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Finite and Infinite Gap Jacobi Matrices

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Abstract. The present paper reviews the theory of bounded Jacobi matrices whose essential spectrum is a finite gap set, and it explains how the theory can be extended to also cover a large number of infinite gap sets. Two of the central results are generalizations of Denisov–Rakhmanov's theorem and Szegő's theorem, including asymptotics of the associated orthogonal polynomials. When the essential spectrum is an interval, the natural limiting object J_0 has constant Jacobi parameters. As soon as gaps occur, ℓ say, the complexity increases and the role of J_0 is taken over by an ℓ-dimensional isospectral torus of periodic or almost periodic Jacobi matrices.

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1. Introduction

Let $d\mu$ be a probability measure on $\mathbb{R}$ with moments of all orders, that is,

$$\int_{\mathbb{R}} |x|^n d\mu(x) < \infty \text{ for all } n \geq 0. \tag{1.1}$$

When $d\mu$ is nontrivial (i.e., $\mathrm{supp}(d\mu)$ is infinite), we can apply the Gram–Schmidt process to $1, x, x^2, \ldots$ and obtain a sequence $\{P_n\}_{n\geq 0}$ of orthonormal polynomials

$$\langle P_n, P_m \rangle := \int_{\mathbb{R}} P_n(x) P_m(x) d\mu(x) = \delta_{nm}, \tag{1.2}$$

where each P_n has positive leading coefficient and is of degree n. It is a basic fact that such polynomials satisfy a three-term recurrence relation of the form

$$xP_n(x) = a_{n+1}P_{n+1}(x) + b_{n+1}P_n(x) + a_nP_{n-1}(x), \quad n \geq 0 \tag{1.3}$$

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with $a_n = \langle P_{n-1}, xP_n \rangle > 0$ and $b_n = \langle P_{n-1}, xP_{n-1} \rangle \in \mathbb{R}$ for $n \geq 1$ (by convention, $P_{-1}(x) \equiv 0$). To see this, simply expand xP_n in terms of $P_0, P_1, \ldots, P_{n+1}$ and use the orthogonality relation (1.2). Note also that

$$P_n(x) = \frac{1}{a_1 \cdots a_n}\Big(x^n - (b_1 + \cdots + b_n)x^{n-1} + \cdots\Big) \text{ for } n \geq 1. \tag{1.4}$$

The spectral theorem for orthonormal polynomials (also known as Favard's theorem) states that *for any pair of sequences* $\{a_n, b_n\}_{n=1}^\infty \in (0,\infty)^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$*, there exists a probability measure* $d\mu$ *on* $\mathbb{R}$ *such that the polynomials generated by* (1.3)*, with* $P_0(x) = 1$*, satisfy the orthogonality relation* (1.2). In general, this measure of orthogonality need not be unique. But when the recurrence coefficients are bounded, say $a_n, |b_n| \leq C$, then $d\mu$ is indeed unique and $\mathrm{supp}(d\mu)$ is contained in $[-3C, 3C]$. Conversely, if $d\mu$ has compact support, then the associated recurrence coefficients are bounded by

$$\max_{x \in \mathrm{supp}(d\mu)} |x| < \infty$$

and the polynomials are dense in $L^2(d\mu)$. We shall henceforth assume that $\mathrm{supp}(d\mu)$ is compact.

The three-term recurrence relation (1.3) links orthogonal polynomials to Jacobi matrices, that is, tridiagonal matrices of the form

$$J = \begin{pmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & a_2 & & \\ & a_2 & b_3 & a_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \tag{1.5}$$

with $a_n > 0$ and $b_n \in \mathbb{R}$. In fact, the matrix J in (1.5) represents the operator of multiplication by the identity function x in the Hilbert space $L^2(d\mu)$ with respect to the orthonormal basis $\{P_n\}_{n\geq 0}$. When J is viewed as an operator on $\ell^2(\mathbb{N})$, its spectrum $\sigma(J)$ coincides with $\mathrm{supp}(d\mu)$ and we shall refer to $d\mu$ as the spectral measure of J.

In spectral theory for orthogonal polynomials, one studies the relation between nontrivial probability measures $d\mu$ satisfying (1.1) on one hand and pairs of sequences $\{a_n, b_n\}_{n=1}^\infty \in (0,\infty)^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ on the other hand. The aim of the present paper is to give a general view of the situation where $d\mu$ is compactly supported and the recurrence coefficients (also known as Jacobi parameters) are bounded sequences. As already mentioned, there is a one-one correspondence between these two classes of objects and we shall focus on results that explain how qualitative features of the Jacobi parameters, say, are reflected in the measure of orthogonality, and vice versa.

