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Globally Bisingular Elliptic Operators

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Abstract. The main goal of this work is to extend the notion of bisingular pseudo-differential operators, already introduced on compact manifolds, to Shubin type operators on $\mathbb{R}^n = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$, $n_1 + n_2 = n$. First, we prove global calculus (an analogue of the Γ calculus in the work of Shubin) for such operators, we introduce the notion of bisingular globally elliptic operators and we derive estimates for the action in anisotropic weighted Sobolev spaces, recently introduced by Gramchev, Pilipović, Rodino. Next, we investigate the complex powers of such operators and we demonstrate a Weyl type theorem for the spectral counting function of positive self-adjoint unbounded bisingular globally elliptic operators. The crucial ingredient for the proof is the use of the spectral zeta function. For particular classes of operators, defined as polynomials of $P_1 \times P_2$, $P_1 \times I_{\mathbb{R}^{n_2}}$, $I_{\mathbb{R}^{n_1}} \times P_2$, P_j being globally elliptic in $\mathbb{R}^{n_j}$, $j = 1, 2$, we are able to estimate and, in some cases, calculate explicitly the lower-order term in the asymptotic expansion of the spectral function.

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1. Introduction

Let us recall the expression of a Shubin type differential operators with polynomial coefficients in $\mathbb{R}^n$ ([36], see also [4, 18, 28]):

$$P = \sum_{|\alpha|+|\beta|\leq m} c_{\alpha\beta} x^\beta D_x^\alpha, \quad D^\alpha = (-i)^{|\alpha|} \partial_x^\alpha. \tag{1.1}$$

We assume that P is an L^2- self-adjoint operator and satisfies the global ellipticity condition

$$p_m(x,\xi) = \sum_{|\alpha|+|\beta|=m} c_{\alpha\beta} x^\beta \xi^\alpha \neq 0 \quad \text{for} \ \ (x,\xi) \neq (0,0). \tag{1.2}$$

This guarantees the existence of a basis of orthonormal eigenfunctions u_j, $j \in \mathbb{N}$, with eigenvalues λ_j, $\lim_{j\to\infty} |\lambda_j| = +\infty$, see Shubin [36] for the asymptotics of the counting function. If $u \in L^2(\mathbb{R}^n)$, or $u \in \mathcal{S}'(\mathbb{R}^n)$, then

$$u = \sum_{j=1}^{\infty} a_j u_j, \quad a_j = (u, u_j)_{L^2(\mathbb{R}^n)}, \quad j = 1, 2, \ldots, \tag{1.3}$$

with convergence in $L^2(\mathbb{R}^n)$ or $\mathcal{S}'(\mathbb{R}^n)$.

Let us address to [5–7, 10, 13, 22] for further information on the regularity of the eigenfunctions. Basic examples of operators in the class considered in this paper are tensor products of Shubin operators. Namely, let $P(x, D)$ be a linear partial differential operator with polynomial coefficients of the form

$$\begin{aligned} P(x, D) &= P_1(x_1, D_{x_1}) P_2(x_2, D_{x_2}) \\ &= \left(\sum_{|\alpha|+|\beta|\le m_1} c^1_{\alpha\beta} x_1^\beta D^\alpha_{x_1} \right) \left(\sum_{|\alpha|+|\beta|\le m_2} c^2_{\alpha\beta} x_2^\beta D^\alpha_{x_2} \right), \end{aligned} \tag{1.4}$$

$x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}$, so that P_1 and P_2 are self-adjoint, invertible and globally elliptic on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, that is (1.2) holds for both operators. Spectrum and eigenfunctions of P are easily detected from those of P_1, P_2, if we note that

$$u_{(j_1,j_2)}(x_1, x_2) = u^1_{j_1}(x_1) u^2_{j_2}(x_2), \quad j_1, j_2 \in \mathbb{N},$$

is an orthonormal basis of $L^2(\mathbb{R}^{n_1+n_2})$, and

$$P u_{j_1,j_2} = \lambda^1_{j_1} \lambda^2_{j_2} u_{j_1,j_2}, \quad j_1, j_2 \in \mathbb{N}.$$

The study of the counting function is interesting, and challenging. In Section 2 we shall embed example (1.4) into a general pseudo-differential calculus, including also the case when $p_i(x, D) \in G^{m_i}(\mathbb{R}^{n_i})$ with symbol $p_i(x_i, \xi_i)$ in the classes of Shubin

$$|\partial^{\beta_i}_{x_i} \partial^{\alpha_i}_{\xi_i} p(x_i, \xi_i)| \le C \langle x_i, \xi_i \rangle^{m_i - |\alpha_i| - |\beta_i|}, \quad \langle x_i, \xi_i \rangle = (1 + |x_i|^2 + |\xi_i|^2)^{\frac{1}{2}}.$$

In Section 3 we shall introduce a general notion of ellipticity, inspired by (1.4), see Definition 3.1. As a consequence of Theorem 3.11, using a generalization of Tauberian Theorem due to J. Aramaki [1], we will be able to study the counting function of operators of the form (1.4), see Theorem 3.12. In Section 4 we focus on the tensor product of Hermite-type operators, and we evaluate directly the first term of the asymptotic expansion of the counting function.

Motivations of the present paper, and connection with existing literature, are twofold. On one hand, the case when in (1.4) we have the tensor product of two Hermite operators, or more generally tensor product of real powers of Hermite operators in several distinct variables, is relevant in Probability, see for example [25], and other applications. Tensorized Hermite operators were treated in [12] from a sequential point of view, i.e., basing on eigenfunction expansions. In [12] the authors observed also a connection with the twisted Laplacian of Wong [37], which was proved to be unitarily equivalent to the tensor product of the one-dimensional Hermite operator and the identity operator. Similar ideas are present in the subsequent papers [9, 13, 15, 16, 26, 27]. On the other hand, the structure of our pseudo-differential class is strictly connected with the pioneering work [31], and [29] where similar operators were dubbed as bisingular operators. Recently, bisingular operators on compact manifolds were studied in [2] and [28], see also [3] for analogue results in the SG-setting. In particular, our results of Section 3 can be seen as a version of [2] for global operators on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. In conclusion, we may also observe that our class of symbols, in the case of zero orders, is included in the Hörmader class $S^0_{0,0}(\mathbb{R}^{n_1+n_2})$. Hence our Theorem 3.3 enters the very general results of [23], see also [24] and [32], where necessary and sufficient condition for the Fredholm property were expressed in terms of invertibility of limit operators. Let us address in particular to Theorem 1.1 in the recent paper [33].

2. Γ calculus for bisingular operators

Definition 2.1. We define $\Gamma^{m_1,m_2}(\mathbb{R}^{n_1+n_2})$, $m_1 \in \mathbb{R}, m_2 \in \mathbb{R}$, as the subset of $C^\infty(\mathbb{R}^{2n_1+2n_2})$ functions such that for all multiindex α_i, β_i $(i = 1, 2)$ there exists a constant C so that

$$|\partial^{\beta_1}_{x_1}\partial^{\beta_2}_{x_2}\partial^{\alpha_1}_{\xi_1}\partial^{\alpha_2}_{\xi_2} a(x_1, x_2, \xi_1, \xi_2)| \leq C\langle x_1, \xi_1\rangle^{m_1-|\alpha_1|-|\beta_1|}\langle x_2, \xi_2\rangle^{m_2-|\alpha_2|-|\beta_2|}, \quad (2.1)$$

for all x_1, ξ_1, x_2, ξ_2.

