2.54}$$

for $i = 1, \dots, n$; this means measurability in x and continuity in the other variables. The further ultimate requirement is integrability of all integrands on the left-hand

[10] Here we use the fact that the set of test functions is sufficiently rich, namely that $W_0^{1,p}(\Omega)$ is dense in $L^1(\Omega)$; cf. Theorem 1.25 and the well-known fact that $C_0^\infty(\Omega)$ is dense in $L^1(\Omega)$.

[11] Here the important fact is that the set $\{v|_{\Gamma_N};\ v \in V\}$ is dense in $L^1(\Gamma_N)$. This is guaranteed by the assumption that Γ_N is open in Γ.

side of (2.51). This, and some continuity requirements needed further, lead us to assume the growth conditions on the nonlinearities a, b, and c:

$$|a(x,r,s)| \le \gamma(x) + C|r|^{(p^*-\epsilon)/p'} + C|s|^{p-1} \qquad \text{for some } \gamma\in L^{p'}(\Omega)\ , \tag{2.55a}$$

$$|\,b(x,r)\,| \ \le\ \gamma(x) + C|r|^{p^\#-\epsilon-1} \qquad \text{for some } \gamma\in L^{p^{\#\prime}}(\Gamma), \tag{2.55b}$$

$$|c(x,r,s)| \le \gamma(x) + C|r|^{p^*-\epsilon-1} + C|s|^{p/p^{*\prime}} \qquad \text{for some } \gamma\in L^{p^{*\prime}}(\Omega). \tag{2.55c}$$

Let us recall the notation of the prime denoting the conjugate exponents (i.e., e.g., $p' = p/(p-1)$, cf. (1.20)) and the continuous (resp. compact) embedding $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ (resp. $W^{1,p}(\Omega) \Subset L^{p^*-\epsilon}(\Omega)$ with $\epsilon>0$), cf. Theorem 1.20. Moreover, the trace operator $u \mapsto u|_\Gamma$ maps $W^{1,p}(\Omega)$ into $L^{p^\#}(\Gamma)$ continuously and into $L^{p^\#-\epsilon}(\Gamma)$ compactly, cf. Theorem 1.23. For p^* and $p^\#$ see (1.34) and (1.37).

Convention 2.26. For $p > n$, the terms $|r|^{+\infty}$ occurring in (2.55) are to be understood such that $|a(x,\cdot,s)|$, $|b(x,\cdot)|$, and $|c(x,\cdot,s)|$ may have arbitrary fast growth if $|r| \to \infty$.

In view of Theorem 1.27, the growth conditions (2.55) are designed so that respectively

$$\mathscr{N}_a : W^{1,p}(\Omega)\times L^p(\Omega;\mathbb{R}^n) \to L^{p'}(\Omega;\mathbb{R}^n) \ \text{ is (weak}\times\times\text{norm,norm)-continuous,} \tag{2.56a}$$

$$u \mapsto \mathscr{N}_b(u|_\Gamma) : W^{1,p}(\Omega) \to L^{p^{\#\prime}}(\Gamma) \qquad \text{is (weak,norm)-continuous,} \tag{2.56b}$$

$$\mathscr{N}_c : W^{1,p}(\Omega)\times L^p(\Omega;\mathbb{R}^n) \to L^{p^{*\prime}}(\Omega) \text{ is (weak}\times\text{norm,norm)-continuous.} \tag{2.56c}$$

In particular, for $u, v \in W^{1,p}(\Omega)$, the integrands $a(x,u,\nabla u)\cdot\nabla v$ and $c(x,u,\nabla u)v$ occurring in (2.51) belong to $L^1(\Omega)$ while $b(x,u|_\Gamma)v|_\Gamma$ belongs to $L^1(\Gamma)$.

Furthermore, we will also suppose the right-hand side qualification:

$$g \in L^{p^{*\prime}}(\Omega), \qquad h \in L^{p^{\#\prime}}(\Gamma). \tag{2.57}$$

Note that (2.57) ensures $gv \in L^1(\Omega)$ and $hv|_\Gamma \in L^1(\Gamma)$ for $v \in W^{1,p}(\Omega)$, hence (2.51) has a good sense. Moreover, we must qualify $u_{\rm D}$ occurring in the Dirichlet boundary condition (2.49a). The simplest way is to assume

$$\exists w \in W^{1,p}(\Omega) :\ u_{\rm D} = w|_\Gamma. \tag{2.58}$$

Then, considering V from (2.52) equipped by the norm (1.30b) denoted simply by $\|\cdot\|$, we define $A : W^{1,p}(\Omega) \to V^*$ and $f \in V^*$ simply by

$$\big\langle A(u), v\big\rangle := \text{left-hand side of (2.51)}, \tag{2.59}$$

$$\big\langle f, v\big\rangle \ :=\ \text{right-hand side of (2.51)}. \tag{2.60}$$

Moreover, referring to (2.58), let us define $A_0 : V \to V^*$ by

$$A_0(u) = A(u+w). \tag{2.61}$$

Note that A_0 has again the form of A from (2.51) but the nonlinearities a, b, and c are respectively replaced by a_0, b_0, and c_0 given by $a_0(x,r,s) := a(x,r+w(x),s+\nabla w(x))$, $b_0(x,r) := b(x,r+w(x))$, and $c_0(x,r,s) := c(x,r+w(x),s+\nabla w(x))$, and these nonlinearities satisfy (2.54)–(2.55) if $w \in W^{1,p}(\Omega)$ and if the original nonlinearities a, b, and c satisfy (2.54)–(2.55). Note also that for zero (or none) Dirichlet boundary conditions, one can assume $w = 0$ in (2.58) and then $A_0 \equiv A|_V$ (or simply $A_0 \equiv A$).

Note that, indeed, $f \in V^*$ because of the obvious estimate

$$\begin{aligned}\|f\|_* &= \sup_{\|v\|\le 1}\Big(\int_\Omega gv\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} hv\,\mathrm{d}S\Big) \le \sup_{\|v\|\le 1}\Big(\|g\|_{L^{p^*\prime}(\Omega)}\|v\|_{L^{p^*}(\Omega)} \\ &\quad + \|h\|_{L^{p^\#\prime}(\Gamma_\mathrm{N})}\|v\|_{L^{p^\#}(\Gamma_\mathrm{N})}\Big) \le N_1\|g\|_{L^{p^*\prime}(\Omega)} + N_2\|h\|_{L^{p^\#\prime}(\Gamma_\mathrm{N})} \end{aligned} \tag{2.62}$$

where N_1 is the norm of the embedding operator $W^{1,p}(\Omega) \to L^{p^*}(\Omega)$ and N_2 is the norm of the trace operator $v \mapsto v|_{\Gamma_\mathrm{N}} : W^{1,p}(\Omega) \to L^{p^\#}(\Gamma_\mathrm{N})$. By similar arguments, (2.54) and (2.55) ensures $A(u) \in V^*$, cf. Lemma 2.31 below.

Proposition 2.27 (Shift for non-zero Dirichlet condition)**.** *The abstract equation* (2.6) *for* A_0 *has a solution* $u_0 \in V$*, i.e.* $A_0(u_0) = f$*, if and only if* $u = u_0 + w \in W^{1,p}(\Omega)$ *is the weak solution to the boundary-value problem* (2.45) *and* (2.49) *in accord to Definition* 2.24*.*

Proof. We obviously have $f = A_0(u_0) = A_0(u-w) = A(u-w+w) = A(u)$, hence the assertion immediately follows by the definition (2.59)–(2.60). □

Remark 2.28 (Why both u and v are from V)**.** In principle, Definition 2.24 could work with $v \in Z := W^{1,\infty}(\Omega)$, or even with v's smoother; the selectivity Proposition 2.25 would hold as far as density of Z in V would be preserved, as used in Section 2.5 below. The choice of v's from the same space where the solution u is supposed to live, i.e. here V, is related to the setting $A : V \to Z^*$ which is fitted with the pseudomonotone-mapping concept only for $Z = V$.

Remark 2.29 (Why both Γ_D and Γ_N are assumed open)**.** In principle, Definition 2.24 as well as the existence Theorem 2.36 below could work for Γ_D and Γ_N only measurable. However, we would lose the connection to the original problem, cf. Proposition 2.25: indeed, one can imagine Γ_D measurable dense in Γ and Γ_N of a positive measure. Then, for $p > n$, $v|_\Gamma \in C(\Gamma)$ and the condition $v|_{\Gamma_\mathrm{D}} = 0$ would imply $v|_\Gamma = 0$, so that the Γ_N-integrals in (2.51) vanish and the Newton boundary condition on Γ_N in (2.49b) would be completely eliminated.

Remark 2.30 (Integral balance)**.** The equation (2.45) is a differential alternative to the integral balance

$$\int_O c(x,u,\nabla u) - g(x)\,\mathrm{d}x = \int_{\partial O} a(x,u,\nabla u)\cdot\nu\,\mathrm{d}S \tag{2.63}$$

for any test volume $O \subset \Omega$ with $\bar{O} \subset \Omega$ and a smooth boundary ∂O with the normal $\nu = \nu(x)$. Obviously, one is to identify c as the balanced quantity (depending on u and ∇u) while a as a flux of this quantity[12], and then (2.63) just says that the overall production of this quantity over the arbitrary test volume O is balanced by the overall flux through the boundary ∂O, cf. Figure 4. The philosophy that integral form (2.63) of physical laws is more natural than their differential form (2.45) was pronounced already by David Hilbert[13]. The weak formulation (2.51) implicitly includes, besides information about the boundary conditions, also (2.63). Indeed, it suffices to take v in (2.51) as some approximation of the characteristic function χ_O (which itself does not belong to $W^{1,p}(\Omega)$, however), e.g. v_ε with $v_\varepsilon(x) := (1 - \mathrm{dist}(x, O)/\varepsilon)^+$, and then to pass $\varepsilon \searrow 0$. This limit passage is, however, legal only if $x \mapsto a(x, u, \nabla u)$ is sufficiently regular near ∂O or, in a general case, it holds only in some "generic" sense; cf. e.g. Exercise 2.63.

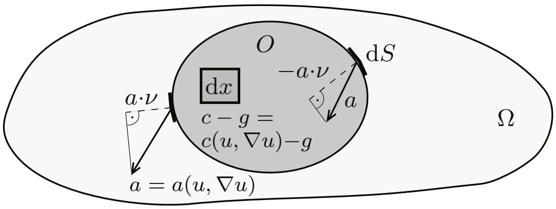

Figure 4. Illustration to balancing the normal flux $a{\cdot}\nu$ through the boundary of a test volume O and the production c inside this volume.

2.4.3 Pseudomonotonicity, coercivity, existence of solutions

In view of Theorem 2.6 with Proposition 2.27, we are to show pseudomonotonicity of $A_0 : V \to V^*$. For simplicity, we can prove it for A as $W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$, which, by Lemma 2.11(ii), implies pseudomonotonicity of $A_0 : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$, and then obviously also of $A_0 : V \to V^*$. Let us prove (2.3a) and (2.3b) respectively in the following lemmas.

Lemma 2.31 (Boundedness of A)**.** *The assumptions* (2.54) *and* (2.55) *ensure* (2.3a)*, i.e. $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ bounded.*

Proof. We prove $A\big(\{u \in W^{1,p}(\Omega);\ \|u\| \le \rho\}\big)$ bounded in $W^{1,p}(\Omega)^*$ for any $\rho > 0$. Here, $\|\cdot\|$ and $\|\cdot\|_*$ will denote the norms in $W^{1,p}(\Omega)$ and $W^{1,p}(\Omega)^*$, respectively. Indeed, we can estimate

[12] In concrete situations, the dependence of a on ∇u may result from a (nonlinear) Fick's, Fourier's, or Darcy's law.

[13] Explicitly, it can be found in his famous Mathematical problems [202, 19th problem]: "Has not every ... variational problem a solution, provided ... if need be that the notion of a solution shall be suitably extended?"

$$
\begin{aligned}
\sup_{\|u\|\le\rho}\|A(u)\|_* &= \sup_{\|u\|\le\rho}\sup_{\|v\|\le 1}\langle A(u),v\rangle \\
&= \sup_{\|u\|\le\rho}\sup_{\|v\|\le 1}\int_\Omega a(u,\nabla u)\cdot\nabla v + c(u,\nabla u)v\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} b(u)v\,\mathrm{d}S \\
&\le \sup_{\|u\|\le\rho}\sup_{\|v\|\le 1}\big\|a(u,\nabla u)\big\|_{L^{p'}(\Omega;\mathbb{R}^n)}\big\|\nabla v\big\|_{L^p(\Omega;\mathbb{R}^n)} \\
&\qquad + \big\|c(u,\nabla u)\big\|_{L^{p^*\prime}(\Omega)}\big\|v\big\|_{L^{p^*}(\Omega)} + \big\|b(u)\big\|_{L^{p^\#\prime}(\Gamma_\mathrm{N})}\big\|v\big\|_{L^{p^\#}(\Gamma_\mathrm{N})} \\
&\le \sup_{\|u\|\le\rho}\big\|a(u,\nabla u)\big\|_{L^{p'}(\Omega;\mathbb{R}^n)} + N_1\big\|c(u,\nabla u)\big\|_{L^{p^*\prime}(\Omega)} + N_2\big\|b(u)\big\|_{L^{p^\#\prime}(\Gamma_\mathrm{N})} \qquad (2.64)
\end{aligned}
$$

where N_1 and N_2 are as in (2.62). In view of (2.55), it is bounded uniformly for u ranging over a bounded set in $W^{1,p}(\Omega)$. $\square$

Further, we still have to strengthen our data qualification. The crucial assumption we must make for pseudomonotonicity of A is the so-called *monotonicity in the main part*:

$$\forall(\text{a.a.})\ \ x\in\Omega\ \ \forall r\in\mathbb{R}\ \ \forall s,\tilde{s}\in\mathbb{R}^n: \qquad \big(a(x,r,s)-a(x,r,\tilde{s})\big)\cdot(s-\tilde{s})\ge 0. \quad (2.65)$$

To cover as many situations as possible, we distinguish three cases in accordance with whether $c(x,r,\cdot)$ is constant, linear, or nonlinear, respectively.

Lemma 2.32 (The implication (2.3b))**.** *Let the assumptions* (2.54) *and* (2.55) *be valid, let a satisfy* (2.65)*, and let one of the following three cases hold: c is independent of s, i.e. for some $\widetilde{c}:\Omega\times\mathbb{R}\to\mathbb{R}$,*

$$c(x,r,s)=\widetilde{c}(x,r)\,, \qquad (2.66)$$

or c is linearly dependent on s, i.e. for some $\overline{c}:\Omega\times\mathbb{R}\to\mathbb{R}^n$,

$$c(x,r,s)=\overline{c}(x,r)\cdot s, \qquad (2.67)$$

or c is generally dependent on s but the strict monotonicity "in the main part" and coercivity of $a(x,r,\cdot)$ hold and the growth of $c(x,\cdot,\cdot)$ is further restricted:

$$(a(x,r,s)-a(x,r,\tilde{s}))\cdot(s-\tilde{s})=0 \quad\Longrightarrow\quad s=\tilde{s}, \qquad (2.68\text{a})$$

$$\forall s_0\in\mathbb{R}^n:\quad \lim_{|s|\to\infty}\frac{a(x,r,s)\cdot(s-s_0)}{|s|}=+\infty \text{ uniformly for } r \text{ bounded}, \qquad (2.68\text{b})$$

$$\exists\gamma\in L^{p^{*\prime}+\epsilon}(\Omega)\,\exists C\in\mathbb{R}:\ |c(x,r,s)|\le\gamma(x)+C|r|^{p^*-\epsilon-1}+C|s|^{(p-\epsilon)/p^{*\prime}} \qquad (2.68\text{c})$$

with Convention 2.26 *in mind. Then $A:W^{1,p}(\Omega)\to W^{1,p}(\Omega)^*$ satisfies* (2.3b)*.*

Remark 2.33. Obviously, (2.66) together with the growth condition (2.55c) imply $|\widetilde{c}(x,r)|\le\gamma(x)+C|r|^{p^*-\epsilon-1}$ with γ as in (2.55c). A bit more difficult is to realize that (2.67) together with the growth condition (2.55c) imply that $\overline{c} : \Omega \times \mathbb{R} \to \mathbb{R}^n$ has to satisfy

$$|\overline{c}(x,r)| \le \gamma(x) + C|r|^{p^*/q-\epsilon_1} \quad \text{with} \;\; \gamma \in L^{q+\epsilon_1}(\Omega) \;\; \text{and some} \;\; \epsilon_1 > 0, \\ \text{where} \;\; q = \begin{cases} \dfrac{np}{np-2n+p} & \text{if } p < n, \\ p' & \text{if } p \ge n. \end{cases} \tag{2.69}$$

This condition together with the structural condition (2.67) now guarantees

$$\mathscr{N}_c : W^{1,p}(\Omega) \times L^p(\Omega;\mathbb{R}^n) \to L^{(p^*-\epsilon)'}(\Omega) \text{ is (weak}\times\text{weak,weak)-continuous.} \tag{2.70}$$

Eventually, note that the growth condition (2.68c) strengthens (2.55c) and is designed so that, for some $\epsilon > 0$ (depending on ϵ used in (2.68c)),

$$\mathscr{N}_c : W^{1,p}(\Omega) \times L^p(\Omega;\mathbb{R}^n) \to L^{p^{*\prime}+\epsilon}(\Omega) \;\text{ is (weak}\times\text{norm,norm)-continuous.} \tag{2.71}$$

Proof of Lemma 2.32. Let us take $u_k \rightharpoonup u$ in $W^{1,p}(\Omega)$ and assume that

$$\limsup_{k\to\infty} \langle A(u_k), u_k - u\rangle \le 0. \tag{2.72}$$

We are to show that $\liminf_{k\to\infty} \langle A(u_k), u_k - v\rangle \ge \langle A(u), u - v\rangle$ for any $v \in W^{1,p}(\Omega)$. To distinguish between the highest and the lower-order terms, we define $B(w,u) \in W^{1,p}(\Omega)^*$ by

$$\big\langle B(w,u), v\big\rangle := \int_\Omega a(x,w,\nabla u)\cdot\nabla v + c(x,w,\nabla w)v \,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} b(x,w)v \,\mathrm{d}S \tag{2.73}$$

for $u, w \in W^{1,p}(\Omega)$; recall the Convention 2.23. Obviously, $A(u) = B(u,u)$.

Let us put $u_\varepsilon = (1-\varepsilon)u + \varepsilon v$, $\varepsilon \in [0,1]$. Monotonicity (2.65) implies $\langle B(u_k,u_k) - B(u_k,u_\varepsilon), u_k - u_\varepsilon\rangle \ge 0$. Then, just by simple algebra,

$$\varepsilon\big\langle A(u_k), u - v\big\rangle \;\ge\; -\big\langle A(u_k), u_k - u\big\rangle \\ + \big\langle B(u_k,u_\varepsilon), u_k - u\big\rangle + \varepsilon\big\langle B(u_k,u_\varepsilon), u - v\big\rangle. \tag{2.74}$$

Let us assume, for a moment, that we have proved

$$\lim_{k\to\infty} \big\langle B(u_k,v), u_k - u\big\rangle = 0, \tag{2.75}$$

$$\text{w-}\lim_{k\to\infty} B(u_k,v) = B(u,v) \qquad \text{(the weak limit in } W^{1,p}(\Omega)^*\text{)}, \tag{2.76}$$

and use them here for $v = u_\varepsilon$ to pass successively to the limit in the right-hand-side terms of (2.74). Using also (2.72), we thus obtain

$$\varepsilon \liminf_{k\to\infty} \big\langle A(u_k), u - v\big\rangle \;\ge\; -\limsup_{k\to\infty} \big\langle A(u_k), u_k - u\big\rangle + \lim_{k\to\infty} \big\langle B(u_k,u_\varepsilon), u_k - u\big\rangle \\ +\; \varepsilon \lim_{k\to\infty} \big\langle B(u_k,u_\varepsilon), u - v\big\rangle \;\ge\; \varepsilon\big\langle B(u,u_\varepsilon), u - v\big\rangle.$$

Divide it by $\varepsilon > 0$. Then the limit passage $\varepsilon \to 0$ gives $u_\varepsilon \to u$ strongly so that we get $B(u, u_\varepsilon) \to B(u, u)$ even strongly[14], which results in $\liminf_{k\to\infty}\langle A(u_k), u - v\rangle \geq \langle B(u, u), u - v\rangle = \langle A(u), u - v\rangle$.

Then, by using the monotonicity in the main part (2.65) once again, now as $\langle B(u_k, u_k) - B(u_k, u), u_k - u\rangle \geq 0$, and by using also (2.75) now with $v = u$, we can claim that

$$\begin{aligned}\liminf_{k\to\infty}\big\langle A(u_k), u_k{-}v\big\rangle &\geq \liminf_{k\to\infty}\big\langle A(u_k), u_k{-}u\big\rangle + \liminf_{k\to\infty}\big\langle A(u_k), u{-}v\big\rangle \\ &= \lim_{k\to\infty}\big\langle B(u_k, u), u_k{-}u\big\rangle + \liminf_{k\to\infty}\big\langle B(u_k, u_k){-}B(u_k, u), u_k{-}u\big\rangle \\ &\quad + \liminf_{k\to\infty}\big\langle A(u_k), u - v\big\rangle \geq \langle A(u), u - v\rangle, \end{aligned} \tag{2.77}$$

which is just the conclusion of (2.3b).

Thus it remains to prove (2.75) and (2.76). Since $u_k \rightharpoonup u$ in $W^{1,p}(\Omega) \Subset L^{p^*-\epsilon}(\Omega)$, we have $u_k \to u$ in $L^{p^*-\epsilon}(\Omega)$. Similarly, $u_k|_\Gamma \to u|_\Gamma$ in $L^{p^\#-\epsilon}(\Gamma)$. Then, by the continuity of the Nemytskiĭ mappings induced by $a(\cdot, \nabla v)$ and b, we get $a(u_k, \nabla v) \to a(u, \nabla v)$ in $L^{p'}(\Omega; \mathbb{R}^n)$, cf. (2.56a), and $b(u_k) \to b(u)$ in $L^{p^{\#\prime}}(\Gamma)$; cf. (2.56b) together with (1.36b); recall again Convention 2.23. Hence, realizing that $\nabla(u_k{-}u) \rightharpoonup 0$ in $L^p(\Omega; \mathbb{R}^n)$ and $(u_k{-}u)|_\Gamma \rightharpoonup 0$ in $L^{p^\#}(\Gamma_{\mathrm{N}})$, one gets

$$\int_\Omega a(u_k, \nabla v) \cdot \nabla(u_k - u)\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} b(u_k)(u_k - u)\,\mathrm{d}S \to 0. \tag{2.78}$$

By the same reasons, for any $z \in W^{1,p}(\Omega)$, we have also

$$\int_\Omega a(u_k, \nabla v){\cdot}\nabla z\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} b(u_k)z\,\mathrm{d}S \to \int_\Omega a(u, \nabla v)z\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} b(u)z\,\mathrm{d}S. \tag{2.79}$$

As to the term c, we will distinguish the above suggested three cases.

The case (2.66): By the continuity of the Nemytskiĭ mappings induced by $\widetilde{c}$, one has $\widetilde{c}(u_k) \to \widetilde{c}(u)$ in $L^{p^{*\prime}}(\Omega)$. Therefore, realizing that $u_k - u \rightharpoonup 0$ in $L^{p^*}(\Omega)$, one gets $\int_\Omega \widetilde{c}(u_k)(u_k - u)\,\mathrm{d}x \to 0$. Adding it with (2.78), one gets

$$\begin{aligned}\langle B(u_k, v), u_k - u\rangle \;:=\; &\int_\Omega \Big(a(u_k, \nabla v) \cdot \nabla(u_k - u) \\ &\quad + \;\widetilde{c}(u_k)(u_k - u)\Big)\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} b(u_k)(u_k - u)\,\mathrm{d}S \to 0,\end{aligned} \tag{2.80}$$

which proves (2.75). Similarly, $\int_\Omega \widetilde{c}(u_k)z\mathrm{d}x \to \int_\Omega \widetilde{c}(u_k)z\mathrm{d}x$, which, together with (2.79), gives just (2.76).

The case (2.67): Here we have a certain reserve in the growth, cf. (2.70), and can thus exploit the compactness of the embedding $W^{1,p}(\Omega) \Subset L^{p^*-\epsilon}(\Omega)$ to use $u_k \to u$

[14] Here we use the continuity of the Nemytskiĭ mapping $\mathscr{N}_{a\circ u}$ with $a\circ u : (x, s) \mapsto a(x, u(x), s)$.

strongly in $L^{p^*-\epsilon}(\Omega)$. Also, we can use $\overline{c}(u_k) \to \overline{c}(u)$ in $L^{q+\epsilon_1}(\Omega)$ with q from (2.69) and some $\epsilon_1 > 0$ (depending on ϵ); note that $(q+\epsilon_1)^{-1} + p^{-1} + (p^* - \epsilon)^{-1} \leq 1$ if ϵ is small enough depending on the chosen ϵ_1. As $\nabla u_k \rightharpoonup \nabla u$ in $L^p(\Omega;\mathbb{R}^n)$, we can pass to the limit in the c-term:

$$\int_\Omega \overline{c}(u_k) \cdot \nabla u_k (u_k - u)\, \mathrm{d}x \to 0. \tag{2.81}$$

Adding it with (2.78), one gets (2.75). Similarly, $\int_\Omega \overline{c}(u_k) \cdot \nabla u_k z \,\mathrm{d}x \to \int_\Omega \overline{c}(u_k) \cdot \nabla u z \,\mathrm{d}x$, which, together with (2.79), gives just (2.76).

The case (2.68): We already showed that $u_k \to u$ in $L^{p^*-\epsilon}(\Omega)$. In view of the boundedness (2.71) of $\{c(u_k, \nabla u_k)\}_{k\in\mathbb{N}}$ in $L^{p^{*\prime}+\epsilon}(\Omega)$, we obviously have

$$\int_\Omega c(u_k, \nabla u_k)(u_k - u)\,\mathrm{d}x \;\to\; 0\ . \tag{2.82}$$

Adding it with (2.78), one gets (2.75).

To prove (2.76), we need to show a convergence of ∇u_k to ∇u in a better mode than the weak one only. Let us denote

$$\mathfrak{a}_k(x) := \big(a(x, u_k(x), \nabla u_k(x)) - a(x, u_k(x), \nabla u(x))\big) \cdot \nabla\big(u_k(x) - u(x)\big). \tag{2.83}$$

By the monotonicity (2.65), it holds

$$\begin{aligned} 0 \;\leq\; & \limsup_{k\to\infty} \int_\Omega \mathfrak{a}_k(x)\,\mathrm{d}x = \limsup_{k\to\infty} \big\langle B(u_k, u_k) - B(u_k, u), u_k - u\big\rangle \\ =\; & \limsup_{k\to\infty} \big\langle A(u_k), u_k - u\big\rangle - \lim_{k\to\infty} \big\langle B(u_k, u), u_k - u\big\rangle \leq 0; \end{aligned} \tag{2.84}$$

note that the last limit superior is non-positive by assumption while the last limit equals zero by (2.75) with $v := u$. This implies that $\mathfrak{a}_k \to 0$ in the measure so that we can select a subsequence such that

$$\mathfrak{a}_k(x) \to 0 \tag{2.85}$$

for a.a. $x \in \Omega$. As $u_k \to u$ strongly in $L^{p^*-\epsilon}(\Omega)$, by Proposition 1.13(ii)–(iii) we can further select a subsequence that also

$$u_k(x) \to u(x) \tag{2.86}$$

for a.a. $x \in \Omega$. Take $x \in \Omega$ such that both (2.85) and (2.86) hold and also $\nabla u(x)$, $\nabla u_k(x)$, $k \in \mathbb{N}$, and $\gamma(x)$ from (2.55a) are finite, and $a(x,\cdot,\cdot)$ is continuous. If the sequence $\{\nabla u_k(x)\}_{k\in\mathbb{N}}$ would be unbounded, then the coercivity (2.68b) used for $s_0 = \nabla u(x)$ would yield $\limsup_{k\to\infty}(a(x,u_k(x),\nabla u_k(x)) - a(x,u_k(x),s_0)) \cdot (\nabla u_k(x) - s_0) = +\infty$, which would contradict (2.85). Therefore, we can take a suitable $s \in \mathbb{R}^n$ and a (for a moment sub-) sequence such that $\nabla u_k(x) \to s$ in $\mathbb{R}^n$.