Throughout, we shall write the probability measure $d\mu$ as

$$d\mu = f(x)dx + d\mu_{\mathrm{s}}, \tag{1.6}$$

with $d\mu_{\rm s}$ singular to dx. Rather than $\sigma(J)$, many of the results are more suitably formulated in terms of $\sigma_{\rm ess}(J)$, the essential spectrum of J. By definition,

$$\sigma_{\rm ess}(J) := \{x \in \sigma(J) \mid x \text{ is } \textit{not} \text{ an isolated eigenvalue of } J\}. \tag{1.7}$$

As regards proofs, in particular, a key role is played by the m-function (or Stieltjes transform of $d\mu$) defined by

$$m(z) := m_\mu(z) = \int \frac{d\mu(x)}{x-z}, \quad z \in \mathbb{C} \setminus \operatorname{supp}(d\mu). \tag{1.8}$$

This analytic function is known to be a Nevanlinna–Pick function (i.e., $\operatorname{Im} m(z) \gtrless 0$ for $\operatorname{Im} z \gtrless 0$) and we have

$$m(z) = -1/z + \mathcal{O}(z^{-2}) \tag{1.9}$$

near ∞. In fact, one can write down the Laurent expansion of m_μ around ∞ in terms of the moments of $d\mu$. More importantly, the boundary values $m(x+i0) := \lim_{\varepsilon\downarrow 0} m(x+i\varepsilon)$ exist for a.e. $x \in \mathbb{R}$ and

$$\frac{1}{\pi} \operatorname{Im} m_\mu(x+i\varepsilon)\, dt \xrightarrow{w} d\mu \ \text{ as } \varepsilon \downarrow 0. \tag{1.10}$$

To be even more specific,

$$f(x) = \frac{1}{\pi} \operatorname{Im} m_\mu(x+i0) \ \text{ a.e. on } \mathbb{R} \tag{1.11}$$

and

$$\mu_{\rm s}\big(\{x\}\big) = \lim_{\varepsilon\to 0} \varepsilon \operatorname{Im} m_\mu(x+i\varepsilon) \ \text{ for all } x \in \mathbb{R}. \tag{1.12}$$

So isolated mass points of $d\mu$ (or isolated eigenvalues of J) are poles of m.

The simplest compact subsets of $\mathbb{R}$ that have positive measure are intervals of the form $[\alpha, \beta]$ with $-\infty < \alpha < \beta < \infty$. In Section 2, we shall consider the situation when $\sigma_{\rm ess}(J)$ has this form and without loss of generality we may assume that $-\alpha = \beta = 2$. The associated Jacobi parameters are often – but not always – close to 1 and 0 as $n \to \infty$. Orthogonal polynomials on a compact interval are intimately related to Jacobi parameters that are asymptotically constant. As we shall see, the theory is well developed and many precise results are available.

In Section 3, we generalize our studies to finite gap sets $\mathfrak{e}$, that is, finite unions of closed intervals. When $\mathfrak{e}$ is the union of two or more disjoint intervals, the complement $\overline{\mathbb{C}} \setminus \mathfrak{e}$ is no longer simply connected. This is to be overcome by using the universal covering map. Perhaps more seriously, the structure of the Jacobi parameters changes. They are no longer asymptotically constant but rather asymptotically periodic or almost periodic. The natural limit point (viz., the free Jacobi matrix) also has to be replaced by an ℓ-dimensional torus, where ℓ counts the number of gaps in $\mathfrak{e}$.

Finally, in Section 4 we consider infinite gap sets of Parreau–Widom type. This notion of regular compact sets includes Cantor sets of positive measure, among others. The theory is less developed, but many results that hold for finite gap sets can be extended to the infinite gap setting.

2. Perturbations of the free Jacobi matrix

The most natural choice of Jacobi parameters is

$$a_n \equiv 1 \text{ and } b_n \equiv 0. \tag{2.1}$$

As is well known, the associated orthogonal polynomials are Chebyshev of the 2nd kind

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = 2\cos\theta.$$

They are orthogonal on the interval $[-2,2]$ with respect to the semicircle law $f_0(x) = \sqrt{4-x^2}/2\pi$. We shall follow the standard terminology and refer to

$$J_0 = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \tag{2.2}$$

as the free Jacobi matrix.