We define

$$\Gamma^{-\infty,-\infty}(\mathbb{R}^{n_1,n_2}) = \bigcap_{m_1,m_2\in\mathbb{R}^2} \Gamma^{m_1,m_2}(\mathbb{R}^{n_1+n_2})$$

as the set of smoothing symbols.

Definition 2.2. A linear operator $A : C^\infty_c(\mathbb{R}^{n_1+n_2}) \to C^\infty(\mathbb{R}^{n_1+n_2})$ is a globally bisingular operator if it can be written in this way[1]

$$\begin{aligned} A(u)(x_1, x_2) &= \mathrm{Op}(a)(u)(x_1, x_2) \\ &= \iint e^{ix_1\cdot\xi_1+ix_2\cdot\xi_2} a(x_1, x_2, \xi_1, \xi_2)\hat{u}(\xi_1, \xi_2) đ\xi_1 đ\xi_2 \end{aligned} \quad (2.2)$$

where $a \in \Gamma^{m_1,m_2}(\mathbb{R}^{n_1+n_2})$. We define $G^{m_1,m_2}(\mathbb{R}^{n_1+n_2})$ as the set of operators as in (2.2) with symbol in $\Gamma^{m_1,m_2}(\mathbb{R}^{n_1+n_2})$.

The $\mathcal{S}$-continuity of globally bisingular operators is immediate, we just have to check all seminorms. More interesting is the Sobolev continuity.

Theorem 2.3. *An operator $A \in G^{m_1,m_2}(\mathbb{R}^{n_1+n_2})$ can be extended for every $s_1 \in \mathbb{R}, s_2 \in \mathbb{R}$ continuously as an operator*

$$A : Q^{s_1,s_2}(\mathbb{R}^{n_1+n_2}) \to Q^{s_1-m_1,s_2-m_2}(\mathbb{R}^{n_1+n_2}).$$

[1] $đ\xi_i = (2\pi)^{-n_i} d\xi_i$.

Where, for positive integers s_1, s_2, we define $Q^{s_1,s_2}(\mathbb{R}^{n_1+n_2})$ as the space of all $u \in L^2(\mathbb{R}^{n_1+n_2})$ such that

$$\|u\|_{Q^{s_1,s_2}} = \sum_{\substack{|\alpha_1|+|\beta_1|\le s_1,\\ |\alpha_2|+|\beta_2|\le s_2}} \|x_1^{\beta_1} x_2^{\beta_2} D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} u\|_{L^2}.$$

For general s_1, s_2, we set

$$\begin{aligned} Q^{s_1,s_2}(\mathbb{R}^{n_1+n_2}) = \{ & u \in \mathcal{S}'(\mathbb{R}^{n_1+n_2}) \mid \\ & u = \mathrm{Op}(\langle x_1, \xi_1\rangle^{-s_1}\langle x_2, \xi_2\rangle^{-s_2})(v),\ v \in L^2(\mathbb{R}^{n_1+n_2})\}. \end{aligned}$$

The proof of Theorem 2.3 follows by the remark that $\Gamma^{0,0}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \subseteq \Gamma^0_0(\mathbb{R}^{n_1+n_2})$. Then we use the well-known results of L^2-continuity and the definition of $Q^{s_1,s_2}(\mathbb{R}^{n_1+n_2})$. We prove now that globally bisingular operators form an algebra.

Theorem 2.4. *Let $A \in G^{m_1,m_2}(\mathbb{R}^{n_1+n_2})$ and $B \in G^{l_1,l_2}(\mathbb{R}^{n_1+n_2})$ then $A \circ B \in G^{m_1+l_1,m_2+l_2}(\mathbb{R}^{n_1+n_2})$.*

Proof. With a simple evaluation we obtain

$$(A \circ B)u(x_1, x_2) = \iint e^{ix_1\xi_1 + ix_2\xi_2} c(x_1, x_2, \xi_1, \xi_2)\hat{u}(\xi_1, \xi_2) đ\xi_1 đ\xi_2,$$

where

$$\begin{aligned} c(x_1, x_2, \xi_1, \xi_2) = \int e^{-i\mu_1 - i\mu_2} a(x_1, x_2, \eta_1, \eta_2) b(y_1, y_2, \xi_1, \xi_2) dy_1 dy_2 đ\eta_1 đ\eta_2 \\ \mu_1 = \langle y_1 - x_1, \eta_1 - \xi_1\rangle, \quad \mu_2 = \langle y_2 - x_2, \eta_2 - \xi_2\rangle. \end{aligned} \tag{2.3}$$

We divide ab into four parts, for a fixed integer $N > 0$:

$$a(x_1, x_2, \eta_1, \eta_2) b(y_1, y_2, \xi_1, \xi_2) = (ab)_1^N + (ab)_2^N + (ab)_3^N + r_N,$$

where

$$\begin{aligned} (ab)_1^N = & \sum_{|\beta_1|+|\alpha_1|<2N} \frac{1}{\beta_1!\alpha_1!} (y_1 - x_1)^{\beta_1} (\eta_1 - \xi_1)^{\alpha_1} \partial_{\eta_1}^{\alpha_1} a(x_1, x_2, \xi_1, \eta_2) \\ & \partial_{y_1}^{\beta_1} b(x_1, y_2, \xi_1, \xi_2), \\ (ab)_2^N = & \sum_{|\beta_2|+|\alpha_2|<2N} \frac{1}{\beta_2!\alpha_2!} (y_2 - x_2)^{\beta_2} (\eta_2 - \xi_2)^{\alpha_2} \partial_{\eta_2}^{\alpha_2} a(x_1, x_2, \eta_1, \xi_2) \\ & \partial_{y_2}^{\beta_2} b(y_1, x_2, \xi_1, \xi_2), \\ (ab)_3^N = & - \sum_{\substack{|\alpha_1|+|\beta_1|<2N\\ |\alpha_2|+|\beta_2|<2N}} \frac{1}{\beta_1!\beta_2!\alpha_1!\alpha_2!} (y_1 - x_1)^{\beta_1} (y_2 - x_2)^{\beta_2} (\eta_1 - \xi_1)^{\alpha_1} \\ & (\eta_2 - \xi_2)^{\beta_2} \partial_{\eta_1}^{\alpha_1} \partial_{\eta_2}^{\alpha_2} a(x_1, x_2, \xi_1, \xi_2) \partial_{y_1}^{\beta_1} \partial_{y_2}^{\beta_2} b(x_1, x_2, \xi_1, \xi_2), \end{aligned}$$

$$r_N = \sum_{\substack{|\alpha_1|+|\beta_1|<2N \\ |\alpha_2|+|\beta_2|<2N}} \frac{1}{\beta_1!\beta_2!\alpha_1!\alpha_2!}(y_1-x_1)^{\beta_1}(y_2-x_2)^{\beta_2}$$
$$(\eta_1-\xi_1)^{\alpha_1}(\eta_2-\xi_2)^{\alpha_2}\int_0^1\int_0^1(1-t_1)^{N-1}(1-t_2)^{N-1}$$
$$\partial_{\eta_1}^{\alpha_1}\partial_{\eta_2}^{\alpha_2}a(x_1,x_2,\xi_1+t_1(\eta_1-\xi_1),\xi_2+t_2(\eta_2-\xi_2))$$
$$\partial_{y_1}^{\beta_1}\partial_{y_2}^{\beta_2}b(x_1+t_1(y_1-x_1),x_2+t_2(y_2-x_2),\xi_1,\xi_2)dt_1dt_2.$$