By (2.85) and (2.86) and the continuity of $a(x,\cdot,\cdot)$, cf. (2.54), we can pass to the limit in (2.83), which yields

$$\big(a(x,u(x),s)-a(x,u(x),\nabla u(x))\big)\cdot\big(s-\nabla u(x)\big)=0. \tag{2.87}$$

By the strict monotonicity (2.68a), we get $s=\nabla u(x)$. As s is determined uniquely, even the whole sequence $\{\nabla u_k(x)\}_{k\in\mathbb{N}}$ converges to s.[15] Then

$$c(u_k,\nabla u_k)\to c(u,\nabla u)\quad\text{a.e. in }\Omega. \tag{2.88}$$

By Hölder's inequality, for any measurable $S\subset\Omega$, we can estimate

$$\int_S\big|c(u_k,\nabla u_k)-c(u,\nabla u)\big|^{p^{*\prime}}\mathrm{d}x\le\big\|c(u_k,\nabla u_k)-c(u,\nabla u)\big\|_{L^{p^{*\prime}+\epsilon}(\Omega)}\mathrm{meas}_d(S)^{1+p^{*\prime}/\epsilon}. \tag{2.89}$$

Further, we realize that the sequence $\{c(u_k,\nabla u_k)-c(u,\nabla u)\}_{k\in\mathbb{N}}$ is bounded in $L^{p^{*\prime}+\epsilon}(\Omega)$ thanks to the assumption (2.68c). Thus (2.89) verifies the equi-absolute-continuity of the collection $\{|c(u_k,\nabla u_k)-c(u,\nabla u)|^{p^{*\prime}}\}_{k\in\mathbb{N}}$, cf. (1.28). By Dunford-Pettis' theorem 1.16(ii), $\{|c(u_k,\nabla u_k)-c(u,\nabla u)|^{p^{*\prime}}\}_{k\in\mathbb{N}}$ is also uniformly integrable and, since it converges to 0 a.e. due to (2.88), by Vitali's theorem 1.17, $|c(u_k,\nabla u_k)-c(u,\nabla u)|^{p^{*\prime}}\to 0$ in $L^1(\Omega)$, i.e.

$$c(u_k,\nabla u_k)\to c(u,\nabla u)\qquad\text{in }L^{p^{*\prime}}(\Omega). \tag{2.90}$$

As the limit $c(u,\nabla u)$ is determined uniquely, even the whole sequence (not only that one selected for (2.85)–(2.86)) must converge. Then (2.76) follows by joining (2.90) with (2.79). $\square$

Note that, as always $p^{*\prime}+\epsilon>1$, (2.90) also proves that

$$c(u_k,\nabla u_k)\rightharpoonup c(u,\nabla u)\qquad\text{in }L^{p^{*\prime}+\epsilon}(\Omega). \tag{2.91}$$

By the same technique one can also prove $a(u_k,\nabla u_k)\rightharpoonup a(u,\nabla u)$ weakly in $L^{p'}(\Omega;\mathbb{R}^n)$. We however did not need this fact in the above proof.

Remark 2.34 (Critical growth in lower-order terms)**.** The above theorem and its proof permits various modifications: If $b(x,\cdot)$ is monotone, then the splitting (2.73) can involve $b(u)$ instead of $b(w)$, which allows for borderline growth of b, i.e. (2.55b) with $\epsilon=0$. Similarly, if $c=\widetilde{c}(x,r)$ as in (2.66) but with $\widetilde{c}(x,\cdot)$ is monotone, then (2.73) can involve $c(u)$ instead of $c(w,\nabla w)$, and (2.55c) with $\epsilon=0$ suffices. Modification of the basic space V in these cases would allow for even a super-critical growth, cf. (2.128). The growth restriction can also be eliminated if a maximum principle, guaranting L^∞-estimates, is at our disposal, which unfortunately can be expected only in special cases like the equation $\Delta u=|\nabla u|^2$ or (6.73) below.

[15] The fact that we do not need to select a subsequence at every x in question is important because the set of all such x's should have the full measure in Ω and thus cannot be countable.

Lemma 2.35 (The coercivity (2.5))**.** *Let the following coercivity hold:*

$$\exists\varepsilon_1,\varepsilon_2>0,\ k_1\in L^1(\Omega):\ a(x,r,s)\cdot s+c(x,r,s)r\ge\varepsilon_1|s|^p+\varepsilon_2|r|^q-k_1(x),\tag{2.92a}$$
$$\exists c_1<+\infty\ \exists k_2\in L^1(\Gamma):\qquad b(x,r)r\ge -c_1|r|^{q_1}-k_2(x),\tag{2.92b}$$

for some $1<q_1<q\le p$*. Then* $A:W^{1,p}(\Omega)\to W^{1,p}(\Omega)^*$ *is coercive.*

Proof. We use the Poincaré inequality in the form (1.55), i.e. $\|u\|_{W^{1,p}(\Omega)}\le C_{\mathrm P}\big(\|\nabla u\|_{L^p(\Omega;\mathbb R^n)}+\|u\|_{L^q(\Omega)}\big)$, which implies

$$\begin{aligned}\|u\|^q_{W^{1,p}(\Omega)}&\le 2^{q-1}C^q_{\mathrm P}\big(\|\nabla u\|^q_{L^p(\Omega;\mathbb R^n)}+\|u\|^q_{L^q(\Omega)}\big)\\&\le C_{p,q}\big(1+\|\nabla u\|^p_{L^p(\Omega;\mathbb R^n)}+\|u\|^q_{L^q(\Omega)}\big).\end{aligned}\tag{2.93}$$

Also, by Young's inequality and boundedness of the trace operator[16] $u\mapsto u|_\Gamma: W^{1,p}(\Omega)\to L^q(\Gamma)$ (let N denote its norm), we use the estimate

$$\big\|u\big\|^{q_1}_{L^{q_1}(\Gamma)}=\int_\Gamma|u|^{q_1}\mathrm dS\le\int_\Gamma\varepsilon|u|^q+C_\varepsilon\mathrm dS\le\varepsilon N^q\big\|u\big\|^q_{W^{1,p}(\Omega)}+C_\varepsilon\mathrm{meas}_{n-1}(\Gamma)\tag{2.94}$$

with $\varepsilon>0$ arbitrarily small and $C_\varepsilon<+\infty$ chosen accordingly; cf. (1.22) with $q/q_1>1$ in place of p. Then (2.92) implies the estimate

$$\begin{aligned}\big\langle A(u),u\big\rangle&\ge\int_\Omega\big(\varepsilon_1|\nabla u|^p+\varepsilon_2|u|^q-k_1\big)\,\mathrm dx-\int_\Gamma\big(c_1|u|^{q_1}+k_2\big)\,\mathrm dS\\&\ge\min(\varepsilon_1,\varepsilon_2)\bigg(\frac{\|u\|^q_{W^{1,p}(\Omega)}}{C_{p,q}}-1\bigg)-\big\|k_1\big\|_{L^1(\Omega)}\\&\qquad-\varepsilon N^q\big\|u\big\|^q_{W^{1,p}(\Omega)}-C_\varepsilon\mathrm{meas}_{n-1}(\Gamma)-\big\|k_2\big\|_{L^1(\Gamma)}.\end{aligned}\tag{2.95}$$

When one chooses $\varepsilon<\min(\varepsilon_1,\varepsilon_2)/(C_{p,q}N^q)$ and realizes that $q>1$, the coercivity (2.5) of A, i.e. $\lim_{\|u\|_{W^{1,p}(\Omega)}\to\infty}\langle A(u),u\rangle=+\infty$, is shown. □

Theorem 2.36 (Leray-Lions [257])**.** *Let* (2.54), (2.55), (2.57), (2.58), (2.65), *and* (2.92) *be valid and at least one of the conditions* (2.66) *or* (2.67) *or* (2.68) *be valid, then the boundary-value problem* (2.45)–(2.49) *has a weak solution.*

Proof. Lemmas 2.31, 2.32, and 2.35 proved $A:W^{1,p}(\Omega)\to W^{1,p}(\Omega)^*$ pseudomonotone and coercive. These properties are inherited by $A_0:V\to V^*$, cf. also Lemma 2.11(ii). Then we use Theorem 2.6 with Proposition 2.27. □

Remark 2.37 (Coercivity (2.68b))**.** Note that the coercivity (2.92a) together with (2.55a) and (2.68c) imply the coercivity (2.68b) because

$$a(x,r,s)\cdot(s-s_0)\ge\varepsilon_1|s|^p+\varepsilon_2|r|^q-k_1(x)-c(x,r,s)r-a(x,r,s)\cdot s_0\tag{2.96}$$

[16] Note that always $q\le p<p^\#$.

for such $x \in \Omega$ that $k_1(x)$ is finite. Realizing that $s \mapsto -c(x,r,s)r$ has a maximal decay as $-|s|^{(p-\epsilon)/p^{*\prime}}$ due to (2.68c) and $s \mapsto -a(x,r,s) \cdot s_0$ maximal decay as $-|s|^{p-1}$ due to (2.55a), the estimate (2.96) shows that $s \mapsto a(x,r,s) \cdot (s - s_0)$ has the p-growth uniformly with respect to r bounded because $\epsilon > 0$ and $p^{*\prime} \geq 1$.

Remark 2.38 (Necessity of monotonicity of $a(x,r,\cdot)$)**.** Boccardo and Dacorogna [55] showed that monotonicity of $a(x,r,\cdot)$ is necessary for pseudomonotonicity of the mapping $A(u) = -\mathrm{div}\, a(x,u,\nabla u)$.

Remark 2.39 (Necessity of Leray-Lions' condition (2.65), (2.68))**.** If a lower-order term c is present, the necessity of strict monotonicity of $a(x,r,\cdot)$ for the pseudomonotonicity was shown by Gossez and Mustonen [184].[17] It is worth observing that, for $c(x,r,\cdot)$ not affine, the mapping $u \mapsto c(u,\nabla u) : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$, although representing a lower-order term, is neither totally continuous[18] nor pseudomonotone but it is still compact, cf. Exercise 2.64, and, when added to $u \mapsto -\mathrm{div}\, a(u,\nabla u)$, it may result in a pseudomonotone mapping.

Remark 2.40 (General right-hand sides)**.** The functional $f : v \mapsto \int_\Omega gv\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} hv\,\mathrm{d}S$ we considered, cf. (2.60), is not the general form of a functional $f \in W^{1,p}(\Omega)^*$. In fact, $W^{1,p}(\Omega)^*$ would allow g and h to be certain distributions on Ω and Γ, respectively. For example, if $p > n$, we have a dense and continuous embedding $W^{1,p}(\Omega) \subset C(\bar{\Omega})$ (resp. the trace operator $W^{1,p}(\Omega) \to C(\Gamma)$), henceforth the functional $f : v \mapsto \int_{\bar{\Omega}} v\,\mu(\mathrm{d}x) + \int_\Gamma v\,\eta(\mathrm{d}S)$ with *measures* $\mu \in \mathscr{M}(\bar{\Omega})$ and $\eta \in \mathscr{M}(\Gamma)$ still belongs to $W^{1,p}(\Omega)^*$. Since, in the case $p > n$, it holds that $p^* = p^\# = +\infty$, we, for the sake of simplicity, have considered (and will consider) only those measures μ and η which are absolutely continuous[19] in the presented text, except Sect. 3.2.5 below.

Convention 2.41 (Coercivity and a-priori estimates)**.** The coercivity estimate (2.95) is just the so-called basic a-priori estimate, obtained by the test by solution u itself. Contrary to (2.95), it is routine to organize the terms having a positive sign in the left-hand side (and to estimate them from below typically by Poincaré-type inequalities) while the other terms are put on the right-hand side (and to estimate them from above, e.g., by Hölder and Young inequalities).

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[17] Indeed, the mere monotonicity in the main part, i.e. (2.54), and (2.55), (2.57), (2.58), and (2.65), cannot be sufficient for the pseudomonotonicity of A. The counterexample is as follows: take $c(x,r,s) \equiv c(s)$ with some $c : \mathbb{R}^n \to \mathbb{R}$ nonlinear, i.e. $\exists s_1, s_2 \in \mathbb{R}^n :\ \frac{1}{2}c(s_1) + \frac{1}{2}c(s_2) \neq c(\frac{1}{2}s_1 + \frac{1}{2}s_2)$ and take $a = 0$ at least on the line segment $[s_1, s_2]$. Then take a sequence $\{u_k\}_{k\in\mathbb{N}}$ such that ∇u_k is faster and faster oscillating between s_1 and s_2 (cf. Figure 3) on p.20 and $u_k(x) \to (\frac{1}{2}s_1 + \frac{1}{2}s_2) \cdot x$.

[18] Indeed, the mapping $u \mapsto c(u,\nabla u)$ is not totally continuous because it need not map weakly convergent sequences on strongly convergent ones. An example is as follows: take u_k with an oscillating gradient if k odd and affine $u_k(x) = (\frac{1}{2}s_1 + \frac{1}{2}s_2) \cdot x$ if k even, so that again $\{u_k\}_{k\in\mathbb{N}}$ converges weakly to this affine function but $\{c(\nabla u_k)\}_{k\in\mathbb{N}}$ does not converge at all if, e.g., $c(s) = |2s - s_1 - s_2|$.

[19] Those measures are known to have densities $g \in L^1(\Omega)$ and $h \in L^1(\Gamma)$, respectively.

2.4.4 Higher-order equations

The generalization of the 2nd-order equation to equations involving $2k$-order derivatives, $k \geq 2$, is often desirable. The corresponding boundary-value problems then involve k-boundary conditions, called either the Dirichlet one if they involve only derivatives up to $(k{-}1)$-order or the Neumann or the Newton one if they involve also derivatives of the order between k and $2k{-}1$. We present here briefly only quasilinear equations of the 4th order in a special[20] divergence form

$$\operatorname{div}\Big(\operatorname{div}\big(a(x,u,\nabla u,\nabla^2 u)\big)\Big) + c(x,u,\nabla u,\nabla^2 u) = g \tag{2.97}$$

in Ω, with $a : \Omega\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n\times n}\to\mathbb{R}^{n\times n}$ and $c : \Omega\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n\times n}\to\mathbb{R}$. Here $\nabla^2 u := \big[\frac{\partial^2}{\partial x_i\partial x_j}u\big]_{i,j=1}^n$. More in detail, (2.97) means

$$\sum_{i,j=1}^n \frac{\partial^2}{\partial x_i\partial x_j} a_{ij}\big(x,u,\nabla u,\nabla^2 u\big) + c\big(x,u,\nabla u,\nabla^2 u\big) = g. \tag{2.98}$$

Formulation of natural boundary conditions is more difficult than for the 2nd-order case. The weak formulation is created by multiplying (2.97) by a test function v, by integration over Ω, and by using Green's formula twice. Like in (2.50), this gives

$$\begin{aligned}\int_\Omega a(x,u,\nabla u,\nabla^2 u):\nabla^2 v + \big(c(x,u,\nabla u,\nabla^2 u) - g\big)v\,\mathrm{d}x\\ = \int_\Gamma a(x,u,\nabla u,\nabla^2 u) : (\nu\otimes\nabla v) - \operatorname{div}\big(a(x,u,\nabla u,\nabla^2 u)\big)\cdot\nu\, v\;\mathrm{d}S.\end{aligned} \tag{2.99}$$

From this we can see that we must now cope with two boundary terms. In view of this, the *Dirichlet boundary conditions* look as

$$u|_\Gamma = u_{\rm D} \qquad \text{and} \qquad \frac{\partial u}{\partial\nu}\Big|_\Gamma = u'_{\rm D} \quad \text{on } \Gamma \tag{2.100}$$

with $u_{\rm D}$ and $u'_{\rm D}$ given. The weak formulation then naturally works with $v\in V := W_0^{2,p}(\Omega) = \{v\in W^{2,p}(\Omega);\ v|_\Gamma = \frac{\partial v}{\partial\nu}|_\Gamma = 0\}$ with $p > 1$ an exponent related to qualification of the highest-order nonlinearity $a(x,r,s,\cdot)$. This choice makes both boundary terms in (2.99) zero; note that $v|_\Gamma = 0$ makes also the tangential derivative of v zero at a.a. $x\in\Gamma$ hence $\frac{\partial v}{\partial\nu}|_\Gamma = 0$ yields $\nabla v(x) = 0$ on Γ.

By this argument, $v|_\Gamma = 0$ makes $\nabla v = \frac{\partial v}{\partial\nu}\nu$ on Γ and allows us to write

$$a(x,u,\nabla u,\nabla^2 u):(\nu\otimes\nabla v) = a(x,u,\nabla u,\nabla^2 u):\Big(\nu\otimes\frac{\partial v}{\partial\nu}\nu\Big) = \big(\nu^\top a(x,u,\nabla u,\nabla^2 u)\nu\big)\frac{\partial v}{\partial\nu}$$

and suggests that we formulate *Dirichlet/Newton boundary conditions* as

$$u|_\Gamma = u_{\rm D} \qquad \text{and} \qquad \nu^\top a(x,u,\nabla u,\nabla^2 u)\,\nu + b(x,u,\nabla u) = h \ \text{ on } \Gamma \tag{2.101}$$

[20] See Exercises 2.98 and 4.32 for a more general case.

with $u_{\rm D}$ and h given and $b:\Gamma\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$. This choice with $v|_\Gamma=0$ converts the boundary terms in (2.99) to $\int_\Gamma(h-b(x,u,\nabla u))\frac{\partial v}{\partial\nu}{\rm d}S$, which turns (2.99) just into the integral identity

$$\int_\Omega a(x,u,\nabla u,\nabla^2 u):\nabla^2 v+c(x,u,\nabla u,\nabla^2 u)v\,{\rm d}x\\ +\int_\Gamma b(x,u,\nabla u)\frac{\partial v}{\partial\nu}{\rm d}S=\int_\Omega gv\,{\rm d}x+\int_\Gamma h\frac{\partial v}{\partial\nu}{\rm d}S \tag{2.102}$$

forming the weak formulation provided the test-function space V is taken as $\{v\in W^{2,p}(\Omega);\ v|_\Gamma=0\}$.

If $v|_\Gamma$ is not fixed to zero, one must use a general decomposition $\nabla v=\frac{\partial v}{\partial\nu}\nu+\nabla_{\rm S}v$ on Γ with $\nabla_{\rm S}v=\nabla v-\frac{\partial v}{\partial\nu}\nu$ being the *tangential gradient* of v. On a smooth boundary Γ, one can use another (now $(n{-}1)$-dimensional) Green-type formula along the tangential spaces:[21]

$$\begin{aligned}&\int_\Gamma a(x,u,\nabla u,\nabla^2 u){:}(\nu\otimes\nabla v)\,{\rm d}S\\ &\quad=\int_\Gamma\Big(\nu^\top a(x,u,\nabla u,\nabla^2 u)\nu\Big)\frac{\partial v}{\partial\nu}+a(x,u,\nabla u,\nabla^2 u){:}(\nu\otimes\nabla_{\rm S}v)\,{\rm d}S\\ &\quad=\int_\Gamma\Big(\nu^\top a(x,u,\nabla u,\nabla^2 u)\nu\Big)\frac{\partial v}{\partial\nu}-{\rm div}_{\rm S}\big(a(x,u,\nabla u,\nabla^2 u)\nu\big)v\\ &\qquad\qquad+\big({\rm div}_{\rm S}\nu\big)\big(\nu^\top a(x,u,\nabla u,\nabla^2 u)\nu\big)v\,{\rm d}S\end{aligned} \tag{2.103}$$

where ${\rm div}_{\rm S}:={\rm Tr}(\nabla_{\rm S})$ with ${\rm Tr}(\cdot)$ being the trace of a $(n{-}1)\times(n{-}1)$-matrix denotes the $(n{-}1)$-dimensional *surface divergence* so that ${\rm div}_{\rm S}\nu$ is (up to a factor $-\frac12$) the mean curvature of the surface Γ. Substituting it into (2.99), one obtains

$$\begin{aligned}&\int_\Omega a(x,u,\nabla u,\nabla^2 u):\nabla^2 v+\big(c(x,u,\nabla u,\nabla^2 u)-g\big)v\,{\rm d}x\\ &=\int_\Gamma\Big(\nu^\top a(x,u,\nabla u,\nabla^2 u)\nu\Big)\frac{\partial v}{\partial\nu}-\Big({\rm div}\big(a(x,u,\nabla u,\nabla^2 u)\big){\cdot}\nu\\ &\quad+{\rm div}_{\rm S}\big(a(x,u,\nabla u,\nabla^2 u)\nu\big)-\big({\rm div}_{\rm S}\nu\big)\big(\nu^\top a(x,u,\nabla u,\nabla^2 u)\nu\big)\Big)v\,{\rm d}S.\end{aligned} \tag{2.104}$$

This allows us to cast a natural *higher-order Dirichlet/Newton boundary condition*:

$$\frac{\partial u}{\partial\nu}\Big|_\Gamma=u_{\rm D}'\qquad\text{and} \tag{2.105a}$$

$$\begin{aligned}&{\rm div}\big(a(x,u,\nabla u,\nabla^2 u)\big){\cdot}\nu+{\rm div}_{\rm S}\big(a(x,u,\nabla u,\nabla^2 u)\nu\big)\\ &\qquad-\big({\rm div}_{\rm S}\nu\big)\big(\nu^\top a(x,u,\nabla u,\nabla^2 u)\nu\big)+b(x,u,\nabla u)=h\qquad\text{on }\Gamma.\end{aligned} \tag{2.105b}$$

[21] This "surface" Green-type formula reads as $\int_\Gamma w{:}((\nabla_{\rm S}v)\otimes\nu)\,{\rm d}S=\int_\Gamma({\rm div}_{\rm S}\nu)(w{:}(\nu\otimes\nu))v-{\rm div}_{\rm S}(w{\cdot}\nu)v\,{\rm d}S$. In the vectorial variant, this is used in mechanics of complex (also called nonsimple) continua, cf. [153, 337, 407]. For even $2k$-order problems with $k>2$ see also [244].

The underlying Banach space is then considered as $V = \{v\in W^{2,p}(\Omega);\ \frac{\partial v}{\partial \nu} = 0\}$ and the weak formulation is based on the integral identity:

$$\int_\Omega a(x,u,\nabla u,\nabla^2 u):\nabla^2 v + c(x,u,\nabla u,\nabla^2 u)v\,\mathrm{d}x \\ + \int_\Gamma b(x,u,\nabla u)v\,\mathrm{d}S = \int_\Omega gv\,\mathrm{d}x + \int_\Gamma hv\,\mathrm{d}S. \qquad (2.106)$$

Eventually, the formula (2.104) reveals also the natural form of *Newton*-type *boundary conditions*:

$$\mathrm{div}\big(a(x,u,\nabla u,\nabla^2 u)\big)\cdot\nu + \mathrm{div}_{\mathrm{S}}\big(a(x,u,\nabla u,\nabla^2 u)\nu\big) \\ - \big(\mathrm{div}_{\mathrm{S}}\nu\big)\big(\nu^\top a(x,u,\nabla u,\nabla^2 u)\nu\big) + b_0(x,u,\nabla u) = h_0 \quad \text{and} \qquad (2.107\text{a})$$

$$\nu^\top a(x,u,\nabla u,\nabla^2 u)\,\nu + b_1(x,u,\nabla u) = h_1 \quad \text{on } \Gamma. \qquad (2.107\text{b})$$

The underlying Banach space can then be considered as $V = W^{2,p}(\Omega)$. The resulting weak formulation of the boundary-value problem (2.97)–(2.107) then employs the integral identity:

$$\int_\Omega a(x,u,\nabla u,\nabla^2 u):\nabla^2 v + c(x,u,\nabla u,\nabla^2 u)v\,\mathrm{d}x \\ + \int_\Gamma b_0(x,u,\nabla u)v + b_1(x,u,\nabla u)\frac{\partial v}{\partial \nu}\,\mathrm{d}S = \int_\Omega gv\,\mathrm{d}x + \int_\Gamma h_0 v + h_1\frac{\partial v}{\partial \nu}\,\mathrm{d}S. \quad (2.108)$$

For nonsmooth boundaries, these arguments based on formula (2.104) are no longer valid however and additional boundary terms can be seen; cf. [338] for boundaries with edges.

We will modify the Leray-Lions' Theorem 2.36 for the case of the Dirichlet conditions (2.100). Let us write naturally[22] $p^{**} := (p^*)^*$ and $p^{*\#} := (p^*)^\#$. For simplicity, the assumptions are not the most general in the following assertion, whose proof, paraphrasing that of Theorem 2.36, is omitted here.

Proposition 2.42 (Existence for Dirichlet problem). *Let* $a(x,r,s,\cdot) : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ *be strictly monotone,*

$$\exists k\in L^1(\Omega),\ 1<q\le p:\ a(x,r,s,S){:}S + c(x,r,s,S)r \ge \varepsilon|S|^p + \varepsilon|r|^q - k(x), \quad (2.109\text{a})$$

$$\exists \gamma\in L^{p'}(\Omega): \quad |a(x,r,s,S)| \le \gamma(x) + C|r|^{(p^{**}-\epsilon)/p'} \\ + C|s|^{(p^*-\epsilon)/p'} + C|S|^{p-1}, \quad (2.109\text{b})$$

$$\exists \gamma\in L^{p^{**\prime}+\epsilon}(\Omega): \quad |c(x,r,s,S)| \le \gamma(x) + C|r|^{p^{**}-\epsilon-1} \\ + C|s|^{(p^*-\epsilon)/p^{**\prime}} + C|S|^{(p-\epsilon)/p^{**\prime}}, \quad (2.109\text{c})$$

[22] This means $p^{**} = np/(n-2p)$ if $p < n/2$ or $p^{**} < +\infty$ if $p = n/2$ or $p^{**} = +\infty$ if $p > n/2$, cf. Corollary 1.22 for $k = 2$. For $p^{*\#} = (np-p)/(n-2p)$ if $p < n/2$, cf. Exercise 2.70.

with some $C\in\mathbb{R}^+$ *and* $\varepsilon,\epsilon>0$ *and again the Convention* 2.26 *(now concerning* $p^{**}=+\infty$ *for* $p>n/2$*), and let* $u_{\rm D}=v|_\Gamma$ *and* $u'_{\rm D}=\frac{\partial v}{\partial\nu}$ *for some* $v\in W^{2,p}(\Omega)$*, and* $g\in L^{p^{**\prime}}(\Omega)$*. Then the boundary-value problem* (2.97) *with* (2.100) *has a weak solution, i.e.* (2.99) *holds for all* $v\in W^{2,p}_0(\Omega)$ *together with the boundary conditions* (2.97).