If $a_n \to 1$ and $b_n \to 0$, then $J = \{a_n, b_n\}_{n=1}^\infty$ is a compact perturbation of J_0 and hence $\sigma_{\mathrm{ess}}(J) = [-2,2]$ by Weyl's theorem. There may be points in $\mathrm{supp}(d\mu) \setminus [-2,2]$, but these are all isolated mass points that can only accumulate at ± 2. Moreover, a result of Nevai [14] states that the ratio $P_{n+1}(x)/P_n(x)$ has a limit for $x \notin \sigma(J)$.

The condition $\sigma_{\mathrm{ess}}(J) = [-2,2]$, on the other hand, is by itself not strong enough to imply $a_n \to 1$ and $b_n \to 0$ (see, e.g., [21, Section 1.4] for a counterexample). An extra condition is needed and for $d\mu$ as in (1.6), the Denisov–Rakhmanov theorem [9] states that *if* $\sigma_{\mathrm{ess}}(J) = [-2,2]$ *and* $f(x) > 0$ *a.e. on* $[-2,2]$*, then* $a_n \to 1$ *and* $b_n \to 0$. Denoting by J_n the n times stripped Jacobi matrix (i.e., the matrix obtained from J by removing the first n rows and columns), the above conclusion can also be formulates as $J_n \to J_0$ strongly.

The more detailed spectral analysis involves the rate of convergence of the Jacobi parameters. Of particular interest are the cases of Hilbert–Schmidt and trace-class perturbations of J_0. A deep result of Killip and Simon [12] classifies the spectral measures of all Jacobi matrices $J = \{a_n, b_n\}_{n=1}^\infty$ for which

$$\sum_{n=1}^\infty (a_n - 1)^2 + b_n^2 < \infty. \tag{2.3}$$

They all have

$$\mathrm{supp}(d\mu) = [-2,2] \cup \{x_k\},$$

where $\{x_k\}$ is a countable set of isolated mass points, possibly empty, and are precisely those probability measures of the form (1.6) that satisfy

$$\int_{-2}^2 \log f(x) \sqrt{4-x^2}\, dx > -\infty \tag{2.4}$$

and

$$\sum_k \big(|x_k| - 2\big)^{3/2} < \infty. \tag{2.5}$$

The proof of Killip–Simon's theorem relies on sum rules, obtained from a factorization of the m-function. More precisely, one shows that

$$M(z) := -m(z + 1/z), \quad |z| < 1 \tag{2.6}$$

is a meromorphic Herglotz function and hence of the form $M = B \cdot O$, where B is an alternating Blaschke product and O an outer function (see [18] for details). The sum rules now result from computing the Taylor coefficients of $\log(M(z)/z)$ in two different ways.

Note that

$$\phi(z) := z + 1/z \tag{2.7}$$

is the unique conformal mapping of the unit disk $\mathbb{D}$ onto $\overline{\mathbb{C}} \backslash [-2, 2]$ for which $\phi(0) = \infty$ and $\lim_{z\to 0} z\phi(z) = 1$. The use of ϕ in the theory of orthogonal polynomials goes back at least to Szegő.

Compared to (2.3), the a priori stronger condition

$$\sum_{n=1}^{\infty} |a_n - 1| + |b_n| < \infty \tag{2.8}$$

was conjectured by Nevai [13] and later proven by Killip and Simon [12] to imply the Szegő condition, that is,

$$\int_{-2}^{2} \frac{\log f(x)}{\sqrt{4 - x^2}} dx > -\infty. \tag{2.9}$$

In turn, (2.9) is closely related to

$$a_1 \cdots a_n \not\to 0 \tag{2.10}$$

and

$$\sum_k \big(|x_k| - 2\big)^{1/2} < \infty. \tag{2.11}$$

What is known as Szegő's theorem states that *if* (2.11) *holds, then* (2.9) *is equivalent to* (2.10). Moreover, (2.9)–(2.10) implies (2.11) so as formulated by Simon and Zlatoš [22], *any two imply the third*. In the setting of Szegő's theorem (i.e., when (2.9)–(2.11) hold), the product in (2.10) has a positive limit, (2.3) is satisfied, and both of the series

$$\sum_{n=1}^{\infty} (a_n - 1), \quad \sum_{n=1}^{\infty} b_n \tag{2.12}$$

are conditionally convergent. Furthermore, a result of Peherstorfer and Yuditskii [15] states that

