Chapter 2
The Gamma Function

In what follows, we introduce the classical Gamma function in Sect. 2.1. It is essentially understood to be a generalized factorial. However, there are many further applications, e.g., as part of probability distributions (see, e.g., Evans et al. 2000). The main properties of the Gamma function are explained in this chapter (for a more detailed discussion the reader is referred to, e.g., Artin (1964), Lebedev (1973), Müller (1998), Nielsen (1906), and Whittaker and Watson (1948) and the references therein). We briefly consider Euler's Beta function in Sect. 2.2 and use it to recursively compute the volume of the $(q-1)$-dimensional unit sphere $\mathbb{S}^{q-1} \subset \mathbb{R}^q$. As outstanding property of the Gamma function the Stirling formula is verified in Sect. 2.3. It leads us to the so-called duplication formula (Lemma 2.3.3) which will simplify a lot of calculations in later chapters. The extension of the Gamma function to complex values is studied in Sect. 2.4. In doing so, we introduce Pochhammer's factorial and Euler's constant γ. Moreover, we establish product representations for the Gamma function as well as for trigonometric functions. In Sect. 2.5 the incomplete Gamma and Beta functions are briefly presented in form of some exercises and their relation to probability distributions is indicated.

2.1 Definition and Functional Equation

For real values $x > 0$, we consider the integrals

$$\int_0^1 \mathrm{e}^{-t} t^{x-1} \, \mathrm{d}t, \tag{2.1.1}$$

and

$$\int_1^\infty \mathrm{e}^{-t} t^{x-1} \, \mathrm{d}t. \tag{2.1.2}$$

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In order to show the convergence of (2.1.1) we observe that $0 < \mathrm{e}^{-t}t^{x-1} \le t^{x-1}$ holds true for all $t \in (0,1]$. Therefore, for $\varepsilon > 0$ sufficiently small, we have

$$\int_{\varepsilon}^{1} \mathrm{e}^{-t}t^{x-1}\,\mathrm{d}t \le \int_{\varepsilon}^{1} t^{x-1}\,\mathrm{d}t = \left.\frac{t^x}{x}\right|_{\varepsilon}^{1} = \frac{1}{x} - \frac{\varepsilon^x}{x}. \tag{2.1.3}$$

Consequently, for all $x > 0$, the integral (2.1.1) is convergent. To assure the convergence of (2.1.2) we observe that

$$\mathrm{e}^{-t}t^{x-1} = \frac{1}{\sum_{k=0}^{\infty}\frac{t^k}{k!}}t^{x-1} \le \frac{1}{\frac{t^n}{n!}}t^{x-1} = \frac{n!}{t^{n-x+1}} \tag{2.1.4}$$

for all $n \in \mathbb{N}$ and $t \ge 1$. This shows us that

$$\int_{1}^{A} \mathrm{e}^{-t}t^{x-1}\,\mathrm{d}t \le n!\int_{1}^{A} \frac{1}{t^{n-x+1}}\,\mathrm{d}t = n!\left.\frac{t^{-n+x}}{x-n}\right|_{1}^{A} = \frac{n!}{x-n}\left(\frac{1}{A^{n-x}} - 1\right) \tag{2.1.5}$$

provided that A is sufficiently large and n is chosen such that $n \ge x+1$. Thus, the integral (2.1.2) is convergent which we summarize in the following lemma.

Lemma 2.1.1. *For all $x > 0$, the integral*

$$\int_{0}^{\infty} \mathrm{e}^{-t}t^{x-1}\,\mathrm{d}t \tag{2.1.6}$$

is convergent.

Definition 2.1.2. The function $x \mapsto \Gamma(x),\ x > 0$, defined by

$$\Gamma(x) = \int_{0}^{\infty} \mathrm{e}^{-t}t^{x-1}\,\mathrm{d}t \tag{2.1.7}$$

is called the *Gamma function* (see Fig. 2.3 (right) for an illustration).

Obviously, we have the following properties:

(i) Γ is positive for all $x > 0$,
(ii) $\Gamma(1) = \int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}t = 1$.

We can use integration by parts to obtain for $x > 0$:

$$\begin{aligned}\Gamma(x+1) &= \int_{0}^{\infty} \mathrm{e}^{-t}t^{x}\,\mathrm{d}t = -\mathrm{e}^{-t}t^x\big|_0^\infty - \int_{0}^{\infty} (-\mathrm{e}^{-t})xt^{x-1}\,\mathrm{d}t \\ &= x\int_{0}^{\infty} \mathrm{e}^{-t}t^{x-1}\,\mathrm{d}t = x\,\Gamma(x).\end{aligned} \tag{2.1.8}$$

Lemma 2.1.3. *The Gamma function Γ satisfies the functional equation*

$$\Gamma(x+1) = x\,\Gamma(x),\ x > 0. \tag{2.1.9}$$

Moreover, by iteration for $x > 0$ and $n \in \mathbb{N}$,

$$\Gamma(x+n) = (x+n-1)\cdot \ldots \cdot (x+1)x\,\Gamma(x) = \prod_{i=1}^{n}(x+i-1)\Gamma(x), \tag{2.1.10}$$

$$\Gamma(n+1) = \left(\prod_{i=1}^{n} i\right)\Gamma(1) = \prod_{i=1}^{n} i = n!. \tag{2.1.11}$$

In other words, the Gamma function can be interpreted as an extension of factorials.

Lemma 2.1.4. *The Gamma function Γ is differentiable for all $x > 0$ and we have*

$$\Gamma'(x) = \int_0^\infty \mathrm{e}^{-t}\ln(t)t^{x-1}\,\mathrm{d}t. \tag{2.1.12}$$

Proof. For $x > |h| > 0$, we use the formula $t^y = \mathrm{e}^{\ln(t)y}$, $y > 0$, $t > 0$:

$$\Gamma(x+h) = \int_0^\infty \mathrm{e}^{-t}t^{x+h-1}\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-t+\ln(t)(x+h-1)}\,\mathrm{d}t. \tag{2.1.13}$$

By Taylor's formula we find $0 < \vartheta < 1$ such that

$$\begin{aligned}\Gamma(x+h) - \Gamma(x) &= \int_0^\infty \mathrm{e}^{-t}\mathrm{e}^{-\ln(t)}\left(\mathrm{e}^{\ln(t)(x+h)} - \mathrm{e}^{\ln(t)x}\right)\,\mathrm{d}t \qquad (2.1.14)\\ &= \int_0^\infty \mathrm{e}^{-t}\mathrm{e}^{-\ln(t)}\left(h\ln(t)t^x + \tfrac{1}{2}h^2(\ln(t))^2t^{x+\vartheta h}\right)\,\mathrm{d}t\\ &= h\int_0^\infty \mathrm{e}^{-t}\ln(t)t^{x-1}\,\mathrm{d}t + \tfrac{1}{2}h^2\int_0^\infty \mathrm{e}^{-t}(\ln(t))^2t^{x+\vartheta h-1}\,\mathrm{d}t.\end{aligned}$$

This gives us the differentiability of Γ if the second integral is bounded. Consider the following estimate (we employ that $(\ln(t))^2 \leq t^2$ for $t \geq 1$ and that $\mathrm{e}^{-t} \leq 1$ for $t \in [0,1]$)