Dividing the integral (2.3) in four parts, one defines

$$c_i^N = \int e^{-i\mu_1-i\mu_2}(ab)_i^N dy_1dy_2 đ\eta_1 đ\eta_2,$$
$$R_N = \int e^{-i\mu_1-i\mu_2}r_N dy_1dy_2 đ\eta_1 đ\eta_2.$$

Now, we only focus on c_1^N. Notice that

$$(y_1-x_1)^{\beta_1}e^{-i\langle y_1-x_1,\eta_1-\xi_1\rangle} = (-i)^{\beta_1}D_{\eta_1}^{\beta_1}e^{-i\langle y_1-x_1,\eta_1-\xi_1\rangle}, \tag{2.4}$$
$$(\eta_1-\xi_1)^{\alpha_1}e^{-i\langle y_1-x_1,\eta_1-\xi_1\rangle} = (-i)^{\alpha_1}D_{y_1}^{\alpha_1}e^{-i\langle y_1-x_1,\eta_1-\xi_1\rangle}. \tag{2.5}$$

If $\alpha_1 \neq \beta_1$, there exists an index i such that, for example, $(\alpha_1)_i > (\beta_1)_i$. So, using relation (2.5) and integrating by parts, we derive $(\alpha_1)_i$ times w.r.t. y_1 the expression $(y_1-x_1)^{\beta_1}$, and, since $(\alpha_1)_i > (\beta_1)_i$, the derivative is zero. Clearly the same scheme can be used if $(\alpha_1)_i < (\beta_1)_i$ using (2.4). This implies that we can restrict ourself to consider the case $\alpha_1 = \beta_1$, so we will just write α_1. Now, integrating by parts and using relation (2.5), we get

$$c_1^N = \frac{1}{\alpha!}\int\int e^{-i\langle y_2-x_2,\eta_2-\xi_2\rangle}\sum_{|\alpha_1|<N}\partial_{\xi_1}^{\alpha_1}a(x_1,x_2,\xi_1,\eta_2) \\ D_{x_1}^{\alpha_1}b(x_1,y_2,\xi_1,\xi_2)dy_2d\eta_2. \tag{2.6}$$

The expression (2.6) can be written in the form

$$c_1^N = \sum_{|\alpha_1|<N}\frac{1}{\alpha_1!}\partial_{\xi_1}^{\alpha_1}a \circ_2 D_{x_1}^{\alpha_1}b,$$

where the symbol $\circ_2$ means the composition of the operators acting on $\mathbb{R}^{n_2}$. With the same scheme we can prove that

$$c_2^N = \sum_{|\alpha_2|<N}\frac{1}{\alpha_2!}\partial_{\xi_2}^{\alpha_2}a \circ_1 D_{x_2}^{\alpha_2}b.$$

Integrating by parts two times, we get

$$c_3^N = -\sum_{\substack{|\alpha_1|<N \\ |\alpha_2|<N}}\frac{1}{\alpha_1!\alpha_2!}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}aD_{x_1}^{\alpha_1}D_{x_2}^{\alpha_2}b.$$

We have now to analyze the remainder. Consider this identity

$$\langle y_1,\eta_1\rangle^{2M}\langle y_2,\eta_2\rangle^{2M}(1-\Delta_{y_1}-\Delta_{\eta_1})^M(1-\Delta_{y_2}-\Delta_{\eta_2})^M e^{-i\mu_1-i\mu_2}=e^{-i\mu_1-i\mu_2}. \tag{2.7}$$

By Peetre inequality, we have

$$\begin{aligned}|r_N|\le \langle x_1,\xi_1\rangle^{m_1+l_1-2N}\langle x_2,\xi_2\rangle^{m_2+l_2-2N}\langle y_1-x_1\rangle^{|l_1|+2N}\langle y_2-x_2\rangle^{|l_2|+2N}\\ \langle \eta_1-\xi_1\rangle^{|m_1|+2N}\langle \eta_2-\xi_2\rangle^{|m_2|+2N}.\end{aligned}$$

Using (2.7) with M big enough and integrating by parts, we prove that $R_N\in\Gamma^{m_1+l_2-2N,m_2+l_2-2N}(\mathbb{R}^{n_1+n_2})$. □

Remark 2.5. It is useful to write c in this way

$$c\sim\sum_{j=0}^{\infty}c_{m_1+l_1-2j,m_2+l_2-2j},$$

where

$$\begin{aligned}c_{m_1+l_1-2j,m_2+l_2-2j}=c^1_{m_1+l_1-2j,m_2+l_2-2j}+c^2_{m_1+l_1-2j,m_2+l_2-2j}\\ +c^3_{m_1+l_1-2j,m_2+l_2-2j},\end{aligned}$$

and

$$\begin{aligned}c^1_{m_1+l_1-2j,m_2+l_2-2j}&=\sum_{|\alpha_1|=j}\frac{1}{\alpha_1!}\Big(\partial^{\alpha_1}_{\xi_1}a\circ_2 D^{\alpha_1}_{x_1}b-\sum_{|\alpha_2|\le j}\frac{1}{\alpha_2!}\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}aD^{\alpha_1}_{x_1}D^{\alpha_2}_{x_2}b\Big),\\ c^2_{m_1+l_1-2j,m_2+l_2-2j}&=\sum_{|\alpha_2|=j}\frac{1}{\alpha_2!}\Big(\partial^{\alpha_2}_{\xi_2}a\circ_1 D^{\alpha_2}_{x_2}b-\sum_{|\alpha_1|\le j}\frac{1}{\alpha_1!}\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}aD^{\alpha_1}_{x_1}D^{\alpha_2}_{x_2}b\Big),\\ c^3_{m_1+l_1-2j,m_2+l_2-2j}&=\sum_{|\alpha_1|=|\alpha_2|=j}\frac{1}{\alpha_1!\alpha_2!}\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}aD^{\alpha_1}_{x_1}D^{\alpha_2}_{x_2}.\end{aligned}$$

In the following, we will study a subclass of globally bisingular operators, namely operators with homogeneous principal part.