$$z^n P_n(z + 1/z) \to \frac{B(z)D(z)}{1 - z^2} \tag{2.13}$$

uniformly on compact subsets of $\mathbb{D}$, where B is the Blaschke product

$$B(z) = \prod_k \frac{|z_k|}{z_k} \frac{z_k - z}{1 - z_k z}, \quad z_k = \frac{1}{2}\Big(x_k - \sqrt{x_k^2 - 4}\Big)$$

and D the outer function

$$D(z) = \exp\bigg\{\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log\bigg(\frac{|\sin\theta|}{\pi f(2\cos\theta)}\bigg)\frac{d\theta}{4\pi}\bigg\}.$$

This type of power asymptotic behavior is known as Szegő asymptotics. Note that since

$$U_n(z + 1/z) = \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}} \sim \frac{z^{-n}}{1 - z^2},$$

we can replace z^n by $1/U_n(z + 1/z)$ on the left-hand side in (2.13) if the factor $1 - z^2$ on the right-hand side is removed too.

While the Szegő condition implies Szegő asymptotics, as has long been known, it is not a necessary condition. Examples for which (2.11) fails and yet the left-hand side of (2.13) has a limit are given by Damanik and Simon in [8]. More importantly, [8] proves that $z^n P_n(z + 1/z)$ has a limit for all $z \in \mathbb{D}$ if and only if (2.3) holds and the series in (2.12) are conditionally convergent. The right-hand side of (2.13), however, is only correct when (2.9) holds.

3. Finite gap Jacobi matrices

In this section, we shall consider Jacobi matrices $J = \{a_n, b_n\}_{n=1}^\infty$ for which $\sigma_{\mathrm{ess}}(J)$ is a finite gap set, that is, a set of the form

$$\mathfrak{e} = \bigcup_{j=1}^{\ell+1} \big[\alpha_j, \beta_j\big], \quad \alpha_1 < \beta_1 < \alpha_2 < \cdots < \beta_{\ell+1}. \tag{3.1}$$

Apart from a single interval, such a finite union of closed intervals is the simplest type of compact sets in $\mathbb{R}$ with positive measure (and no isolated points). Note that ℓ counts the number of gaps in $\mathfrak{e}$ and when $\ell \geq 1$, two questions arise:

- Is there a natural choice of J that can serve as a limit point, like J_0 did for the interval $[-2, 2]$?
- What replaces the conformal mapping ϕ in (2.7) when $\overline{\mathbb{C}} \setminus \mathfrak{e}$ is no longer simply connected?

The answer to the first question is negative. There is no single J that will take over the role of J_0. Even when $\mathfrak{e}$ only has one gap, say $\mathfrak{e} = [-2, -1] \cup [1, 2]$, there are several sequences of periodic Jacobi parameters with period 2 (i.e., $a_{n+2} = a_n$ and $b_{n+2} = b_n$ for all n) leading to the right spectrum, namely $\mathfrak{e}$. And it seems impossible to pick out one that should be more natural than all the others. In fact, the Denisov–Rakhmanov theorem is known to fail when $[-2, 2]$ is replaced by a finite gap set with at least one gap. The Jacobi parameters need not approach a single point. Rather, they approach a set which is topologically a circle (or a 1-dimensional torus) when $\ell = 1$.

For a general finite gap set $\mathfrak{e}$ as in (3.1), Simon [19,20] suggested to introduce the so-called isospectral torus $\mathcal{T}_{\mathfrak{e}}$ of dimension ℓ. The structure of this limiting object is carefully described in [4]. It consists of all Jacobi matrices whose m-function is a minimal Herglotz function on the two-sheeted Riemann surface $\mathcal{S}$ associated with $\mathfrak{e}$. Loosely speaking, one can think of $\mathcal{S}$ as two copies of $\mathbb{C} \setminus \mathfrak{e}$ glued together suitably. Alternatively, $\mathcal{T}_{\mathfrak{e}}$ is the collection of all two-sided Jacobi matrices $J = \{a_n, b_n\}_{n=-\infty}^{\infty}$ that have spectrum $\mathfrak{e}$ and are reflectionless on $\mathfrak{e}$ (see, e.g., [17,23] for more details).