For the Newton boundary conditions (2.107), the analog of the existence assertion looks as follows:

Proposition 2.43 (Existence for Newton problem). *Let a, c, and g be as in Proposition 2.42 and satisfy (2.109), and let* b_0 *and* b_1 *satisfy*

$$\exists k\in L^1(\Gamma):\qquad b_0(x,r,s)r+b_1(x,r,s)(s\cdot\nu(x))\ge -k(x),\tag{2.110a}$$

$$\exists\gamma\in L^{p^{*\#\prime}}(\Gamma):\qquad \big|b_0(x,r,s)\big|\le\gamma(x)+C|r|^{p^{*\#}-\epsilon-1}+C|s|^{(p^\#-\epsilon)/p^{*\#}},\tag{2.110b}$$

$$\exists\gamma\in L^{p^{\#\prime}}(\Gamma):\qquad \big|b_1(x,r,s)\big|\le\gamma(x)+C|r|^{(p^{*\#}-\epsilon)/p^{\#\prime}}+C|s|^{p^\#-\epsilon-1}\tag{2.110c}$$

with some $C\in\mathbb{R}^+$ *and* $\epsilon>0$*, and let* $h_0\in L^{p^{*\#\prime}}(\Gamma)$ *and* $h_1\in L^{p^{\#\prime}}(\Gamma)$*. Then the boundary-value problem* (2.97) *with* (2.107) *has a weak solution, i.e.* (2.108) *holds for all* $v\in W^{2,p}_0(\Omega)$*.*

The modification for other boundary conditions (2.101) or (2.105) can easily be cast and is left as an exercise.

As pointed out before, one should care about consistency and selectivity of the definitions of weak solutions. Consistency is guaranteed by the derivation of the weak formulation itself. Let us illustrate the selectivity, i.e. an analog of Proposition 2.25, on the most complicated case of the Newton boundary-value problem:

Proposition 2.44 (Selectivity of the weak-solution definition). *Let* Γ *be smooth,* $a\in C^2(\bar\Omega\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n\times n};\mathbb{R}^{n\times n})$*,* $c\in C^0(\bar\Omega\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n\times n})$*, and* $b_0,b_1\in C^0(\bar\Gamma\times\mathbb{R}\times\mathbb{R}^n)$*,* $g\in C(\bar\Omega)$*, and* $h_0,h_1\in C(\Gamma)$*. Then any weak solution* $u\in C^4(\bar\Omega)$ *of the boundary-value problem* (2.97) *with* (2.107) *is the also classical solution.*

Proof. Put $v\in V=W^{2,p}(\Omega)$ into (2.108) and use Green's formula (1.54) twice, as well as the surface Green formula (2.103). One gets

$$\begin{aligned}&\int_\Omega\Big({\rm div}^2a(x,u,\nabla u)+c(x,u,\nabla u)-g\Big)v\,{\rm d}x\\&\quad+\int_\Gamma\Big({\rm div}\big(a(x,u,\nabla u,\nabla^2u)\big)\cdot\nu+{\rm div}_{\rm S}\big(a(x,u,\nabla u,\nabla^2u)\nu\big)\\&\qquad-\big({\rm div}_{\rm S}\nu\big)\big(\nu^\top a(x,u,\nabla u,\nabla^2u)\nu\big)+b_0(x,u,\nabla u)-h_0\Big)v\,{\rm d}S\\&\qquad+\int_\Gamma\Big(\nu^\top a(x,u,\nabla u,\nabla^2u)\,\nu+b_1(x,u,\nabla u)-h_1\Big)\frac{\partial v}{\partial\nu}\,{\rm d}S=0.\end{aligned}\tag{2.111}$$

Considering v with a compact support in Ω, one has $v|_\Gamma = 0 = \frac{\partial v}{\partial\nu}$ and both boundary integrals in (2.111) vanish. As v is otherwise arbitrary, one deduces that (2.97) holds a.e., and hence even everywhere in Ω due to the assumed smoothness of a and c. Hence, the first integral in (2.111) vanishes. Then, put a more general $v \in V$ into (2.111) but still such that $\frac{\partial v}{\partial\nu} = 0$. Thus the second boundary integral in (2.111) vanishes. From the first boundary integral, we recover the boundary condition (2.107a).[23] Due to the assumed smoothness of a and continuity of b_0 and h_0, (2.107a) holds pointwise. Finally, we can take $v \in V$ fully general. Knowing already that the first and the second integral in (2.111) vanish, from the last integral we can recover the remaining boundary condition (2.107b).[24] $\square$

Remark 2.45 (Other boundary conditions)**.** The above four combinations of boundary conditions still do not represent the whole class of variationally consistent boundary conditions for equation (2.97). For $\alpha_0, \alpha_1 \in L^\infty(\Gamma)$, one can consider a combined condition composed from (2.101) and (2.105), namely

$$\alpha_1\frac{\partial u}{\partial\nu} + \alpha_0 u = u_{\rm D} \qquad \text{and} \tag{2.112a}$$

$$\begin{aligned}&\alpha_1\Big(\mathrm{div}\big(a(x,u,\nabla u,\nabla^2 u)\big)\cdot\nu + \mathrm{div}_{\rm S}\big(a(x,u,\nabla u,\nabla^2 u)\nu\big)\\ &\qquad\quad - \big(\mathrm{div}_{\rm S}\nu\big)\big(\nu^\top a(x,u,\nabla u,\nabla^2 u)\nu\big)\Big)\\ &\qquad\quad + \alpha_0\nu^\top a(x,u,\nabla u,\nabla^2 u)\,\nu + b(x,u,\nabla u) = h \qquad \text{on } \Gamma.\end{aligned} \tag{2.112b}$$

The underlying Banach space is then $V = \{v \in W^{2,p}(\Omega);\ \alpha_1\frac{\partial u}{\partial\nu} + \alpha_0 u = 0\}$ and the weak formulation is again (2.106) with $b = b_2/\alpha_1$ and $h = h_2/\alpha_1$ provided $\alpha_1 \neq 0$. Alternatively, for $\alpha_0 \neq 0$ one can rather pursue the weak formulation based on (2.102).

Example 2.46 (p-biharmonic operator)**.** A concrete choice of a from (2.97)

$$a_{ij}(x,r,s,S) := \begin{cases} \big|\sum_{k=1}^n S_{kk}\big|^{p-2}\sum_{k=1}^n S_{kk} & \text{for } i=j,\\ 0 & \text{for } i\neq j,\end{cases} \tag{2.113}$$

converts $\mathrm{div}\,\mathrm{div}\,a(x,u,\nabla u,\nabla^2 u)$ into the so-called p-biharmonic operator $\Delta(|\Delta u|^{p-2}\Delta u)$. Applying Green's formula twice to this operator tested by v yields the identity

[23] Here the important fact is that the set $\{v|_\Gamma;\ v \in V,\ \frac{\partial v}{\partial\nu} = 0\}$ is still dense in $L^1(\Gamma)$. Indeed, any $v \in W^{1,2}(\Omega)$ can be modified to u_ε so that $(v_\varepsilon - v)|_\Gamma$ is small but $\frac{\partial}{\partial\nu}v_\varepsilon = 0$ on Γ. To outline this procedure, first we rectify Γ locally so that we can consider a half-space, cf. Fig. 8 on p. 91 below, then extend v by reflection of v with respect to Γ, and eventually mollify the extended v.

[24] Here the important fact is that the set $\{\frac{\partial v}{\partial\nu};\ v \in V\}$ is dense in $L^1(\Gamma)$, which can be seen by a local rectification of Γ and by an explicit construction of v in the vicinity of Γ with a given smooth $\frac{\partial v}{\partial\nu}$ and, e.g., zero trace on Γ.

$$\begin{aligned}\int_\Omega & \Delta\big(|\Delta u|^{p-2}\Delta u\big) v \,\mathrm{d}x \\ &= -\int_\Omega \nabla\big(|\Delta u|^{p-2}\Delta u\big)\cdot\nabla v \,\mathrm{d}x + \int_\Gamma \frac{\partial}{\partial\nu}\big(|\Delta u|^{p-2}\Delta u\big)\, v \,\mathrm{d}S \\ &= \int_\Omega |\Delta u|^{p-2}\Delta u \Delta v \,\mathrm{d}x + \int_\Gamma \frac{\partial}{\partial\nu}\big(|\Delta u|^{p-2}\Delta u\big)\, v - |\Delta u|^{p-2}\Delta u \frac{\partial v}{\partial\nu}\,\mathrm{d}S, \quad (2.114)\end{aligned}$$

from which, besides the Dirichlet conditions (2.100), one can pose naturally also Dirichlet/Newton conditions (2.101) now in the form

$$u|_\Gamma = u_{\mathrm{D}} \qquad \text{and} \qquad |\Delta u|^{p-2}\Delta u + b(x,u,\nabla u) = h, \tag{2.115}$$

or the higher Dirichlet/Newton conditions (2.105) now in the simpler form

$$\frac{\partial u}{\partial\nu}\Big|_\Gamma = u'_{\mathrm{D}} \qquad \text{and} \qquad \frac{\partial}{\partial\nu}\big(|\Delta u|^{p-2}\Delta u\big) + b(x,u,\nabla u) = h, \tag{2.116}$$

or also the Newton condition (2.107) now in the simpler form

$$\frac{\partial}{\partial\nu}\big(|\Delta u|^{p-2}\Delta u\big) + b_0(x,u,\nabla u) = h_0, \quad |\Delta u|^{p-2}\Delta u + b_1(x,u,\nabla u) = h_1. \tag{2.117}$$

Note that (2.116) and (2.117) do not contain the $\mathrm{div}_{\mathrm{S}}$-terms because, instead of $\nu\otimes\nabla v$ in (2.99), one has $\nu\cdot\nabla v = \frac{\partial v}{\partial\nu}$ in (2.114). The pointwise coercivity (2.109a) cannot be satisfied for (2.113), however, and the coercivity of A on V must rely on a delicate interplay with the boundary conditions. E.g., for Dirichlet conditions (2.100) with $u_{\mathrm{D}} = 0 = u'_{\mathrm{D}}$ and for $p = 2$, one has by using Green's formula twice $\langle A(u),u\rangle = \int_\Omega |\Delta u|^2 \mathrm{d}x = -\int_\Omega \nabla u\cdot\nabla\Delta u \mathrm{d}x = -\int_\Omega \nabla u\cdot\mathrm{div}(\nabla^2 u)\,\mathrm{d}x = \int_\Omega |\nabla^2 u|^2 \mathrm{d}x$, which thus controls $\nabla^2 u$ in $L^2(\Omega;\mathbb{R}^{n\times n})$. Another example is the Newton's condition (2.117) with $b_0(x,r,s) = \beta_0(x)r$, $b_1(x,r,s) = -\beta_1(x)(s\cdot\nu)$, and $p = 2$, one has $\langle A(u),u\rangle = \int_\Omega |\Delta u|^2 \mathrm{d}x + \int_\Gamma \beta_0 u^2 + \beta_1(\frac{\partial}{\partial\nu}u)^2 \mathrm{d}S$. This is a continuous quadratic form on $W^{2,2}(\Omega)$ and for the Poincaré-like inequality $\langle A(u),u\rangle \geq C_{\mathrm{P}}\|u\|^2_{W^{2,2}(\Omega)}$ it suffices to guarantee that $\langle A(u),u\rangle = 0$ implies $u = 0$. This can be done by assuming $\beta_0,\beta_1 \geq 0$, and β_0 or β_1 positive on a "sufficiently large part" of Γ.[25]

2.5 Weakly continuous mappings, semilinear equations

In case that A is coercive and, instead of being pseudomonotone, is weakly continuous, we can prove existence of a solution to $A(u) = f$ much more easily. Although the assumption of the weak continuity is restrictive, such mappings enjoy still a considerably large application area. Here, we can even advantageously generalize the concept for mappings $A : V \to Z^*$ for some Banach space $Z \subset V$ densely so that $Z^* \supset V^*$. If $V_k \subset Z$ for any $k \in \mathbb{N}$, we can modify (2.5) and then Theorem 2.6:

[25] Here, a certain caution is advisable: e.g. for Ω a square $[0,1]^2$, it is not sufficient if $\beta_0(\cdot) = 1$ on the sides with $x_1 = 0$ and $x_2 = 0$ and otherwise β_0 and β_1 vanishes because of existence of a non-vanishing function $u(x) = x_1 x_2$ for which $\langle A(u),u\rangle = 0$.

Proposition 2.47 (Existence). *If a weakly continuous mapping $A : V \to Z^*$ is coercive in the modified sense*

$$\lim_{\substack{\|v\|_V \to \infty \\ v \in Z}} \frac{\langle A(v), v\rangle_{Z^* \times Z}}{\|v\|_V} = +\infty, \tag{2.118}$$

and if $f \in V^*$, then the equation $A(u) = f$ has a solution.

Proof. The technique of the proof of Theorem 2.6 allows for a very simple modification: instead of (2.14), we consider the Galerkin identity (2.8) as $\langle A(u_k) - f, v_k\rangle_{Z^* \times Z} = 0$ for $v_k \in V_k$ such that $v_k \to v$ in Z, and make a direct limit passage. Note that (2.13) looks now as

$$\zeta(\|u_k\|_V)\big\|u_k\big\|_V \leq \big\langle A(u_k), u_k\big\rangle_{Z^* \times Z} = \big\langle f, u_k\big\rangle_{Z^* \times Z} = \big\langle f, u_k\big\rangle_{V^* \times V} \leq \big\|f\big\|_{V^*}\big\|u_k\big\|_V$$

and again yields $\{u_k\}_{k \in \mathbb{N}}$ bounded in V because $f \in V^*$. □

Confining ourselves again to the 2nd-order problems as in Sections 2.4.1–2.4.3, we can easily use this concept for the special case when $a(x, r, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ and $c(x, r, \cdot) : \mathbb{R}^n \to \mathbb{R}$ are affine, we will call such problems as *semilinear* although sometimes this adjective needs still $a(x, \cdot, s)$ constant as in (0.1). So, here

$$a_i(x, r, s) := \sum_{j=1}^{n} a_{ij}(x, r)s_j + a_{i0}(x, r), \quad i = 1, \dots, n, \tag{2.119a}$$

$$c(x, r, s) := \sum_{j=1}^{n} c_j(x, r)s_j + c_0(x, r), \tag{2.119b}$$

with $a_{ij}, c_j : \Omega \times R \to \mathbb{R}$ Carathéodory mappings whose growth is now to be designed to induce the Nemytskiĭ mappings $\mathscr{N}_{(a_{i1},\dots,a_{in})}, \mathscr{N}_{(c_1,\dots,c_n)} : L^{2^*-\epsilon}(\Omega) \to L^2(\Omega; \mathbb{R}^n)$ and $\mathscr{N}_{a_{i0}}, \mathscr{N}_{c_0} : L^{2^*-\epsilon}(\Omega) \to L^1(\Omega)$ with $\epsilon > 0$. Besides, the boundary nonlinearity $b : \Gamma \times \mathbb{R} \to \mathbb{R}$ is now to induce the Nemytskiĭ mapping $\mathscr{N}_b : L^{2^\#-\epsilon}(\Gamma) \to L^1(\Gamma)$. This means, for $i, j = 1, \dots, n$,

$$\begin{aligned}
&\exists \gamma_1 \in L^2(\Omega),\ C \in \mathbb{R}: && |a_{ij}(x, r)| \leq \gamma_1(x) + C|r|^{(2^*-\epsilon)/2}, \\
& && |c_j(x, r)| \leq \gamma_1(x) + C|r|^{(2^*-\epsilon)/2}, && (2.120a)\\
&\exists \gamma_2 \in L^1(\Omega),\ C \in \mathbb{R}: && |a_{i0}(x, r)| \leq \gamma_2(x) + C|r|^{2^*-\epsilon}, \\
& && |c_0(x, r)| \leq \gamma_2(x) + C|r|^{2^*-\epsilon}, && (2.120b)\\
&\exists \gamma_3 \in L^1(\Gamma),\ C \in \mathbb{R}: && |b(x, r)| \leq \gamma_3(x) + C|r|^{2^\#-\epsilon}. && (2.120c)
\end{aligned}$$

The exponent $p = 2$ is natural because $a(x, r, \cdot)$ has now a linear growth. Note that these requirements just guarantee that all integrals in (2.51) have a good sense if $v \in W^{1,\infty}(\Omega) =: Z$. Again, Convention 2.26 on p. 46 is considered.

Lemma 2.48 (Weak continuity of A). *Let* (2.119)–(2.120) *hold. Then A is weakly* continuous as a mapping $W^{1,2}(\Omega) \to W^{1,\infty}(\Omega)^*$.*

Proof. Having a weakly convergent sequence $\{u_k\}_{k\in\mathbb{N}}$ in $W^{1,2}(\Omega)$, this sequence converges strongly in $L^{2^*-\epsilon}(\Omega)$. Then, by the continuity of the Nemytskiĭ mappings $\mathscr{N}_{(a_{i1},\dots,a_{in})}, \mathscr{N}_{(c_1,\dots,c_n)} : L^{2^*-\epsilon}(\Omega) \to L^2(\Omega;\mathbb{R}^n)$ and $\mathscr{N}_{a_{i0}}, \mathscr{N}_{c_0} : L^{2^*-\epsilon}(\Omega) \to L^1(\Omega)$, it holds that

$$\lim_{k\to\infty}\int_\Omega \sum_{i=1}^n\Big(\sum_{j=1}^n a_{ij}(u_k)\frac{\partial u_k}{\partial x_j}+a_{i0}(u_k)\Big)\frac{\partial v}{\partial x_i}+\Big(\sum_{j=1}^n c_j(u_k)\frac{\partial u_k}{\partial x_j}+c_0(u_k)\Big)v\,\mathrm{d}x$$
$$=\int_\Omega \sum_{i=1}^n\Big(\sum_{j=1}^n a_{ij}(u)\frac{\partial u}{\partial x_j}+a_{i0}(u)\Big)\frac{\partial v}{\partial x_i}+\Big(\sum_{j=1}^n c_j(u)\frac{\partial u}{\partial x_j}+c_0(u_k)\Big)v\,\mathrm{d}x$$

for $k\to\infty$ and any $v \in W^{1,\infty}(\Omega)$. Also $u_k|_\Gamma \to u|_\Gamma$ in $L^{2^\#-\epsilon}(\Gamma)$, and, by (2.120c), we have convergence in the boundary term $\int_\Gamma b(u_k)v\,\mathrm{d}S \to \int_\Gamma b(u)v\,\mathrm{d}S$. $\square$

Proposition 2.49 (Existence of weak solutions). *Let* (2.119)–(2.120) *hold, $g\in L^{2^{*\prime}}(\Omega)$, $h\in L^{2^{\#\prime}}(\Gamma)$, and, for some $\varepsilon>0$, $\gamma_1\in L^2(\Omega)$, $\gamma_2\in L^1(\Omega)$, $\gamma_3\in L^1(\Gamma)$, and for a.a. $x\in\Omega$ (resp. $x\in\Gamma$ for* (2.121b)*) and all $(r,s)\in\mathbb{R}^{1+n}$, it holds that*

$$\sum_{i=1}^n\Big(\sum_{j=1}^n a_{ij}(x,r)s_j+a_{i0}(x,r)\Big)s_i+\Big(\sum_{j=1}^n c_j(x,r)s_j+c_0(x,r)\Big)r$$
$$\ge \varepsilon|s|^2+\varepsilon|r|^2-\gamma_1(x)|s|-\gamma_2(x), \qquad (2.121a)$$
$$b(x,r)r \ge -\gamma_3(x). \qquad (2.121b)$$

Then the boundary-value problem (2.45) *with* (2.49) *has a weak solution in the sense of Definition* 2.24 *using now $v \in W^{1,\infty}(\Omega)$.*

Proof. We can use the abstract Proposition 2.47 now with $V := W^{1,2}(\Omega)$, $Z := W^{1,\infty}(\Omega)$, and V_k some finite-dimensional subspaces of $W^{1,\infty}(\Omega)$ satisfying (2.7).[26] The coercivity (2.118) is implied by (2.121) by routine calculations.[27] Then we use Lemma 2.48 and Proposition 2.47. $\square$

Remark 2.50 (Conventional weak solutions). Let, in addition to the assumptions of Proposition 2.49, also the growth condition (2.55) with $p = 2$ hold. Then the solution obtained in Proposition 2.49 allows for $v \in W^{1,2}(\Omega)$ in Definition 2.24.

[26] Such subspaces always exists, e.g. one can imagine subspaces as in Example 2.67.

[27] We have $\langle A(v),v\rangle=\int_\Omega\sum_{i=1}^n\big(\sum_{j=1}^n a_{ij}(v)\frac{\partial}{\partial x_j}v+a_{i0}(v)\big)\frac{\partial}{\partial x_i}v+\big(\sum_{j=1}^n c_j(v)\frac{\partial}{\partial x_j}v+c_0(v)\big)v\,\mathrm{d}x+\int_\Gamma b(v)v\,\mathrm{d}S \ge \varepsilon\|\nabla v\|^2_{L^2(\Omega;\mathbb{R}^n)}-\frac{1}{2\varepsilon}\|\gamma_1\|^2_{L^2(\Omega)}-\frac{\varepsilon}{2}\|\nabla v\|^2_{L^2(\Omega;\mathbb{R}^n)}-\|\gamma_2\|_{L^1(\Omega)}-\gamma_3\|v\|^1_{L^1(\Gamma)}$.

2.6 Examples and exercises

This section contains both exercises to make the above presented theory more complete and some examples of analysis of concrete semi- and quasi-linear equations. The exercises will mostly be accompanied by brief hints in the footnotes.

2.6.1 General tools

Exercise 2.51 (*Banach's selection principle)*. Assuming the sequential compactness of closed bounded intervals in $\mathbb{R}$ is known, prove Banach's Theorem 1.7 by a suitable diagonalization procedure.[28]

Exercise 2.52 (*Uniform convexity of Hilbert spaces)*. For V being a Hilbert space, prove the assertion of Theorem 1.2 directly.[29] Using (1.4), prove that any Hilbert space is uniformly convex.[30]

Exercise 2.53 (*Pseudomonotonicity)*. Assuming (2.3a), show that (2.3b) is equivalent to[31]

$$\left.\begin{array}{c} u_k \rightharpoonup u, \\ \limsup\limits_{k\to\infty}\big\langle A(u_k), u_k{-}u\big\rangle \le 0 \end{array}\right\} \quad\Rightarrow\quad \left\{\begin{array}{c} \text{w-}\lim\limits_{k\to\infty} A(u_k) = A(u), \\ \lim\limits_{k\to\infty}\big\langle A(u_k), u_k\big\rangle = \big\langle A(u), u\big\rangle. \end{array}\right. \tag{2.122}$$

Exercise 2.54 (Weakening of pseudomonotonicity)**.** Modify the proof of Brézis Theorem 2.6 for A coercive, bounded, demicontinuous, and satisfying[32]

$$\left.\begin{array}{l} u_k \rightharpoonup u \quad\ \ \& \quad\ \ A(u_k) \rightharpoonup f \\ \limsup\limits_{k\to\infty}\big\langle A(u_k), u_k\big\rangle \le \langle f, u\rangle \end{array}\right\} \quad\Rightarrow\quad f = A(u). \tag{2.123}$$

[28]Hint: Consider a sequence $\{f_k\}_{k\in\mathbb{N}}$ bounded in V^* and a countable dense subset $\{v_k\}_{k\in\mathbb{N}}$ in V, take v_1 and select an infinite subset $A_1 \subset \mathbb{N}$ such that the sequence of real numbers $\{\langle f_k, v_1\rangle\}_{k\in A_1}$ converges in $\mathbb{R}$ to some $f(v_1)$, then take v_2 and select an infinite subset $A_2 \subset A_1$ such that $\{\langle f_k, v_2\rangle\}_{k\in A_2}$ converges to some $f(v_2)$, etc. for v_3, v_4, $\ldots$. Then make a diagonalization procedure by taking l_k the first number in A_k which is greater than k. Then $\{\langle f_{l_k}, v_i\rangle\}_{k\in\mathbb{N}}$ converge to $f(v_i)$ for all $i \in \mathbb{N}$. Show that f is linear on $\mathrm{span}(\{v_i\}_{i\in\mathbb{N}})$ and bounded because $|f(v_i)| \le \lim_{k\to\infty} |\langle f_{l_k}, v_i\rangle| \le \limsup_{k\to\infty} ||f_k||_*||v_i||$, and finally extend f on the whole V^* just by continuity.

[29]Hint: $\|u_k\|\to\|u\|$ and $u_k\rightharpoonup u$ imply $\|u_k-u\|^2 = \|u_k\|^2 + (u-2u_k, u) \to \|u\|^2 + (u-2u, u) = 0$.

[30]Hint: Realize that $\|u\| = 1 = \|v\|$ and $\|u - v\| \ge \varepsilon$ in (1.5) imply $\frac{1}{2}\|u + v\| = \sqrt{(u, v) - \|u-v\|^2/4} \le \sqrt{1 - \varepsilon^2/4} \le 1 - \delta$ provided $0 < \delta \le 1$ solves $\delta^2 - 2\delta + \varepsilon^2/4 = 0$. Such δ exists if $0 < \varepsilon \le 2$, while for $\varepsilon > 2$ the implication (1.5) is trivial.

[31]Hint: (2.122)$\Rightarrow$(2.3b) is trivial. The converse implication: by (2.3a), assume $A(u_k) \rightharpoonup f$ (a subsequence), then $0 \ge \limsup_{k\to\infty}\langle A(u_k), u_k{-}u\rangle = \limsup_{k\to\infty}\langle A(u_k), u_k\rangle - \langle f, u\rangle$ implies $\langle A(u), u - v\rangle \le \liminf_{k\to\infty}\langle A(u_k), u_k{-}v\rangle \le \limsup_{k\to\infty}\langle A(u_k), u_k\rangle - \langle f, v\rangle \le \langle f, u{-}v\rangle$, from which $A(u) = f$, hence $A(u_k) \rightharpoonup f$ (the whole sequence), and eventually (2.3b) for $v = 0$ yields

$$\langle A(u), u\rangle \le \liminf_{k\to\infty}\langle A(u_k), u_k\rangle \le \limsup_{k\to\infty}\langle A(u_k), u_k\rangle \le \lim_{k\to\infty}\langle A(u_k), u\rangle = \langle A(u), u\rangle.$$

[32]Hint: Modify Step 4 of the proof of Theorem 2.6: as both $\{u_k\}_{k\in N}$ and A are bounded, $A(u_k) \rightharpoonup \chi$ (as a subsequence) and, from (2.8), $\chi = f$, hence $A(u_k) \rightharpoonup f$ (the whole sequence) and, again by (2.8), $\langle A(u_k), u_k\rangle = \langle f, u_k\rangle \to \langle f, u\rangle$. Then by (2.123) $f = A(u)$.

Show that any pseudomonotone A satisfies (2.123).[33]

Exercise 2.55 (Tikhonov-type modification[34] of Schauder's Theorem 1.9)**.** Assuming a reflexive separable Banach space $V \Subset V_1$, show that a weakly continuous mapping $M : V \to V$ which maps a ball B in V into itself has a fixed point.[35]

Exercise 2.56 (Direct method for A weakly continuous)**.** Assume $A : V \to V^*$ weakly continuous, V Hilbert, and modify the Brézis Theorem 2.6 by using directly Schauder fixed-point Theorem 1.9 without approximating the problem.[36]

Exercise 2.57. Try to make a limit passage in (2.38)–(2.39) simultaneously in i and l by considering $i = l$. Realize why it was necessary to make the double limit $\lim_{l\to\infty} \lim_{i\to\infty}$ instead of $\lim_{l=i\to\infty}$ in the proof of Proposition 2.17.