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-t}(\ln(t))^2t^{x+\vartheta h-1}\,\mathrm{d}t &= \int_0^1 \mathrm{e}^{-t}(\ln(t))^2t^{x+\vartheta h-1}\,\mathrm{d}t + \int_1^\infty \mathrm{e}^{-t}(\ln(t))^2t^{x+\vartheta h-1}\,\mathrm{d}t\\ &\leq \int_0^1 (\ln(t))^2t^{x+\vartheta h-1}\,\mathrm{d}t + \int_1^\infty \mathrm{e}^{-t}t^2t^{x+\vartheta h-1}\,\mathrm{d}t\\ &\leq \Gamma(2+x+\vartheta h) + \frac{2}{(x+\vartheta h)^3} < \infty. \qquad (2.1.15)\end{aligned}$$

This provides us with the desired result. □

An analogous proof can be given to show that Γ is infinitely often differentiable for all $x > 0$ and

$$\Gamma^{(k)}(x) = \int_0^\infty \mathrm{e}^{-t}(\ln(t))^k t^{x-1}\,\mathrm{d}t,\ k \in \mathbb{N}. \tag{2.1.16}$$

Lemma 2.1.5 (Gauß' Expression of the Second Logarithmic Derivative). *For $x > 0$,*

$$\left(\Gamma'(x)\right)^2 < \Gamma(x)\Gamma''(x). \tag{2.1.17}$$

Equivalently, we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 \ln(\Gamma(x)) = \frac{\Gamma''(x)}{\Gamma(x)} - \left(\frac{\Gamma'(x)}{\Gamma(x)}\right)^2 > 0, \tag{2.1.18}$$

i.e., $x \mapsto \ln(\Gamma(x))$, $x > 0$, is a convex function or Γ is logarithmic convex.

Proof. We start with

$$\begin{aligned}\left(\Gamma'(x)\right)^2 &= \left(\int_0^\infty \mathrm{e}^{-t}\ln(t)t^{x-1}\,\mathrm{d}t\right)^2 \\ &= \left(\int_0^\infty \mathrm{e}^{-\frac{t}{2}}t^{\frac{x-1}{2}}\ln(t)\mathrm{e}^{-\frac{t}{2}}t^{\frac{x-1}{2}}\,\mathrm{d}t\right)^2. \end{aligned} \tag{2.1.19}$$

The Cauchy–Schwarz inequality yields (note that equality cannot occur since the two functions are linearly independent):

$$\begin{aligned}\left(\Gamma'(x)\right)^2 &< \int_0^\infty \left(\mathrm{e}^{-\frac{t}{2}}t^{\frac{x-1}{2}}\right)^2\,\mathrm{d}t \int_0^\infty \left(\mathrm{e}^{-\frac{t}{2}}t^{\frac{x-1}{2}}\ln(t)\right)^2\,\mathrm{d}t \\ &= \int_0^\infty \mathrm{e}^{-t}t^{x-1}\,\mathrm{d}t \int_0^\infty \mathrm{e}^{-t}t^{x-1}(\ln(t))^2\,\mathrm{d}t = \Gamma(x)\ \Gamma''(x). \end{aligned} \tag{2.1.20}$$

Moreover, we find with the help of (2.1.20) that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln(\Gamma(x)) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\Gamma'(x)}{\Gamma(x)} = \frac{\Gamma''(x)\Gamma(x) - (\Gamma'(x))^2}{(\Gamma(x))^2} > 0, \tag{2.1.21}$$

which yields (2.1.18). □

Note that $\ln(\Gamma(\cdot))$ is convex, i.e., for $t \in [0,1]$

$$\begin{aligned}\ln\left(\Gamma(tx + (1-t)y)\right) &\le t\ln(\Gamma(x)) + (1-t)\ln(\Gamma(y)) \\ &= \ln\left(\Gamma^t(x)\right) + \ln\left(\Gamma^{1-t}(y)\right) \\ &= \ln\left(\Gamma^t(x)\cdot\Gamma^{1-t}(y)\right) \end{aligned} \tag{2.1.22}$$

which is equivalent to $\Gamma(tx + (1-t)y) \le \Gamma^t(x)\cdot\Gamma^{1-t}(y)$ with $x, y > 0$.

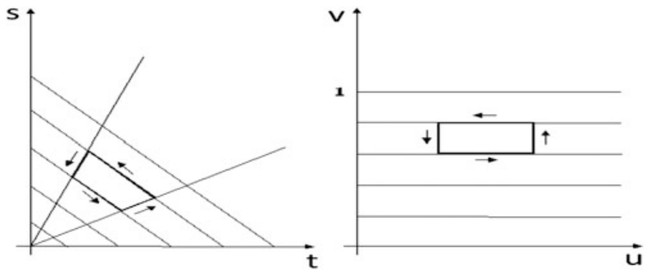

Fig. 2.1 The illustration of the coordinate transformation relating the Beta and the Gamma functions

2.2 Euler's Beta Function

Next, we notice that for $x, y > 0$, the integral

$$\int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t \tag{2.2.1}$$

is convergent.

Definition 2.2.1. The function $(x, y) \mapsto B(x, y)$, $x, y > 0$, defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t \tag{2.2.2}$$

is called the *Euler Beta function*.

For $x, y > 0$, we see that

$$\Gamma(x)\Gamma(y) = \int_0^\infty \mathrm{e}^{-t}t^{x-1}\,\mathrm{d}t \int_0^\infty \mathrm{e}^{-s}s^{y-1}\,\mathrm{d}s = \int_0^\infty\int_0^\infty \mathrm{e}^{-(t+s)}t^{x-1}s^{y-1}\,\mathrm{d}t\,\mathrm{d}s. \tag{2.2.3}$$

Note that the transition from one-dimensional to two-dimensional integrals is permitted by Fubini's theorem. We make a coordinate transformation (see Fig. 2.1) as follows:

$$t = u(1-v), \qquad 0 \le u < \infty, \tag{2.2.4}$$

$$s = uv, \qquad 0 \le v \le 1. \tag{2.2.5}$$

It is not difficult to verify that the functional determinant of the coordinate transformation is given by

$$\frac{\partial(t,s)}{\partial(u,v)} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u(1-v) + uv = u \ge 0. \tag{2.2.6}$$

Thus, we find

$$\begin{aligned}\int_0^\infty \int_0^\infty e^{-(t+s)} t^{x-1} s^{y-1}\, dt\, ds &= \int_0^1 \int_0^\infty e^{-u} (u(1-v))^{x-1} (uv)^{y-1} u\, du\, dv \\ &= \int_0^1 \int_0^\infty e^{-u} u^{x+y-2} (1-v)^{x-1} v^{y-1} u\, du\, dv \\ &= \int_0^\infty e^{-u} u^{x+y-1}\, du \int_0^1 v^{y-1} (1-v)^{x-1}\, dv. \end{aligned} \tag{2.2.7}$$

This leads us to the following theorem:

Theorem 2.2.2. *For $x, y > 0$,*

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{2.2.8}$$

In particular,

$$\begin{aligned} B\left(\frac{1}{2}, \frac{1}{2}\right) &= \int_0^1 t^{-\frac{1}{2}} (1-t)^{-\frac{1}{2}}\, dt = 2 \int_0^1 (1-u^2)^{-\frac{1}{2}}\, du \\ &= 2 \arcsin(1) = 2\frac{\pi}{2} = \pi. \end{aligned} \tag{2.2.9}$$

Therefore, we have

$$\frac{\Gamma^2\left(\frac{1}{2}\right)}{\Gamma(1)} = \pi. \tag{2.2.10}$$

This shows that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = \int_0^\infty e^{-t} t^{-\frac{1}{2}}\, dt. \tag{2.2.11}$$

Other types of integrals can be derived from

$$\int_0^\infty e^{-t^\alpha}\, dt \overset{u=t^\alpha}{=} \frac{1}{\alpha} \int_0^\infty e^{-u} u^{\frac{1}{\alpha}-1}\, du = \frac{1}{\alpha} \Gamma\left(\frac{1}{\alpha}\right), \quad \alpha > 0. \tag{2.2.12}$$

Lemma 2.2.3. *For $\alpha > 0$,*

$$\int_0^\infty e^{-t^\alpha}\, dt = \Gamma\left(\frac{\alpha+1}{\alpha}\right). \tag{2.2.13}$$

In particular, Lemma 2.2.3 yields for $\alpha = 2$

$$\int_0^\infty e^{-t^2}\, dt = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}. \tag{2.2.14}$$

Moreover, we have

$$\int_0^\infty t^{x-1} e^{-t^\alpha}\, dt = \frac{1}{\alpha}\Gamma\left(\frac{x}{\alpha}\right), \quad x, \alpha > 0, \tag{2.2.15}$$

and

$$\int_0^\infty t^{x-1} e^{-\alpha t^2}\, dt = \frac{1}{2}\alpha^{-\frac{x}{2}}\Gamma\left(\frac{x}{2}\right), \quad x, \alpha > 0. \tag{2.2.16}$$

Within the notational framework of polar coordinates (see (6.1.17) and (6.1.18) for details) we are now prepared to give the well-known calculation of the area $\|\mathbb{S}^{q-1}\|$ of the unit sphere $\mathbb{S}^{q-1}$ in $\mathbb{R}^q$: By definition, we set $\|\mathbb{S}^0\| = 2$. Clearly, $\mathbb{S}^1$ is the unit circle in $\mathbb{R}^2$, i.e., $\mathbb{S}^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$. Hence, its area is equal to

$$\|\mathbb{S}^1\| = 2\pi. \tag{2.2.17}$$

Furthermore, $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ is the unit sphere in $\mathbb{R}^3$. Thus, its area is known to be equal to

$$\|\mathbb{S}^2\| = 4\pi. \tag{2.2.18}$$

We are interested in deriving the area of the sphere $\mathbb{S}^{q-1}$ in $\mathbb{R}^q$ $(q > 3)$:

$$\|\mathbb{S}^{q-1}\| = \int_{\mathbb{S}^{q-1}} dS_{(q-1)}(\xi_{(q)}). \tag{2.2.19}$$

In terms of *spherical coordinates* (6.1.17) and (6.1.18) in $\mathbb{R}^q$ the surface element $dS_{(q-1)}(\cdot)$ of the sphere $\mathbb{S}^{q-1}$ in $\mathbb{R}^q$ admits the representation

$$\begin{aligned} dS_{(q-1)}\left(\xi_{(q)}\right) = {} & dS_{(q-2)}\left(\sqrt{1-t^2}\,\xi_{(q-1)}\right)\, dt \\ & + (-1)^{q-1} t\, dV_{(q-1)}\left(\sqrt{1-t^2}\,\xi_{(q-1)}\right). \end{aligned} \tag{2.2.20}$$

Now, we notice that

$$\begin{aligned} dV_{(q-1)}\left(\sqrt{1-t^2}\,\xi_{(q-1)}\right) &= -t(1-t^2)^{\frac{q-3}{2}}\, dt\, dS_{(q-2)}\left(\xi_{(q-1)}\right) \\ &= (-1)^{q-1} t(1-t^2)^{\frac{q-3}{2}}\, dS_{(q-2)}\left(\xi_{(q-1)}\right)\, dt. \end{aligned} \tag{2.2.21}$$

In addition, it is not difficult to see that

$$\mathrm{d}S_{(q-2)}\left(\sqrt{1-t^2}\xi_{(q-1)}\right) = (1-t^2)^{\frac{q-1}{2}}\ \mathrm{d}S_{(q-2)}\left(\xi_{(q-1)}\right). \tag{2.2.22}$$

Combining our results we are led to the identity

$$\begin{aligned}&\mathrm{d}S_{(q-1)}\left(t\varepsilon^q + \sqrt{1-t^2}\xi_{(q-1)}\right) \\ &\quad = (1-t^2)^{\frac{q-3}{2}}\left(1-t^2+t^2\right)\ \mathrm{d}S_{(q-2)}\left(\xi_{(q-1)}\right)\ \mathrm{d}t,\end{aligned} \tag{2.2.23}$$

where we have used the decomposition $\xi_{(q)} = t\varepsilon^q\sqrt{1-t^2}\xi_{(q-1)}$. Note that $\varepsilon^1,\ldots,\varepsilon^q$ is the canonical orthonormal system in $\mathbb{R}^q$. In brief, we obtain

$$\mathrm{d}S_{(q-1)}\left(\xi_{(q)}\right) = (1-t^2)^{\frac{q-3}{2}}\ \mathrm{d}S_{(q-2)}\left(\xi_{(q-1)}\right)\ \mathrm{d}t, \tag{2.2.24}$$

such that

$$\begin{aligned}\|\mathbb{S}^{q-1}\| &= \int_{-1}^{1}\int_{\mathbb{S}^{q-2}}(1-t^2)^{\frac{q-3}{2}}\ \mathrm{d}S_{(q-2)}(\xi_{(q-1)})\ \mathrm{d}t \\ &= \|\mathbb{S}^{q-2}\|\int_{-1}^{1}(1-t^2)^{\frac{q-3}{2}}\ \mathrm{d}t.\end{aligned} \tag{2.2.25}$$

For the computation of the remaining integral it is helpful to use some facts known from the Gamma function as well as Euler's Beta function. More explicitly,

$$\begin{aligned}\int_{-1}^{1}(1-t^2)^{\frac{q-3}{2}}\ \mathrm{d}t &= 2\int_0^1(1-t^2)^{\frac{q-3}{2}}\ \mathrm{d}t \overset{t^2=v}{=} \int_0^1 v^{-\frac{1}{2}}(1-v)^{\frac{q-3}{2}}\ \mathrm{d}v \\ &= B\left(\frac{1}{2},\frac{q-1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{q-1}{2}\right)}{\Gamma\left(\frac{q}{2}\right)} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{q-1}{2}\right)}{\Gamma\left(\frac{q}{2}\right)}.\end{aligned} \tag{2.2.26}$$

By recursion we get the following lemma from (2.2.26):