The isospectral torus is invariant under coefficient stripping, a very useful fact. If J' is a point on $\mathcal{T}_{\mathfrak{e}}$, then the Jacobi parameters $\{a_n', b_n'\}_{n=1}^{\infty}$ are periodic or almost periodic sequences, depending on whether the intervals in $\mathfrak{e}$ all have rational harmonic measure (i.e., whether $\mu_{\mathfrak{e}}\big([\alpha_j, \beta_j]\big) \in \mathbb{Q}$ for all j, where $d\mu_{\mathfrak{e}}$ is the equilibrium measure of $\mathfrak{e}$). We say that $\mathfrak{e}$ is periodic if all $[\alpha_j, \beta_j]$ have rational harmonic measure. The spectral measure of J' is also very regular. It is purely absolutely continuous on $\mathfrak{e}$ with a density that satisfies the Szegő condition (see (3.3) below). Besides, it has at most one mass point in each of the ℓ gaps in $\mathfrak{e}$ and no other singular part. For later use, we pick $J^\sharp$ to be a suitable reference point on $\mathcal{T}_{\mathfrak{e}}$, namely a Jacobi matrix whose spectral measure has no singular part at all.

A remarkable result of Remling [17] generalizes the Denisov–Rakhmanov theorem to finite gap sets. It states that *if $\sigma_{\mathrm{ess}}(J) = \mathfrak{e}$ and $f(x) > 0$ a.e. on $\mathfrak{e}$, then the orbit of J under coefficient stripping approaches the isospectral torus $\mathcal{T}_{\mathfrak{e}}$.* The sequence of J_n's need not have a limit, but any of its accumulation points (essentially right limits) lie on $\mathcal{T}_{\mathfrak{e}}$. In order to ensure convergence to some point on the isospectral torus and not only the torus as a set, stronger assumptions on J are needed.

We say that a Jacobi matrix $J = \{a_n, b_n\}_{n=1}^{\infty}$ with spectral measure $d\mu$ of the form (1.6) belongs to the Szegő class for $\mathfrak{e}$ if

$$\operatorname{supp}(d\mu) = \mathfrak{e} \cup \{x_k\},$$

where $\{x_k\}$ is a countable set of isolated mass points satisfying the Blaschke condition

$$\sum_k \operatorname{dist}(x_k, \mathfrak{e})^{1/2} < \infty \tag{3.2}$$

and f obeys the Szegő condition

$$\int_{\mathfrak{e}} \frac{\log f(x)}{\operatorname{dist}(x, \mathbb{R} \setminus \mathfrak{e})^{1/2}} dx > -\infty. \tag{3.3}$$

It is proven in [5] that *when* (3.2) *holds,* (3.3) *is equivalent to*

$$\frac{a_1 \cdots a_n}{\operatorname{Cap}(\mathfrak{e})^n} \not\to 0. \tag{3.4}$$

In fact, just as for Szegő's theorem on $[-2,2]$, *any two of* (3.2)–(3.4) *imply the third.* While the sequence in (3.4) no longer has a limit, it turns out to be asymptotically periodic/almost periodic.

Another result of Christiansen, Simon, and Zinchenko [5] states that *if J belongs to the Szegő class for $\mathfrak{e}$, there is a unique point $J' \in \mathcal{T}_\mathfrak{e}$ so that*

$$|a_n - a_n'| + |b_n - b_n'| \to 0. \tag{3.5}$$

Equivalently, this means that $J_n - J_n' \to 0$ strongly (i.e., the orbit of J under coefficient stripping approaches the orbit of J' on $\mathcal{T}_\mathfrak{e}$). To explain which point on the torus to pick and to make a statement about the asymptotics of P_n, we first need to answer the second question.

In short, the role of ϕ is taken over by the universal covering map of $\mathbb{D}$ onto $\Omega := \overline{\mathbb{C}} \setminus \mathfrak{e}$. This is the standard tool for 'lifting' functions on multiply connected domains to the unit disk. The universal covering map $\psi : \mathbb{D} \to \Omega$ is only locally one-to-one and each point in Ω has infinitely many preimages in $\mathbb{D}$. These are related to one another through a Fuchsian group Γ of Möbius transformations,

$$\psi(z) = \psi(w) \iff \exists\gamma \in \Gamma : \; z = \gamma(w).$$

We fix ψ uniquely by also requiring that $\psi(0) = \infty$ and $\lim_{z\to 0} z\psi(z) > 0$. Γ is isomorphic to the fundamental group $\pi_1(\Omega)$ and hence a free group on ℓ generators, say $\gamma_1, \ldots, \gamma_\ell$.