Exercise 2.58. Assuming $1 \le q \le p < +\infty$, evaluate the norms of the continuous embeddings $L^\infty(\Omega) \subset L^p(\Omega) \subset L^q(\Omega)$.[37]

Exercise 2.59 (Interpolation of Lebesgue spaces)**.** Prove (1.23) by using Hölder's inequality.[38]

Exercise 2.60 (Continuity of Nemytskiĭ mappings)**.** Show that the Nemytskiĭ mapping $\mathscr{N}_a$ with a satisfying (1.48) is a bounded continuous mapping $L^{p_1}(\Omega) \times$

[33] Hint: The premise of (2.123) and the pseudomonotonicity implies $\limsup_{k\to\infty}\langle A(u_k), u_k - u\rangle = \limsup_{k\to\infty}\langle A(u_k), u_k\rangle - \lim_{k\to\infty}\langle A(u_k), u\rangle \le \langle f, u\rangle - \langle f, u\rangle = 0$ so that, by (2.3b), $\langle A(u), u - v\rangle \le \liminf_{k\to\infty}\langle A(u_k), u_k - v\rangle \le \limsup_{k\to\infty}\langle A(u_k), u_k - v\rangle \le \langle f, u - v\rangle$ for any $v \in V$, from which $f = A(u)$ indeed follows.

[34] Tikhonov [413] proved a bit more general assertion, known now as Tikhonov's theorem: a continuous mapping from a compact subset of a locally convex space into itself has a fixed point.

[35] Hint: Consider B endowed with a weak topology, realize that $u_k \to u$ in V_1 and $u_k \in B$ implies $u_k \rightharpoonup u$ in B, hence $M(u_k) \rightharpoonup M(u)$ in V and then also $M(u_k) \to M(u)$ in V_1, and then use Schauder's Theorem 1.9.

[36] Hint: Repeat Step 2 of the proof of Brézis Theorem 2.6 directly for V instead of V_k. Use the weak topology on $\{v \in V;\ \|v\| \le \varrho\}$, and realize that I_k is to be omitted while J_k^{-1} is to be weakly continuous (which really is due to its demicontinuity, cf. Corollary 3.3 below, and its linearity, cf. Remark 3.10). Also use Exercise 2.55.

[37] Hint: Estimate

$$\|u\|_{L^p(\Omega)} = \sqrt[p]{\int_\Omega |u|^p \mathrm{d}x} \le \sqrt[p]{\int_\Omega \operatorname*{ess\,sup}_{\xi\in\Omega} |u(\xi)|^p \mathrm{d}x} = \sqrt[p]{\|u\|^p_{L^\infty(\Omega)} \int_\Omega 1 \mathrm{d}x} = N\|u\|_{L^\infty(\Omega)}$$

with $N = \big(\mathrm{meas}_n(\Omega)\big)^{1/p}$ being the norm of the embedding $L^\infty(\Omega) \subset L^p(\Omega)$. Likewise, by Hölder's inequality,

$$\|u\|^q_{L^q(\Omega)} = \int_\Omega 1 \cdot |u|^q \mathrm{d}x \le \sqrt[(p/q)']{\int_\Omega 1 \mathrm{d}x}\ \sqrt[p/q]{\int_\Omega |u|^p \mathrm{d}x} = N^q \|u\|^q_{L^p(\Omega)}$$

with $N = \big(\mathrm{meas}_n(\Omega)\big)^{(p-q)/(pq)}$.

[38] Hint: Use Hölder's inequality for

$$\int_\Omega |v|^p \mathrm{d}x = \int_\Omega |v|^{\lambda p}|v|^{(1-\lambda)p} \mathrm{d}x \le \Big(\int_\Omega |v|^{\lambda p\alpha} \mathrm{d}x\Big)^{1/\alpha} \Big(\int_\Omega |v|^{(1-\lambda)p\beta} \mathrm{d}x\Big)^{1/\beta}$$

with a suitable $\alpha = p_1/(\lambda p)$ and $\beta = p_2/((1-\lambda)p)$, namely $\alpha^{-1} + \beta^{-1} = 1$ which just means that p satisfies the premise in (1.23).

$L^{p_2}(\Omega;\mathbb{R}^n) \to L^{p_0}(\Omega)$.[39]

Exercise 2.61. Show that p_3 in Theorem 1.27 indeed cannot be $+\infty$: find some a satisfying (1.48) for $p_1, p_2 < +\infty$ and $p_3 = +\infty$ such that $\mathscr{N}_a$ is not continuous.[40]

Exercise 2.62. Show that, for any $c : \mathbb{R}^{m_1} \to \mathbb{R}^{m_2}$ not affine, the Nemytskiĭ mapping $\mathscr{N}_c : u \mapsto c(u)$ is not weakly continuous; modify Figure 3 on p.20.[41]

Exercise 2.63 (Integral balance (2.63))**.** Consider the test volume in the integral balance (2.63) as a ball $O = \{x;\ |x - x_0| \leq \varrho\}$ and derive (2.63) for a.a. ϱ by a limit passage in the weak formulation (2.45) tested by $v = v_\varepsilon$ with $v_\varepsilon(x) := (1 - \mathrm{dist}(x, O)/\varepsilon)^+$ provided the basic data qualification (2.55a,c) is fulfilled.[42]

Exercise 2.64. Show that the mapping $u \mapsto c(u, \nabla u)$ is compact, i.e. it maps bounded sets in $W^{1,p}(\Omega)$ into relatively compact sets in $W^{1,p}(\Omega)^*$, cf. Remark 2.39. For this, specify a growth assumption on c.[43]

Exercise 2.65. By using (2.56), show that $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ defined by (2.59) is demicontinuous. Note that no monotonicity of this A is needed, contrary to an abstract case addressed in Lemma 2.16.

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[39] Hint: Take $u_k \to u$ in $L^{p_1}(\Omega)$ and $y_k \to y$ in $L^{p_2}(\Omega;\mathbb{R}^n)$, then take subsequences converging a.e. on Ω. Then, by continuity of $a(x,\cdot,\cdot)$ for a.a. $x \in \Omega$, $\mathscr{N}_a(u_k, y_k) \to \mathscr{N}_a(u, y)$ a.e., and by Proposition 1.13(i), in measure, too. Due to the obvious estimate

$$\big|a(x,u_k,y_k) - a(x,u,y)\big|^{p_0} \leq 6^{p_0-1}\Big(\gamma^{p_0}(x) + C\big|u_k(x)\big|^{p_1} + C\big|u(x)\big|^{p_1} + C\big|y_k(x)\big|^{p_2} + C\big|y(x)\big|^{p_2}\Big)$$

for a.a. $x \in \Omega$, show that $\{|a(x,u_k,y_k) - a(x,u,y)|^{p_0}\}_{k\in\mathbb{N}}$ is equi-absolutely continuous since strongly convergent sequences are; use e.g. Theorem 1.16(i)$\Rightarrow$(iii). Eventually combine these two facts to get $\int_\Omega |a(x,u_k,y_k) - a(x,u,y)|^{p_0}\,\mathrm{d}x \to 0$ and realize that, as the limit $\mathscr{N}_a(u,y)$ is determined uniquely, eventually the whole sequence converges.

[40] Hint: For example, $a(x,r,s) = r/(1+|r|)$ and $u_k = \chi_{A_k}$, a characteristic function of a set A_k, $\mathrm{meas}_n(A_k) > 0$, $\lim_{k\to\infty}\mathrm{meas}_n(A_k) = 0$, and realize that $\|u_k\|_{L^p(\Omega)} = (\mathrm{meas}_n(A_k))^{1/p} \to 0$ but $\|\mathscr{N}_a(u_k)\|_{L^\infty(\Omega)} = 1/2 \not\to 0 = \|\mathscr{N}_a(0)\|_{L^\infty(\Omega)}$.

[41] Hint: Take $r_1, r_2 \in \mathbb{R}^{m_1}$ such that $c(\frac{1}{2}r_1 + \frac{1}{2}r_2) \neq \frac{1}{2}c(r_1) + \frac{1}{2}c(r_1)$ and a sequence of functions oscillating faster and faster between r_1 and r_2 (instead of 1 and -1 as used on Figure 3).

[42] Hint: Putting $x_0 = 0$ without any loss of generality, realizing that $\nabla v_\varepsilon(x) = -\varepsilon^{-1}x/|x|$ if $\varrho < |x| < \varrho + \varepsilon$ otherwise $\nabla v_\varepsilon(x) = 0$ a.e. and that $\nu(x) = x/|x|$, the limit passage

$$\begin{aligned} 0 \;=\; & \int_{|x|\leq\varrho} c(x,u,\nabla u) - g(x)\,\mathrm{d}x + \int_{\varrho\leq|x|\leq\varrho+\varepsilon}\bigg(\Big(c(x,u,\nabla u) - g(x)\Big)\Big(1 - \frac{|x|-\varrho}{\varepsilon}\Big) \\ & - \frac{1}{\varepsilon}a(x,u,\nabla u)\cdot\frac{x}{|x|}\bigg)\,\mathrm{d}x \to \int_{|x|\leq\varrho} c(x,u,\nabla u) - g(x)\,\mathrm{d}x - \int_{|x|=\varrho} a(x,u,\nabla u)\cdot\nu\,\mathrm{d}S \end{aligned}$$

holds at every right Lebesgue point of the function $f : \varrho \mapsto \varrho^{-1}\int_{|x|=\varrho} a(x,u,\nabla u)\cdot x\,\mathrm{d}S$, i.e. at every ϱ such that $f(\varrho) = \lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon}\int_\varrho^{\varrho+\varepsilon} f(\xi)\mathrm{d}\xi$. As f is locally integrable thanks to the growth conditions (2.55a,c), it is known that, for a.a. ϱ, it enjoys this property.

[43] Hint: It suffices to design the growth condition so that $\mathscr{N}_c$ maps $L^{p^*}(\Omega) \times L^p(\Omega;\mathbb{R}^n)$ into $L^{(p^*-\epsilon)'}(\Omega)$ which is compactly embedded into $W^{1,p}(\Omega)^*$.

Exercise 2.66 (V-coercivity)**.** Consider, instead of (2.92),

$$\exists\varepsilon_1, k_0{>}0,\ k_1\in L^1(\Omega):\ a(x,r,s)\cdot s + c(x,r,s)r \geq \varepsilon_1|s|^p - k_0|r|^{q_1} - k_1(x), \quad (2.124\text{a})$$
$$\exists\varepsilon_2>0,\quad k_2\in L^1(\Gamma):\qquad b(x,r)r \geq \varepsilon_2\chi_{\Gamma_\mathrm{N}}(x)|r|^q - k_2(x), \quad (2.124\text{b})$$

for some $1 < q_1 < q \leq p$ and $\mathrm{meas}_{n-1}(\Gamma_\mathrm{N}) > 0$, and prove Lemma 2.35 by using the Poincaré inequality in the form (1.56). Likewise, formulate similar conditions for the case of mixed Dirichlet/Newton conditions (2.49) and use (1.57) to show coercivity of the shifted operator $A_0 = A(\cdot + w)$ with $w|_{\Gamma_\mathrm{D}} = u_\mathrm{D}$ on V from (2.52).

Example 2.67 (Finite-element method)**.** As an example for the finite-dimensional space V_k used in *Galerkin's method* in the concrete case $V = W^{1,p}(\Omega)$, the reader can think of $\mathscr{T}_k$ as a simplicial partition of a polyhedral domain $\Omega \subset \mathbb{R}^n$, i.e. $\mathscr{T}_k$ is a collection of n-dimensional simplexes having mutually disjoint interiors and covering $\bar{\Omega}$; if $n = 2$ or 3, it means a triangulation or a "tetrahedralization" as on Figure 5a or 5b, respectively. Then, one can consider $V_k := \{v \in W^{1,p}(\Omega);\ \forall S \in \mathscr{T}_k:\ v|_S \text{ is affine}\}$. A canonical base of V_k is formed by "hat" functions vanishing at all mesh points except one; cf. Figure 5a.[44] Nested triangulations, i.e. each triangulation $\mathscr{T}_{k+1}$ is a refinement of $\mathscr{T}_k$, obviously imply $V_k \subset V_{k+1}$ which we have used in (2.7).

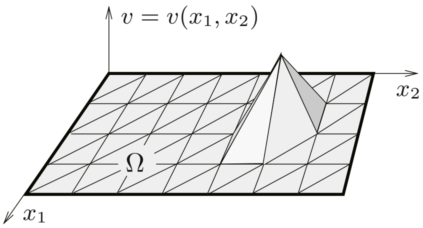

Figure 5*a. Triangulation of a polygonal domain* $\Omega \subset \mathbb{R}^2$ *and one of the piece-wise affine 'hat-shaped' base functions.*

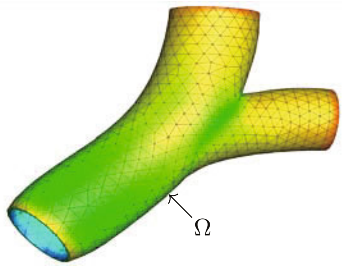

Figure 5*b. A fine* 3*-dimensional tetrahedral mesh on a complicated (but still simply connected) Lipschitz domain* $\Omega \subset \mathbb{R}^3$*; courtesy of M.Mádlík.*

This is the so-called P1-finite-element method. Often, higher-order polynomials are used for the base functions, sometimes in combination with non-simplectic meshes. For non-polyhedral domains, one can use a rectification of the curved boundary by a certain homeomorphism as on Figure 8 on p. 91. Efficient software packages based on finite-element methods are commercially available, including routines for automatic mesh generation on complicated domains, as illustrated on Figure 5b.

Exercise 2.68. Assuming $n = 1$ and $\lim_{k\to\infty} \max_{S\in\mathscr{T}_k} \mathrm{diam}(S) = 0$, prove density of $\bigcup_{k\in\mathbb{N}} V_k$ in $W^{1,p}(\Omega)$, cf. (2.7), for V_k from Example 2.67.[45]

[44] In such a base, the local character of differential operators is reflected in the Galerkin scheme that, e.g., linear differential operators result in matrices which are sparse.

[45] Hint: By density Theorem 1.25, take $v \in W^{2,\infty}(\Omega)$ and $v_k \in V_k$ such that $v_k(x) = v(x)$

Remark 2.69. To ensure (2.7) if $n \geq 2$, a qualification of the triangulation is necessary; usually, for some $\varepsilon > 0$, one requires that always $\mathrm{diam}(S)/\varrho_S \geq \varepsilon$ with denoting ϱ_S the radius of a ball contained in S.

Exercise 2.70 (Traces of higher-order Sobolev spaces)**.** Generalize the trace Theorem 1.23 for $W^{2,p}(\Omega)$, and identify the integrability exponent for traces of functions from $W^{2,p}(\Omega)$, namely $p^{*\#} := (p^*)^\#$, as $(np-p)/(n-2p)$ if $2p < n$, otherwise, its integrability is arbitrarily large if $2p = n$ or in $L^\infty(\Gamma)$ if $2p > n$. Continue by induction for $W^{k,p}(\Omega)$, $k \geq 3$.[46]

2.6.2 Semilinear heat equation of type $-\mathrm{div}(\mathbb{A}(x,u)\nabla u) = g$

Here we focus on a heat equation where, from physical reasons, the heat-transfer coefficients depend typically on temperature but not on its gradient, giving rise to a semilinear equation as investigated in Section 2.5. Moreover, we speak about a *critical growth* of the particular nonlinearity when (2.55) would be fulfilled only if $\epsilon = 0$. Here we will meet the situation when even $\epsilon = -1$ in (2.55b) is needed (and by replacing the conventional Sobolev space $W^{1,p}(\Omega)$ by (2.128) eventually allowed) for $b(x,\cdot)$; this is reported as a *super-critical growth.*

Example 2.71 (*Nonlinear heat equation*)**.** The steady-state heat transfer in a nonhomogeneous anisotropic nonlinear[47] medium with a boundary condition controlling the heat flux through two mechanisms, convection and Stefan-Boltzmann-type radiation[48] as outlined on Figure 6a, is described by the following boundary-value problem

$$\left.\begin{aligned} -\mathrm{div}\big(\mathbb{A}(x,u)\nabla u\big) &= g(x) && \text{on } \Omega, \\ \nu^\top \mathbb{A}(x,u)\nabla u &= \underbrace{b_1(x)(\theta - u)}_{\substack{\text{convective}\\ \text{heat flux}}} + \underbrace{b_2(x)(\theta^4 - |u|^3 u)}_{\substack{\text{radiative}\\ \text{heat flux}}} && \text{on } \Gamma, \end{aligned}\right\} \qquad (2.125)$$

at every $x \in \bar{\Omega}$ which is a mesh point of the partition $\mathscr{T}_k$, and $\|\nabla v_k - \nabla v\|_{L^\infty(\Omega;\mathbb{R}^n)} \leq \mathrm{diam}(S)\|\nabla^2 v\|_{L^\infty(\Omega;\mathbb{R}^{n\times n})}$; as $n = 1$, each S is an interval here.

[46] Hint: For $W^{2,p}(\Omega)$ with $2p < n$, consider $W^{2,p}(\Omega) \subset W^{1,p^*}(\Omega)$ and apply Theorem 1.23 for $p^* = np/(n-p)$ instead of p. By induction, $u \mapsto u|_\Gamma : W^{k,p}(\Omega) \to L^{(np-p)/(n-kp)}(\Gamma)$ if $kp < n$.

[47] The adjective "nonhomogeneous" refers to spatial dependence of the material properties, here $\mathbb{A}$. The adjective "anisotropic" means that $\mathbb{A} \neq \mathbb{I}$ in general, i.e. the heat flux is not necessarily parallel with the temperature gradient and applies typically in single-crystals or in materials with a certain ordered structure, e.g. laminates. The adjective "nonlinear" is related here to a temperature dependence of $\mathbb{A}$ which applies especially when the temperature range is large. E.g. heat conductivity in conventional steel varies by tens of percents when temperature ranges hundreds degrees; cf. [358].

[48] Recall the Stefan-Boltzmann radiation law: the heat flux is proportional to $u^4 - \theta^4$ where θ is the absolute temperature of the outer space. In room temperature, the convective heat transfer, proportional to $u - \theta$ through the coefficient b_1, is usually dominant. Yet, for example, in steel-manufacturing processes the radiative heat flux becomes quickly dominant when temperature rises, say, above 1000 K and definitely cannot be neglected; cf. [358].

with

$u =$ temperature in a thermally conductive body occupying Ω,
$\theta =$ temperature of the environment,
$\mathbb{A} = [a_{ij}]_{i,j=1}^n =$ a symmetric heat-conductivity matrix, $\mathbb{A}:\Omega\times\mathbb{R}\to\mathbb{R}^{n\times n}$, i.e.
$a_i(x,r,s) = \sum_{j=1}^n a_{ij}(x,r)s_j$,
$\mathbb{A}(x,u)\nabla u = \big(\sum_{j=1}^n a_{ij}(u)\frac{\partial}{\partial x_j}u\big)_{i=1}^n$ =the heat flux,
$\nu^\top\mathbb{A}(x,u)\nabla u = \sum_{i,j=1}^n a_{ij}(u)\nu_i\frac{\partial}{\partial x_j}u$ =the heat flux through the boundary,
$b_1, b_2 =$ coefficients of convective and radiative heat transfer through Γ,
$g =$ volume heat source.

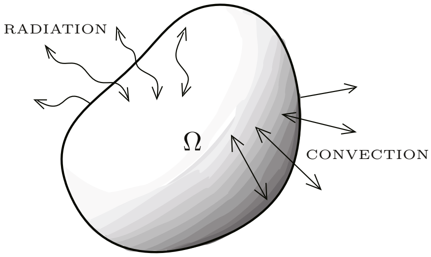

Figure 6*a. Illustration of a heat-transfer problem in a 3-dimensional body* $\Omega\subset\mathbb{R}^3$.

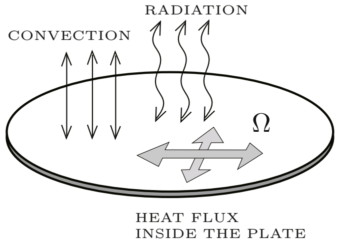

Figure 6*b. Illustration of a heat-transfer problem in a 2-dimensional plate* $\Omega\subset\mathbb{R}^2$.

In the setting (2.48), $b(x,r) = b_1(x)r + b_2(x)|r|^3r$ and $h(x) = [b_1\theta + b_2\theta^4](x)$. We assume $\theta \geq 0$, $b_1(x) \geq \underline{b}_1 > 0$ and $b_2(x) \geq \underline{b}_2 > 0$, $b_1 \in L^{5/3}(\Gamma)$ and $b_2 \in L^\infty(\Gamma)$, and the operator A is defined by

$$\langle A(u),v\rangle := \int_\Omega (\nabla v)^\top\mathbb{A}(x,u)\nabla u\,\mathrm{d}x + \int_\Gamma \big(b_1(x)u + b_2(x)|u|^3u\big)v\,\mathrm{d}S. \tag{2.126}$$

It should be emphasized that no monotonicity of A with respect to the L^2-inner product can be expected if $\mathbb{A}(x,\cdot)$ is not constant.[49]

Exercise 2.72 (Pseudomonotone-operator approach)**.** Check the assumptions (2.55) and (2.65) in Section 2.4[50] as well as the coercivity (2.124)[51]. Realize, in

[49] This means $\int_\Omega(\nabla u_1 - \nabla u_2)^\top(\mathbb{A}(x,u_1)\nabla u_1 - \mathbb{A}(x,u_2)\nabla u_2)\,\mathrm{d}x < 0$ may occur.

[50] Hint: The assumption (2.55a) reads here as $|\sum_{j=1}^n a_{ij}(x,r)s_j| \leq \gamma(x) + C|r|^{(p^*-\epsilon)/p'} + C|s|^{p-1}$ with $\gamma \in L^{p'}(\Omega)$. This requires $p \geq 2$. The assumption of monotonicity in the main part (2.65) just requires that $\mathbb{A}(x,r) = [a_{ij}(x,r)]$ is positive semi-definite for all r and a.a. $x \in \Omega$, i.e. $s^\top\mathbb{A}(x,r)s \geq 0$. The assumption (2.55b) for the "physical" dimension $n = 3$ and for $p = 2$ yields $p^\# = (np{-}p)/(n{-}p) = 4$, cf. (1.37). This just agrees with the 4-power growth of the Stefan-Boltzmann law at least in the sense that the traces $|u|^3u$ are in $L^1(\Gamma)$ if $u \in W^{1,2}(\Omega)$. Yet (2.55b) admits only $(3{-}\epsilon)$-power growth of $b(x,\cdot)$ which does not fit with the 4-power growth of Stefan-Boltzmann law.

[51] Hint: The coercivity assumption (2.124a) requires here $s^\top\mathbb{A}(x,r)s \geq \varepsilon_1|s|^p - k_1$, which requires, besides uniform positive definiteness of $\mathbb{A}$, also $p \leq 2$. Altogether, $p = 2$ is ultimately needed. Note that $p = 2$ and (2.55a) need $\mathbb{A}(x,r)$ bounded, i.e. $|a_{ij}(x,r)| \leq C$ for any $i,j = 1,\dots,n$. The condition (2.124b) holds trivially with $k_2 = 0$.

particular, that $p = 2$ is needed and the qualification (2.57) of (g, h) means

$$g \in \begin{cases} L^1(\Omega), \\ L^{1+\epsilon}(\Omega), \\ L^{2n/(n+2)}(\Omega), \end{cases} \qquad h \in \begin{cases} L^1(\Gamma) & \text{if } n = 1, \\ L^{1+\epsilon}(\Gamma) & \text{if } n = 2, \\ L^{2-2/n}(\Gamma) & \text{if } n \geq 3. \end{cases} \tag{2.127}$$

In view of this, the pseudo-monotone approach has disadvantages:

✓ direct usage of Leray-Lions Theorem 2.36 on the conventional Sobolev space $W^{1,2}(\Omega)$ is limited to $n \leq 2$ or to $b_2 \equiv 0$,

✓ if $n > 1$, an artificial integrability of g and h is needed, contrary to the physically natural requirement of a finite energy of heat sources, i.e. $g \in L^1(\Omega)$, $h \in L^1(\Gamma)$,

✓ $\mathbb{A}$ must be bounded.

Modify the setting of Section 2.4.3 by replacing $W^{1,p}(\Omega)$ by

$$V = \big\{v \in W^{1,2}(\Omega);\ v|_\Gamma \in L^5(\Gamma)\big\} \tag{2.128}$$

and show that V becomes a reflexive Banach space densely containing $C^\infty(\bar{\Omega})$ if equipped with the norm $\|v\| := \|v\|_{W^{1,2}(\Omega)} + \big\|v|_\Gamma\big\|_{L^5(\Gamma)}$.[52] Show that $A : V \to V^*$ defined by (2.126) is bounded and coercive. Make a limit passage though the monotone boundary term by Minty's trick instead of the compactness.

Exercise 2.73 (Weak-continuity approach)**.** Use V from (2.128) and $Z = W^{1,\infty}(\Omega)$, assume

$$\exists \varepsilon_1 > 0 : \qquad s^\top \mathbb{A}(x, r)s \geq \varepsilon_1 |s|^2, \tag{2.129a}$$

$$\exists \gamma \in L^2(\Omega),\ \epsilon > 0 : \qquad |\mathbb{A}(x, r)| \leq \gamma(x) + C|r|^{(2^*-\epsilon)/2}, \tag{2.129b}$$

and show that $A : V \to Z^*$ defined by (2.126) is weakly continuous; use the fact that $L^4(\Gamma)$ is an interpolant between $L^2(\Gamma)$ and $L^5(\Gamma)$.[53]

Exercise 2.74 (*Galerkin method*)**.** Consider V_k a finite-dimensional subspace of $W^{1,\infty}(\Omega)$ nested for $k \to \infty$ with a dense union in $W^{1,2}(\Omega)$ and traces dense in

[52] Hint: For $n \leq 2$, simply $V = W^{1,2}(\Omega)$. For $n \geq 3$, any Cauchy sequence $\{v_k\}_{k\in\mathbb{N}}$ in V has a limit v in $W^{1,2}(\Omega)$ and $\{v_k|_\Gamma\}_{k\in\mathbb{N}}$ converges in $L^{p^\#}(\Gamma)$ to $v|_\Gamma$, and simultaneously has some limit w in $L^5(\Gamma)$ but necessarily $v|_\Gamma = w$. As V is (isometrically isomorphic to) a closed subspace in a reflexive Banach space $W^{1,2}(\Omega) \times L^5(\Gamma)$, it is itself reflexive. Density of smooth functions can be proved by standard mollifying procedure.

[53] Hint: Take u_k such that $u_k \rightharpoonup u$ in $W^{1,2}(\Omega)$ and $u_k|_\Gamma \rightharpoonup u|_\Gamma$ in $L^5(\Gamma)$. Use $W^{1,2}(\Omega) \Subset L^{2^*-\epsilon}(\Omega)$ and then $\mathbb{A}(u_k) \to \mathbb{A}(u)$ in $L^2(\Omega; \mathbb{R}^{n\times n})$ and $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^2(\Omega; \mathbb{R}^n)$, and, for $v \in W^{1,\infty}(\Omega) =: Z$, pass to the limit $\int_\Omega (\nabla u_k)^\top \mathbb{A}(x, u_k)\nabla v \,\mathrm{d}x \to \int_\Omega (\nabla u)^\top \mathbb{A}(x, u)\nabla v \,\mathrm{d}x$. By compactness of the trace operator, $u_k|_\Gamma \to u|_\Gamma$ in $L^{p^\#-\epsilon}(\Gamma) \subset L^2(\Gamma)$, realize that $u_k|_\Gamma \to u|_\Gamma$ in $L^4(\Gamma)$ because

$$\big\|u_k|_\Gamma - u|_\Gamma\big\|_{L^4(\Gamma)} \leq \big\|u_k|_\Gamma - u|_\Gamma\big\|_{L^5(\Gamma)}^{5/6} \big\|u_k|_\Gamma - u|_\Gamma\big\|_{L^2(\Gamma)}^{1/6} \to 0.$$

Then $|u_k|^3 u_k|_\Gamma \to |u|^3 u|_\Gamma$ in $L^1(\Gamma)$, and $\int_\Gamma |u_k|^3 u_k v \,\mathrm{d}S \to \int_\Gamma |u|^3 uv \,\mathrm{d}S$ for any $v \in L^\infty(\Gamma)$.