Lemma 2.2.4. *For $q \geq 2$,*

$$\|\mathbb{S}^{q-1}\| = 2\frac{\pi^{\frac{q}{2}}}{\Gamma\left(\frac{q}{2}\right)}. \tag{2.2.27}$$

The area of the sphere $\mathbb{S}_R^{q-1}(y)$ with center $y \in \mathbb{R}^q$ and radius $R > 0$ is given by

$$\|\mathbb{S}_R^{q-1}(y)\| = \|\mathbb{S}^{q-1}\|\ R^{q-1} = 2\frac{\pi^{\frac{q}{2}}}{\Gamma\left(\frac{q}{2}\right)}R^{q-1}. \tag{2.2.28}$$

Furthermore, using $\xi_{(q)} = \frac{x}{|x|}$, the volume of the ball $\mathbb{B}_R^q(y)$ with center $y \in \mathbb{R}^q$ and radius $R > 0$ is given by

$$\|\mathbb{B}_R^q(y)\| = \int_{\mathbb{B}_R^q(y)} \mathrm{d}V_{(q)}(x) = \int_{r=0}^{R} \left(\int_{\mathbb{S}_r^{q-1}(y)} \mathrm{d}S_{(q-1)}(\xi_{(q)}) \right) \mathrm{d}r \tag{2.2.29}$$
$$= 2\frac{\pi^{\frac{q}{2}}}{\Gamma\left(\frac{q}{2}\right)} \int_0^R r^{q-1}\,\mathrm{d}r = \frac{\pi^{\frac{q}{2}}}{\Gamma\left(\frac{q}{2}+1\right)} R^q.$$

2.3 Stirling's Formula

Next, we are interested in the behavior of the Gamma function Γ for large positive values x. This provides us with the so-called Stirling's formula, a result which we apply to verify the helpful duplication formula and to extend the Gamma function in Sect. 2.4.

Theorem 2.3.1 (Stirling's Formula). *For $x > 0$,*

$$\left| \frac{\Gamma(x)}{\sqrt{2\pi}x^{x-1/2}\mathrm{e}^{-x}} - 1 \right| \leq \sqrt{\frac{2}{\pi x}}. \tag{2.3.1}$$

Proof. Regard x as fixed and substitute

$$t = x(1+s), \quad -1 \leq s < \infty, \quad \frac{\mathrm{d}t}{\mathrm{d}s} = x \tag{2.3.2}$$

in the defining integral of the Gamma function. We obtain

$$\Gamma(x) = \int_0^\infty \mathrm{e}^{-t} t^{x-1}\,\mathrm{d}t = \int_{-1}^\infty \mathrm{e}^{-x-xs} x^{x-1} (1+s)^{x-1} x\,\mathrm{d}s \tag{2.3.3}$$
$$= x^x \mathrm{e}^{-x} \int_{-1}^\infty (1+s)^{x-1} \mathrm{e}^{-xs}\,\mathrm{d}s = x^x \mathrm{e}^{-x} I(x).$$

Our aim is to verify that $I(x)$ satisfies

$$\left| I(x) - \sqrt{\frac{2\pi}{x}} \right| \leq \frac{2}{x}. \tag{2.3.4}$$

such that

$$\left| \frac{\Gamma(x)}{x^x \mathrm{e}^{-x}} - \sqrt{\frac{2\pi}{x}} \right| \leq \frac{2}{x}, \quad \text{i.e.,} \quad \left| \frac{\Gamma(x)}{x^{x-1/2}\mathrm{e}^{-x}\sqrt{2\pi}} - 1 \right| \leq \sqrt{\frac{2}{x\pi}}. \tag{2.3.5}$$

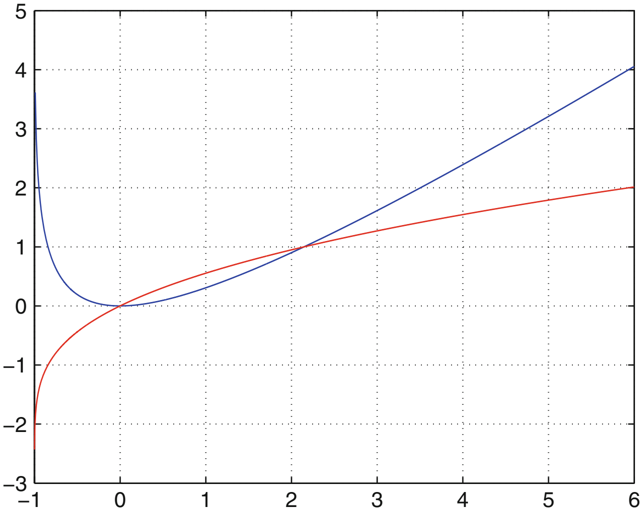

Fig. 2.2 The functions $s \mapsto u^2(s) = s - \ln(1+s)$ (*blue*) and $u(s)$ defined by (2.3.7) (*red*)

For that purpose we write

$$(1+s)^x \mathrm{e}^{-xs} = \exp\left(-x(s - \ln(1+s))\right) = \mathrm{e}^{-xu^2(s)}, \tag{2.3.6}$$

where (cf. Fig. 2.2)

$$u(s) = \begin{cases} |s - \ln(1+s)|^{\frac{1}{2}} & , \quad s \in [0, \infty), \\ -|s - \ln(1+s)|^{\frac{1}{2}} & , \quad s \in (-1, 0). \end{cases} \tag{2.3.7}$$

We set up the Taylor expansion of u^2 for $s \in (-1, \infty)$ at 0:

$$\begin{aligned} u^2(s) &= u^2(0) + \frac{\mathrm{d}u^2}{\mathrm{d}s}(0)s + \frac{\mathrm{d}^2u^2}{\mathrm{d}s^2}(\vartheta s)\frac{s^2}{2} \\ &= 0 + \left(1 - \frac{1}{1+0}\right)s + \frac{1}{(1+s\vartheta)^2}\frac{s^2}{2}, \end{aligned} \tag{2.3.8}$$

where $\vartheta \in (0, 1)$. Therefore,

$$u^2(s) = \frac{s^2}{2}\frac{1}{(1+s\vartheta)^2} \tag{2.3.9}$$

with $0 < \vartheta < 1$. We interpret ϑ as a uniquely defined function of s, i.e., $\vartheta : s \mapsto \vartheta(s)$, such that

$$\frac{u(s)}{s} = \frac{1}{\sqrt{2}} \frac{1}{(1 + s\vartheta(s))} \tag{2.3.10}$$

is a positive continuous function for $s \in (-1, \infty)$ with the property

$$\begin{aligned} \left| \frac{u(s)}{s} - \frac{1}{\sqrt{2}} \right| &= \left| \frac{1}{\sqrt{2}} \left(\frac{1}{1 + s\vartheta(s)} - 1 \right) \right| = \frac{1}{\sqrt{2}} \left| \frac{s\vartheta(s)}{1 + s\vartheta(s)} \right| \\ &= \left| \frac{s\vartheta(s)u(s)}{s} \right| = |\vartheta(s)| \cdot |u(s)| \leq |u(s)|. \end{aligned} \tag{2.3.11}$$