To get a better picture of Γ, we introduce the open set

$$\mathbb{F} := \big\{ z \in \mathbb{D} : |\gamma'(z)| < 1 \;\text{ for all } \gamma \neq \mathrm{id} \big\}. \tag{3.6}$$

This is a fundamental domain for Γ, that is, no two points of $\mathbb{F}$ are equivalent under Γ and $\overline{\mathbb{F}}$ contains at least one point from each Γ-orbit. Geometrically, $\mathbb{F}$ is symmetric in the real line and consists of the unit disk with 2ℓ orthocircles (and their interior) removed. The circular arcs in the upper (or lower) half-disk, say $C_1, \ldots, C_\ell$, are in one-one correspondence with the gaps in $\mathfrak{e}$ under the covering map ψ. In fact, one can take the generator γ_j to be reflection in C_j following complex conjugation.

The multiplicative group of characters on Γ, denoted Γ^*, turns out to play an important role. Since an element in Γ^* is determined from its values on the generators of Γ, we can think of Γ^* as an ℓ-dimensional torus. The point is that $\mathcal{T}_\mathfrak{e}$ and Γ^* are homeomorphic. To get hold of a homeomorphism between these two ℓ-dimensional tori, we first introduce the Jost function of an element in the Szegő class. Let $d\mu^\sharp = f^\sharp(x)dx$ be the spectral measure of $J^\sharp$, our reference point on $\mathcal{T}_\mathfrak{e}$. For J in the Szegő class of $\mathfrak{e}$, we define its Jost function by

$$u(z;J) = \prod_k B(z,z_k) \exp\bigg\{ \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log\bigg(\frac{f^\sharp(\psi(e^{i\theta}))}{f(\psi(e^{i\theta}))} \bigg) \frac{d\theta}{4\pi} \bigg\}, \quad z \in \mathbb{D} \tag{3.7}$$

where $\{z_k\}$ are the unique points in $\overline{\mathbb{F}}$ with $\operatorname{Im} z_k \geq 0$ and $\psi(z_k) = x_k$. This analytic function turns out to be character automorphic, that is, there exists $\chi_J \in \Gamma^*$ such that

$$u(\gamma(\cdot);J) = \chi_{_J}(\gamma)u(\,\cdot\,;J) \ \text{ for all } \gamma \in \Gamma. \tag{3.8}$$

Most importantly, the map

$$\mathcal{T}_{\mathfrak{e}} \ni J \longrightarrow \chi_{_J} \in \Gamma^*, \tag{3.9}$$

essentially the Abel map, is a homeomorphism (see, e.g., [4] for details).

We are now able to explain which point J' on $\mathcal{T}_{\mathfrak{e}}$ is the right one for (3.5) to hold: *Take the unique point for which* $\chi_{_{J'}} = \chi_{_J}$. This fact is proven in [5] by use of Remling's theorem, the homeomorphism (3.9), and a technical lemma stating that strong convergence to a point on the torus implies convergence of the associated characters. We repeat the proof here as it merely takes a few lines.

For contradiction, suppose that

$$|a_n - a_n'| + |b_n - b_n'| \not\to 0.$$

Then there is a subsequence $\{n_k\}$ so that J and J' have different right limits, say $K \neq K'$. Due to Remling's theorem, both K and K' lie on $\mathcal{T}_{\mathfrak{e}}$, and we have

$$\chi_{_{J_{n_k}}} \longrightarrow \chi_{_K} \ \text{ and } \ \chi_{_{J'_{n_k}}} \longrightarrow \chi_{_{K'}}$$

since $J_{n_k} \to K$ and $J'_{n_k} \to K'$ strongly. As $\chi_{_J} = \chi_{_{J'}}$, we also have $\chi_{_{J_n}} = \chi_{_{J'_n}}$ so that $\chi_{_K} = \chi_{_{K'}}$. This contradicts the fact that $K \neq K'$.

The Jost function also enters the picture in connection with the asymptotic behavior of P_n. With P_n' the orthonormal polynomials associated with J' (not to be confused with the derivative), we have

$$\frac{P_n(\psi(z))}{P_n'(\psi(z))} \longrightarrow \frac{u(z;J)}{u(z;J')} \tag{3.10}$$

uniformly on compact subsets of $\mathbb{F}$, the fundamental domain for Γ. This result should be compared with (2.13) and the fact that $u(z;J_0) = 1$.

