Chapter 2
Global Attractors for Autonomous Evolution Equations

2.1 Kolmogorov ε-Entropy and Its Asymptotics in Functional Spaces

We start with the definition of Kolmogorov ε-entropy, via which we define the fractal dimension of a compact set in a metric space. We will use these two concepts in the sequel.

Definition 2.1 Let K be a (pre)compact set in a metric space M. Then, due to Hausdorff's criteria, it can be covered by a finite number of ε-balls in M. Let $N_\varepsilon(K, M)$ be the minimal number of ε-balls that cover K. Then, we can call Kolmogorov's ε-entropy of K the logarithm of this number;

$$\mathbb{H}_\varepsilon(K, M) := \log_2 N_\varepsilon(K, M).$$

We now give several examples of typical asymptotics for the ε-entropy.

Example 2.1 We assume that $K = [0, 1]^n$ and $M = \mathbb{R}^n$ (more generally, K is an n-dimensional compact Lipschitz manifold of the metric space M). Then

$$\mathbb{H}_\varepsilon(K, M) = \big(n + \overline{\overline{o}}(1)\big) \log_2 \frac{1}{\varepsilon} \quad \text{as } \varepsilon \to 0.$$

This example justifies the definition of the fractal dimension.

Definition 2.2 The fractal dimension $\dim_F(K, M)$ is defined as

$$\dim_F(K, M) := \limsup_{\varepsilon \to 0} \frac{\mathbb{H}_\varepsilon(K, M)}{\log_2 1/\varepsilon}.$$

Hence, for a compact n-dimensional Lipschitz manifold K in a metric space M, $\dim_F(K, M) = n$.

The following example shows that, for sets that are not manifolds, the fractal dimension may be a non-integer.

M. Efendiev, *Evolution Equations Arising in the Modelling of Life Sciences*, International Series of Numerical Mathematics 163,
DOI 10.1007/978-3-0348-0615-2_2,

Example 2.2 Let K be a standard ternary Cantor set in $M = [0, 1]$. Then $\dim_F(K, M) = \frac{\ln 2}{\ln 3} < 1$.

Proof Let K be the Cantor set obtained from the segment $[0, 1]$ by the sequential removal of the centre thirds. First we remove all the points between $1/3$ and $2/3$. Then the centre thirds $(1/9, 2/9)$ and $(7/9, 8/9)$ of the remaining segments $[0, 1/3]$ and $[2/3, 1]$ are deleted. After that the centre parts $(1/27, 2/27)$, $(7/27, 8/27)$, $(19/27, 20/27)$ and $(25/27, 26/27)$ of the four remaining segments $[0, 1/9]$, $[2/9, 1/3]$, $[2/3, 7/9]$ and $[8/9, 1]$, respectively, are deleted. If we continue this process to infinity, it will lead to the standard Cantor set K. Next we calculate its fractal dimension. We emphasize that $K = \bigcap_{M=0}^{\infty} \theta_m$, where $\theta_0 = [0, 1]$, $\theta_1 = [0, 1/3] \cup [2/3, 1]$, $\theta_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$ and so on. Each of the sets θ_m can be considered as a union of 2^m segments of length 3^{-m}. In particular, the cardinality of the covering of the set K with segments of length 3^{-m} is equal to 2^m. Consequently $\dim_F(K, [0, 1]) = \lim_{m\to\infty} \frac{\ln 2^m}{\ln 3^m} = \frac{\ln 2}{\ln 3}$. It is not difficult to show that

(1) if $K_1 \subseteq K_2$, then $\dim_F(K_1, M) \le \dim_F(K_2, M)$,
(2) $\dim_F(K_1 \cup K_2, M) \le \max\{\dim_F(K_1, M);\ \dim_F(K_2, M)\}$,
(3) $\dim_F(K_1 \times K_2, M \times M) \le \dim_F(K_1, M) + \dim_F(K_2, M)$,
(4) let g be a Lipschitzian mapping of one metric space M_1 into another M_2. Then $\dim_F(g(K), M_2) \le \dim_F(K, M_1)$. □

The next example gives the typical behaviour of the entropy in classes of functions with finite smoothness.