$L^5(\Gamma)$, and thus in V from (2.128), too. Then the Galerkin method for (2.125) is defined by:

$$\int_\Omega (\nabla u_k)^\top \mathbb{A}(x,u_k)\nabla v - gv\,\mathrm{d}x + \int_\Gamma \Big((b_1+b_2|u_k|^3)u_k - h\Big)v\,\mathrm{d}S = 0 \tag{2.130}$$

for all $v\in V_k$. Assume $g \in L^{2^{*\prime}}(\Omega)$, $h = b_1\theta+b_2\theta^4$ with $\theta \in L^5(\Gamma)$, see Example 2.71, and assuming existence of u_k, show the a-priori estimate in V from (2.128) by putting $v := u_k$ into (2.130).[54] Then, using the linearity of $s \mapsto a(x,r,s) = \mathbb{A}(x,r)s$, make the limit passage directly in the Galerkin identity (2.130) by using the weak continuity as in Exercise 2.73.

Exercise 2.75 (Strong convergence)**.** Assume again $\mathbb{A}$ bounded as in Exercise 2.72 and, despite the lack of d-monotonicity of $u \mapsto \mathrm{div}\big(\mathbb{A}(x,u)\nabla u\big)$, use strong monotonicity of $u \mapsto \mathrm{div}\big(\mathbb{A}(x,v)\nabla u\big)$ for v fixed, and show $u_k \to u$ in $W^{1,2}(\Omega)$; make the limit passage in the boundary term by compactness[55] if $n \le 2$, or treat it by

[54] Hint: By Hölder's and Young's inequalities, this yields the estimate

$$\begin{aligned}
\varepsilon_1\int_\Omega |\nabla u_k|^2\mathrm{d}x + \underline{b}_1\int_\Gamma |u_k|^2\mathrm{d}S + \underline{b}_2\int_\Gamma |u_k|^5\mathrm{d}S &\le \int_\Omega (\nabla u_k)^\top \mathbb{A}(x,u_k)\nabla u_k\mathrm{d}x\\
&\quad + \int_\Gamma b_1|u_k|^2 + b_2|u_k|^5\mathrm{d}S = \int_\Omega g u_k\mathrm{d}x + \int_\Gamma (b_1\theta + b_2\theta^4)u_k\mathrm{d}S\\
&\le \|g\|_{L^{p^*\prime}(\Omega)}\|u_k\|_{L^{p^*}(\Omega)} + \Big(\|b_1\|_{L^{5/3}(\Gamma)}\|\theta\|_{L^5(\Gamma)} + \|b_2\|_{L^\infty(\Gamma)}\|\,|\theta|^4\|^{5/4}_{L^{5/4}(\Gamma)}\Big)\|u_k\|_{L^5(\Gamma)}\\
&\le \frac{1}{4\varepsilon}N^2\|g\|^2_{L^{p^*\prime}(\Omega)} + \varepsilon\|u_k\|^2_{W^{1,2}(\Omega)}\\
&\quad + C\Big(\|b_1\|_{L^{5/3}(\Gamma)}\|\theta\|_{L^5(\Gamma)} + \|b_2\|_{L^\infty(\Gamma)}\|\theta\|^4_{L^5(\Gamma)}\Big)^{5/4} + \frac12\underline{b}_2\|u_k\|^5_{L^5(\Gamma)}
\end{aligned}$$

where N is the norm of the embedding $W^{1,2}(\Omega) \subset L^{2^*}(\Omega)$ and C is a sufficiently large constant, namely $C = 2^9/(5^5\underline{b}_2)$, and $\varepsilon < \min(\varepsilon_1, \underline{b}_1)/C_\mathrm{P}$ with C_P the constant from the Poincaré inequality (1.56) with $p = 2 = q$. Then use (1.56) for the estimate of the left-hand side from below and absorb the right-hand-side terms with u_k in the left-hand side.

[55] Hint: Abbreviate $b(u) = (b_1{+}b_2|u|^3)u$ and $\mathfrak{a}_k := \nabla(u_k{-}u)^\top\mathbb{A}(u_k)\nabla(u_k{-}u)$, cf. (2.83). Then, use the Galerkin identity (2.130), i.e. $\int_\Omega\nabla(u_k{-}v_k)^\top\mathbb{A}(u_k)\nabla u_k\mathrm{d}x = \int_\Gamma b(u_k)(v_k{-}u_k)\,\mathrm{d}S$, to get

$$\begin{aligned}
\int_\Omega \mathfrak{a}_k\,\mathrm{d}x &= \int_\Omega \nabla(u_k{-}u)^\top\big(\mathbb{A}(u_k)\nabla u_k - \mathbb{A}(u)\nabla u\big) - \nabla(u_k{-}u)^\top\big(\mathbb{A}(u_k) - \mathbb{A}(u)\big)\nabla u\,\mathrm{d}x\\
&= \int_\Omega \nabla(u_k{-}v_k)^\top\big(\mathbb{A}(u_k)\nabla u_k - \mathbb{A}(u)\nabla u\big) + \nabla(v_k{-}u)^\top\big(\mathbb{A}(u_k)\nabla u_k{-}\mathbb{A}(u)\nabla u\big)\\
&\qquad - \nabla(u_k{-}u)^\top\big(\mathbb{A}(u_k) - \mathbb{A}(u)\big)\nabla u\,\mathrm{d}x = \int_\Gamma \big(b(u){-}b(u_k)\big)(u_k{-}v_k)\,\mathrm{d}S\\
&\qquad + \int_\Omega \nabla(v_k{-}u)^\top\big(\mathbb{A}(u_k)\nabla u_k - \mathbb{A}(u)\nabla u\big) - \nabla(u_k{-}u)^\top\big(\mathbb{A}(u_k) - \mathbb{A}(u)\big)\nabla u\,\mathrm{d}x
\end{aligned}$$

for any $v_k \in V_k$. In particular, take $v_k \to u$ in $W^{1,2}(\Omega)$. For $n \le 2$, use compactness of the trace operator $W^{1,2}(\Omega) \to L^{p^\#-\epsilon}(\Gamma) \subset L^5(\Gamma)$ and push the first right-hand-side term to zero. Furthermore, use $\nabla v_k \to \nabla u$ in $L^2(\Omega;\mathbb{R}^n)$ and $\mathbb{A}(u_k)\nabla u_k - \mathbb{A}(u)\nabla u$ bounded in $L^2(\Omega;\mathbb{R}^n)$ to push the second term to zero. Finally, push the last expression to zero when using $\big(\mathbb{A}(u_k) - \mathbb{A}(u)\big)\nabla u \to 0$ in $L^2(\Omega;\mathbb{R}^n)$ (note that one cannot rely on $\mathbb{A}(u_k) \to \mathbb{A}(u)$ in $L^\infty(\Omega;\mathbb{R}^{n\times n})$, monotonicity if $n = 3$ when this term has the *super-critical growth*.[56] Note that, for $n = 3$, the super-critical growth of the boundary term is such that, although being formally a lower-order term, it behaves like a highest-order term and must be treated by its monotonicity.[57]

Exercise 2.76 (*Comparison principle*)**.** Put $v := u^- = \min(u, 0)$ into the integral identity (2.51) for the case of (2.125). Show that non-negativity of heat sources, i.e. $h = b_1\theta + b_2\theta^4 \geq 0$ and $g \geq 0$, implies the non-negativity of temperature, i.e. $u \geq 0$.[58] Assume $g = 0$ and $0 \leq \theta(\cdot) \leq \theta_{\max}$ for a constant $\theta_{\max} > 0$ and use $v := (u - \theta_{\max})^+$ in (2.51) to show that $u(\cdot) \leq \theta_{\max}$ almost everywhere.

Exercise 2.77 (Mixed boundary conditions)**.** Perform the analysis by the Galerkin method of the mixed Dirichlet/Newton boundary-value problem[59]

---

however) and when assuming $v_k \to u$ in $W^{1,2}(\Omega)$. Then $\varepsilon_1\|\nabla u_k - \nabla u\|^2_{L^2(\Omega;\mathbb{R}^d)} \leq \int_\Omega \mathfrak{a}_k \mathrm{d}x \to 0$ and, as $u_k \to u$ in $L^2(\Omega)$ by Rellich-Kondrachov's theorem 1.21, $u_k \to u$ in $W^{1,2}(\Omega)$.

[56] Hint: Use identity (2.130) and the previous notation of $\mathfrak{a}_k$ and b to write

$$
\begin{aligned}
&\int_\Omega \mathfrak{a}_k \,\mathrm{d}x + \int_\Gamma \big(b(u_k) - b(u)\big)(u_k - u)\,\mathrm{d}S \\
&\quad = \int_\Omega \nabla(u_k - v_k)^\top\big(\mathbb{A}(u_k)\nabla u_k - \mathbb{A}(u)\nabla u\big) + \nabla(v_k - u)^\top\big(\mathbb{A}(u_k)\nabla u_k - \mathbb{A}(u)\nabla u\big) \\
&\qquad - \nabla(u_k - u)^\top\big(\mathbb{A}(u_k) - \mathbb{A}(u)\big)\nabla u\,\mathrm{d}x + \int_\Gamma \big(b(u_k) - b(u)\big)(u_k - v_k) + \big(b(u_k) - b(u)\big)(v_k - u)\,\mathrm{d}S \\
&\quad = \int_\Gamma -b(u)(u_k - v_k) + (b(u_k) - b(u))(v_k - u)\,\mathrm{d}S + \int_\Omega -\nabla(u_k - v_k)^\top\mathbb{A}(u)\nabla u \\
&\qquad + \nabla(v_k - u)^\top\big(\mathbb{A}(u_k)\nabla u_k - \mathbb{A}(u)\nabla u\big) - \nabla(u_k - u)^\top\big(\mathbb{A}(u_k) - \mathbb{A}(u)\big)\nabla u\,\mathrm{d}x \\
&\quad = I_k^{(1)} + I_k^{(2)} + I_k^{(3)} + I_k^{(4)} + I_k^{(5)}.
\end{aligned}
$$

Assume $v_k \to u$ in $W^{1,2}(\Omega)$ and $v_k|_\Gamma \to u|_\Gamma$ in $L^5(\Gamma)$. Use $b(u) \in L^{5/4}(\Gamma)$ and $u_k - v_k \rightharpoonup 0$ in $L^5(\Gamma)$ to show $I_k^{(1)} \to 0$. Use $\{b(u_k)\}_{k\in\mathbb{N}}$ bounded in $L^{5/4}(\Gamma)$ and $v_k - u \to 0$ in $L^5(\Gamma)$ to show $I_k^{(2)} := \int_\Gamma (b(u_k) - b(u))(v_k - u)\,\mathrm{d}S \to 0$. Push the remaining terms $I_k^{(3)}$, $I_k^{(4)}$, and $I_k^{(5)}$ as before. Altogether, conclude $u_k \to u$ in $W^{1,2}(\Omega)$. Moreover, conclude also $\|u_k - u\|_{L^5(\Gamma)} \to 0$.

[57] Hint: Realize the difficulties in pushing $\int_\Gamma (b(u) - b(u_k))(u_k - v_k)\,\mathrm{d}S$ to zero if $n \geq 3$ because we have $\{u_k\}_{k\in\mathbb{N}}$ and $\{b(u_k)\}_{k\in\mathbb{N}}$ only bounded in $L^5(\Gamma)$ and $L^{5/4}(\Gamma)$, respectively, but no strong convergence can be assumed in these spaces.

[58] Hint: Note that $u^- \in W^{1,2}(\Omega)$ if $u \in W^{1,2}(\Omega)$ so $v := u^-$ is a legal test, cf. Proposition 1.28, and then $\int_\Omega (\nabla u)^\top \mathbb{A}(u)\nabla u^-\,\mathrm{d}x = \int_\Omega (\nabla u^-)^\top \mathbb{A}(u)\nabla u^-\,\mathrm{d}x$ due to (1.50). By this way, come to the estimate

$$
\begin{aligned}
\varepsilon_1 \int_\Omega |\nabla u^-|^2 \mathrm{d}x + \underline{b}_1 \int_\Gamma (u^-)^2 \mathrm{d}S \leq \int_\Omega (\nabla u)^\top \mathbb{A}(u)\nabla u^-\,\mathrm{d}x \\
+ \int_\Gamma b_1(u^-)^2 + b_2|u^-|^5 \mathrm{d}S = \int_\Omega g u^-\,\mathrm{d}x + \int_\Gamma h u^-\,\mathrm{d}S \leq 0.
\end{aligned}
$$

By the Poincaré inequality (1.56), we get $\|u^-\|_{W^{1,2}(\Omega)} = 0$, hence $u^- = 0$ a.e. in Ω.

[59] Hint: Instead of (2.128), use $V = \big\{v \in W^{1,2}(\Omega);\ v|_{\Gamma_\mathrm{N}} \in L^5(\Gamma_\mathrm{N}),\ v|_{\Gamma_\mathrm{D}} = 0\big\}$, define Galerkin's approximate solution u_k with approximate Dirichlet conditions $u_k|_{\Gamma_\mathrm{D}} = u_\mathrm{D}^k$, and derive the a-priori estimate by a test $v := u_k - w_k$ where $w_k \in V_k$, a finite-dimensional subspace of V, is chosen so that $w_k|_{\Gamma_\mathrm{N}} = u_\mathrm{D}^k \to u_\mathrm{D}$ in $L^{2^\#}(\Gamma_\mathrm{D})$ and the sequence $\{w_k\}_{k\in\mathbb{N}}$ is bounded in V.

$$\left.\begin{aligned} -\mathrm{div}\big(\mathbb{A}(x,u)\nabla u\big) &= g(x) && \text{in } \Omega, \\ \nu^\top \mathbb{A}(x,u)\nabla u &= b_1(x)(\theta - u) + b_2(x)(\theta^4 - |u|^3 u) && \text{on } \Gamma_{\mathrm{N}}, \\ u|_{\Gamma_{\mathrm{D}}} &= u_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}}. \end{aligned}\right\} \tag{2.131}$$

Exercise 2.78 (Heat-conductive plate)**.** Perform the analysis by Galerkin's method of the problem

$$\left.\begin{aligned} -\mathrm{div}\big(\mathbb{A}(x,u)\nabla u\big) &= c_1(x)(\theta - u) + c_2(x)(\theta^4 - |u|^3 u) && \text{in } \Omega, \\ u|_{\Gamma} &= u_{\mathrm{D}} && \text{on } \Gamma. \end{aligned}\right\} \tag{2.132}$$

In the case $n = 2$, this problem has an interpretation of a plate conducting heat in tangential direction with normal-direction temperature variations neglected, and being cooled/heated by convection and radiation and with fixed temperature on the boundary as outlined on Figure 6b. Consider $n \leq 3$, use the conventional Sobolev space $W_0^{1,2}(\Omega)$, define Galerkin's approximate solution u_k with approximate Dirichlet conditions $u_k|_\Gamma = u_\Gamma^k$, and derive the a-priori estimate by a test by $v := u_k - w_k$ with w_k as in Exercise 2.77.

Example 2.79 (*Special nonlinear media*)**.** Let us consider again the nonlinear heat-transfer problem (2.125) with $\mathbb{A}(x,r) = [a_{ij}(x,r)]$ in the special form

$$a_{ij}(x,r) = b_{ij}(x)\kappa(r) \tag{2.133}$$

with $\mathbb{B} = [b_{ij}] : \Omega \to \mathbb{R}^{n\times n}$ and $\kappa : \mathbb{R} \to \mathbb{R}^+$. Then the so-called *Kirchhoff transformation* employs the primitive function $\widehat{\kappa} : \mathbb{R} \to \mathbb{R}$ to κ, i.e. defined by

$$\widehat{\kappa}(r) := \int_0^r \kappa(\varrho)\mathrm{d}\varrho\,, \tag{2.134}$$

and transforms the nonlinearity of (2.125) inside Ω to the (already nonlinear) boundary conditions. Indeed, $\mathbb{B}(x)\nabla\widehat{\kappa}(u) = \mathbb{B}(x)\kappa(u)\nabla u = \mathbb{A}(x,u)\nabla u$ and $\mathbb{B}(x)\frac{\partial}{\partial\nu}\widehat{\kappa}(u) = \mathbb{B}(x)\kappa(u)\frac{\partial}{\partial\nu}u = \mathbb{A}(x,u)\frac{\partial}{\partial\nu}u$ and, by a substitution $w = \widehat{\kappa}(u)$, one transfers the nonlinearity from the equation on Ω to the boundary conditions which has been nonlinear even originally anyhow due to the Stefan-Boltzmann radiation term. Thus one gets the following semilinear equation for w:

$$\left.\begin{aligned} -\mathrm{div}\big(\mathbb{B}(x)\nabla w\big) &= g && \text{in } \Omega, \\ \mathbb{B}(x)\frac{\partial w}{\partial\nu} + \Big(b_1 + b_2|\widehat{\kappa}^{-1}(w)|^3\Big)\widehat{\kappa}^{-1}(w) &= h && \text{on } \Gamma. \end{aligned}\right\} \tag{2.135}$$

We assume $\mathbb{B} : \Omega \to \mathbb{R}^{n\times n}$ measurable, bounded, and $\mathbb{B}(\cdot)$ uniformly positive definite in the sense $\xi^\top\mathbb{B}(x)\xi \geq \beta|\xi|^2$ for all $\xi \in \mathbb{R}^n$ and some $\beta > 0$. Further, we assume $\kappa(\cdot) \geq \varepsilon > 0$ measurable and bounded; note that this implies $\widehat{\kappa}$ to be continuous and increasing, and one-to-one with $\widehat{\kappa}^{-1}$ Lipschitz continuous, in particular having a linear growth. Furthermore, g and h satisfy (2.127). Again, ultimately $p = 2$, and one can show the coercivity. As the function $\mathbb{R} \to \mathbb{R} : r \mapsto$

$b(x,r) := \big(b_1(x) + b_2(x)|\widehat{\kappa}^{-1}(r)|^3\big)\,\widehat{\kappa}^{-1}(r)$ is monotone for a.a. $x \in \Gamma$, we can use the monotonicity technique. Then there is just one weak solution $w \in W^{1,2}(\Omega)$. By Proposition 1.28, $u = \widehat{\kappa}^{-1}(w) \in W^{1,2}(\Omega)$ and this u solves the original problem in the weak sense. Moreover, $(g,h) \mapsto u : L^{p^{*\prime}}(\Omega) \times L^{p^{\#\prime}}(\Gamma) \to W^{1,2}(\Omega)$ is (norm×norm,norm)-continuous. Note that the heat-conductivity coefficient κ need not be assumed continuous.[60]

Example 2.80 (*Heat transfer with advection*)**.** The heat equation in moving homogeneous isotropic media, i.e. with advection by a prescribed velocity, say $\vec{v} = \vec{v}(x)$, is

$$-\mathrm{div}\big(\kappa(u)\nabla u\big) + \mathfrak{c}(u)\vec{v}\cdot\nabla u = g, \tag{2.136}$$

where $\mathfrak{c}$ is the heat capacity dependent on temperature. Let us consider, for simplicity, constant Dirichlet boundary conditions and use the Kirchhoff transformation (2.134), i.e. put $w = \widehat{\kappa}(u)$ and using $\nabla\widehat{\kappa}^{-1}(w) = \nabla w/\kappa(\widehat{\kappa}^{-1}(w))$, to arrive at

$$\left.\begin{aligned} -\Delta w + \frac{C(w)\vec{v}\cdot\nabla w}{K(w)} &= g \quad \text{in } \Omega,\\ w &= 0 \quad \text{on } \Gamma \end{aligned}\right\} \tag{2.137}$$

where we abbreviated $\kappa(\widehat{\kappa}^{-1}(w)) =: K(w)$ and $\mathfrak{c}(\widehat{\kappa}^{-1}(w)) =: C(w)$; note that we can shift $\widehat{\kappa}$ by a constant so that $w = 0$ can be considered on Γ. Note that the pointwise coercivity (2.92a) for $p = 2 \geq q > 1$ is violated if $c(x,r,s) = C(r)\vec{v}(x)\cdot s/K(r)$. Assume the velocity field $\vec{v} \in C^1(\bar{\Omega};\mathbb{R}^n)$ as divergence free, which corresponds to a motion of an incompressible medium, cf. also the equations (6.26c), (12.44c), or (12.95b) below, one can consider an alternative setting with $\mathfrak{c}(u)\vec{v}\cdot\nabla u = \mathrm{div}(\vec{v}\,\widehat{\mathfrak{c}}(u)) = \mathrm{div}(\vec{v}\,\widehat{\mathfrak{c}}(\kappa^{-1}(w)))$ with $\widehat{\mathfrak{c}}$ the primitive function of $\mathfrak{c}$. This leads to $a(x,r,s) = s + \vec{v}(x)\widehat{\mathfrak{c}}(\kappa^{-1}(r))$ which again need not satisfy (2.92a).

Exercise 2.81 (*Uniqueness*)**.** Show uniqueness of the weak solution w to (2.135), and thus of u, as well. Try to show uniqueness in the general case (2.125) and realize the difficulties if smallness of $\|u\|_{W^{1,\infty}(\Omega)}$ is not guaranteed.[61]

Exercise 2.82. Assume $\mathrm{div}\,\vec{v} \leq 0$ in Example 2.80 and show the coercivity of the respective A (in spite of this lack of any pointwise coercivity pointed out in Example 2.80) by derivation of an a-priori estimate again by a test by w.[62]

[60] A discontinuity of κ can indeed occur during various phase transformations, cf. [358] for a discontinuity in the heat-conductivity coefficient κ within a recrystallization in steel.

[61] The uniqueness holds even for a general case (2.125) but the proof is rather technical, cf. [243].

[62] Hint: For N the norm of the embedding $W^{1,2}(\Omega) \subset L^{2^*}(\Omega)$, use Green's Theorem 1.31 to estimate

$$\begin{aligned}\int_\Omega |\nabla w|^2\,\mathrm{d}x &\leq \int_\Omega |\nabla w|^2 - (\mathrm{div}\,\vec{v})\Big(\int_0^{w(x)}\frac{C(\xi)}{K(\xi)}\mathrm{d}\xi\Big)\mathrm{d}x = \int_\Omega |\nabla w|^2 + \vec{v}\cdot\nabla\Big(\int_0^{w(x)}\frac{C(\xi)}{K(\xi)}\mathrm{d}\xi\Big)\mathrm{d}x\\ &= \int_\Omega |\nabla w|^2 + \frac{(\vec{v}\cdot\nabla w)C(w)}{K(w)}\,\mathrm{d}x = \int_\Omega gw\,\mathrm{d}x \leq N\|w\|_{W^{1,2}(\Omega)}\|g\|_{L^{2^*\prime}(\Omega)}.\end{aligned}$$

Furthermore, assuming Lipschitz continuity of κ, show uniqueness of a solution to (2.137) if $\vec{v}$ is small enough in the L^∞-norm.[63]

2.6.3 Quasilinear equations of type $-\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big)+c(u,\nabla u)=g$

Here we will address quasilinear equations (2.45) with $a(x,r,\cdot)$ or $c(x,r,\cdot)$ nonlinear so that a limit passage in approximate solutions cannot be made by using mere weak convergence in ∇u and compactness in lower-order terms, unlike in semilinear equations scrutinized in Section 2.6.2. As a "training" quasilinear differential operator in the divergence form, we will frequently use

$$\Delta_p u := \mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) \tag{2.138}$$

called the *p-Laplacean*; hence the usual Laplacean is what is called here 2-Laplacean. For $p > 2$ one gets a degenerate nonlinearity, while for $p < 2$ a singular one, cf. Figure 9 on p.128 below. Note that, by using the formula $\mathrm{div}(vw) = v\,\mathrm{div}\,w + \nabla v\cdot w$, (2.138) can equally be written in the form

$$\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) = |\nabla u|^{p-2}\Delta u + (p-2)|\nabla u|^{p-4}(\nabla u)^\top\nabla^2 u\,\nabla u. \tag{2.139}$$

Example 2.83 (d-monotonicity of p-Laplacean)**.** To be more specific, $A = -\Delta_p$ will be understood here as a mapping $W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ corresponding to a Neumann-boundary-value problem, i.e.

$$\langle A(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,\mathrm{d}x \tag{2.140}$$

for any $v \in W^{1,p}(\Omega)$. For $p > 1$, the p-Laplacean is always *d-monotone* in the sense (2.1) with respect to the seminorm $|u| := \|\nabla u\|_{L^p(\Omega;\mathbb{R}^n)}$, i.e.

$$\int_\Omega \big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\big)\cdot(\nabla u - \nabla v)\,\mathrm{d}x \geq \big(d(|u|) - d(|v|)\big)\big(|u| - |v|\big)$$

[63] Hint: Realizing that also $[C/K](\cdot)$ is Lipschitz continuous, with ℓ denoting the Lipschitz constant, we have

$$\begin{aligned}
&\int_\Omega \vec{v}\cdot\Big(\frac{C(w_1)\nabla w_1}{K(w_1)} - \frac{C(w_2)\nabla w_2}{K(w_2)}\Big)(w_1 - w_2)\,\mathrm{d}x\\
&\quad= \int_\Omega \vec{v}\Big(\frac{C(w_1)}{K(w_1)} - \frac{C(w_2)}{K(w_2)}\Big)\cdot\nabla w_1(w_1 - w_2)\,\mathrm{d}x + \int_\Omega \frac{C(w_2)\vec{v}\cdot\nabla(w_1 - w_2)}{K(w_2)}(w_1 - w_2)\,\mathrm{d}x\\
&\quad\leq \|\vec{v}\|_{L^\infty(\Omega;\mathbb{R}^n)}\Big\|\frac{C(w_1)}{K(w_1)} - \frac{C(w_2)}{K(w_2)}\Big\|_{L^4(\Omega)}\|\nabla w_1\|_{L^2(\Omega;\mathbb{R}^n)}\|w_1 - w_2\|_{L^4(\Omega)}\\
&\qquad+ \|\vec{v}\|_{L^\infty(\Omega;\mathbb{R}^n)}\Big\|\frac{C(w_2)}{K(w_2)}\Big\|_{L^4(\Omega)}\|\nabla w_1 - \nabla w_2\|_{L^2(\Omega;\mathbb{R}^n)}\|w_1 - w_2\|_{L^4(\Omega)}\\
&\quad\leq \|\vec{v}\|_{L^\infty(\Omega;\mathbb{R}^n)}\Big(\|\nabla w_1\|_{L^2(\Omega;\mathbb{R}^n)}\ell N^2 + N\frac{\max\mathfrak{c}(\cdot)}{\min\kappa(\cdot)}\mathrm{meas}_n(\Omega)^{1/4}\Big)\|w_1 - w_2\|^2_{W^{1,2}(\Omega)},
\end{aligned}$$

where N is the norm of the embedding $W^{1,2}(\Omega) \subset L^4(\Omega)$ valid for $n \leq 3$. For $\|\vec{v}\|_{L^\infty(\Omega;\mathbb{R}^n)}$ small enough, conclude that $w_1 = w_2$.

with $d(\xi)=\xi^{p-1}$, which can be proved simply by Hölder's inequality as follows:

$$\begin{aligned}\int_\Omega &\big(|y|^{p-2}y-|z|^{p-2}z\big)\cdot(y-z)\,\mathrm{d}x\\
&=\|y\|^p_{L^p(\Omega;\mathbb{R}^n)}-\int_\Omega\Big(|y|^{p-2}y\cdot z+|z|^{p-2}z\cdot y\Big)\mathrm{d}x+\|z\|^p_{L^p(\Omega;\mathbb{R}^n)}\\
&\geq\|y\|^p_{L^p(\Omega;\mathbb{R}^n)}-\big\||y|^{p-2}y\big\|_{L^{p'}(\Omega;\mathbb{R}^n)}\|z\|_{L^p(\Omega;\mathbb{R}^n)}\\
&\qquad\qquad-\big\||z|^{p-2}z\big\|_{L^{p'}(\Omega;\mathbb{R}^n)}\big\|y\big\|_{L^p(\Omega;\mathbb{R}^n)}+\big\|z\big\|^p_{L^p(\Omega;\mathbb{R}^n)}\\
&=\big\|y\big\|^p_{L^p(\Omega;\mathbb{R}^n)}-\big\|y\big\|^{p-1}_{L^p(\Omega;\mathbb{R}^n)}\|z\|_{L^p(\Omega;\mathbb{R}^n)}\\
&\qquad\qquad-\big\|z\big\|^{p-1}_{L^p(\Omega;\mathbb{R}^n)}\big\|y\big\|_{L^p(\Omega;\mathbb{R}^n)}+\big\|z\big\|^p_{L^p(\Omega;\mathbb{R}^n)}\\
&=\Big(\big\|y\big\|^{p-1}_{L^p(\Omega;\mathbb{R}^n)}-\big\|z\big\|^{p-1}_{L^p(\Omega;\mathbb{R}^n)}\Big)\Big(\big\|y\big\|_{L^p(\Omega;\mathbb{R}^n)}-\big\|z\big\|_{L^p(\Omega;\mathbb{R}^n)}\Big).\end{aligned}\tag{2.141}$$

For $p\geq 2$, from the algebraic inequality[64]

$$\big(|s|^{p-2}s-|\widetilde{s}|^{p-2}\widetilde{s}\big)\cdot\big(s-\widetilde{s}\big)\geq c(n,p)|s-\widetilde{s}|^p\tag{2.142}$$

with some $c(n,p)>0$, we obtain a uniform monotonicity on $W^{1,p}_0(\Omega)$ in the sense (2.2) with $\zeta(z)=z^{p-1}$ (or with respect to the seminorm $\|\nabla\cdot\|_{L^p(\Omega;\mathbb{R}^n)}$ on $W^{1,p}(\Omega)$):

$$\begin{aligned}\big\langle A(u)-A(v),u-v\big\rangle&=\int_\Omega\big(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\big)\cdot\nabla(u-v)\,\mathrm{d}x\\
&\geq c(n,p)\int_\Omega|\nabla u-\nabla v|^p\mathrm{d}x.\end{aligned}$$

It should be emphasized that, for $p<2$, one has only $\langle A(u)-A(v),u-v\rangle\geq(p-1)\int_\Omega\max(1+|\nabla u|,1+|\nabla v|)^{p-2}|\nabla u-\nabla v|^2\mathrm{d}x$.[65]

Exercise 2.84 (Monotonicity of p-Laplacean)**.** Realize that (2.138) corresponds to $a_i(x,r,s)=|s|^{p-2}s_i$ and verify the strict monotonicity (2.65) and (2.68a).[66]

Exercise 2.85 (Strong convergence in $c(\nabla u)$)**.** Consider the Dirichlet boundary-value problem

$$\left.\begin{aligned}-\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big)+c(x,\nabla u)&=g&&\text{in }\Omega,\\ u&=0&&\text{on }\Gamma.\end{aligned}\right\}\tag{2.143}$$

For some $\epsilon>0$, assume the growth condition

$$\exists\gamma\in L^{(p^*-\epsilon)'}(\Omega)\ C\in\mathbb{R}\ \ \forall(\text{a.a.})x\in\Omega\ \forall s\in\mathbb{R}^n:\ \ \big|c(x,s)\big|\leq\gamma(x)+C|s|^{p-1-\epsilon}.\tag{2.144}$$

[64] See DiBenedetto [120, Sect.I.4] or Hu and Papageorgiou [209, Part.I,Sect.3.1].