From $u^2(s) = s - \ln(1 + s)$ follows that

$$2u \, \mathrm{d}u = \frac{s}{1 + s} \, \mathrm{d}s. \tag{2.3.12}$$

Obviously, $s : u \mapsto s(u), u \in \mathbb{R}$, is of class $\mathrm{C}^{(1)}(\mathbb{R})$ and thus,

$$I(x) = \int_{-1}^{\infty} (1 + s)^{x-1} \mathrm{e}^{-xs} \, \mathrm{d}s = 2 \int_{-\infty}^{+\infty} \mathrm{e}^{-xu^2} \frac{u}{s(u)} \, \mathrm{d}u. \tag{2.3.13}$$

We are able to deduce that

$$\begin{aligned} \left| I(x) - 2\sqrt{2} \int_0^{\infty} \mathrm{e}^{-xu^2} \, \mathrm{d}u \right| &= \left| 2 \int_{-\infty}^{\infty} \mathrm{e}^{-xu^2} \frac{u}{s(u)} \, \mathrm{d}u - \sqrt{2} \int_{-\infty}^{+\infty} \mathrm{e}^{-xu^2} \, \mathrm{d}u \right| \\ &= \left| 2 \int_{-\infty}^{\infty} \mathrm{e}^{-xu^2} \left(\frac{u}{s(u)} - \frac{1}{\sqrt{2}} \right) \mathrm{d}u \right| \\ &\leq 2 \int_{-\infty}^{\infty} \mathrm{e}^{-xu^2} \left| \frac{u}{s(u)} - \frac{1}{\sqrt{2}} \right| \mathrm{d}u \\ &\leq 2 \int_{-\infty}^{\infty} \mathrm{e}^{-xu^2} |u| \, \mathrm{d}u = 4 \int_0^{\infty} \mathrm{e}^{-xu^2} u \, \mathrm{d}u. \end{aligned} \tag{2.3.14}$$

Note that we can use the integral (2.2.16) with $\alpha = x$ and $x = 1$,

$$\int_0^{\infty} u^{1-1} \mathrm{e}^{-xu^2} \, \mathrm{d}u = \int_0^{\infty} \mathrm{e}^{-xu^2} \, \mathrm{d}u = \tfrac{1}{2} \, x^{-1/2} \Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2\sqrt{x}}, \tag{2.3.15}$$

as well as with $\alpha = x$ and $x = 2$ in (2.2.16)

$$\int_0^{\infty} u^{2-1} \mathrm{e}^{-xu^2} \, \mathrm{d}u = \int_0^{\infty} \mathrm{e}^{-xu^2} u \, \mathrm{d}u = \tfrac{1}{2} \, x^{-1} \Gamma(1) = \frac{1}{2x}. \tag{2.3.16}$$

This yields:

$$\left| I(x) - 2\sqrt{2}\,\frac{1}{2}\sqrt{\frac{\pi}{x}} \right| = \left| I(x) - \sqrt{\frac{2\pi}{x}} \right| \le 4\,\frac{1}{2x} = \frac{2}{x}. \tag{2.3.17}$$

This leads to the desired result. □

Remark 2.3.2. Stirling's formula can be rewritten in the form

$$\lim_{x\to\infty} \frac{\Gamma(x)}{\sqrt{2\pi}x^{x-1/2}\mathrm{e}^{-x}} = 1. \tag{2.3.18}$$

An immediate application is the limit relation

$$\lim_{x\to\infty} \frac{\Gamma(x+a)}{x^a\Gamma(x)} = 1, \quad a > 0. \tag{2.3.19}$$

This can be seen from Stirling's formula by

$$\lim_{x\to\infty} \frac{\Gamma(x+a)}{\sqrt{2\pi}(x+a)^{x+a-\frac{1}{2}}\mathrm{e}^{-x-a}} = 1 \tag{2.3.20}$$

due to the relation

$$(x+a)^{x+a-\frac{1}{2}} = x^{x+a-\frac{1}{2}}(1+\tfrac{a}{x})^{x+a-\frac{1}{2}} \tag{2.3.21}$$

and the limits

$$\lim_{x\to\infty} \frac{(1+\frac{a}{x})^x}{\mathrm{e}^a} = 1, \quad \lim_{x\to\infty}(1+\tfrac{a}{x})^{a-\frac{1}{2}} = 1. \tag{2.3.22}$$

Next, we prove the so-called *Legendre relation* or *duplication formula*.

Lemma 2.3.3 (Duplication Formula). *For $x > 0$ we have*

$$2^{x-1}\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right) = \sqrt{\pi}\,\Gamma(x). \tag{2.3.23}$$

Proof. We consider the function $x \mapsto \Phi(x)$, $x > 0$, defined by

$$\Phi(x) = \frac{2^{x-1}\Gamma(\frac{x}{2})\Gamma(\frac{x+1}{2})}{\Gamma(x)} \tag{2.3.24}$$

for $x > 0$. Setting $x + 1$ instead of x we find the following functional equation for the numerator

$$2^x\Gamma\left(\frac{x+1}{2}\right)\Gamma\left(\frac{x}{2}+1\right) = 2^{x-1}x\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right), \tag{2.3.25}$$

such that the numerator satisfies the same functional equation as the denominator. This means $\Phi(x+1) = \Phi(x), x > 0$. By repetition we get for all $n \in \mathbb{N}$ and x fixed $\Phi(x+n) = \Phi(x)$. We let n tend to ∞. For the numerator of $\Phi(x+n)$ we then find by use of the result in Remark 2.3.2, i.e., by using (2.3.19) twice, that

$$\lim_{n\to\infty} \frac{2^{x+n-1}\Gamma(\frac{x+n}{2})\Gamma(\frac{x+n+1}{2})}{2^{x+n-1}\left(\frac{n}{2}\right)^{\frac{x}{2}}\left(\frac{n}{2}\right)^{\frac{x+1}{2}}\left(\Gamma(\frac{n}{2})\right)^2} = 1. \tag{2.3.26}$$

For the denominator we consider

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\Gamma(x+n)}{2^{x+n-1}\left(\frac{n}{2}\right)^{\frac{x}{2}}\left(\frac{n}{2}\right)^{\frac{x+1}{2}}\left(\Gamma(\frac{n}{2})\right)^2} \\
&= \lim_{n\to\infty} \frac{\Gamma(x+n)}{2^{x+n-1}\left(\frac{n}{2}\right)^{x+\frac{1}{2}}(\frac{n}{2})^{n-1}\mathrm{e}^{-n}2\pi\frac{\left(\Gamma(\frac{n}{2})\right)^2}{(\frac{n}{2})^{n-1}\mathrm{e}^{-n}2\pi}} \\
&= \lim_{n\to\infty} \frac{\Gamma(x+n)}{2^{x+n}\left(\frac{n}{2}\right)^{n+x-\frac{1}{2}}\mathrm{e}^{-n}\pi}\left(\frac{\left(\Gamma(\frac{n}{2})\right)^2}{(\frac{n}{2})^{n-1}\mathrm{e}^{-n}2\pi}\right)^{-1} \\
&= \lim_{n\to\infty} \frac{\Gamma(x+n)}{\sqrt{2\pi}n^{n+x-\frac{1}{2}}\mathrm{e}^{-n}\sqrt{\pi}}\left(\frac{\left(\Gamma(\frac{n}{2})\right)^2}{(\frac{n}{2})^{n-1}\mathrm{e}^{-n}2\pi}\right)^{-1} \\
&= \frac{1}{\sqrt{\pi}},
\end{aligned} \tag{2.3.27}$$