Definition 2.6. A symbol $a\in\Gamma^{m_1,m_2}(\mathbb{R}^{n_1+n_2})$ has homogeneous principal part if

i) there exists a function $a_{m_1,\cdot}(x_1,x_2,\xi_1,\xi_2)$ homogeneous w.r.t. (x_1,ξ_1) of order m_1 such that

$$a-\psi_1(x_1,\xi_1)a_{m_1,\cdot}\in\Gamma^{m_1-1,m_2}(\mathbb{R}^{n_1+n_2}),$$

ψ_1 cut-off function at the origin, and the operator $a(x_1,x_2,\xi_1,D_2)$, with (x_1,ξ_1) *frozen*, is a classical global operator in $\mathbb{R}^{n_2}$;

ii) there exists $a_{\cdot,m_2}$ homogeneous w.r.t. (x_2,ξ_2) of order m_2 such that

$$a-\psi_2(x_2,\xi_2)a_{\cdot,m_2}\in\Gamma^{m_1,m_2-1}(\mathbb{R}^{n_1,n_2}),$$

ψ_2 cut-off function at the origin, and the operator $a(x_1,x_2,D_1,\xi_2)$, with (x_2,ξ_2) *frozen*, is a classical global operator in $\mathbb{R}^{n_1}$;

iii) there exists a function $a_{m_1,m_2}(x_1, x_2, \xi_1, \xi_2)$ bihomogeneous w.r.t. (x_1, ξ_1) of order m_1 and w.r.t. (x_2, ξ_2) of order m_2, such that a_{m_1,m_2} is equal to the principal symbol of $a_{m_1,\cdot}(x_1, x_2, \xi_1, D_2)$ and of $a_{\cdot,m_2}(x_1, x_2, D_1, \xi_2)$ and

$$a - \psi_1(x_1, \xi_1)a_{m_1,\cdot} - \psi_2(x_2, \xi_2)(a_{\cdot,m_2}) + \psi_1(x_1, \xi_1)\psi_2(x_2, \xi_2)a_{m_1,m_2}$$

belongs to $\Gamma^{m_1-1,m_2-1}(\mathbb{R}^{n_1+n_2})$.

In the following, the class of symbols with homogeneous principal part is written as $\Gamma^{m_1,m_2}_{pr}(\mathbb{R}^{n_1+n_2})$, and the operators with homogeneous principal symbol as $G^{m_1,m_2}_{pr}(\mathbb{R}^{n_1+n_2})$. We introduce three functions associated to an operator $A \in G^{m_1,m_2}_{pr}(\mathbb{R}^{n_1+n_2})$:[2]

$$\begin{aligned}
\sigma_1^{m_1}(A) : T^*(\mathbb{R}^{n_1}) \setminus \{0\} &\to G^{m_2}_{cl}(\mathbb{R}^{n_2}) \\
(x_1, \xi_1) &\mapsto a_{m_1,\cdot}(x_1, x_2, \xi_1, D_2), \\
\sigma_2^{m_2}(A) : T^*(\mathbb{R}^{n_2}) \setminus \{0\} &\to G^{m_1}_{cl}(\mathbb{R}^{n_1}) \\
(x_2, \xi_2) &\mapsto a_{\cdot,m_2}(x_1, x_2, D_1, \xi_2), \\
\sigma^{m_1,m_2}(A) : T^*(\mathbb{R}^{n_1}) \setminus \{0\} \times T^*(\mathbb{R}^{n_2}) \setminus \{0\} &\to \mathcal{H}^{m_1,m_2}_{\xi_1,\xi_2}(\mathbb{R}^{n_1+n_2}) \\
(x_1, x_2, \xi_1, \xi_2) &\mapsto a_{m_1,m_2}(x_1, x_2, \xi, \xi_2).
\end{aligned}$$

3. Globally elliptic bisingular operators and the Weyl formula

Definition 3.1. Let $A \in G^{m_1,m_2}_{pr}(\mathbb{R}^{n_1+n_2})$, A is globally elliptic bisingular operator if there exist constants R_1, R_2 such that

i) the operator

$$a_{m_1,\cdot}(x_1, x_2, \xi_1, D_2) : \mathcal{S}(\mathbb{R}^{n_2}) \to \mathcal{S}(\mathbb{R}^{n_2})$$

is invertible for every $(x_1, \xi_1) \in T^*\mathbb{R}^{n_1} \setminus \{0\}$;

ii) the operator

$$a_{\cdot,m_2}(x_1, x_2, D_1, \xi_2) : \mathcal{S}(\mathbb{R}^{n_1}) \to \mathcal{S}(\mathbb{R}^{n_1})$$

is invertible for every $(x_2, \xi_2) \in T^*\mathbb{R}^{n_2} \setminus \{0\}$;

iii) there exists a positive constant C such that

$$\begin{aligned}
|a_{m_1,m_2}(x_1, x_2, \xi_1, \xi_2)| \geq C\langle x_1, \xi_1\rangle^{m_1}\langle x_2, \xi_2\rangle^{m_2}&, \\
\forall |x_i|^2 + |\xi_i|^2 \geq R_i, i = 1, 2.&
\end{aligned} \tag{3.1}$$

Since $a_{m_1,m_2}(x_1, x_2, \xi_1, \xi_2)$ is bihomogeneous it is enough to require that (3.1) is fulfilled for $(x_1, \xi_1) \in T^*\mathbb{R}^{n_1} \setminus \{0\}$, $(x_2, \xi_2) \in T^*\mathbb{R}^{n_2} \setminus \{0\}$.

Remark 3.2. If an operator $A \in G^{m_1,m_2}_{pr}(\mathbb{R}^{n_1+n_2})$ satisfies item *iii)* of Definition 3.1 then both the operators $a_{m_1,\cdot}(x_1, x_2, \xi_1, D_2)(x_2, \xi_2) \in G^{m_2}(\mathbb{R}^{n_2})$ and $a_{\cdot,m_2}(x_1, x_2, D_1, \xi_2)(x_1, \xi_1) \in G^{m_1}(\mathbb{R}^{n_1})$ are elliptic Shubin type operators. If moreover A satisfies items *i)* and *ii)* one can prove that both $a_{m_1,\cdot}(x_1, x_2, \xi_1, D_2)(x_2, \xi_2)$

[2] $\mathcal{H}^{m_1,m_2}_{\xi_1,\xi_2}(\mathbb{R}^{n_1+n_2})$ is the set of homogeneous function of order m_i w.r.t. ξ_i.

and $a_{\cdot,m_2}(x_1,x_2,D_1,\xi_2)(x_1,\xi_1)$ are injective Fredholm operator with zero index, therefore invertible operators also in the scale of Q^s spaces. Thus, in Definition 3.1, it is equivalent to require the invertibility of the operators on the Schwartz spaces or on the Sobolev spaces Q^s. For this reason, in the following we will not specify the space in which the operators are assumed to be invertible.

Theorem 3.3. *If an operator A is globally elliptic bisingular then it is a Fredholm operator.*

Proof. It is a consequence of Theorem 2.3. From Remark 2.5, if A is elliptic one can define an operator B as the operator with symbol

$$b = \psi_1(x_1,\xi_1)a^{-1}_{m_1,\cdot} + \psi_2(x_2,\xi_2)a^{-1}_{\cdot,m_2} - \psi_1(x_1,\xi_1)\psi_2(x_2,\xi_2)a^{-1}_{m_1,m_2}.$$

Applying the calculus, one can check that B is an inverse of A modulo compact operator. □

Using a Neumann series procedure, by Theorem 3.3, one can prove that, if an operator is globally elliptic bisingular, then there exists an inverse modulo smoothing operators. So we have this immediate corollary:

Corollary 3.4. *Let $A \in G^{m_1,m_2}_{pr}(\mathbb{R}^{n_1+n_2})$ be globally elliptic then*

i) *if $Au \in Q^{s_1,s_2}(\mathbb{R}^{n_1+n_2})$ then $u \in Q^{s_1+m_1,s_2+m_2}(\mathbb{R}^{n_1+n_2})$;*
ii) *if $Au \in \mathcal{S}(\mathbb{R}^{n_1+n_2})$ then $u \in \mathcal{S}(\mathbb{R}^{n_1+n_2})$.*

Our aim is now to study the counting function of positive self-adjoint globally bisingular operators. We will use Tauberian techniques, so we need to define complex powers of globally bisingular operators.