Along the lines of [8], Christiansen, Simon, and Zinchenko [6] set out to find weaker assumptions than the Szegő condition that still imply Szegő asymptotics (in the sense that the left-hand side of (3.10) has a limit). At first sight, it may look like

$$\sum_{n=1}^{\infty}(a_n - a_n')^2 + (b_n - b_n')^2 < \infty \tag{3.11}$$

and conditional convergence of

$$\sum_{n=1}^{\infty}(a_n - a_n'), \quad \sum_{n=1}^{\infty}(b_n - b_n') \tag{3.12}$$

will be sufficient. But a more careful analysis shows that the periodicity/almost periodicity has to be taken into account and one needs to replace the conditional convergence with a more involved set of assumptions involving the harmonic measures $\mu_{\mathfrak{e}}([\alpha_j,\beta_j])$ for all j. The reader is referred to [6] for more details.

The generalized Nevai conjecture has recently been proved in the finite gap setting by Frank and Simon [10]. They answer in the affirmative that *if* $J = \{a_n, b_n\}_{n=1}^\infty$ *is a Jacobi matrix with spectral measure* $d\mu$ *of the form* (1.6) *and*

$$\sum_{n=1}^\infty |a_n - a_n'| + |b_n - b_n'| < \infty \tag{3.13}$$

for some point J' *on* $\mathcal{T}_{\mathfrak{e}}$*, then the Szegő condition* (3.3) *holds.* Hence there is some understanding of ℓ^1-convergence to $\mathcal{T}_{\mathfrak{e}}$. Among other things, [10] relies on an improved Birman–Schwinger bound in the gaps of $\mathfrak{e}$.

The situation of ℓ^2-convergence to $\mathcal{T}_{\mathfrak{e}}$, on the other hand, is much less understood. Whether or not the Killip–Simon theorem can be proved for all finite gap sets is still an open question. That it is true when $\mathfrak{e}$ is periodic has proven by Damanik, Killip, and Simon [7]. The ingenious idea of [7] is to handle the periodic case by use of matrix orthogonal polynomials. But this method only applies to periodic $\mathfrak{e}$. The proof of Killip–Simon's theorem for $[-2, 2]$ relies among other things on the explicit form of ϕ. The universal covering map, in turn, is much more complicated. Even if one succeeds in finding ψ explicitly, the expression at hand will still be too difficult to work with. New insight is needed to really understand the concept of ℓ^2-convergence to the isospectral torus.

4. Infinite gap Jacobi matrices

Every compact set $\mathsf{E} \subset \mathbb{R}$ can be written in the form

$$\mathsf{E} = [\alpha, \beta] \setminus \bigcup\nolimits_j (\alpha_j, \beta_j), \tag{4.1}$$

where $\cup_j$ is a countable union of disjoint open subintervals of $[\alpha, \beta]$. We shall refer to (α_j, β_j) as a 'gap' in E and now mainly focus on the situation of infinitely many gaps. In order to develop the theory, a few restrictions have to be put on E. But among others, there will still be room for Cantor sets of positive measure.

First of all, we shall always assume that $|\mathsf{E}| > 0$ to allow for an absolutely continuous part of $d\mu$. This in particular implies that the logarithmic capacity of E, denoted $\mathrm{Cap}(\mathsf{E})$, is positive so that the domain $\Omega = \overline{\mathbb{C}} \setminus \mathsf{E}$ has a Green's function. We denote by g the Green's function for Ω with pole at ∞. This function is known to be positive and harmonic on Ω, and

$$g(z) = \log|z| + \gamma(\mathsf{E}) + o(1)$$

near ∞, where $e^{-\gamma(\mathsf{E})} = \mathrm{Cap}(\mathsf{E})$.

To avoid dealing with isolated points in the essential spectrum, we assume that E is regular, that is,

$$\lim_{\Omega \ni z \to x} g(z) = 0 \ \text{ for all } x \in \mathsf{E}. \tag{4.2}$$

Hence g has precisely one critical point in each gap of E. Denoting by c_j the critical point in (α_j, β_j), we impose the so-called Parreau–Widom condition,

$$\sum\nolimits_j g(c_j) < \infty. \tag{4.3}$$

While Widom was interested in Riemann surfaces with sufficiently many analytic functions, the notion becomes useful to us as the equilibrium measure $d\mu_{\mathsf{E}}$ of E turns out to be absolutely continuous (see, e.g., [2] for a detailed proof). Moreover, the m-function for measures supported on E is of bounded characteristic when lifted to $\mathbb{D}$.