since Stirling's formula yields that

$$\lim_{n\to\infty} \frac{\Gamma(\frac{n}{2})}{\sqrt{2\pi}(\frac{n}{2})^{\frac{n}{2}-\frac{1}{2}}\mathrm{e}^{-\frac{n}{2}}} = 1, \tag{2.3.28}$$

i.e.,

$$\lim_{n\to\infty} \frac{\left(\Gamma(\frac{n}{2})\right)^2}{2\pi(\frac{n}{2})^{n-1}\mathrm{e}^{-n}} = 1, \tag{2.3.29}$$

and by the same arguments as in Remark 2.3.2 (set a in (2.3.20) to x and x in (2.3.20) to n) we find that

$$\lim_{n\to\infty} \frac{\Gamma(x+n)}{\sqrt{2\pi}n^{n+x-\frac{1}{2}}\mathrm{e}^{-n}} = 1. \tag{2.3.30}$$

Therefore, we get for every $x > 0$ and all $n \in \mathbb{N}$,

$$\Phi(x) = \Phi(x+n) = \lim_{n\to\infty} \Phi(x+n) = \sqrt{\pi}. \tag{2.3.31}$$

A periodic function with this property must be constant. This proves the lemma. □

A generalization of the Legendre relation ("duplication formula") is the *Gauß multiplication formula* that can be verified by analogous arguments.

Lemma 2.3.4. *For $x > 0$ and $n \geq 2$,*

$$\Gamma\left(\frac{x}{n}\right)\Gamma\left(\frac{x+1}{n}\right)\cdot\ldots\cdot\Gamma\left(\frac{x+n-1}{n}\right)n^x = (2\pi)^{\frac{n-1}{2}}\sqrt{n}\,\Gamma(x). \quad (2.3.32)$$

2.4 Pochhammer's Factorial

Thus far, the Gamma function Γ is defined for positive values, i.e., $x \in \mathbb{R}_{>0}$. We are interested in an extension of Γ to the real line $\mathbb{R}$ (or even to the complex plane $\mathbb{C}$) if possible.

Definition 2.4.1. The so-called *Pochhammer factorial* $(x)_n$ with $x \in \mathbb{R}$ and $n \in \mathbb{N}$ is defined by

$$(x)_n = x(x+1)\ldots(x+n-1) = \prod_{i=1}^{n}(x+i-1). \quad (2.4.1)$$

For $x > 0$, it is clear that

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} \quad (2.4.2)$$

or

$$\frac{(x)_n}{\Gamma(x+n)} = \frac{1}{\Gamma(x)}. \quad (2.4.3)$$

The left-hand side is defined for $x > -n$ and gives the same value for all $n \in \mathbb{N}$ with $n > -x$. We may use this relation to define $\frac{1}{\Gamma(x)}$ for all $x \in \mathbb{R}$, and we see that this function vanishes for $x = 0, -1, -2, \ldots$ (see Fig. 2.3 (left)). We know that the Gamma integral is absolutely convergent for $x \in \mathbb{C}$ with $\mathrm{Re}(x) > 0$ and represents a holomorphic function for all $x \in \mathbb{C}$ with $\mathrm{Re}(x) > 0$. Moreover, the Pochhammer factorial $(x)_n$ can be defined for all complex x. Because of (2.4.3) with n chosen sufficiently large, we have a definition of $\frac{1}{\Gamma(x)}$ for all $x \in \mathbb{C}$. This is summarized in the following lemma.

Lemma 2.4.2. *The Γ-function is a meromorphic function that has simple poles in $0, -1, -2, \ldots$ (see Fig. 2.3 (right)). The reciprocal $x \mapsto \frac{1}{\Gamma(x)}$ is an entire analytic function (see Fig. 2.3 (left) for an illustration).*

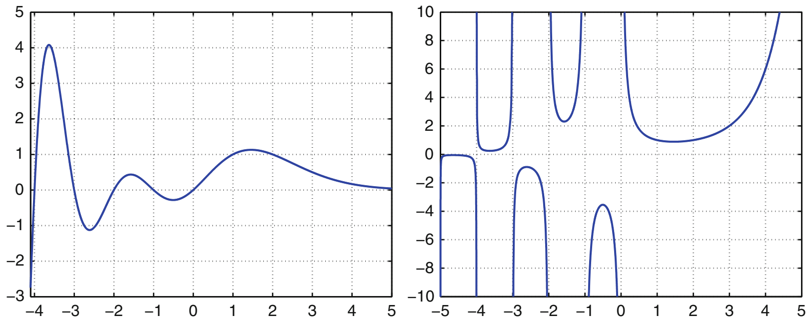

Fig. 2.3 The reciprocal of the Gamma function (*left*) and the Gamma function (*right*) on the real line $\mathbb{R}$

Lemma 2.4.3. *For $x \in \mathbb{C}$,*

$$\frac{1}{\Gamma(x)} = \lim_{n\to\infty} n^{-x} x \prod_{k=1}^{n-1} \left(1 + \tfrac{x}{k}\right). \tag{2.4.4}$$

Proof. Because of (2.4.3) the identity

$$\frac{(x)_n}{\Gamma(n)} \frac{\Gamma(n)}{\Gamma(x+n)} = \frac{1}{\Gamma(x)} \tag{2.4.5}$$

is valid for all $x \in \mathbb{C}$ with $\mathrm{Re}(x) > -n$. Furthermore it is easy to see that

$$\frac{(x)_n}{\Gamma(n)} = x \frac{(x+1)(x+2)\dots(x+n-1)}{1\cdot 2\dots(n-1)} = x \prod_{k=1}^{n-1} \left(1 + \frac{x}{k}\right). \tag{2.4.6}$$

Combining the two and multiplying with n^{-x} we find that

$$\frac{\Gamma(x+n)}{n^x \Gamma(n)\Gamma(x)} = n^{-x} x \prod_{k=1}^{n-1} \left(1 + \frac{x}{k}\right). \tag{2.4.7}$$

Now, we use (2.3.19) with $x = n$ and $a = x$, i.e.,

$$\lim_{n\to\infty} \frac{\Gamma(x+n)}{\Gamma(n) n^x} = 1, \quad x > 0, \tag{2.4.8}$$

on the left-hand side and obtain for $x > 0$,

$$\frac{1}{\Gamma(x)} = \lim_{n\to\infty} \frac{\Gamma(x+n)}{n^x \Gamma(n)} \frac{1}{\Gamma(x)} = \lim_{n\to\infty} n^{-x} x \prod_{k=1}^{n-1} \left(1 + \frac{x}{k}\right), \tag{2.4.9}$$

which proves Lemma 2.4.3 for $x > 0$. To determine if this limit is also defined for $x \leq 0$ we consider once again $s - \ln(1+s)$ (cf. Fig. 2.2) with $-1 < s < \infty$ from (2.3.9) in the proof of Theorem 2.3.1:

$$0 \leq s - \ln(1+s) = \frac{s^2}{2}\frac{1}{(1+\vartheta s)^2} \quad , \quad \vartheta = \vartheta(s) \in (0,1). \tag{2.4.10}$$