Example 2.3 Let V be a smooth bounded domain of $\mathbb{R}^n$ and let K be the unit ball in the Sobolev space $W^{l_1,p_1}(V)$ and M be another Sobolev space $W^{l_2,p_2}(V)$ such that the embedding $W^{l_1,p_1} \subset W^{l_2,p_2}$ is compact, i.e.

$$l_1 > l_2 \ge 0, \quad \frac{l_1}{n} - \frac{1}{p_1} > \frac{l_2}{n} - \frac{1}{p_2}.$$

Then, the entropy $\mathbb{H}_\varepsilon(K, M)$ has the following asymptotics (see [128]):

$$C_1\left(\frac{1}{\varepsilon}\right)^{n/(l_1-l_2)} \le \mathbb{H}_\varepsilon(K, M) \le C_2\left(\frac{1}{\varepsilon}\right)^{n/(l_1-l_2)}.$$

Finally, the last example shows the typical behaviour of the entropy in classes of analytic functions.

Example 2.4 Let $V_1 \subset V_2$ be two bounded domains of $\mathbb{C}^n$. We assume that K is the set of all analytic functions ϕ in V_2 such that $\|\phi\|_{C(V_2)} \le 1$ and that $M = C(V_1)$. Then

$$C_1(\log_2 1/\varepsilon)^{n+1} \le \mathbb{H}_\varepsilon(K|_{V_1}, M) \le C_2(\log_2 1/\varepsilon)^{n+1}$$

(see [73]).

2.2 Global Attractors and Finite-Dimensional Reduction

It is well-known that one of the main concepts of the modern theory of DS in infinite dimensions is that of a *global attractor*. We give below its definition for an abstract semigroup $S(t)$ acting on a metric space Φ, although, without loss of generality, the reader may think that $(S(t), \Phi)$ is just a DS associated with one of the PDEs described in the introduction.

To this end, we first recall that a subset K of the phase space Φ is an attracting set of the semigroup $S(t)$ if it attracts the images of all the *bounded* subsets of Φ, i.e., for every bounded set B and every $\varepsilon > 0$, there exists a time T (depending in general on B and ε) such that the image $S(t)B$ belongs to the ε-neighbourhood of K if $t \geq T$. This property can be rewritten in the equivalent form

$$\lim_{t\to\infty} \operatorname{dist}_H\big(S(t)B, K\big) = 0,$$

where $\operatorname{dist}_H(X, Y) := \sup_{x\in\mathbb{X}} \inf_{y\in\mathbb{Y}} d(x, y)$ is the non-symmetric Hausdorff distance between subsets of Φ.

We now give the definition of a global attractor, following Babin-Vishik (see [13, 23, 41, 124]).

Definition 2.3 A set $\mathcal{A} \subset \Phi$ is a global attractor for the semigroup $S(t)$ if

(1) $\mathcal{A}$ is *compact* in Φ;
(2) $\mathcal{A}$ is *strictly invariant*: $S(t)\mathcal{A} = \mathcal{A}$, for all $t \geq 0$;
(3) $\mathcal{A}$ is an *attracting* set for the semigroup $S(t)$.

Thus, the second and third properties guarantee that a global attractor, if it exists, is unique and that the DS reduced to the attractor contains all the nontrivial dynamics of the initial system. Furthermore, the first property indicates that the reduced phase space $\mathcal{A}$ is indeed "thinner" than the initial phase space Φ (we recall that, in infinite dimensions, a compact set cannot contain, e.g., balls and should thus be nowhere dense).

In most applications, one can use the following attractor's existence theorem.

Theorem 2.1 *Let a DS $(S(t), \Phi)$ possess a compact attracting set and the operators $S(t) : \Phi \to \Phi$ be continuous for every fixed t. Then, this system possesses the global attractor $\mathcal{A}$ which is generated by all the trajectories of $S(t)$ which are defined for all $t \in \mathbb{R}$ and are globally bounded.*

The strategy for applying this theorem to concrete equations of mathematical physics is the following. In a first step, one verifies a so-called *dissipative* estimate which has usually the form

$$\big\|S(t)u_0\big\|_\Phi \leq Q\big(\|u_0\|_\Phi\big)e^{-\alpha t} + C_*, \quad u_0 \in \Phi, \tag{2.1}$$

where $\|\cdot\|_\Phi$ is a norm in the function space Φ and the positive constants α and C_* and the monotonic function Q are independent of t and $u_0 \in \Phi$ (usually, this estimate follows from energy estimates and is sometimes even used in order to "define" a dissipative system). This estimate obviously gives the existence of an attracting set for $S(t)$ (e.g., the ball of radius $2C_*$ in Φ), which is however *non-compact* in Φ. In order to overcome this problem, one usually derives, in a second step, a *smoothing* property for the solutions, which can be formulated as follows:

$$\|S(1)u_0\|_{\Phi_1} \leq Q_1\big(\|u_0\|_\Phi\big), \quad u_0 \in \Phi, \tag{2.2}$$

where Φ_1 is another function space which is *compactly* embedded into Φ. In applications, Φ is usually the space $L^2(\Omega)$ of square integrable functions, Φ_1 is the Sobolev space $H^1(\Omega)$ of the functions u such that u and $\nabla_x u$ belong to $L^2(\Omega)$ and estimate (2.2) is a classical smoothing property for solutions of parabolic equations (for parabolic equations in unbounded domains and for hyperbolic equations, a slightly more complicated *asymptotic* smoothing property should be used instead of (2.2), see Sect. 3.2 of the monograph [41] and the references therein.

Since the continuity of the operators $S(t)$ usually poses no difficulty (if the uniqueness is proven), then the above scheme gives indeed the existence of the global attractor for most of the PDE of mathematical physics in bounded domains.

Remark 2.1 As was shown in [6] the assumption that $S(t) : \Phi \to \Phi$ is continuous for every fixed t can be replaced by the closedness of the graph $\{(u_0, S(t)u_0), u_0 \in \Phi\}$.

Remark 2.2 Although the global attractor has usually a very complicated geometric structure, there exists one exceptional class of DS for which the global attractor has a relatively simple structure which is completely understood, namely the DS having a global Lyapunov function. We recall that a continuous function $\mathcal{L} : \Phi \to \mathbb{R}$ is a global Lyapunov function if

(1) $\mathcal{L}$ is non-increasing along the trajectories, i.e. $\mathcal{L}(S(t)u_0) \leq \mathcal{L}(u_0)$, for all $t \geq 0$;
(2) $\mathcal{L}$ is *strictly* decreasing along all non-equilibrium solutions, i.e. $\mathcal{L}(S(t)u_0) = \mathcal{L}(u_0)$ for some $t > 0$ and u_0 implies that u_0 is an equilibrium of $S(t)$.

It is well-known that, if a DS possesses a global Lyapunov function, then, at least under the generic assumption that the set $\mathcal{R}$ of equilibria is finite, every trajectory $u(t)$ *stabilizes* to one of these equilibria as $t \to +\infty$. Moreover, every complete bounded trajectory $u(t)$, $t \in \mathbb{R}$, belonging to the attractor is a heteroclinic orbit joining two equilibria. Thus, the global attractor $\mathcal{A}$ can be described as follows [6, 79]:

$$\mathcal{A} = \bigcup_{u_0 \in \mathcal{R}} \mathcal{M}^+(u_0),$$

where $\mathcal{M}^+(u_0)$ is the so-called unstable set of the equilibrium u_0 (which is generated by all heteroclinic orbits of the DS which start from the given equilibrium $u_0 \in \mathcal{A}$). It is also known that, if the equilibrium u_0 is *hyperbolic* (generic assumption [6]), then the set $\mathcal{M}^+(u_0)$ is a κ-dimensional submanifold of Φ, where κ is the instability index of u_0. Thus, under the generic hyperbolicity assumption on the equilibria, the attractor $\mathcal{A}$ of a DS having a global Lyapunov function is a finite union of smooth finite-dimensional submanifolds of the phase space Φ. These attractors are called *regular* (following Babin-Vishik (see [13]).

It is also worth emphasizing that, in contrast to general global attractors, regular attractors are robust under perturbations. Moreover, in some cases, it is also possible to verify the so-called *transversality* conditions (for the intersection of stable and unstable manifolds of the equilibria) and, thus, verify that the DS considered is a Morse-Smale system. In particular, this means that the dynamics restricted to the regular attractor $\mathcal{A}$ are also preserved (up to homeomorphisms) under perturbations.

In the sequel we will apply Theorem 2.1 or Remark 2.1 (whenever it will be necessary) to a class of PDEs arising in mathematical physics. We especially emphasize that one of the challenging questions in the theory of attractors is, in which sense are the dynamics on the global attractor finite-dimensional? As already mentioned, the global attractor is usually not a manifold, but has a rather complicated geometric structure. So, it is natural to use the definitions of dimensions adopted for the study of fractal sets here. We restrict ourselves to the so-called fractal (or box-counting, entropy) dimension, although other dimensions (e.g., Hausdorff, Lyapunov, etc.) are also used in the attractors' theory. Here the so-called Mané theorem (which can be considered as a generalization of the classical Whitney embedding theorem for fractal sets) plays an important role in the finite-dimensional reduction theory (see [124]).