[65] See Málek et al. [268, Sect.5.1.2].

[66] Hint: Like (2.141), $(|s|^{p-2}s-|\tilde{s}|^{p-2}\tilde{s})\cdot(s-\tilde{s})=|s|^p-|s|^{p-2}s\cdot\tilde{s}-|\tilde{s}|^{p-2}\tilde{s}\cdot s+|\tilde{s}|^p\geq|s|^p-|s|^{p-1}|\tilde{s}|-|\tilde{s}|^{p-1}|s|+|\tilde{s}|^p=(|s|^{p-1}-|\tilde{s}|^{p-1})(|s|-|\tilde{s}|)$, hence (2.65) holds. If $(|s|^{p-2}s-|\tilde{s}|^{p-2}\tilde{s})\cdot(s-\tilde{s})=0$, then $|s|=|\tilde{s}|$, and if $s\neq\tilde{s}$, then $|s|^2>s\cdot\tilde{s}$ hence $|s|^p-|s|^{p-2}s\cdot\tilde{s}>0$, and similarly $|\tilde{s}|^p-|\tilde{s}|^{p-2}\tilde{s}\cdot s>0$, hence $(|s|^{p-2}s-|\tilde{s}|^{p-2}\tilde{s})\cdot(s-\tilde{s})>0$, a contradiction, proving (2.68a).

Formulate Galerkin's approximation[67] and prove the a-priori estimate in $W_0^{1,p}(\Omega)$ by testing the Galerkin identity by $v = u_k$[68] and prove strong convergence of $\{u_k\}$ in $W_0^{1,p}(\Omega)$ by using d-monotonicity of $-\Delta_p$, following the scheme of Proposition 2.20 with Remark 2.21 simplified by having boundedness guaranteed explicitly through Lemma 2.31 instead of the Banach-Steinhaus principle through (2.36) and (2.42).[69] Further, considering $p = 2$, formulate a Lipschitz-continuity condition like (2.157) in Exercise 2.90 that would guarantee (uniform) monotonicity of the underlying mapping A.

Exercise 2.86 (Weak convergence in $c(\nabla u)$)**.** Consider the boundary-value problem (2.143) in a more general form:

$$\left.\begin{array}{rcll} -\mathrm{div}\, a(x,\nabla u) + c(x,\nabla u) &=& g & \text{in } \Omega, \\ u &=& 0 & \text{on } \Gamma, \end{array}\right\} \tag{2.145}$$

with $a(x,\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ strictly monotone. The Galerkin approximation looks as

$$\int_\Omega a(\nabla u_k)\cdot\nabla v + \big(c(\nabla u_k) - g\big) v \,\mathrm{d}x = 0 \qquad \forall v \in V_k. \tag{2.146}$$

Assuming coercivity $a(x,s)\cdot s \ge \varepsilon_a |s|^p$ and the growth (2.144), prove the a-priori

[67] See (2.146) below for $a(x,s) = |s|^{p-2}s$.

[68] Hint: Use Hölder's inequality between $L^{p/(p-1-\epsilon)}(\Omega)$ and $L^q(\Omega)$ with $q = p/(1+\epsilon)$ to estimate

$$\begin{aligned} \|u_k\|^p_{W_0^{1,p}(\Omega)} &= \|\nabla u_k\|^p_{L^p(\Omega;\mathbb{R}^n)} = \int_\Omega \big(g - c(\nabla u_k)\big) u_k \,\mathrm{d}x \le \int_\Omega \big(|g| + \gamma + C|\nabla u_k|^{p-1-\epsilon}\big)|u_k| \,\mathrm{d}x \\ &\le \big\||g| + \gamma\big\|_{L^{p^*\prime}(\Omega)} \big\|u_k\big\|_{L^{p^*}(\Omega)} + C\|\nabla u_k\|^{p-1-\epsilon}_{L^p(\Omega)} \|u_k\|_{L^q(\Omega)} \\ &\le N_{p^*} \big\||g| + \gamma\big\|_{L^{p^*\prime}(\Omega)} \big\|u_k\big\|_{W_0^{1,p}(\Omega)} + C N_q \big\|u_k\big\|^{p-\epsilon}_{W_0^{1,p}(\Omega)} \end{aligned}$$

with N_q the norm of the embedding $W^{1,p}(\Omega) \subset L^q(\Omega)$, and N_{p^*} with an analogous meaning.

[69] Hint: Take a subsequence $u_k \rightharpoonup u$ in $W_0^{1,p}(\Omega)$. Use the norm $\|v\|_{W_0^{1,p}(\Omega)} := \|\nabla v\|_{L^p(\Omega;\mathbb{R}^n)}$ and, by (2.141) and using still the abbreviation $a(\nabla v) = |\nabla v|^{p-2}\nabla v$, estimate

$$\begin{aligned} \Big(\|u_k\|^{p-1}_{W_0^{1,p}(\Omega)} - \|v\|^{p-1}_{W_0^{1,p}(\Omega)}\Big)\Big(\|u_k\|_{W_0^{1,p}(\Omega)} - \|v\|_{W_0^{1,p}(\Omega)}\Big) &\le \int_\Omega \big(a(\nabla u_k) - a(\nabla v)\big)\cdot\nabla(u_k - v)\,\mathrm{d}x \\ &= \int_\Omega a(\nabla u_k)\cdot\nabla(u_k - v_k) + a(\nabla u_k)\cdot\nabla(v_k - v) - a(\nabla v)\cdot\nabla(u_k - v)\,\mathrm{d}x \\ &= \int_\Omega \big(g - c(\nabla u_k)\big)(u_k - v_k) + a(\nabla u_k)\cdot\nabla(v_k - v) - a(\nabla v)\cdot\nabla(u_k - v)\,\mathrm{d}x \end{aligned}$$

with $v_k \in V_k$. Assume $v_k \to v$. For $v = u$, $u_k - v_k \to u - u = 0$ in $L^{p^*-\epsilon}(\Omega)$ because of the compact embedding $W_0^{1,p}(\Omega) \Subset L^{p^*-\epsilon}(\Omega)$, and then $\int_\Omega c(\nabla u_k)(u_k - v_k)\,\mathrm{d}x \to 0$; note that $\{c(\nabla u_k)\}_{k\in\mathbb{N}}$ is bounded in $L^{(p^*-\epsilon)'}(\Omega)$. Push the other terms to zero, too. Conclude that $u_k \to u$ in $W_0^{1,p}(\Omega)$. Then, having got the strong convergence $\nabla u_k \to \nabla u$, pass to the limit directly in the Galerkin identity.

estimate by testing (2.146) by $v = u_k$.[70] Then prove weak convergence of the Galerkin method as in (2.84).[71]

Exercise 2.87. Modify Exercises 2.85 and 2.86 for non-zero Dirichlet or Newton boundary conditions.

Exercise 2.88 (Monotone case I)**.** Consider the boundary-value problem (2.45)–(2.49) in the special case $a_i(x,r,s) := a_i(x,s)$ and $c(x,r,s) := c(x,r)$, i.e.

$$\left.\begin{array}{rcll} -\mathrm{div}\, a(\nabla u) + c(u) &=& g & \text{on } \Omega, \\ \nu\cdot a(\nabla u) + b(u) &=& h & \text{on } \Gamma_{\mathrm{N}}, \\ u|_{\Gamma_{\mathrm{D}}} &=& u_{\mathrm{D}} & \text{on } \Gamma_{\mathrm{D}}. \end{array}\right\} \tag{2.147}$$

Assume that $a(x,\cdot)$, $b(x,\cdot)$, and $c(x,\cdot)$ are monotone, coercive (say $a(x,s)\cdot s \geq |s|^p$, $b(x,0) = 0$, $c(x,0) = 0$, and $\mathrm{meas}_{n-1}(\Gamma_{\mathrm{D}}) > 0$) with basic growth conditions, i.e.

$$\begin{aligned} &\big(a(x,s) - a(x,\tilde{s})\big)\cdot(s-\tilde{s}) \geq 0, \\ &\exists \gamma_a \in L^{p'}(\Omega),\ C_a \in \mathbb{R}: \quad |a(x,s)| \leq \gamma_a(x) + C_a|s|^{p-1}, && \text{(2.148a)} \\ &\big(b(x,r) - b(x,\tilde{r})\big)(r-\tilde{r}) \ \geq\ 0, \\ &\exists \gamma_b \in L^{p^{\#\prime}}(\Gamma),\ C_b \in \mathbb{R}: \quad |b(x,r)| \leq \gamma_b(x) + C_b|r|^{p^{\#}-1}, && \text{(2.148b)} \\ &\big(c(x,r) - c(x,\tilde{r})\big)(r-\tilde{r}) \ \geq\ 0, \\ &\exists \gamma_c \in L^{p^{*\prime}}(\Omega),\ C_c \in \mathbb{R}: \quad |c(x,r)| \leq \gamma_c(x) + C_c|r|^{p^{*}-1}, && \text{(2.148c)} \end{aligned}$$

and prove a-priori estimates[72] and the convergence of Galerkin's approximations by Minty's trick.[73]

---

[70] Hint: Estimate

$$\varepsilon_a\|u_k\|^p_{W^{1,p}_0(\Omega)} = \varepsilon_a\|\nabla u_k\|^p_{L^p(\Omega;\mathbb{R}^n)} \leq \int_\Omega a(\nabla u_k)\cdot\nabla u_k \mathrm{d}x \leq \int_\Omega \big(g - c(\nabla u_k)\big)u_k \mathrm{d}x$$

and finish it as in Exercise 2.85.

[71] Hint: Prove $\lim_{k\to\infty}\int_\Omega (a(\nabla u_k) - a(\nabla u))\cdot\nabla(u_k - u)\,\mathrm{d}x = 0$ as in Exercise 2.85. Then, for a selected subsequence, deduce $c(\nabla u_k) \to c(\nabla u)$ a.e. in Ω by the same way as done in (2.88), and similarly also $a(\nabla u_k) \to a(\nabla u)$ a.e. in Ω. Then prove $a(\nabla u_k) \rightharpoonup a(\nabla u)$ in $L^{p'}(\Omega)$ and $c(\nabla u_k) \rightharpoonup c(\nabla u)$ in $L^{p^{*\prime}+\epsilon}(\Omega)$ and pass to the limit directly in (2.146) for any $v \in \bigcup_{h>0} V_k$ without using Minty's trick. Finally, extend the resulted identity by continuity for any $v \in W^{1,p}(\Omega)$.

[72] Hint: Denoting $\bar{u_{\mathrm{D}}} \in W^{1,p}(\Omega)$ an extension of u_{D} test the Galerkin identity determining $u_k \in V_k$ by $v := u_k - \bar{u}_k$ where $\bar{u}_k|_\Gamma \to u_{\mathrm{D}}$ in $W^{1,p}(\Omega)|_\Gamma$ for $k \to \infty$, $\{\bar{u}_k\}_{k\in\mathbb{N}}$ bounded in $W^{1,p}(\Omega)$, $\bar{u}_k \in V_k$, V_k a finite-dimensional subspace of $W^{1,p}(\Omega)$. Arrive to

$$\begin{aligned} \int_\Omega |\nabla u_k|^p \mathrm{d}x &\leq \int_\Omega a(\nabla u_k)\cdot\nabla u_k + c(u_k)u_k \mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} b(u_k)u_k \mathrm{d}S \\ &= \int_\Omega a(\nabla u_k)\cdot\nabla\bar{u}_k + c(u_k)\bar{u}_k + g(u_k - \bar{u}_k)\,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} b(u_k)\bar{u}_k + h(u_k - \bar{u}_k)\,\mathrm{d}S \end{aligned}$$

and then get u_k estimated in $W^{1,p}(\Omega)$ by Hölder's inequality and Poincaré's inequality (1.57). Alternatively, use the a-priori shift as in Proposition 2.27.

[73] Hint: For $v \in W^{1,p}(\Omega)$, use $v_k \to v$ in $W^{1,p}(\Omega)$, $v_k \in V_k$, $u_k|_{\Gamma_{\mathrm{D}}} = v_k|_{\Gamma_{\mathrm{D}}}$, the monotonicity

Exercise 2.89 (Monotone case II). Consider $A:W^{1,\max(2,p)}(\Omega) \to W^{1,\max(2,p)}(\Omega)^*$ given by

$$\big\langle A(u), v\big\rangle = \int_\Omega (1+|\nabla u|^{p-2})\nabla u \cdot \nabla v + c(u)v \,\mathrm{d}x + \int_\Gamma b(u)v \,\mathrm{d}S \tag{2.149}$$

so that the equation $A(u) = f$ with f from (2.60) corresponds to the boundary-value problem for the *regularized p-Laplacean*:

$$\left.\begin{array}{ll} -\mathrm{div}\big((1+|\nabla u|^{p-2})\nabla u\big) + c(x,u) = g & \text{in } \Omega, \\ \big(1+|\nabla u|^{p-2}\big)\dfrac{\partial u}{\partial \nu} + b(x,u) = h & \text{on } \Gamma. \end{array}\right\} \tag{2.150}$$

Assume $c(x,\cdot)$ strongly monotone and $b(x,\cdot)$ either increasing or, if decreasing at a given point r, then being locally Lipschitz continuous with a constant ℓ_b^-:

$$\big(c(x,r) - c(x,\tilde r)\big)(r-\tilde r) \geq \ \varepsilon_c (r-\tilde r)^2, \tag{2.151}$$

$$\big(b(x,r) - b(x,\tilde r)\big)(r-\tilde r) \geq -\ell_b^- (r-\tilde r)^2. \tag{2.152}$$

Show that A can be monotone even if $b(x,\cdot)$ is not monotone; assume that[74]

$$\ell_b^- \leq N^{-2} \min(1,\varepsilon_c). \tag{2.153}$$

Show further strong monotonicity of A with respect to the $W^{1,2}$-norm if (2.153) holds as a strict inequality.

Exercise 2.90 (Monotone case III). Let $A : W^{1,\max(2,p)}(\Omega) \to W^{1,\max(2,p)}(\Omega)^*$ be given by

$$\big\langle A(u), v\big\rangle = \int_\Omega \big(1+|\nabla u|^{p-2}\big)\nabla u \cdot \nabla v + c(\nabla u)v \,\mathrm{d}x + \int_\Gamma b(u)v \,\mathrm{d}S. \tag{2.154}$$

---

and Galerkin's identity

$$0 \leq \int_\Omega \big(a(\nabla u_k) - a(\nabla v_k)\big)\cdot\nabla(u_k{-}v_k) + \big(c(u_k) - c(v_k)\big)(u_k{-}v_k)\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} \big(b(u_k) - b(v_k)\big)$$
$$\times(u_k{-}v_k)\,\mathrm{d}S = \int_\Omega \big(g - c(v_k)\big)(u_k{-}v_k) - a(\nabla v_k)\cdot\nabla(u_k{-}v_k)\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} \big(h - b(v_k)\big)(u_k{-}v_k)\,\mathrm{d}S$$
$$\to \int_\Omega \big(g - c(v)\big)(u{-}v) - a(\nabla v)\cdot\nabla(u{-}v)\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} \big(h - b(v)\big)(u{-}v)\,\mathrm{d}S$$

and then put $v := u \pm \varepsilon w$, divide it by $\varepsilon > 0$, and pass $\varepsilon \to 0$.

[74] Hint: Indeed,

$$\big\langle A(u) - A(v), u-v\big\rangle = \int_\Omega |\nabla u - \nabla v|^2 + \ \big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\big)\cdot(\nabla u - \nabla v)$$
$$+ \big(c(u) - c(v)\big)(u-v)\,\mathrm{d}x + \int_\Gamma \big(b(u) - b(v)\big)(u-v)\,\mathrm{d}S$$
$$\geq \int_\Omega |\nabla u - \nabla v|^2 + \varepsilon_c (u-v)^2 \mathrm{d}x - \int_\Gamma \ell_b^- (u-v)^2 \mathrm{d}S$$
$$\geq \min(1,\varepsilon_c)\|u{-}v\|^2_{W^{1,2}(\Omega)} - \ell_b^- \|u{-}v\|^2_{L^2(\Gamma)} \geq \big(\min(1,\varepsilon_c) - \ell_b^- N^2\big)\|u{-}v\|^2_{W^{1,2}(\Omega)}.$$

Note that the equation $A(u) = f$ with f from (2.60) corresponds to the boundary-value problem

$$\left.\begin{aligned} -\mathrm{div}\big((1+|\nabla u|^{p-2})\nabla u\big) + c(x,\nabla u) &= g \quad \text{ for } x\in\Omega,\\ \big(1+|\nabla u|^{p-2}\big)\frac{\partial u}{\partial\nu} + b(x,u) &= h \quad \text{ for } x\in\Gamma. \end{aligned}\right\} \tag{2.155}$$

Assume $b(x,\cdot)$ strongly monotone and $c(x,\cdot)$ Lipschitz continuous, i.e.

$$\big(b(x,r)-b(x,\tilde r)\big)(r-\tilde r) \ge \varepsilon_b|r-r|^2, \tag{2.156}$$

$$\big|c(x,s)-c(x,\tilde s)\big| \le \ell_c|s-\tilde s|, \tag{2.157}$$

and show monotonicity of A if ℓ_c is sufficiently small, despite that $u\mapsto c(\nabla u)$ alone would not allow for any monotone structure.[75] In particular, if ℓ_c is small enough, realize that A is strictly monotone and uniqueness of the solution follows.

Exercise 2.91 (Monotone case IV: *advection*)**.** Consider a special case of (2.155) with $c(x,s) := \vec v(x)\cdot s$ with $\vec v:\Omega\to\mathbb{R}^n$ being a prescribed velocity field. Assume $\mathrm{div}\,\vec v \le 0$ (as in Exercise 2.82) and $\vec v|_\Gamma\cdot\nu \ge 0$, and show that A enjoys the monotonicity[76] even if there is no point-wise monotonicity.

Exercise 2.92. Consider the following boundary-value problem:

$$\left.\begin{aligned} -\mathrm{div}\Big(|\nabla u|^{p-2}\nabla u + a_0(x,u)\Big) &= g \quad \text{ in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial\nu} + b_0(x,u) &= h \quad \text{ on } \Gamma. \end{aligned}\right\} \tag{2.158}$$

[75] Hint: Estimate

$$\begin{aligned}\langle A(u)-A(v),u-v\rangle &= \int_\Omega \Big((1+|\nabla u|^{p-2})\nabla u - (1+|\nabla v|^{p-2})\nabla v\Big)\cdot\nabla(u-v)\\ &\qquad + \big(c(\nabla u)-c(\nabla v)\big)(u-v)\,\mathrm{d}x + \int_\Gamma \big(b(u)-b(v)\big)(u-v)\,\mathrm{d}S\\ &\ge \|\nabla u-\nabla v\|^2_{L^2(\Omega;\mathbb{R}^n)} - \|c(\nabla u)-c(\nabla v)\|_{L^2(\Omega)}\|u-v\|_{L^2(\Omega)} + \varepsilon_b\|u-v\|^2_{L^2(\Gamma)}\\ &\ge \|\nabla u-\nabla v\|^2_{L^2(\Omega;\mathbb{R}^n)} - \ell_c\|\nabla u-\nabla v\|_{L^2(\Omega;\mathbb{R}^n)}\|u-v\|_{L^2(\Omega)} + \varepsilon_b\|u-v\|^2_{L^2(\Gamma)}\\ &\ge \Big(1-\frac{\ell_c^2}{2\delta}\Big)\|\nabla u-\nabla v\|^2_{L^2(\Omega;\mathbb{R}^n)} + \varepsilon_b\|u-v\|^2_{L^2(\Gamma)} - \frac{\delta}{2}\|u-v\|^2_{L^2(\Omega)}\\ &\ge C_{\mathrm{P}}^{-1}\min\Big(1-\frac{\ell_c^2}{2\delta},\varepsilon_b\Big)\|u-v\|^2_{W^{1,2}(\Omega)} - \frac{\delta}{2}N^2\|u-v\|^2_{W^{1,2}(\Omega)},\end{aligned}$$

with N the norm of the embedding $W^{1,2}(\Omega)\subset L^2(\Omega)$ and C_{P} the constant from the Poincaré inequality (1.56) with $p=2=q$ and $\Gamma_{\mathrm{N}}=\Gamma$. If ℓ_c is so small that there is some $\delta>0$ such that

$$\min\Big(1-\frac{\ell_c^2}{2\delta},\varepsilon_b\Big) \ge \frac{\delta}{2}N^2C_{\mathrm{P}},$$

the monotonicity of A follows.

[76] Hint: By using Green's formula, the monotonicity of this linear term is based on the estimate:

$$\int_\Omega(\vec v\cdot\nabla u)u\,\mathrm{d}x = \frac12\int_\Omega \vec v\cdot\nabla u^2\mathrm{d}x = \frac12\int_\Gamma(\vec v\cdot\nu)u^2\mathrm{d}S - \frac12\int_\Omega(\mathrm{div}\,\vec v)\,u^2\mathrm{d}x \ge 0.$$

Assume the basic growth condition: $|a_0(x,r)| \le \gamma(x) + C|r|^{p^*/p'}$ for some $\gamma \in L^{p'}(\Omega)$ and formulate a definition of the weak solution; denote: $b(x,r) := b_0(x,r) - a_0(x,r)\cdot\nu(x)$. Prove that $u \mapsto \mathrm{div}(a_0(x,u)) : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ is a totally continuous mapping (which allows us to use Theorem 2.6 with Corollary 2.12 to get the existence of a weak solution). Further prove the a-priori estimate by testing by u.[77] Prove convergence of the Galerkin approximation via Minty's trick, and alternatively strong convergence and direct limit passage without Minty's trick. Show uniqueness of the weak solution for Lipschitz continuous $a_0(x,\cdot)$ with a small Lipschitz constant. Make the modification for the Dirichlet boundary condition.[78]

Example 2.93 (Banach fixed-point technique)**.** Consider the boundary-value problem (2.147)) and assume the strong monotonicity of $a(x,\cdot)$ and, e.g., of $c(x,\cdot)$ but no monotonicity of $b(x,\cdot)$, i.e.

$$\big(a(x,s) - a(x,\tilde{s})\big)\cdot(s-\tilde{s}) \ge \varepsilon_a|s-\tilde{s}|^2, \tag{2.159a}$$
$$\big(c(x,r) - c(x,\tilde{r})\big)(r-\tilde{r}) \ge \varepsilon_c(r-\tilde{r})^2, \tag{2.159b}$$

and the Lipschitz continuity

$$|a(x,s) - a(x,\tilde{s})| \le \ell_a|s-\tilde{s}|, \tag{2.160a}$$
$$-\ell_b^-(r-\tilde{r})^2 \le \big(b(x,r) - b(x,\tilde{r})\big)(r-\tilde{r}) \le \ell_b^+(r-\tilde{r})^2, \tag{2.160b}$$
$$|c(x,r) - c(x,\tilde{r})| \le \ell_c|r-\tilde{r}|, \tag{2.160c}$$

with some $\ell_b^+ \ge \ell_b^- \ge 0$; note that $b(x,\cdot)$ is Lipschitz continuous with the constant ℓ_b^+. Then one can use the Banach fixed-point Theorem 1.12 technique based on the contractiveness of the mapping T_ε from (2.43) where the Lipschitz constant ℓ of A can be estimated as:[79]

---

[77] Hint: Realize that

$$\int_\Omega |\nabla u|^p \mathrm{d}x + \int_\Gamma b_0(u)u\,\mathrm{d}S = \int_\Omega -a_0(u)\cdot\nabla u + gu\,\mathrm{d}x + \int_\Gamma \big(a_0(u)\cdot\nu + h\big)\,u\,\mathrm{d}S.$$

Assume $b_0(x,r)r \ge |r|^p$, and estimate u by assuming further $|a_0(x,r)| \le \gamma(x) + C|r|^{p-1-\epsilon}$.