Therefore, we can put $s = \frac{1}{k}$ and estimate the right-hand side with its maximum, i.e., with $\vartheta = 0$:

$$0 \leq \tfrac{1}{k} - \ln\left(1 + \tfrac{1}{k}\right) \leq \tfrac{1}{2k^2}. \tag{2.4.11}$$

This immediately proves that $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \left(\frac{1}{k} - \ln\left(1+\frac{1}{k}\right)\right)$ exists and is positive. Moreover,

$$\begin{aligned}\lim_{n\to\infty} \sum_{k=1}^{n} \left(\tfrac{1}{k} - \ln\left(1+\tfrac{1}{k}\right)\right) &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\tfrac{1}{k} - \ln(k+1) + \ln(k)\right) \\ &= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \tfrac{1}{k} - \ln(n+1)\right) = \gamma,\end{aligned} \tag{2.4.12}$$

where γ denotes *Euler's constant*

$$\gamma = \lim_{m\to\infty} \left(\sum_{k=1}^{m-1} \frac{1}{k} - \ln m\right) \approx 0.577215665\ldots. \tag{2.4.13}$$

Assume now that $x \in \mathbb{R}$. If $k \geq 2|x|$, then

$$0 \leq \tfrac{x}{k} - \ln\left(1+\tfrac{x}{k}\right) < \tfrac{x^2}{k^2} \tag{2.4.14}$$

and

$$\prod_{k=1}^{n-1}\left(1+\frac{x}{k}\right) = \prod_{k=1}^{n-1}\left(1+\frac{x}{k}\right) e^{\frac{x}{k}} e^{-\frac{x}{k}} = \prod_{j=1}^{n-1} e^{\frac{x}{j}} \prod_{k=1}^{n-1}\left(1+\frac{x}{k}\right) e^{-\frac{x}{k}}. \tag{2.4.15}$$

For $k \geq k_0 = \lfloor 2|x| \rfloor$, we obtain by multiplying (2.4.14) with -1 and applying the exponential function to it:

$$1 \geq \left(1+\tfrac{x}{k}\right) e^{-\frac{x}{k}} > e^{-\frac{x^2}{k^2}}, \tag{2.4.16}$$

which shows that

$$\lim_{n\to\infty} \prod_{k=1}^{n-1} \left(1 + \tfrac{x}{k}\right) e^{-\frac{x}{k}} = \prod_{k=1}^{\infty} \left(1 + \tfrac{x}{k}\right) e^{-\frac{x}{k}} \tag{2.4.17}$$

exists for all x. Furthermore,

$$\begin{aligned} \prod_{j=1}^{n-1} e^{\frac{x}{j}} &= \exp\left(x \sum_{j=1}^{n-1} \tfrac{1}{j}\right) = \exp\left(x \sum_{j=1}^{n-1} \tfrac{1}{j} - x\ln(n) + x\ln(n)\right) \\ &= n^x \exp\left(x \sum_{j=1}^{n-1} \tfrac{1}{j} - x\ln(n)\right), \end{aligned} \tag{2.4.18}$$

where

$$\lim_{n\to\infty} \exp\left(x \sum_{j=1}^{n-1} \tfrac{1}{j} - x\ln(n)\right) = e^{\gamma x}. \tag{2.4.19}$$

Therefore, $\lim_{n\to\infty} n^{-x} x \prod_{k=1}^{n-1} \left(1 + \tfrac{x}{k}\right)$ exists for all $x \in \mathbb{R}$ and it holds that

$$\lim_{n\to\infty} n^{-x} x \prod_{k=1}^{n} \left(1 + \tfrac{x}{k}\right) = x e^{x\gamma} \prod_{k=1}^{\infty} \left(1 + \tfrac{x}{k}\right) e^{-\frac{x}{k}}, \tag{2.4.20}$$

where the infinite product is also convergent for all $x \in \mathbb{R}$. By similar arguments these results can be extended for all $x \in \mathbb{C}$. □

The proof of Lemma 2.4.3 also shows us the following lemma.

Lemma 2.4.4. *For $x \in \mathbb{C}$,*

$$\frac{1}{\Gamma(x)} = x e^{\gamma x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{k}\right) e^{-\frac{x}{k}}. \tag{2.4.21}$$

Let us consider the expression

$$Q(x) = \frac{1}{\pi} \Gamma(x)\Gamma(1-x)\sin(\pi x), \tag{2.4.22}$$

which has no singularities and is holomorphic for all $x \in \mathbb{C}$. It is not difficult to show that

$$Q(x) = \frac{1}{\pi x}\Gamma(1+x)\Gamma(1-x)\sin(\pi x) \tag{2.4.23}$$

$$= \frac{1}{\pi(1-x)}\Gamma(x)\Gamma(2-x)\sin(\pi x). \tag{2.4.24}$$

Obviously, by using (2.4.23) for $x = 0$ and (2.4.24) for $x = 1$, we obtain

$$Q(0) = \Gamma(1)\Gamma(1)\lim_{x\to 0}\frac{\sin(\pi x)}{\pi x} = 1, \tag{2.4.25}$$

$$Q(1) = \Gamma(1)\Gamma(1)\lim_{x\to 1}\frac{\sin(\pi x)}{\pi(1-x)} = 1. \tag{2.4.26}$$

In the interval [0, 1] the function Q is positive and twice continuously differentiable. With the duplication formula (Lemma 2.3.3) we get

$$Q\left(\frac{x}{2}\right)Q\left(\frac{x+1}{2}\right) = Q(x), \tag{2.4.27}$$

which is easily verified. Setting $R(x) = \ln(Q(x))$, we see that

$$R\left(\frac{x}{2}\right) + R\left(\frac{x+1}{2}\right) = R(x). \tag{2.4.28}$$

By differentiation we obtain

$$\frac{1}{4}R''\left(\frac{x}{2}\right) + \frac{1}{4}R''\left(\frac{x+1}{2}\right) = R''(x). \tag{2.4.29}$$

As the second order derivative R'' is continuous on the compact interval [0, 1], there is a value $\xi \in [0, 1]$ such that

$$|R''(\xi)| \geq |R''(x)|,\ \ x \in [0, 1]. \tag{2.4.30}$$

Therefore, we obtain from (2.4.29)

$$|R''(\xi)| \leq \frac{1}{4}\left|R''\left(\frac{\xi}{2}\right)\right| + \frac{1}{4}\left|R''\left(\frac{\xi+1}{2}\right)\right| \leq \frac{1}{2}|R''(\xi)|, \tag{2.4.31}$$

which implies $|R''(\xi)| = 0$, i.e., $R''(x) = 0$. From $R(1) = R(0) = 0$ we then deduce $R(x) = 0$. Therefore, $Q(x) = 1$. This result can be written in the form

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}. \tag{2.4.32}$$

It establishes an identity between the meromorphic functions $\Gamma(\cdot)$, $\Gamma(1-\cdot)$, and $(\sin \pi \cdot)^{-1}$. Altogether, we have

$$\frac{1}{\Gamma(x)} \frac{1}{\Gamma(1-x)} = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right). \tag{2.4.33}$$

In connection with (2.4.32) we obtain

Lemma 2.4.5. *For $x \in \mathbb{C}$,*

$$\sin(\pi x) = \pi x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right). \tag{2.4.34}$$

2.5 Exercises (Incomplete Gamma and Beta Function, Applications in Statistics)

In this section the discussion of the so-called incomplete Gamma functions and their relation to the error functions erf and erfc as well as the incomplete Beta function is left to the reader in the form of some exercises. The results demonstrate an immediate transition to probability distributions in statistics.