The Parreau–Widom condition (4.3) is known to be satisfied for compact sets that are homogeneous in the sense of Carleson [1]. By definition, this means there is an $\varepsilon > 0$ such that

$$\frac{|(x-\delta, x+\delta) \cap \mathsf{E}|}{\delta} \geq \varepsilon \text{ for all } x \in \mathsf{E} \text{ and all } \delta < \operatorname{diam}(\mathsf{E}). \tag{4.4}$$

Carleson introduced this geometric condition to avoid the possibility of certain parts of E to be very thin, compared to Lebesgue measure. To get an explicit example of an infinite gap set which is homogeneous, remove the middle $1/4$ from the interval $[0, 1]$ and continue removing subintervals of length $1/4^n$ from the middle of each of the 2^{n-1} remaining intervals. The set E of what is left in $[0, 1]$ is a Cantor set of length $1/2$, and the reader may check that $|(x-\delta, x+\delta) \cap \mathsf{E}| \geq \delta/4$ for all $x \in \mathsf{E}$ and all $\delta < 1$.

Just as in the finite gap setting, we can make use of the covering space formalism. In fact, the seminal paper [23] of Sodin and Yuditskii deals with infinite gap sets of Parreau–Widom type. Let $J = \{a_n, b_n\}_{n=1}^{\infty}$ be a Jacobi matrix with $\sigma_{\text{ess}}(J) = \mathsf{E}$ and spectral measure $d\mu$ of the form (1.6). Denote by $\{x_k\}$ the possible mass points of $d\mu$ outside E. We say that $d\mu$ (or J) satisfies the Szegő condition if

$$\int_{\mathsf{E}} \log f(x)\, d\mu_{\mathsf{E}}(x) > -\infty. \tag{4.5}$$

As follows at once when recalling the explicit form of $d\mu_{\mathfrak{e}}$ (see, e.g., [21, Chap. 5]), this is the natural way of generalizing (3.3). On condition that

$$\sum\nolimits_k g(x_k) < \infty, \tag{4.6}$$

Sodin and Yuditskii [23] showed that $M := m \circ \psi$ is of bounded characteristic on $\mathbb{D}$ and without a singular inner part. Hence it admits a factorization of the form

$$M(z) = B_\infty(z) \exp\left\{ \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log\bigl|M(e^{i\theta})\bigr| \frac{d\theta}{2\pi} \right\} \tag{4.7}$$

with B_∞ the Blaschke product of zeros and poles, and this paves the way for step-by-step sum rules. Comparing the constant terms in (4.7) and iterating n times lead to

$$\log\left(\frac{a_1 \cdots a_n}{\operatorname{Cap}(\mathsf{E})^n}\right) = \sum_k \bigl(g(x_k) - g(x_{n,k})\bigr) + \frac{1}{2} \int_{\mathsf{E}} \log\left(\frac{f(t)}{f_n(t)}\right) d\mu_{\mathsf{E}}(t), \tag{4.8}$$

where $\{x_{n,k}\}$ are the eigenvalues of J_n outside E and f_n is the absolutely continuous part of its spectral measure. Interpreting the integral on the right-hand side in terms of relative entropies, one can show that *the Szegő condition is equivalent to*

$$\frac{a_1 \cdots a_n}{\mathrm{Cap}(\mathsf{E})^n} \not\to 0 \tag{4.9}$$

provided that (4.6) *holds.* The details are given in [2] and the proof also shows that the sequence in (4.9) is bounded above and below. While one direction is straightforward using (4.8), the other involves some cutting and pasting in the Jacobi matrix before applying (4.8).

For general Parreau–Widom sets, the isospectral torus $\mathcal{T}_{\mathsf{E}}$ will be infinite dimensional and we equip it with the product topology. It is known that Remling's theorem generalizes and one can ask if elements in the Szegő class still approach a point on $\mathcal{T}_{\mathsf{E}}$ and not only the isospectral torus as a set. Provided the Abel map remains a homeomorphism, the same proof as in Section 3 should work. For this to hold, an extra condition on E turns out to be needed. The so-called direct Cauchy theorem has to be valid (see [24], [11]). These and related issues are treated in the upcoming paper [3]. A recent article of Yuditskii [25] points out that Parreau–Widom sets for which the direct Cauchy theorem holds are still more general than homogeneous sets. Asymptotics of orthogonal polynomials on homogeneous sets were treated by Peherstorfer and Yudiskii in [16].

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