[78] Hint: Denoting $\widehat{a}(x,r) = \big(\widehat{a}_1(x,r),\ldots,\widehat{a}_n(x,r)\big)$ the component-wise primitive functions to $a_0(x,r) = \big(a_1(x,r),\ldots,a_n(x,r)\big)$ and realizing that now $u|_\Gamma = u_\mathrm{D}$, by Green's Theorem 1.31, one gets

$$\int_\Omega a_0(x,u)\cdot\nabla u\,\mathrm{d}x = \int_\Omega \sum_{i=1}^n \frac{\partial}{\partial x_i}\widehat{a}_i(x,u)\,\mathrm{d}x = \int_\Gamma \sum_{i=1}^n \widehat{a}_i(x,u)\nu_i(x)\,\mathrm{d}S = \int_\Gamma \widehat{a}(u_\mathrm{D})\cdot\nu\,\mathrm{d}S = \mathrm{const.}$$

[79] Cf. also (4.17) below.

$$
\begin{aligned}
\|A(u)-A(v)\|_{W^{1,2}(\Omega)^*} &= \sup_{\|z\|_{W^{1,2}(\Omega)}\le 1} \langle A(u)-A(v), z\rangle \\
&= \sup_{\|z\|_{W^{1,2}(\Omega)}\le 1} \int_\Omega \big(a(\nabla u)-a(\nabla v)\big)\cdot\nabla z + \big(c(u)-c(v)\big) z\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} \big(b(u)-b(v)\big) z\,\mathrm{d}S \\
&\le \sup_{\|z\|_{W^{1,2}(\Omega)}\le 1} \Big(\ell_a\|\nabla u-\nabla v\|_{L^2(\Omega;\mathbb{R}^n)}\|\nabla z\|_{L^2(\Omega;\mathbb{R}^n)} \\
&\qquad + \ell_c\|u-v\|_{L^2(\Omega)}\|z\|_{L^2(\Omega)} + \ell_b^+\|u-v\|_{L^2(\Gamma_\mathrm{N})}\|z\|_{L^2(\Gamma_\mathrm{N})}\Big) \\
&\le \big(\sqrt{2}\max(\ell_a,\ell_c) + N^2\ell_b^+\big)\|u-v\|_{W^{1,2}(\Omega)} =: \ell\|u-v\|_{W^{1,2}(\Omega)}
\end{aligned}
$$

while the constant δ in the strong monotonicity of A can be estimated as $\langle A(u)-A(v), u-v\rangle \ge \big(\min(\varepsilon_c,\varepsilon_a) - N^2\ell_b^-\big)\|u-v\|^2_{W^{1,2}(\Omega)} =: \delta\|u-v\|^2_{W^{1,2}(\Omega)}$; cf. Exercise 2.89. Then, by Proposition 2.22, T_ε from (2.43) with $J : W^{1,2}(\Omega) \to W^{1,2}(\Omega)^*$ defined by[80]

$$
\langle J(u), v\rangle = \int_\Omega \nabla u\cdot\nabla v + uv \,\mathrm{d}x \tag{2.161}
$$

is a contraction provided $\varepsilon > 0$ satisfies[81]

$$
\varepsilon < 2\frac{\min(\varepsilon_c,\varepsilon_a) - N^2\ell_b^-}{\big(\sqrt{2}\max(\ell_a,\ell_c) + N^2\ell_b^+\big)^2}. \tag{2.162}
$$

Exercise 2.94. Modify the above Example 2.93 for Dirichlet boundary conditions[82] and/or the term $c(\nabla u)$ instead of $c(u)$[83].

Example 2.95 (Limit passage in coefficients)**.** Consider the problem from Example 2.93 modified, for simplicity, as in Exercise 2.94 with zero Dirichlet boundary conditions. Assume $s \mapsto a(x,s)$ and $r \mapsto c(x,r)$ monotone, $a(x,s)\cdot s + c(x,r)\cdot r \ge \varepsilon_0|s|^p - C$, $|a(x,s)| \le \gamma(x) + C|s|^{p-1}$ with $\gamma \in L^{p'}(\Omega)$ and $1 < p \le 2$. Such a problem does not satisfy (2.159) and (2.160a,c). Therefore, we approximate a and c respectively by some a_ε and c_ε which will satisfy both (2.159) and (2.160a,c) and such that $a_\varepsilon(x,\cdot) \to a(x,\cdot)$ uniformly on bounded sets in $\mathbb{R}^n$, and $c_\varepsilon(x,\cdot) \to c(x,\cdot)$ uniformly on bounded sets in $\mathbb{R}$, and such that the collection $\{(a_\varepsilon, c_\varepsilon)\}_{\varepsilon>0}$ is uniformly coercive in the sense

$$
\exists\delta > 0\ \forall\varepsilon > 0 : \qquad a_\varepsilon(x,s)\cdot s + c_\varepsilon(x,r)\cdot r \ge \delta|s|^p - 1/\delta. \tag{2.163}
$$

[80] Note that $\langle J(u), u\rangle = \|u\|^2_{W^{1,2}(\Omega)}$ and also $\|u\|_{W^{1,2}(\Omega)} = \|J(u)\|_{W^{1,2}(\Omega)^*}$ if one considers the standard norm $\|u\|_{W^{1,2}(\Omega)} = \sqrt{\|\nabla u\|^2_{L^2(\Omega;\mathbb{R}^n)} + \|u\|^2_{L^2(\Omega)}}$; cf. Remark 3.15.

[81] Cf. (2.44) on p. 42.

[82] Hint: Instead of (2.161) use $\langle J(u), v\rangle = \int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x$, cf. Proposition 3.14.

[83] Hint: In case of Newton boundary conditions, $b(x,\cdot)$ has to be strongly monotone as in Exercise 2.90.

E.g. one can put $a_\varepsilon(x,\cdot) := \mathscr{Y}^{\mathrm{M}}_{n,\varepsilon}(a(x,\cdot))$ and $c_\varepsilon(x,\cdot) := \mathscr{Y}^{\mathrm{M}}_{1,\varepsilon}(c(x,\cdot))$ where $\mathscr{Y}^{\mathrm{M}}_{n,\varepsilon} : \mathbb{R}^n \to \mathbb{R}^n$ denotes a suitable *modification of Yosida's approximation* $\mathscr{Y}_{n,\varepsilon}:\mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\big[\mathscr{Y}^{\mathrm{M}}_{n,\varepsilon}(f)\big](s) := \Big[\mathscr{Y}_{n,\varepsilon}\Big(f + \frac{\varepsilon^2}{1-\varepsilon}\mathbf{I}_n\Big)\Big](s) \quad \text{with} \tag{2.164a}$$

$$\big[\mathscr{Y}_{n,\varepsilon}(f)\big](s) := \frac{s - \big(\mathbf{I}_n + \varepsilon f\big)^{-1}(s)}{\varepsilon} \tag{2.164b}$$

and $\mathbf{I}_n$ the identity on $\mathbb{R}^n$; cf. also Remark 5.18 below. Unlike the mere Yosida approximation $\mathscr{Y}_{n,\varepsilon}$, the regularization (2.164) turns monotonicity to strong monotonicity; note also that $\mathscr{Y}^{\mathrm{M}}_{n,\varepsilon}(\mathbf{I}_n) = \mathbf{I}_n$.

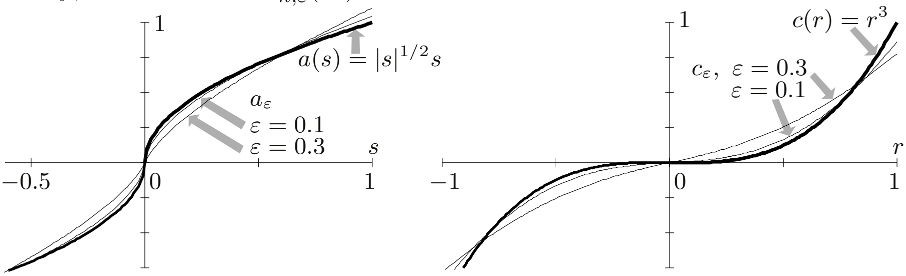

Figure 7. A regularization of the nonlinearities $a(x,s) = |s|^{1/2}s$ and $c(x,r) = r^3$ that makes them both strongly monotone and Lipschitz continuous.

Then we can obtain the weak solution $u_\varepsilon \in W^{1,2}(\Omega)$ of the approximate problem

$$\left.\begin{aligned} -\mathrm{div}\, a_\varepsilon(\nabla u) + c_\varepsilon(u) &= g \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma \end{aligned}\right\} \tag{2.165}$$

constructively by Example 2.93 (modified as in Exercise 2.94). The convergence of $u_\varepsilon \in W^{1,2}_0(\Omega)$ for $\varepsilon \to 0$ relies on an a-priori estimate in $W^{1,p}_0(\Omega)$ which is uniform with respect to $\varepsilon > 0$ due to (2.163), and then a selection of a subsequence $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega)$. Note that, as $p \le 2$, we have $W^{1,p}_0(\Omega) \supset W^{1,2}_0(\Omega)$. Taking $\widetilde v \in W^{1,\infty}_0(\Omega)$ and using monotonicity, we obtain

$$\begin{aligned} 0 &\le \int_\Omega \big(a_\varepsilon(\nabla u_\varepsilon) - a_\varepsilon(\nabla \widetilde v)\big)\cdot(\nabla u_\varepsilon - \nabla \widetilde v) + \big(c_\varepsilon(u_\varepsilon) - c_\varepsilon(\widetilde v)\big)(u_\varepsilon - \widetilde v)\,\mathrm{d}x \\ &= \int_\Omega \big(g - c_\varepsilon(\widetilde v)\big)(u_\varepsilon - \widetilde v) - a_\varepsilon(\nabla \widetilde v)\cdot(\nabla u_\varepsilon - \nabla \widetilde v)\,\mathrm{d}x \\ &\to \int_\Omega \big(g - c(\widetilde v)\big)(u - \widetilde v) - a(\nabla \widetilde v)\cdot(\nabla u - \nabla \widetilde v)\,\mathrm{d}x \end{aligned} \tag{2.166}$$

for $\varepsilon \to 0$, where we used $a_\varepsilon(\nabla \widetilde v) \to a(\nabla \widetilde v)$ in $L^\infty(\Omega;\mathbb{R}^n)$. Then we can pass $\widetilde v$ to $v \in W^{1,p}_0(\Omega)$; by density of $W^{1,\infty}_0(\Omega)$ in $W^{1,p}_0(\Omega)$, cf. Theorem 1.25, v can be considered arbitrary. By continuity of the Nemytskiĭ mappings $\mathscr{N}_a : L^p(\Omega;\mathbb{R}^n) \to L^{p'}(\Omega;\mathbb{R}^n)$ and $\mathscr{N}_c : L^{p^*}(\Omega) \to L^{p^{*\prime}}(\Omega)$, from (2.166) we get $\int_\Omega (g - c(v))(u - v) - a(\nabla v)\cdot(\nabla u - \nabla v)\,\mathrm{d}x \geq 0$. Eventually, by Minty's trick, we conclude that u solves (2.165); cf. Lemma 2.13.

Remark 2.96 (Constructivity)**.** Let us still point out that, by combining the Banach fixed-point iterations as in Example 2.93 with some coefficient approximation as in Example 2.95, one can solve problems as (2.147) under quite weak assumptions rather constructively, without any Brouwer's fixed-point argument, cf. Remark 2.7. In case of strict monotonicity in (2.147), the whole sequence of approximate solutions converges.

Exercise 2.97. Modify Example 2.95 for the case of Newton boundary conditions.

Exercise 2.98. Add a term $\mathrm{div}\big(b(x,u,\nabla u,\nabla^2 u)\big)$ here with $b : \Omega\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n\times n} \to \mathbb{R}^n$ into (2.97) and modify (2.99) and Propositions 2.42 and 2.43.

Exercise 2.99. Realizing that only four out of all six combinations of derivatives up to 3rd-order on the boundary have been used in (2.100), (2.101), (2.105), and (2.107), identify the remaining two combinations and explain why they are not compatible with a consistent and selective weak formulation.[84]

Exercise 2.100 (*Singular higher-order perturbations*)**.** Consider the weak solution $u_\varepsilon \in W^{2,2}(\Omega)\cap W^{1,p}(\Omega)$ of the problem

$$\left.\begin{array}{rcll} \mathrm{div}\big(\varepsilon\mathrm{div}\nabla^2 u - |\nabla u|^{p-2}\nabla u\big) &=& g & \text{in } \Omega,\\ \frac{\partial u}{\partial\nu} = u &=& 0 & \text{on } \Gamma. \end{array}\right\} \tag{2.167}$$

Prove the a-priori estimates

$$\big\|u_\varepsilon\big\|_{W^{1,p}(\Omega)} \leq C, \qquad \big\|u_\varepsilon\big\|_{W^{2,2}(\Omega)} \leq C/\sqrt{\varepsilon}. \tag{2.168}$$

By using Minty's trick based on the monotonicity of the mapping $\varepsilon\mathrm{div}^2\nabla^2 - \Delta_p$, prove the weak convergence $u_\varepsilon \rightharpoonup u$ in $W^{1,p}_0(\Omega)$ to the solution of the boundary-value problem $\mathrm{div}(|\nabla u|^{p-2}\nabla u) + g = 0$ and $u = 0$ on Γ.[85] Alternatively, make

[84] Hint: These two wrong options would exactly over-determine either the first or the second boundary term in (2.104).

[85] Hint: Taking into account the identity $\int_\Omega |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\cdot\nabla v + \varepsilon\nabla^2 u_\varepsilon{:}\nabla^2 v - gv\mathrm{d}x = 0$, use $\|\varepsilon\nabla^2 u_\varepsilon\|_{L^2(\Omega;\mathbb{R}^{n\times n})} = \mathscr{O}(\sqrt{\varepsilon})$ and, for any $v \in W^{2,2}_0(\Omega)\cap W^{1,p}(\Omega)$, show

$$\begin{aligned} 0 &\leq \int_\Omega \big(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon - |\nabla v|^{p-2}\nabla v\big)\cdot\nabla(u_\varepsilon - v) + \varepsilon\big|\nabla^2 u_\varepsilon - \nabla^2 v\big|^2\,\mathrm{d}x\\ &= \int_\Omega g(u_\varepsilon - v) - |\nabla v|^{p-2}\nabla v\cdot\nabla(u_\varepsilon - v) - \varepsilon\nabla^2 v{:}\nabla^2(u_\varepsilon - v)\,\mathrm{d}x \to \int_\Omega g(u-v) - |\nabla v|^{p-2}\nabla v\cdot\nabla(u-v)\,\mathrm{d}x. \end{aligned}$$

Then extend the limit identity by continuity for all $v \in W^{1,p}_0(\Omega)$, and use $v := u \pm \epsilon z$ and accomplish it by Minty's trick.

it by Minty's trick based on the monotonicity of the mapping $-\Delta_p$.[86] Show also the strong convergence $u_\varepsilon \to u$ in $W_0^{1,p}(\Omega)$ by using d-monotonicity of $-\Delta_p$.[87] Modify it by considering also term $c(\nabla u)$ as in Example 2.85 and/or Newton-type boundary conditions, or a quasilinear regularizing term as in Example 2.46.

2.7 Excursion to regularity for semilinear equations

By *regularity* we understand, in general, that the weak solution has some additional differentiability properties as a consequence of some additional qualification of data, i.e. in case of the boundary-value problem (2.45)–(2.49) a certain differentiability of a, b, c, g, and h, and a qualification of Ω as smoothness or restrictions on angles of possible corners. This represents usually a difficult task and there are examples showing that, in case of higher-order equations or systems of equations, any smoothness of the data need not imply an additional smoothness of weak solutions. Regularity theory is a broad and still developing area which determines a lot of investigations in particular in systems of nonlinear equations and in numerical analysis, and the exposition presented below is to be understood as only an absolutely minimal excursion into this area.

We will confine ourselves to $W^{k,2}$-type regularity for semilinear equations and we start with a so-called interior regularity[88] for the linear equation

[86] Hint: Again first for any $v \in W_0^{2,2}(\Omega) \cap W^{1,p}(\Omega)$, calculate

$$\begin{aligned}0 &\le \int_\Omega \big(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon - |\nabla v|^{p-2}\nabla v\big)\cdot\nabla(u_\varepsilon - v)\,\mathrm{d}x\\
&= \int_\Omega g(u_\varepsilon - v) - |\nabla v|^{p-2}\nabla v\cdot\nabla(u_\varepsilon - v) - \varepsilon\nabla^2 u_\varepsilon{:}\nabla^2(u_\varepsilon - v)\,\mathrm{d}x\\
&\le \int_\Omega g(u_\varepsilon - v) - |\nabla v|^{p-2}\nabla v\cdot\nabla(u_\varepsilon - v) + \varepsilon\nabla^2 u_\varepsilon{:}\nabla^2 v\,\mathrm{d}x \to \int_\Omega g(u - v) - |\nabla v|^{p-2}\nabla v\cdot\nabla(u - v)\,\mathrm{d}x.\end{aligned}$$

[87] Hint: Using Example 2.83, for any $v \in W_0^{2,2}(\Omega) \cap W^{1,p}(\Omega)$, show

$$\begin{aligned}&\Big(\|\nabla u_\varepsilon\|^{p-1}_{L^p(\Omega;\mathbb{R}^n)} - \|\nabla u\|^{p-1}_{L^p(\Omega;\mathbb{R}^n)}\Big)\Big(\|\nabla u_\varepsilon\|_{L^p(\Omega;\mathbb{R}^n)} - \|\nabla u\|_{L^p(\Omega;\mathbb{R}^n)}\Big)\\
&\quad\le \int_\Omega \big(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon - |\nabla u|^{p-2}\nabla u\big)\cdot\nabla(u_\varepsilon - u)\,\mathrm{d}x\\
&\quad= \int_\Omega g(u_\varepsilon - v) - \varepsilon\nabla^2 u_\varepsilon{:}\nabla^2(u_\varepsilon - v) - |\nabla u|^{p-2}\nabla u\cdot\nabla(u_\varepsilon - u) + |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\cdot\nabla(v - u)\,\mathrm{d}x\\
&\quad\le \int_\Omega g(u_\varepsilon - v) + \varepsilon\nabla^2 u_\varepsilon{:}\nabla^2 v - |\nabla u|^{p-2}\nabla u\cdot\nabla(u_\varepsilon - u) + |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\cdot\nabla(v - u)\,\mathrm{d}x\\
&\quad\to \int_\Omega g(u - v) + \xi\cdot\nabla(v - u)\,\mathrm{d}x\end{aligned}$$

with some $\xi \in L^{p'}(\Omega;\mathbb{R}^n)$ being a weak limit of (a subsequence) of $|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon$. Pushing $v \to u$ in $W^{1,p}(\Omega)$ makes the last expression arbitrarily close to zero, which shows $\|\nabla u_\varepsilon\|_{L^p(\Omega;\mathbb{R}^n)} \to \|\nabla u\|_{L^p(\Omega;\mathbb{R}^n)}$, hence the strong convergence $u_\varepsilon \to u$.

[88] This means we get estimates only in subdomains of Ω having a positive distance from Γ.

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) = g(x) \qquad \text{on } \Omega \tag{2.169}$$

with nonspecified boundary conditions. By a weak solution to (2.169) we will naturally understand $u \in W^{1,2}(\Omega)$ such that $\int_\Omega (\nabla u)^\top \mathbb{A}\nabla v - gv\,\mathrm{d}x = 0$ for all $v \in W^{1,2}_0(\Omega)$ where $\mathbb{A} : \Omega \mapsto \mathbb{R}^{n\times n} : x \mapsto \mathbb{A}(x) = [a_{ij}(x)]_{i,j=1}^n$.

Proposition 2.101 (Interior $W^{2,2}$-regularity). *Let $\mathbb{A} \in C^1(\Omega;\mathbb{R}^{n\times n})$ satisfy*

$$\exists \delta > 0 \quad \forall \zeta \in \mathbb{R}^n \quad \forall (\text{a.a.})\, x \in \Omega : \qquad \zeta^\top \mathbb{A}(x)\,\zeta \geq \delta|\zeta|^2, \tag{2.170}$$

$g \in L^2_{\mathrm{loc}}(\Omega)$, and let u be a weak solution to (2.169)*. Then $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$. Moreover, for any open sets $O, O_2 \subset \mathbb{R}^n$ satisfying $\bar{O} \subset O_2$ and $\bar{O}_2 \subset \Omega$, it holds that*

$$\|u\|_{W^{2,2}(O)} \leq C\big(\|g\|_{L^2(O_2)} + \|u\|_{L^2(\Omega)}\big) \tag{2.171}$$

with $C = C\big(O, O_2, \|\mathbb{A}\|_{C^1(\Omega;\mathbb{R}^{n\times n})}\big)$.

As the rigorous proof is very technical and not easy to observe, we begin with a heuristic one. Take still an open set O_1 such that $\bar{O} \subset O_1$ and $\bar{O}_1 \subset O_2$, and a smooth "cut-off function" $\zeta : \Omega \to [0,1]$ such that $\chi_O \leq \zeta \leq \chi_{O_1}$. Then, for a test function

$$v := \frac{\partial}{\partial x_k}\Big(\zeta^2 \frac{\partial u}{\partial x_k}\Big) \tag{2.172}$$

with $k = 1, \ldots, n$, by using Green's Theorem 1.30, we have formally the identity

$$\begin{aligned}
&\int_{O_1} \sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_i}\bigg(\frac{\partial}{\partial x_k}\Big(\zeta^2\frac{\partial u}{\partial x_k}\Big)\bigg)\mathrm{d}x \\
&\quad = -\int_{O_1} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_k}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big)\frac{\partial}{\partial x_i}\Big(\zeta^2\frac{\partial u}{\partial x_k}\Big)\mathrm{d}x \\
&\quad = -\int_{O_1} \sum_{i,j=1}^{n} \Big(\frac{\partial a_{ij}}{\partial x_k}\frac{\partial u}{\partial x_j} + a_{ij}\frac{\partial^2 u}{\partial x_j \partial x_k}\Big)\Big(\zeta^2\frac{\partial^2 u}{\partial x_i\partial x_k} + 2\zeta\frac{\partial \zeta}{\partial x_i}\frac{\partial u}{\partial x_k}\Big)\mathrm{d}x.
\end{aligned} \tag{2.173}$$

The identity (2.173) leads to the estimate

$$\begin{aligned}
&\delta\Big\|\zeta\nabla\frac{\partial u}{\partial x_k}\Big\|^2_{L^2(O_1;\mathbb{R}^n)} = \delta\int_{O_1}\sum_{i=1}^{n}\zeta^2\Big|\frac{\partial^2 u}{\partial x_i\partial x_k}\Big|^2\mathrm{d}x \\
&\quad \leq \int_{O_1}\sum_{i,j=1}^{n}\zeta^2 a_{ij}\frac{\partial^2 u}{\partial x_i\partial x_k}\frac{\partial^2 u}{\partial x_j\partial x_k}\mathrm{d}x = -\int_{O_1}\sum_{i,j=1}^{n}\zeta^2\frac{\partial a_{ij}}{\partial x_k}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i\partial x_k} \\
&\qquad\qquad + 2\zeta\frac{\partial\zeta}{\partial x_i}\frac{\partial u}{\partial x_k}\Big(\frac{\partial a_{ij}}{\partial x_k}\frac{\partial u}{\partial x_j} + a_{ij}\frac{\partial^2 u}{\partial x_j\partial x_k}\Big) + g\bigg(\frac{\partial}{\partial x_k}\Big(\zeta^2\frac{\partial u}{\partial x_k}\Big)\bigg)\mathrm{d}x
\end{aligned}$$

$$\begin{aligned}
&\le \int_{O_1} \sum_{i,j=1}^{n} \|a_{ij}\|_{C^1(\Omega)} \Big(\zeta^2 \Big|\frac{\partial u}{\partial x_j}\Big| \Big|\frac{\partial^2 u}{\partial x_i \partial x_k}\Big| \\
&\qquad\qquad + 2\zeta \|\zeta\|_{C^1(\Omega)} \Big|\frac{\partial u}{\partial x_k}\Big| \Big(\Big|\frac{\partial u}{\partial x_j}\Big| + \Big|\frac{\partial^2 u}{\partial x_j \partial x_k}\Big| \Big)\Big) \mathrm{d}x \\
&\qquad\qquad + \|g\|_{L^2(O_1)} \Big\| 2\zeta\nabla\zeta \frac{\partial u}{\partial x_k} + \zeta^2 \nabla \frac{\partial u}{\partial x_k} \Big\|_{L^2(O_1;\mathbb{R}^n)} \\
&\le C_1 \big(\|\nabla u\|_{L^2(O_1;\mathbb{R}^n)} + \|g\|_{L^2(O_1)} \big) \Big(\Big\| \zeta \nabla \frac{\partial u}{\partial x_k} \Big\|_{L^2(O_1;\mathbb{R}^n)} + \Big\| \zeta \frac{\partial u}{\partial x_k} \Big\|_{L^2(O_1)} \Big) \\
&\le \frac{\delta}{2} \Big\| \zeta \nabla \frac{\partial u}{\partial x_k} \Big\|^2_{L^2(O_1;\mathbb{R}^n)} + \Big(\frac{C_1^2}{\delta} + \frac{3}{2} \Big) \big(\|\nabla u\|^2_{L^2(O_1;\mathbb{R}^n)} + \|g\|^2_{L^2(O_1)} \big) \qquad (2.174)
\end{aligned}$$

with C_1 depending on $\|[a_{ij}]_{i,j=1}^n\|_{C^1(\Omega;\mathbb{R}^{n\times n})}$ and $\|\zeta\|_{C^1(\Omega)}$. Then, letting k range over $1,..,n$, we obtain

$$\|u\|_{W^{2,2}(O)} \le C_2 \big(\|g\|_{L^2(O_1)} + \|u\|_{W^{1,2}(O_1)} \big). \qquad (2.175)$$

Finally, using a smooth "cut-off function" $\eta : \Omega \to [0,1]$ such that $\chi_{O_1} \le \eta \le \chi_{O_2}$ and the test-function $v = \eta u$, we get $\delta \|\nabla u\|^2_{L^2(O_1;\mathbb{R}^n)} \le \delta \int_\Omega \eta |\nabla u|^2 \mathrm{d}x \le \int_\Omega \eta g u \,\mathrm{d}x \le \frac{1}{2}\|g\|^2_{L^2(O_2)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)}$, which eventually leads to (2.171). The rigorous proof is, however, more complicated because (2.172) is not a legal test function unless we know that $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$, which is just what we want to prove.