Incomplete Gamma Function

Definition 2.5.1. By definition, we let for $x, a > 0$,

$$\Gamma(a, x) = \int_x^{\infty} e^{-t} t^{a-1}\, dt \tag{2.5.1}$$

and

$$\gamma(a, x) = \Gamma(a) - \Gamma(a, x) = \int_0^x e^{-t} t^{a-1}\, dt. \tag{2.5.2}$$

The functions $\Gamma(\cdot, x)$, $\gamma(\cdot, x)$ are called the *incomplete Gamma functions* related to x.

Exercise 2.5.2. Prove that

$$\gamma(a, x) = x^a \int_0^1 e^{-xt} t^{a-1}\, dt, \tag{2.5.3}$$

$$\Gamma(a, x) = x^a e^{-x} \int_0^{\infty} e^{-xt}\, dt. \tag{2.5.4}$$

Exercise 2.5.3. Show that the so-called *error functions*

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2} \, \mathrm{d}t, \tag{2.5.5}$$

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \mathrm{e}^{-t^2} \, \mathrm{d}t \tag{2.5.6}$$

admit the representations

$$\operatorname{erf}(x) = \frac{1}{\sqrt{\pi}} \gamma(\tfrac{1}{2}, x^2), \tag{2.5.7}$$

$$\operatorname{erfc}(x) = \frac{1}{\sqrt{\pi}} \Gamma(\tfrac{1}{2}, x^2). \tag{2.5.8}$$

Exercise 2.5.4. Verify that

$$\Gamma(n+1, x) = n! \mathrm{e}^{-x} \sum_{m=0}^{n} \frac{x^m}{m!}, \tag{2.5.9}$$

$$\gamma(n+1, x) = n! \left(1 - \mathrm{e}^{-x} \sum_{m=0}^{n} \frac{x^m}{m!} \right) \tag{2.5.10}$$

hold true for all $n \in \mathbb{N}_0$.

Exercise 2.5.5. Prove the following recurrence relations

$$\gamma(a+1, x) = a\gamma(a, x) - x^a \mathrm{e}^{-x}, \tag{2.5.11}$$

$$\Gamma(a+1, x) = a\Gamma(a, x) + x^a \mathrm{e}^{-x}. \tag{2.5.12}$$

Incomplete Beta Function

Definition 2.5.6. The function $(x, y) \mapsto B(x, y, \alpha)$ defined by

$$B(x, y, \alpha) = \int_0^\alpha t^{x-1} (1-t)^{y-1} \, \mathrm{d}t \tag{2.5.13}$$

is called *incomplete Beta function* relative to $\alpha \in (0, 1]$.

Exercise 2.5.7. Show that

$$I(x, y, \alpha) = \frac{B(x, y, \alpha)}{B(x, y)} \tag{2.5.14}$$

satisfies

$$I(x, y, 1) = 1, \tag{2.5.15}$$

$$I(x, y, \alpha) = 1 - I(y, x, 1 - \alpha). \tag{2.5.16}$$

Exercise 2.5.8. Prove the recurrence relation

$$(x + y)I(x, y, \alpha) = xI(x + 1, y, \alpha) + yI(x, y + 1, \alpha). \tag{2.5.17}$$

Exercise 2.5.9. Verify the binomial expansion

$$I(m, n + 1 - m, \alpha) = \sum_{l=m}^{n} \binom{n}{l} \alpha^l (1 - \alpha)^{n-l}, \quad n, m \in \mathbb{N}. \tag{2.5.18}$$

As already announced, the incomplete Gamma and Beta functions possess a variety of applications in probability theory and statistics, from which we mention only two examples. We restrict ourselves to continuous random variables with non-negative realizations.

Definition 2.5.10 (Gamma Distribution). A random variable X with density distribution

$$F(x) = \begin{cases} 0 & , \text{ if } x \le 0, \\ \frac{\alpha^p}{\Gamma(p)} x^{p-1} \mathrm{e}^{-\alpha x} & , \text{ if } x > 0, \end{cases} \tag{2.5.19}$$

is called a *Gamma distribution* with $p > 0$ and $\alpha > 0$.

Clearly, $F(x) \ge 0$ for all $x \in \mathbb{R}$. An easy calculation gives

$$\int_{\mathbb{R}} F(x)\, \mathrm{d}x = 1. \tag{2.5.20}$$

The probability density function of the Gamma distribution reads as follows:

$$x \mapsto \int_0^x F(t)\, \mathrm{d}t = \frac{\alpha^p}{\Gamma(p)} \int_0^x t^{p-1} \mathrm{e}^{-\alpha t}\, \mathrm{d}t = \frac{1}{\Gamma(p)} \int_0^{\alpha x} t^{p-1} \mathrm{e}^{-t}\, \mathrm{d}t = \gamma(p, \alpha x). \tag{2.5.21}$$

The Gamma distribution is widely used as a conjugate prior in Bayesian statistics (for more details see, e.g., Papoulis and Pillai 2002).

Beta Distribution

The Beta distribution is a family of continuous probability distributions defined on the interval (0, 1) and parameterized by two positive values p and q.

Definition 2.5.11. A random variable X with density distribution

$$F(x) = \begin{cases} 0 & \text{, if } x \notin (0,1), \\ \frac{1}{B(p,q)} x^{p-1}(1-x)^{q-1} & \text{, if } x \in (0,1), \end{cases} \tag{2.5.22}$$

is called a *Beta distribution* with $p, q > 0$.

The probability density function of the Beta distribution reads as follows:

$$x \mapsto \int_0^x F(t)\,\mathrm{d}t = \frac{1}{B(p,q)} \int_0^x t^{p-1}(1-t)^{q-1}\,\mathrm{d}t = \frac{B(p,q,x)}{B(p,q)} \tag{2.5.23}$$

for $x \in (0,1)$. The expectation value and the variance of a Beta distributed random variable X corresponding to the parameters p and q are given by

$$\mu = \mathrm{E}(X) = \frac{p}{p+q}, \tag{2.5.24}$$

$$\mathrm{Var}(X) = \mathrm{E}(X - \mu^2) = \frac{pq}{(p+q)^2(p+q+1)}. \tag{2.5.25}$$

Beta distributions are often used in Bayesian inference, since Beta distributions provide a family of conjugate prior distributions for binomial distributions (more details can be found, e.g., in van der Waerden 1969).

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