First we define parameter ellipticity:

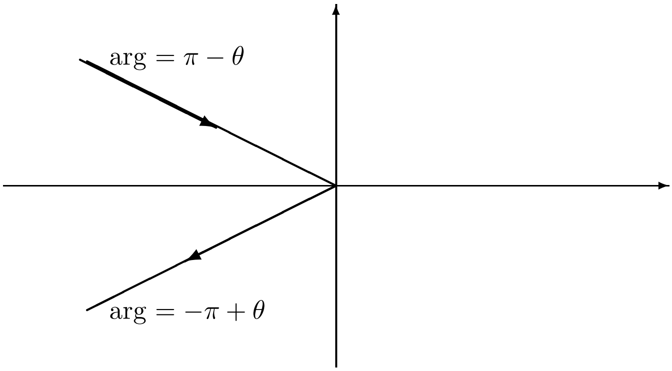
Definition 3.5. Let Λ be a sector of the complex plane and a be a symbol belonging to $\Gamma^{m_1,m_2}_{pr}(\mathbb{R}^{n_1+n_2})$; a is called Λ-elliptic w.r.t. Λ if there exists a constant R such that

i)
$$\sigma^{m_1}_1(A)(x_1,\xi_1) - \lambda I_{\mathbb{R}^{n_2}} \in G^{m_2}_{cl}(\mathbb{R}^{n_2})$$
is invertible for all $|x_1|+|\xi_1| > R$, for all $\lambda \in \Lambda$.

ii)
$$\sigma^{m_2}_2(A)(x_2,\xi_2) - \lambda I_{\mathbb{R}^{n_1}} \in G^{m_1}_{cl}(\mathbb{R}^{n_1})$$
is invertible for all $|x_2|+|\xi_2| > R$, for all $\lambda \in \Lambda$.

iii)
$$\left(\sigma^{m_1,m_2}(A)(x_1,x_2,\xi_1,\xi_2) - \lambda\right)^{-1} \in \Gamma^{-m_1,-m_2}(\mathbb{R}^{n_1+n_2})$$
for all $|x_i|+|\xi_i| > R$, for all $\lambda \in \Lambda$.

In the following, we consider sector of the complex plane Λ with vertex at the origin as in the figure below.



It is an exercise to prove that, if $A \in G^{m_1,m_2}_{pr}(\mathbb{R}^{n_1+n_2})$ is Λ-elliptic, then the operator is sectorial. Follow for example the scheme of Theorem 2 in [2].

We make now some natural assumptions in order to perform the functional calculus.

Assumptions 3.6.

i) $A \in G^{m_1,m_2}_{pr}(\mathbb{R}^{n_1+n_2})$ is Λ-elliptic,
ii) $\sigma(A) \cap \Lambda = \emptyset$, in particular A is invertible.

Remark 3.7. In item ii) of Assumptions 3.6, we assume that the operator is invertible. We have made these assumptions in order to get a simpler theory. It is nevertheless possible to handle functional calculus of operators with non trivial kernel, even with infinite-dimensional kernel, the crucial requirement is that the origin must be an isolated point of the spectrum. Roughly speaking, instead of considering the operator A, one studies the operator $A \circ (I - P_{\ker A})$; $P_{\ker A}$ being the projection into the kernel of A. Clearly this operator is invertible, cf. [8].

Definition 3.8. Let A be a globally bisingular operator that satisfies Assumptions 3.6, we can define

$$A_z := \frac{i}{2\pi} \int_{\partial \Lambda^+_\epsilon} \lambda^z (A - \lambda \mathrm{Id})^{-1} d\lambda, \quad \mathrm{Re}(z) < 0, \tag{3.2}$$

where $\Lambda_\epsilon = \Lambda \cup \{z \in \mathbb{C} \mid |z| < \epsilon\}$. The complex power of A is defined in this way

$$A^z = \begin{cases} A_z & \mathrm{Re}(z) < 0, \\ A_{z-k} \circ A^k & k \in \mathbb{N}, \mathrm{Re}(z-k) < 0. \end{cases}$$

Since the operator A is sectorial, the Dunfort integral in (3.2) converges. As usual, one can prove that the Definition 3.8 does not depend on k.

Theorem 3.9. *If $A \in G^{m_1,m_2}_{pr}(\mathbb{R}^{n_1,n_2})$ fulfils Assumptions 3.6, then $A^z \in G^{m_1 z, m_2 z}(\mathbb{R}^{n_1+n_2})$. Moreover,*[3]

$$\sigma_1^{m_1 z}(A^z)(x_1,\xi_1) = \left(\sigma_1^{m_1}(A)(x_1,\xi_1)\right)^z, \tag{3.3}$$

$$\sigma_2^{m_2 z}(A^z)(x_2,\xi_2) = \left(\sigma_2^{m_2}(A)(x_2,\xi_2)\right)^z, \tag{3.4}$$

$$\sigma^{m_1 z, m_2 z}(A^z)(x_1,x_2,\xi_1,\xi_2) = \left(\sigma^{m_1,m_2}(A)(x_1,x_2,\xi_1,\xi_2)\right)^z, \tag{3.5}$$

where the complex power in (3.3), (3.4) *is the complex power of operators, while in* (3.5) *is the standard complex power of a function.*

We now introduce the ζ-function of suitable bisingular operators, then we will study the meromorphic extension of the ζ-function and we will analyze its first *left* pole. We do not write the proofs of the following statements, they are similar to Theorem 4 and Corollary 1 of [2].

Definition 3.10. Let $A \in G^{m_1,m_2}(\mathbb{R}^{n_1+n_2})$ be a bisingular operator that satisfies Assumptions 3.6, then

$$\zeta(A,z) = \iint_{\mathbb{R}^{n_1+n_2}} K_{A^z}(x_1,x_2,x_1,x_2)dx_1dx_2, \quad \mathrm{Re}(z) < 2\min\left\{-\frac{n_1}{m_1}, -\frac{n_2}{m_2}\right\},$$

where K_{A^z} is the kernel of A^z.