Sketch of the proof of Proposition 2.101*.* We introduce the difference operator D_k^ε defined by

$$\big[\mathrm{D}_k^\varepsilon u\big](x) := \frac{u(x+\varepsilon e_k) - u(x)}{\varepsilon}, \qquad \varepsilon \ne 0, \quad [e_k]_i := \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \ne k, \end{cases} \qquad (2.176)$$

and use the test function

$$v := \mathrm{D}_k^{-\varepsilon} \big(\zeta^2 \mathrm{D}_k^\varepsilon u \big) \qquad (2.177)$$

with $k = 1, \ldots, n$. Note that, contrary to (2.172), now $v \in W_0^{1,2}(\Omega)$ is a legal test function. The analog of Green's Theorem 1.30 is now

$$\begin{aligned}
\int_\Omega v \mathrm{D}_k^{-\varepsilon} w \,\mathrm{d}x &= \int_\Omega v(x) \frac{w(x-\varepsilon e_k) - w(x)}{\varepsilon} \mathrm{d}x \\
&= \frac{1}{\varepsilon} \int_\Omega v(x) w(x - \varepsilon e_k) \,\mathrm{d}x - \frac{1}{\varepsilon} \int_\Omega v(x) w(x) \,\mathrm{d}x \\
&= \frac{1}{\varepsilon} \int_\Omega v(x+\varepsilon e_k) w(x) \,\mathrm{d}x - \frac{1}{\varepsilon} \int_\Omega v(x) w(x) \,\mathrm{d}x = - \int_\Omega w \mathrm{D}_k^\varepsilon v \,\mathrm{d}x \qquad (2.178)
\end{aligned}$$

if $|\varepsilon|$ is smaller than the distance ε_0 of Γ from O_1; note that v vanishes on $\Omega \setminus O_1$. Moreover, by simple algebra, we have the formula

$$\mathrm{D}_k^\varepsilon (vw) = \mathrm{S}_k^\varepsilon v \,\mathrm{D}_k^\varepsilon w + w \,\mathrm{D}_k^\varepsilon v \qquad (2.179)$$

with the "shift" operator S_k^ε defined by $\big[\mathrm{S}_k^\varepsilon v\big](x) := v(x+\varepsilon e_k)$. The analog of (2.173) now reads as

$$\begin{aligned}\int_{O_1}\sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_i}\Big(\mathrm{D}_k^{-\varepsilon}\big(\zeta^2\mathrm{D}_k^\varepsilon u\big)\Big)\mathrm{d}x\\ &= -\int_{O_1}\sum_{i,j=1}^n \mathrm{D}_k^\varepsilon\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big)\frac{\partial}{\partial x_i}\Big(\zeta^2\mathrm{D}_k^\varepsilon u\Big)\mathrm{d}x\\ &= -\int_{O_1}\sum_{i,j=1}^n \Big(\mathrm{D}_k^\varepsilon a_{ij}\frac{\partial u}{\partial x_j}+\mathrm{S}_k^\varepsilon a_{ij}\mathrm{D}_k^\varepsilon\frac{\partial u}{\partial x_j}\Big)\Big(\zeta^2\mathrm{D}_k^\varepsilon\frac{\partial u}{\partial x_i}+2\zeta\frac{\partial\zeta}{\partial x_i}\mathrm{D}_k^\varepsilon u\Big)\mathrm{d}x.\end{aligned}\tag{2.180}$$

We also use that $\|\mathrm{D}_k^\varepsilon v\|_{L^2(\Omega_1)}\le\|\nabla v\|_{L^2(\Omega)}$ if $|\varepsilon|\le\varepsilon_0:=\mathrm{dist}(O_1,\Gamma)$.[89] Then the analog of (2.174) reads as

$$\begin{aligned}\delta\big\|\zeta\,\mathrm{D}_k^\varepsilon\nabla u\big\|^2_{L^2(O_1;\mathbb{R}^n)} &\le \frac{\delta}{2}\big\|\zeta\,\mathrm{D}_k^\varepsilon\nabla u\big\|^2_{L^2(O_1;\mathbb{R}^n)}\\ &\quad+\Big(\frac{C_1^2}{\delta}+\frac{3}{2}\Big)\big(\big\|\nabla u\big\|^2_{L^2(O_1;\mathbb{R}^n)}+\|g\|^2_{L^2(O_1)}\big).\end{aligned}\tag{2.181}$$

Hence the sequence (selected from) $\{\zeta\,\mathrm{D}_k^\varepsilon\nabla u\}_{0<\varepsilon\le\varepsilon_0}$ is bounded in $L^2(O_1;\mathbb{R}^n)$ and converges, possibly as a subsequence, weakly to some w in $L^2(O_1;\mathbb{R}^n)$. In the sense of distributions, it must hold that $w=\zeta\frac{\partial}{\partial x_k}\nabla u$.[90] In particular, $\frac{\partial}{\partial x_k}\nabla u\in L^2(O;\mathbb{R}^n)$ and, if considering $k=1,\dots,n$, we have obtained (2.175). Then (2.171) follows as outlined in the heuristics. $\square$

Proposition 2.102 (Interior $W^{3,2}$-regularity)**.** *Let $\mathbb{A}\in C^1(\Omega;\mathbb{R}^{n\times n})\cap W^{2,q}(\Omega;\mathbb{R}^{n\times n})$ with $q=2^*2/(2^*-2)$ with 2^* from (1.34) satisfy (2.170), and let $g\in W^{1,2}_{\mathrm{loc}}(\Omega)$, and let u be a weak solution to (2.169). Then $u\in W^{3,2}_{\mathrm{loc}}(\Omega)$. Moreover, for any open sets $O,O_2\subset\mathbb{R}^n$ satisfying $\bar O\subset O_2$ and $\bar O_2\subset\Omega$, it holds that*

$$\|u\|_{W^{3,2}(O)}\le C\big(\|g\|_{W^{1,2}(O_2)}+\|u\|_{L^2(\Omega)}\big)\tag{2.182}$$

with $C=C\big(O,\|\mathbb{A}\|_{C^1(\Omega;\mathbb{R}^{n\times n})\cap W^{2,q}(\Omega;\mathbb{R}^{n\times n})}\big)$.

Proof. Applying $\frac{\partial}{\partial x_k}$ to (2.169), we obtain

$$\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial^2 u}{\partial x_j\partial x_k}\Big)=\frac{\partial g}{\partial x_k}-\sum_{i,j=1}^n\Big(\frac{\partial^2 a_{ij}}{\partial x_i\partial x_k}\frac{\partial u}{\partial x_j}+\frac{\partial a_{ij}}{\partial x_k}\frac{\partial^2 u}{\partial x_j\partial x_i}\Big)\tag{2.183}$$

[89] It holds that $[\mathrm{D}_k^\varepsilon v](x)=\int_0^1\frac{\partial}{\partial x_k}(v+\tau\varepsilon e_k)\mathrm{d}\tau$ so that, by Hölder inequality, we obtain $\|\mathrm{D}_k^\varepsilon v\|^2_{L^2(\Omega_1)}=\int_{\Omega_1}\big|\int_0^1\frac{\partial}{\partial x_k}(v+\tau\varepsilon e_k)\mathrm{d}\tau\big|^2\mathrm{d}x\le\int_\Omega\big|\nabla v\big|^2\mathrm{d}x$.

[90] For any $v\in\mathscr{D}(O)$ it holds that $\lim_{\varepsilon\to0}\int_{\Omega_1}(\zeta\,\mathrm{D}_k^\varepsilon\nabla u)v\,\mathrm{d}x=\lim_{\varepsilon\to0}-\int_\Omega\mathrm{D}_k^{-\varepsilon}(\zeta v)\nabla u\,\mathrm{d}x=-\int_\Omega\frac{\partial}{\partial x_k}(\zeta v)\nabla u\,\mathrm{d}x=-\int_O\frac{\partial}{\partial x_k}v\nabla u\,\mathrm{d}x$.

in Ω. Note that, by Proposition 2.101, $u \in W^{2,2}_{\rm loc}(\Omega)$ and therefore (2.183) has indeed a good "weak" sense: $z := \frac{\partial}{\partial x_k}u$ is a weak solution to (2.169) with z instead of u and with $\frac{\partial}{\partial x_k}g - \mathrm{div}\big((\frac{\partial}{\partial x_k}\mathbb{A})\nabla u\big) - (\frac{\partial}{\partial x_k}\mathbb{A})\nabla^2 u \in L^2_{\rm loc}(\Omega)$ instead of g. Hence $\frac{\partial}{\partial x_k}u \in W^{2,2}_{\rm loc}(\Omega)$. $\square$

For linear equations as (2.169) the process suggested in (2.183) can be iterated for $k = 4, \ldots$ to obtain $W^{k,2}_{\rm loc}$-regularity under the assumption that $\mathbb{A} \in C^{k-2}(\Omega;\mathbb{R}^{n\times n}) \cap W^{k-1,q}(\Omega;\mathbb{R}^{n\times n})$ and $g \in W^{k-2,2}(\Omega)$. This differs from nonlinear equations where the regularity has usually a natural bound. Here, we confine ourselves to semilinear equations where results for linear equations can directly be exploited. To be more specific, we will handle the equation

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j} + a_{0i}(u)\Big) + c_0(\nabla u) + |u|^{q-2}u = g(x) \qquad \text{on } \Omega \tag{2.184}$$

again with unspecified boundary conditions. By a weak solution to (2.184) we will naturally understand $u \in W^{1,2}(\Omega)$ such that $\int_\Omega \big((\nabla u)^\top\mathbb{A} + a_0(u)\big)\cdot\nabla v + \big(c_0(\nabla u) + |u|^{q-2}u - g\big)v\,\mathrm{d}x = 0$ for all $v \in W^{1,2}_0(\Omega)$.

Proposition 2.103 (Regularity for semilinear equations).

(i) *Let $\mathbb{A} \in C^1(\Omega;\mathbb{R}^{n\times n})$ satisfy (2.170), let $1 < q \le (2n-2)/(n-2)$ for $n \ge 3$ (or $q > 1$ arbitrary if $n \le 2$), $a_0 : \mathbb{R} \to \mathbb{R}^n$ be Lipschitz continuous, c_0 have at most linear growth, and $g \in L^2_{\rm loc}(\Omega)$. Then any weak solution $u \in W^{1,2}(\Omega)$ to (2.184) satisfies also $u \in W^{2,2}_{\rm loc}(\Omega)$.*

(ii) *Moreover, let, in addition, $\mathbb{A} \in W^{2,\max(2,n+\epsilon)}(\Omega;\mathbb{R}^{n\times n})$ with $\epsilon > 0$ if $n = 2$ (otherwise $\epsilon = 0$ is allowed), and let also $q \ge 2$, $a_0 \in C^2(\mathbb{R};\mathbb{R}^n)$ with*

$$a_0'' : \mathbb{R} \to \mathbb{R}^n \begin{cases} \text{having arbitrary growth} & \text{if } n \le 3, \\ \text{being bounded} & \text{if } n = 4, \\ \equiv 0 & \text{if } n \ge 5, \end{cases} \tag{2.185}$$

$c_0 : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz continuous, and $g \in W^{1,2}_{\rm loc}(\Omega)$. Then any weak solution $u \in W^{1,2}(\Omega)$ to (2.184) belongs also to $W^{3,2}_{\rm loc}(\Omega)$.

Proof. Note that $u \in W^{1,2}(\Omega)$ implies $\mathrm{div}(a_0(u)) = a_0'(u)\nabla u = \sum_{i=1}^n a_{0i}'(u)\frac{\partial}{\partial x_i}u \in L^2(\Omega)$ if $a_0 \in W^{1,\infty}(\mathbb{R};\mathbb{R}^n)$ as assumed, cf. Proposition 1.28. Also, $c_0(\nabla u) \in L^2(\Omega)$ because of the linear growth of c_0, and eventually $|u|^{q-2}u \in L^{2^*/(q-1)}(\Omega) \subset L^2(\Omega)$ if $1 < q \le (2n-2)/(n-2)$ (or $q > 1$ arbitrary if $n \le 2$). Noting also that the exponent $2^*2/(2^*-2)$ equals $\max(2,n)$ if $n \ne 2$, or is greater than 2 if $n = 2$, we can use simply Proposition 2.101 with g being now $g_1 := g - \mathrm{div}(a_0(u)) - c_0(\nabla u) - |u|^{q-2}u \in L^2(\Omega)$. The point (i) is thus proved.

Assuming the additional data qualification as specified in the point (ii), we want to show that $g_1 \in W^{1,2}_{\rm loc}(\Omega)$. For $i = 1, \ldots, n$, we have

$$\frac{\partial g_1}{\partial x_i} = \frac{\partial g}{\partial x_i} - \sum_{j=1}^{n} \Big(a'_{0j}(u) \frac{\partial^2 u}{\partial x_i \partial x_j} + a''_{0j}(u) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \frac{\partial c_0}{\partial s_i}(\nabla u) \frac{\partial^2 u}{\partial x_i \partial x_j} \Big) - (q-1)|u|^{q-2} \frac{\partial u}{\partial x_i}. \tag{2.186}$$

For $u \in W^{1,2}(\Omega)$, we have $|u|^{q-2} \in L^{2^*/(q-2)}(\Omega)$ so that, in general, we do not have $|u|^{q-2}\nabla u \in L^2(\Omega)$ guaranteed. Likewise, the a_0- and c_0-terms also do not live in $L^2(\Omega)$ in general if we do not have some additional information about $u \in W^{1,2}(\Omega)$. However, we can use the already proved assertion (i), i.e. $u \in W^{2,2}_{\rm loc}(\Omega)$; this trick is called a *bootstrap*[91]. Then it is easy to show that $g_1 \in W^{1,2}_{\rm loc}(\Omega)$ hence we can use simply Proposition 2.102 with g being now g_1. □

Having the data qualification $\mathbb{A} \in C^1(\Omega; \mathbb{R}^{n\times n})$ and $a_0 \in W^{1,\infty}(\mathbb{R}; \mathbb{R}^n)$ assumed and the $W^{2,2}_{\rm loc}(\Omega)$-regularity at our disposal, it is then straightforward to check that (2.184) holds not only in the weak sense but even a.e. in Ω. Such a mode of a solution to a differential equation is called a *Carathéodory solution.*

Let us now briefly outline how regularity up to the boundary can be obtained. We will confine ourselves to $W^{2,2}$-regularity and the Newton boundary conditions (2.48) and begin with (2.169). Thus (2.48) reads as

$$\sum_{j=1}^{n} \nu_i a_{ij}(x) \frac{\partial u}{\partial x_j} + b(x,u) = h(x) \qquad \text{on } \ \Gamma. \tag{2.187}$$

Proposition 2.104 ($W^{2,2}$-REGULARITY UP TO BOUNDARY)**.** *Let Ω be of C^2-class, $\mathbb{A} \in C^1(\Omega; \mathbb{R}^{n\times n})$ satisfy* (2.170), *$b \in C^1(\mathbb{R}^n \times \mathbb{R})$ satisfy, for some $b_0 > 0$ and $C \in \mathbb{R}$,*

$$\forall (\text{a.a.}) x \in \Gamma \ \ \forall r_1, r_2 \in \mathbb{R}: \qquad \big(b(x,r_1) - b(x,r_2)\big)(r_1 - r_2) \geq b_0 \big|r_1 - r_2\big|^2, \tag{2.188a}$$

$$\exists \gamma \in L^2(\Gamma) \ \ \forall (\text{a.a.}) x \in \Gamma \ \ \forall r \in \mathbb{R}: \qquad \Big|\frac{\partial b}{\partial x}(x,r)\Big| \leq \gamma(x) + C|r|^{2^\#/2}, \tag{2.188b}$$

$g \in L^2(\Omega)$, $h \in W^{1,2}(\Gamma)$,[92] *and let $u \in W^{1,2}(\Omega)$ be the unique weak solution to the boundary-value problem* (2.169)–(2.187)*. Then $u \in W^{2,2}(\Omega)$. Moreover, if $b(x,r) = b_1(x) r$ with $b_1 \in W^{1,2^\# 2/(2^\# - 2)}(\Gamma)$, then*

$$\|u\|_{W^{2,2}(\Omega)} \leq C\big(\|g\|_{L^2(\Omega)} + \|h\|_{W^{1,2}(\Gamma)}\big) \tag{2.189}$$

with $C = C\big(\Omega, \|\mathbb{A}\|_{C^1(\Omega;\mathbb{R}^{n\times n})}, \|b_1\|_{W^{1,2n-2+\epsilon}(\Gamma)}\big)$.

[91] Often, bootstrap is used not only in the order of differentiation but rather in the integrability, which is not possible here because we present the Hilbertian theory only.

[92] The notation $W^{1,2}(\Gamma)$ for Γ smooth means that, after a local rectification like on Figure 8, the transformed and "smoothly cut" functions belong to $W^{1,2}(\mathbb{R}^{n-1})$. Also $\frac{\partial}{\partial x} b$ in (2.188b) refers to the derivatives in the tangential directions only.

Sketch of the proof. First, as Ω is bounded, Γ is a compact set in $\mathbb{R}^n$, and can be covered by a finite number of open sets which are C^2-diffeomorphical images of the unit ball $B = \{\xi \in \mathbb{R}^n;\ |\xi| \le 1\}$ such that the respective part of Γ is an image of $\{\xi = (\xi_1,\dots,\xi_n) \in B;\ \xi_1 = 0\}$. Thus we rectified locally the boundary Γ, cf. Figure 8.

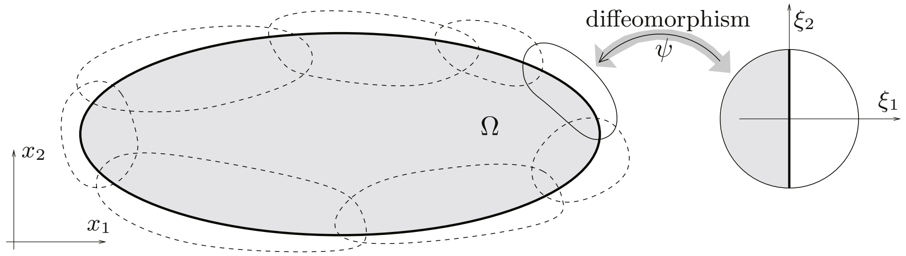

Figure 8. Illustration of finite coverage of $\Gamma \subset \mathbb{R}^2$ and one diffeomorphism rectifying locally a part of Γ.

It is a technical calculation showing that $\tilde u \in W^{1,2}(B_0)$ defined by $\tilde u(\xi) = u(\psi(\xi))$, where $\psi : B_0 := \{\xi \in B;\ x_1 \le 0\} \to \bar\Omega$ is the homeomorphism in question, is a weak solution to an equation like (2.169) but with the coefficients $\mathbb{A}$ transformed but again being continuously differentiable and satisfying (2.170)[93], and the boundary condition (2.187) transforms to a similar condition for $\tilde u|_{\xi_1=0}$. Hence, in fact, it suffices to obtain an estimate like (2.189) only for $\tilde u \in W^{1,2}(B_0)$. For simplicity, we will use the original notation.

We again use the test function (2.177) but now only for $k = 2,\dots,n$, i.e. we use shifts only in the tangential direction, so that we still have (2.178) at our disposal. Now the cut-off function $\zeta : B_0 \to [0,1]$ can be taken as 1 in a semi-ball $\{\xi \in \mathbb{R}^n;\ |\xi| \le 1-\varepsilon_0,\ x_1 \le 0\}$ and vanishing on $\{\xi \in \mathbb{R}^n;\ |\xi| \ge 1-\frac12\varepsilon_0,\ x_1 \le 0\}$ with some ε_0. The heuristical estimate (2.173)–(2.174) now involves also the boundary term $\int_\Gamma (b(x,u)-h)\frac{\partial}{\partial x_k}(\zeta^2\frac{\partial}{\partial x_k}u)\,\mathrm{d}S$ which, in the difference variant, reads and, for $|\varepsilon| \le \frac12\varepsilon_0$, can be estimated as

$$\begin{aligned}\int_\Gamma \big(b(x,u)-h\big)\mathrm{D}_k^{-\varepsilon}\big(\zeta^2\mathrm{D}_k^\varepsilon u\big)\,\mathrm{d}S &= -\int_\Gamma \zeta^2\mathrm{D}_k^\varepsilon\big(b(x,u)-h\big)\mathrm{D}_k^\varepsilon u\,\mathrm{d}S\\
&= -\int_\Gamma \zeta^2\bigg(\frac{b\big(x+\varepsilon e_k, u(x+\varepsilon e_k)\big)-b\big(x+\varepsilon e_k,u(x)\big)}{\varepsilon}\\
&\qquad\qquad -\mathrm{D}_k^\varepsilon h + \frac1\varepsilon\int_0^\varepsilon \frac{\partial b}{\partial r}\big(x+\tau e_k,u(x)\big)\mathrm{d}\tau\bigg)\mathrm{D}_k^\varepsilon u\,\mathrm{d}S\\
&\le -b_0\big\|\zeta\mathrm{D}_k^\varepsilon u\big\|_{L^2(\Gamma)}^2 + \Big(\big\|h\big\|_{W^{1,2}(\Gamma)} + \big\|\gamma+C|u|^{2^\#/2}\big\|_{L^2(\Gamma)}\Big)\big\|\zeta\mathrm{D}_k^\varepsilon u\big\|_{L^2(\Gamma)}\end{aligned}$$

[93] To be more specific, $\tilde u$ satisfies $\sum_{i,j=1}^n \partial(\tilde a_{ij}\partial\tilde u/\partial x_j)\partial x_i = \tilde g$ with the transformed coefficients $\tilde a_{ij}(\xi) = \sum_{k,l=1}^n[a_{kl}\frac{\partial}{\partial x_k}\psi^{-1}\frac{\partial}{\partial x_l}\psi^{-1}](\psi(\xi))$ and $\tilde g(\xi) = g(\psi(\xi))$. The boundary conditions are transformed accordingly, i.e. $\tilde b(\xi,r) = b(\psi(\xi),r)$ and $\tilde h(\xi) = h(\psi(\xi))$.

$$\leq \|h\|^2_{W^{1,2}(\Gamma)} + \frac{1}{b_0}\Big(\|\gamma\|^2_{L^2(\Gamma)} + C^2\|u\|^{2^\#}_{L^{2^\#}(\Gamma)}\Big). \tag{2.190}$$

In this way, we get the local estimates for $\frac{\partial^2}{\partial x_i \partial x_j}u$ for all (i,j) except $i = 1 = j$ meant in the locally rectified coordinate system, cf. Figure 8(right).

The estimate of the normal derivative follows just from the equation itself which has been shown to hold a.e. in Ω. Thus

$$\frac{\partial^2 u}{\partial x_1^2} = \frac{1}{a_{11}}\Big(g - \sum_{i+j>2} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_i}\frac{\partial u}{\partial x_j}\Big) \tag{2.191}$$

from which we get the local L^2-bound for $\frac{\partial^2}{\partial x_1^2}u$ near the boundary because $a_{11}^{-1} \in L^\infty(\Omega)$ due to the uniform ellipticity of $\mathbb{A}$.

For the special case $b(x,r) = b_1(x)r$, the estimate (2.190) can be finalized by $\|[\frac{\partial}{\partial x}b](x,u)\|_{L^2(\Gamma)} \leq \|\frac{\partial}{\partial x}b_1\|_{L^{2^\# 2/(2^\# -2)}(\Gamma)}\|u\|_{L^{2^\#}(\Gamma)}$. This eventually allows us to derive the a-priori estimate (2.189) by summing the (finite number of) the local estimates on the boundary with one estimate on an open set O from Proposition 2.101 and by using the conventional energy estimate $\|u\|_{W^{1,2}(\Omega)}$ and thus also $\|u\|_{L^{2^\#}(\Gamma)}$ in terms of g and h. $\square$

Corollary 2.105 ($W^{2,2}$-REGULARITY FOR SEMILINEAR EQUATION)**.** *Let the assumptions of Propositions* 2.103(i) *and* 2.104 *be satisfied. Then any weak solution u to the equation* (2.184) *with the boundary conditions*

$$\sum_{j=1}^{n} \nu_i\Big(a_{ij}(x)\frac{\partial u}{\partial x_j} + a_{0i}(u)\Big) + b(x,u) = h(x) \qquad \text{on} \;\; \Gamma \tag{2.192}$$

is a Carathéodory solution and belongs also to $W^{2,2}(\Omega)$.

Remark 2.106 (Dirichlet boundary conditions)**.** Alternatively, instead of (2.192), one can think about prescribing $u|_\Gamma = u_{\rm D}$ with $u_{\rm D} = w|_\Gamma$ for some $w \in W^{2,2}(\Omega)$. After a shift by w, cf. Proposition 2.27, one gets a problem for $u_0 = u - w$ with zero Dirichlet condition and a contribution to the right-hand side which is again in $L^2(\Omega)$. The proof of Proposition 2.104 is even simpler because (2.190) simply vanishes.

2.8 Bibliographical remarks

Pseudomonotone mappings have been introduced by Brézis [64].[94] A further reading can involve the books by Nečas [305], Pascali and Sburlan [325], Renardy and

[94] In fact, [64] allows for $A : V \to V_2$ with V_2 "in duality" with V but not necessarily $V_2 = V^*$, and also requires $u \mapsto \langle A(u), u - v\rangle$ to be lower bounded on each compact set in V and for each $v \in V$, which is weaker than (2.3a). In literature, "pseudomonotone" sometimes omit (2.3a) completely, cf. [427, Definition 27.5].

Rogers [349], Růžička [376], and Zeidler [427, Chap.27]. Mere monotone mappings can be found there, too, and also in a lot of further monographs, say [95, 168, 414, 424]. Historically, theory of monotone mappings arises by the works by Browder [73], Minty [286], and Vishik [416].

The mappings weakly continuous when restricted to finite-dimensional subspaces and satisfying (2.123) are called *mappings of the type (M)*, having been invented by Brézis [64], and further generalized e.g. in [201, 227]. This class involves both the pseudomonotone and the weakly continuous mappings[95] but, contrary to those two classes, it is not closed under addition. Mappings of type (M) do not inherit some other nice properties of pseudomonotone mappings, too.[96] As to the weakly continuous mappings, their importance in the context of semilinear equations has been pointed out by Franců [149]. The setting $A : V \to Z^* \supset V^*$ with $V_k \subset Z$ we used in Section 2.5 was used by Hess [201] in the context of the mappings of the type (M), see also [325, Ch.IV, Sect.3.1] or [427, Sect.27.7]. The mappings satisfying (2.23) are called *mappings of the type (S_+)*; this notion has been invented by Browder [76, p.279].

The fruitful Galerkin method originated at the beginning of 20th century [171], being motivated by engineering applications.

Concrete quasilinear partial differential equations in the divergence form has been scrutinized, e.g., by Chen and Wu [92, Chap.5], Fučík and Kufner [159], Gilbarg and Trudinger [178, Chap.11], Ladyzhenskaya and Uraltseva [250, Chap.4], Lions [261, Sect.2.2], Nečas [305], Taylor [402, Chap.14], and Zeidler [427, Chap.27]. For semilinear equations see Pao [324]. Quasilinear equations in a non-divergence form (not mentioned in here) can be found, e.g., in Ladyzhenskaya and Uraltseva [250, Chap.6] or Gilbarg and Trudinger [178, Chap.12]. *Fully nonlinear equations* of the type $a(\Delta u) = g$ (also not mentioned in here) are, e.g., in Chen, Wu [92, Chap.7], Caffarelli, Chabré [86], Dong [126, Chap.9,10], Gilbarg and Trudinger [178, Chap.17].

Regularity results in Sect. 2.7 can easily be generalized for the strongly monotone quasilinear equation of the type (2.147) satisfying (2.159a). More general regularity theory for elliptic equations is exposed, e.g., in the monographs by Bensoussan, Frehse [50], Evans [138], Giaquinta [175], Gilbarg, Trudinger [178], Grisvard [190], Lions, Magenes [262], Ladyzhenskaya, Uraltseva [250], Nečas [302, 305], Renardy, Rogers [349], Skrypnik [387], and Taylor [402]. Besides, this active research area is recorded in thousands of papers; e.g. Agmon, Douglis, and Nirenberg [6] and Nečas [303].

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[95] For the implication "pseudomonotone $\Rightarrow$ type-(M)" see Exercise 2.54 while the implication "weakly continuous $\Rightarrow$ type-(M)" is obvious – note that even $\limsup_{k\to\infty} \langle A(u_k), u_k\rangle \le \langle f, u\rangle$ occurring in (2.123) does not need to have a sense if $A(u_k) \in Z^* \setminus V^*$ or $f \in Z^* \setminus V^*$.

[96] E.g., Φ' of type (M) does not yield weak lower-semicontinuity of Φ, unlike pseudomonotonicity, cf. Theorem 4.4(ii); e.g. $\Phi(u) = -\|u\|^2$ if V is an infinite-dimensional Hilbert space.

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