Theorem 3.11. *Let $A \in G^{m_1,m_2}(\mathbb{R}^{n_1+n_2})$ be an operator that satisfies Assumptions 3.6. Then $\zeta(A,z)$ can be extended as a meromorphic function on $\{z \in \mathbb{C} \mid \mathrm{Re}(z) < 2\min\{-\frac{n_1}{m_1}, -\frac{n_2}{m_2}\} + \epsilon\}$. Moreover, the Laurent coefficients at pole $z_{pole} = 2\min\{-\frac{n_1}{m_1}, -\frac{n_2}{m_2}\}$ depend on $\frac{n_1}{m_1}$ and $\frac{n_2}{m_2}$.*

In the case $\frac{n_1}{m_1} > \frac{n_2}{m_2}$:

$$\lim_{z\to -\frac{2n_1}{m_1}} \left(z + \frac{2n_1}{m_1}\right)\zeta(A,z) = \frac{(2\pi)^{-n_1-n_2}}{m_1}\int_{\mathbb{R}^{2n_2}}\int_{\mathbb{S}^{2n_1-1}} (a_{m_1,\cdot})^{-\frac{2n_1}{m_1}} d\theta_1 dx_2 d\xi_2. \tag{3.6}$$

In the case $\frac{n_2}{m_2} > \frac{n_1}{m_1}$:

$$\lim_{z\to -\frac{2n_2}{m_2}} \left(z + \frac{2n_2}{m_2}\right)\zeta(A,z) = \frac{(2\pi)^{-n_1-n_2}}{m_2}\int_{\mathbb{R}^{2n_1}}\int_{\mathbb{S}^{2n_2-1}} (a_{\cdot,m_2})^{-\frac{2n_2}{m_2}} d\theta_2 dx_1 d\xi_1. \tag{3.7}$$

In the case $\frac{n_1}{m_1} = \frac{n_2}{m_2} = l$:

$$res^2(A) = \lim_{z\to -l}(z+l)^2\zeta(A,z) = \frac{(2\pi)^{-n_1-n_2}}{m_1 m_2}\int_{\mathbb{S}^{2n_2-1}}\int_{\mathbb{S}^{2n_1-1}} (a_{m_1,m_2})^{-l} d\theta_1 d\theta_2, \tag{3.8}$$

$$\lim_{z\to -l}(z+l)\left(\zeta(A,z) - \frac{res^2(A)}{(z+l)^2}\right), = -TR_{1,2}(A) + TR_\theta(A), \tag{3.9}$$

[3] We have just defined symbols $\Gamma^{m_1,m_2}(\mathbb{R}^{n_1+n_2})$ with $m_1, m_2 \in \mathbb{R}^2$. It is nevertheless possible to define the same class with complex numbers z_1, z_2, in the inequality (2.1) instead of m_i we use $\mathrm{Re}(z_i)$.

where

$$TR_{1,2}(A) = (2\pi)^{-n_1-n_2}$$
$$\left(\lim_{\tau\to\infty}\left(\frac{1}{m_1}\int_{|x_2|+|\xi_2|<\tau}\int_{\mathbb{S}^{2n_1-1}}\left((a_{m_1,\cdot})^{-l}d\theta_1 dx_2 d\xi_2 - res^2(A)\log\tau\right)\right.\right.$$
$$\left.\left.+\lim_{\tau\to\infty}\left(\frac{1}{m_2}\int_{|x_1|+|\xi_1|<\tau}\int_{\mathbb{S}^{2n_2-1}}\left((a_{m_2,\cdot})^{-l}d\theta_2 dx_1 d\xi_1 - res^2(A)\log\tau\right)\right)\right),$$

and

$$TR_\theta(A) = \frac{(2\pi)^{-n_1-n_2}}{m_1 m_2}\int_{\mathbb{S}^{2n_2-1}}\int_{\mathbb{S}^{2n_1-1}}(a_{m_1,m_2})^{-l}\log(a_{m_1,m_2})d\theta_1 d\theta_2.$$

Now, applying a generalization of Tauberian Theorem due to J. Aramaki [1], we easily obtain the following:

Theorem 3.12. *Let $A \in G^{m_1,m_2}(\mathbb{R}^{n_1+n_2})$ be self-adjoint and positive, suppose moreover that A satisfies Assumptions 3.6. Then*

$$N_A(\lambda) = \begin{cases} C_1\lambda^l\log\lambda + C_1'\lambda^l + O(\lambda^{l-\delta_1}) & \frac{2n_1}{m_2} = \frac{2n_2}{m_2} = l, \\ C_2\lambda^{2\frac{n_2}{m_2}} + O(\lambda^{2\frac{n_2}{m_2}-\delta_2}) & \frac{2n_2}{m_2} > \frac{2n_1}{m_1}, \\ C_3\lambda^{2\frac{n_1}{m_2}} + O(\lambda^{2\frac{n_1}{m_1}-\delta_3}) & \frac{2n_1}{m_1} > \frac{2n_2}{m_2}, \end{cases}$$

for certain $\delta_i > 0$. It is moreover possible to find the exact value of the constants in terms of $\{a_{m_1,\cdot}, a_{\cdot,m_2}, a_{m_1,m_2}\}$, the principal symbol of A.

$$C_1 = \frac{1}{(2\pi)^{n_1+n_2}2n_1m_2}\int_{\mathbb{S}^{2n_2-1}}\int_{\mathbb{S}^{2n_1-1}}(a_{m_1,m_2})^{-l}d\theta_1 d\theta_2,$$
$$C_1' = \frac{TR_{1,2}(A) - TR_\theta(A)}{l} - \frac{1}{4n_1n_2}\int_{\mathbb{S}^{2n_2-1}}\int_{\mathbb{S}^{2n_1-1}}(a_{m_1,m_2})^{-l}d\theta_1 d\theta_2,$$
$$C_2 = \frac{1}{(2\pi)^{n_1+n_2}2n_2}\int_{\mathbb{R}^{2n_1}}\int_{\mathbb{S}^{2n_2-1}}(a_{\cdot,m_2})^{-\frac{2n_2}{m_2}}d\theta_2 dx_1 d\xi_1,$$
$$C_3 = \frac{1}{(2\pi)^{n_1+n_2}2n_1}\int_{\mathbb{R}^{2n_2}}\int_{\mathbb{S}^{2n_1-1}}(a_{m_1,\cdot})^{-\frac{2n_1}{m_1}}d\theta_1 dx_2 d\xi_2.$$

4. Tensor products of Hermite-type operators

We use the notation in the Introduction. We consider globally elliptic self-adjoint bisingular differential operators of the special form

$$P(x, D_x) = P_1(x_1, D_{x_1})P_2(x_2, D_{x_2}),$$

so that $Pu_{\mathbf{j}} = \lambda^1_{j_1}\lambda^2_{j_2}u_{\mathbf{j}}$, $\mathbf{j} = (j_1, j_2) \in \mathbb{Z}^2_+$, cf. (1.4). Hence we have:

Proposition 4.1. *Let u be a tempered distribution in $\mathbb{R}^{n_1+n_2}$. If*

$$u = \sum_{\mathbf{j}\in\mathbb{Z}^2_+} a_{\mathbf{j}}u_{\mathbf{j}} \quad \textit{in } \mathcal{S}'(\mathbb{R}^{n_1+n_2}),$$

then

$$P(x, D_x)u = \sum_{\mathbf{j}=(j_1,j_2)\in\mathbb{Z}_+^2} \lambda^1_{j_1}\lambda^2_{j_2} a_{\mathbf{j}} u_{\mathbf{j}}, \tag{4.1}$$

where $P(x, D_x)u \in L^2(\mathbb{R}^{n_1+n_2})$ *is equivalent to*

$$\sum_{\mathbf{j}\in\mathbb{Z}_+^2} (\lambda^1_{j_1}\lambda^2_{j_2})^2 |a_{\mathbf{j}}|^2 \quad < \quad +\infty. \tag{4.2}$$

With the notation at the end of Section 2, we obtain

$$\begin{aligned} \sigma_1^{m_1}(P)(x_1, \xi_1) &= p_{m_1}(x_1, \xi_1) P_2(x_2 D_2), \\ \sigma_2^{m_2}(P)(x_2, \xi_2) &= p_{m_2}(x_2, \xi_2) P_1(x_1, D_1), \\ \sigma^{m_1,m_2}(x_1, x_2, \xi_1, \xi_2) &= p_{m_1}(x_1, \xi_1) p_{m_2}(x_2, \xi_2). \end{aligned}$$

Thus, Definition 3.1 before amounts to assume global ellipticity of p_1, p_2, cf. (1.2), and invertibility of $P_1(x_1, D_1)$ and $P_2(x_1, D_2)$ as required in the Introduction. From Corollary 3.4, we have that $Pu \in Q^{s_1-m_1,s_2-m_2}(\mathbb{R}^{n_1+n_2})$ implies $u \in Q^{s_1,s_2}(\mathbb{R}^{n_1+n_2})$, for every $s_1 \in \mathbb{R}, s_2 \in \mathbb{R}$. In particular, if $Pu \in \mathcal{S}(\mathbb{R}^{n_1+n_2})$ then $u \in \mathcal{S}(\mathbb{R}^{n_1+n_2})$.

Assuming further that P_1, P_2 are strictly positive, we may apply Theorem 3.12 to estimate the counting function $N(\lambda)$ of P. By direct calculation, we shall give now more precise results in the case when P_1, P_2 are Hermite-type operators.

We first recall a classical result of L. Dirichlet for the first summatory function of $\tau(n)$, see [20, 21] for an overview on the subject:

$$D(\lambda) = \sum_{\substack{n\le\lambda,\\ n\in\mathbb{N}}} \tau(n) = \sum_{n=1}^{[\lambda]} \tau(n), \qquad \lambda \ge 1, \tag{4.3}$$

where $\tau(n)$ denotes the number of divisors of n and $[\lambda]$ stands for the integer part of λ. In 1849, Dirichlet proved that

$$D(\lambda) = \lambda \ln \lambda + (2\tilde{\gamma} - 1)\lambda + E(\lambda), \qquad \lambda \ge 1, \tag{4.4}$$

where $\tilde{\gamma}$ is the Euler-Mascheroni constant and

$$E(\lambda) = O(\lambda^{1/2}), \quad \lambda \to +\infty. \tag{4.5}$$

It is still an open problem to evaluate the optimal order of the reminder $E(\lambda)$ in the asymptotic expansion (4.4). In 1916 [17] Hardy discovered that $O(\lambda^{\frac{1}{4}})$ is a lower bound. Then a lot of upper bound have been proved, the better one has been given by Huxley in [19]. He proved that $E(\lambda)$ is $O(\lambda^c(\log\lambda)^d)$, where

$$c := \frac{131}{416} \sim 0,3149038462 \quad d := \frac{18627}{8320} + 1 \sim 3,238822115.$$

The conjecture is that the $E(\lambda)$ is $O(\lambda^{\frac{1}{4}})$. One can recast the issue of computing $D(\lambda)$ as a lattice point problem. More precisely, since

$$D(\lambda) = \sum_{n\leq\lambda, n\in\mathbb{N}} \sum_{d|n, d\in\mathbb{N}} 1 = \sum_{\substack{n_1n_2\leq\lambda\\ n_1,n_2\in\mathbb{N}}} 1, \tag{4.6}$$

we readily obtain that $D(\lambda)$ is the number of positive integers lattice points in the first quadrant between the axes and hyperbola $x_1x_2 \leq \lambda$. We will use this result for the proof of the second part of the next proposition.

Proposition 4.2. *Assume that A_i are self-adjoint operators with spectrum $n^{m_i}, n \in \mathbb{N}$, and eigenfunctions u_i^n being an orthonormal basis of $\mathbb{R}^{n_i}$, $i = 1, 2$. Denote by $N(\lambda)$, $\lambda > 0$ the counting function,*

$$N(\lambda) = \mathrm{card}\{(n_1, n_2) : n_1^{m_1} n_2^{m_2} \leq \lambda\}, \quad \lambda > 0,$$

where card *means cardinal number. Then we have the following assertions.*

a) *Let $m_1 > m_2 > 0$. Then*

$$N(\lambda) \sim \zeta\left(\frac{m_1}{m_2}\right)\lambda^{1/m_2} + C(m_1, m_2, \lambda)\lambda^{1/m_1} + O(1), \tag{4.7}$$

where

$$-1 - \frac{m_2}{(m_1 - m_2)} \leq C(m_1, m_2, \lambda) \leq -\frac{m_2}{(m_1 - m_2)}.$$

b) *Let $m = m_1 = m_2$. Then*

$$N(\lambda) = \frac{1}{m}\lambda^{\frac{1}{m}} \ln\lambda + \frac{2\tilde{\gamma} - 1}{m}\lambda + O(\lambda^{\frac{1}{2m}}). \tag{4.8}$$

Proof. a) First we recall from [11] the next identity, for $\alpha > 1, q > 0$:

$$\begin{aligned}\sum_{n=0}^{N} \frac{1}{(n+q)^\alpha} \\ = \sum_{n=0}^{\infty} \frac{1}{(n+q)^\alpha} + \frac{1}{(1-\alpha)(N+q)^{\alpha-1}} + \alpha\sum_{n=N}^{\infty}\int_n^{n+1} \frac{t-n}{(t+q)^{\alpha+1}}dt.\end{aligned} \tag{4.9}$$

One can find that

$$\alpha\sum_{n=N}^{\infty}\int_n^{n+1} \frac{t-n}{(t+q)^{\alpha+1}} = O(1/N^\alpha).$$

Note that with $q = 1$ one obtains the formula for the partial sum of the Riemann zeta function $\zeta(\alpha)$. Set

$$R(\lambda, m_1, m_2) = \sum_{n=1}^{\lambda^{1/m_1}} \left(\frac{\lambda^{1/m_2}}{n^{m_1/m_2}} - \left[\frac{\lambda^{1/m_2}}{n^{m_1/m_2}}\right]\right).$$

Clearly $0 \leq R(\lambda, m_1, m_2) \leq \lambda^{1/m_1}$ but we are not able to find the exact behavior of $R(\lambda, m_1, m_2)$ as $\lambda \to \infty$.

Now we calculate

$$N(\lambda) = \sum_{n_1^{m_1} n_2^{m_2} \leq \lambda} 1 = \sum_{n_1=1}^{[\lambda^{1/m_1}]} \sum_{n_2=1}^{\left[\frac{\lambda^{1/m_2}}{n_1^{m_1/m_2}}\right]} 1.$$

We have

$$\sum_{n=1}^{[\lambda^{1/m_1}]} \left[\frac{\lambda^{1/m_2}}{n^{m_1/m_2}}\right] = \sum_{n=1}^{[\lambda^{1/m_1}]} \frac{\lambda^{1/m_2}}{n^{m_1/m_2}} - R(\lambda, m_1, m_2)$$

and, using (4.9) with $\alpha = \frac{m_1}{m_2}, N = [\lambda^{1/m_1}]$, we obtain

$$\lambda^{1/m_2} \sum_{n=1}^{\lambda^{1/m_1}} \frac{1}{n^{m_1/m_2}} \sim \zeta\left(\frac{m_1}{m_2}\right) \lambda^{1/m_2} - \frac{m_2}{(m_1 - m_2)} \lambda^{1/m_1} + O(1).$$

This implies

$$\sum_{n_1^{m_1} n_2^{m_2} \leq \lambda} 1 \sim \zeta\left(\frac{m_1}{m_2}\right) \lambda^{1/m_2} - \frac{m_2}{(m_1 - m_2)} \lambda^{1/m_1} - R(\lambda, m_1, m_2) + O(1).$$

This and the estimate of R proves (4.7).

b) Since

$$n_1^m n_2^m \leq \lambda \ \text{ is equivalent to } \ n_1 n_2 \leq \lambda^{1/m},$$

we directly obtain $N(\lambda) = D(\lambda^{1/m})$ which gives (4.8). □

Example. An example of on operator that satisfies the hypothesis of Proposition 4.2 is the following. Let $m_1, m_2 \in \mathbb{N}, m_1 > m_2$, $k_1, k_2 > 0$ and

$$A_1 = k_1 \left(-\frac{\partial^2}{\partial_x^2} + x^2\right)^{m_1}, \quad A_2 = k_2 \left(-\frac{\partial^2}{\partial_y^2} + y^2\right)^{m_2}, \quad x, y \in \mathbb{R}.$$

Then, we know that the Hermite basis of $L^2(\mathbb{R}^2)$,

$$h_{j_1, j_2}(x, y) = h_{j_1}(x) h_{j_2}(y), \quad j_1, j_2 = 0, 1, \ldots,$$

is the set of eigenfunctions and that $k_i(2n+1)^{m_i}, n = 0, 1, \ldots$, are the eigenvalues for $A_i,\ i = 1, 2$.

We use the result of Proposition 4.2 to calculate the counting function for $A_1 A_2$. With the transformation $\lambda/(k_1 k_2) \to \lambda$, we can, and we will, assume that $k_1 = k_2 = 1$.

Put

$$I_\lambda = \{n \in \mathbb{N} \cup 0; 1 \leq 2n + 1 \leq [\lambda^{1/m_1}]\},$$

$$I_{\lambda, n_1} = \left\{n_2 \in \mathbb{N} \cup 0; 1 \leq 2n_2 + 1 \leq \left[\frac{\lambda^{1/m_2}}{(2n_1 + 1)^{m_1/m_2}}\right]\right\}.$$

Hence, the cardinal numbers of these sets are not greater than

$$[\lambda^{1/m_1}]/2+1 \text{ and } \frac{1}{2}\left[\frac{\lambda^{1/m_2}}{(2n_1+1)^{m_1/m_2}}\right]+1,$$

respectively. With this notation we have

$$\begin{aligned} N(\lambda) &= \sum_{(2n_1+1)^{m_1}(2n_2+1)^{m_2}\le\lambda} 1 = \sum_{n_1\in I_\lambda}\sum_{n_2\in I_{\lambda,n_1}} 1 \\ &= \frac{1}{2}\sum_{n\in I_\lambda}\left[\frac{\lambda^{1/m_2}}{(2n+1)^{m_1/m_2}}\right]+r_n, \end{aligned}$$

where r_n takes values 0 and 1. We obtain

$$0\le S(\lambda,m_1,m_2)=\sum_{n\in I_\lambda} r_n \le \frac{1}{2}[\lambda^{1/m_1}]+1.$$

Next

$$\frac{1}{2}\sum_{n\in I_\lambda}\left[\frac{\lambda^{1/m_2}}{(2n+1)^{m_1/m_2}}\right]=\frac{1}{2}\sum_{n\in I_\lambda}\frac{\lambda^{1/m_2}}{(2n+1)^{m_1/m_2}}-R(\lambda,m_1,m_2),$$

where

$$0\le R(\lambda,m_1,m_2)\le\frac{1}{2}[\lambda^{1/m_1}]+1.$$

By the proof of the previous proposition, with $r=1$ or $r=0$,

$$\begin{aligned} \sum_{n\in I_\lambda}\frac{1}{(2n+1)^{m_1/m_2}} &= \sum_{t=1}^{\frac{1}{2}[\lambda^{1/m_1}]+r}\frac{1}{t^{m_1/m_2}} \\ &= \zeta\left(\frac{m_1}{m_2}\right)-\frac{m_2}{m_1-m_2}\left(\frac{1}{2}([\lambda^{1/m_1}]+r)\right)^{\frac{m_2-m_1}{m_2}}+O(1/\lambda^{1/m_2}). \end{aligned}$$

This implies

$$\begin{aligned} N(\lambda) &= \frac{1}{2}\lambda^{1/m_2}\left(\zeta\left(\frac{m_1}{m_2}\right)-\frac{m_2}{m_1-m_2}\left(\frac{1}{2}([\lambda^{1/m_1}]+r)\right)^{\frac{m_2-m_1}{m_2}}\right. \\ &\qquad \left.+O(1/\lambda^{1/m_2})\right)-R(\lambda,m_1,m_2)+S(\lambda,m_1,m_2) \\ &= \frac{\lambda^{1/m_2}}{2}\zeta\left(\frac{m_1}{m_2}\right)-\frac{m_2}{m_1-m_2}2^{\frac{m_1-2m_2}{m_2}}\lambda^{1/m_1} \\ &\qquad -R(\lambda,m_1,m_2)+S(\lambda,m_1,m_2)+O(1) \\ &= \frac{\lambda^{1/m_2}}{2}\zeta\left(\frac{m_1}{m_2}\right)+C(\lambda,m_1,m_2)\lambda^{1/m_1}+O(1), \end{aligned}$$

where

$$-\frac{2^{\frac{m_1-2m_2}{m_2}}m_2}{m_1-m_2}-\frac{[\lambda^{1/m_1}]}{2\lambda^{1/m_1}}-\frac{1}{\lambda^{1/m_1}}\leq C(\lambda,m_1,m_2)$$
$$\leq -\frac{2^{\frac{m_1-2m_2}{m_2}}m_2}{m_1-m_2}+\frac{[\lambda^{1/m_1}]}{2\lambda^{1/m_1}}+\frac{1}{\lambda^{1/m_1}}.$$

One can consider in a similar way

$$A_1=k_1(-\Delta_{x_1}+\|x_1\|^2)^{m_1}+r_1, A_2=k_2(-\Delta_{x_2}+\|x_2\|^2)^{m_2}+r_2, x_i\in\mathbb{R}^{n_1},$$

$k_1>k_2>0, r_i>0, i=1,2$; but the computation is much more complicate.

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