Theorem 2.2 *Let Φ be a Banach space and $\mathcal{A}$ be a compact set such that $d_f(\mathcal{A}) < N$ for some $N \in \mathbb{N}$. Then, for "almost all" $2N+1$-dimensional planes L in Φ, the corresponding projector $\Pi_L : \Phi \to L$ restricted to the set $\mathcal{A}$ is a Hölder continuous homeomorphism.*

Thus, if the finite fractal dimensionality of the attractor is established, then, fixing a hyperplane L satisfying the assumptions of the Mané theorem and projecting the attractor $\mathcal{A}$ and the DS $S(t)$ restricted to $\mathcal{A}$ onto this hyperplane ($\bar{\mathcal{A}} := \Pi_L \mathcal{A}$ and $\bar{S}(t) := \Pi_L \circ S(t) \circ \Pi_L^{-1}$), we obtain indeed a reduced DS $(\bar{S}(t), \bar{\mathcal{A}})$ which is defined on a finite-dimensional set $\bar{\mathcal{A}} \subset L \sim \mathbb{R}^{2N+1}$. Moreover, this DS will be Hölder continuous with respect to the initial data.

Remark 2.3 Note that, good estimates on the dimension of the attractors in terms of the physical parameters are crucial for the finite-dimensional reduction described above and (consequently) there exists a highly developed machinery for obtaining such estimates. The best known upper estimates are usually obtained by the so-called volume contraction method which is based on the study of the evolution of infinitesimal k-dimensional volumes in the neighbourhood of the attractor (and, if the DS considered contracts the k-dimensional volumes, then the fractal dimension of the attractor is less than k (see [13, 124])).

Remark 2.4 Lower bounds on the dimension are usually based on the observation that the global attractor always contains the unstable manifolds of the (hyperbolic) equilibria. Thus, the instability index of a properly constructed equilibrium gives a lower bound on the dimension of the attractor (see [13, 41, 124]).

The following Theorem 2.3 plays the decisive role in the study of the dimension of an attractor, which in turn does not require differentiability of the associated semigroup in contrast to (see [13, 23, 124]). We especially emphasize that for a quite large class of degenerate parabolic systems arising in the modelling of life science problems (see Chap. 5 of this book) the associate semigroup is not differentiable. We denote this by $S := S(1)$.

Theorem 2.3 *Let H_1 and H be Banach spaces, H_1 be compactly embedded in H and let $K \Subset H$. Assume that there exists a map $S : K \to K$, such that $S(K) = K$ and that the 'smoothing' property*

$$\big\| S(k_1) - S(k_2) \big\|_{H_1} \le C \| k_1 - k_2 \|_H \tag{2.3}$$

is valid for every $k_1, k_2 \in K$. Then the fractal dimension of K in H is finite and can be estimated in the following way:

$$d_F(K, H) \le \mathbb{H}_{1/4C}\big(B(1, 0, H_1), H\big) \tag{2.4}$$

where C is the same as in (2.3) and $B(1, 0, H_1)$ denotes the unit ball in the space H_1.

Proof Let $\{B(\varepsilon, k_i, H)\}_{i=1}^{N_\varepsilon}$, $k_i \in K$, be some ε-covering of the set K (here and below we denote by $B(\varepsilon, k, V)$ the ε-ball in the space V, centred in k). Then according to (2.3), the system $\{B(C\varepsilon, L(k_i), H_i)\}_{i=1}^{N_\varepsilon}$ of $C\varepsilon$-balls in H_1 covers the set $S(K)$ and consequently (since $S(K) = K$) the same system covers the set K.

Cover now every H_1-ball with radius $C\varepsilon$ by a finite number of $\frac{\varepsilon}{4}$-balls in H. By definition, the minimal number of such balls equals to

$$\begin{aligned} N_{\varepsilon/4}\big(B\big(C\varepsilon, S(k_i), H_1\big), H\big) &= N_{\varepsilon/4}\big(B(C\varepsilon, 0, H_1), H\big) \\ &= N_{1/4C}\big(B(1, 0, H_1), H\big) \equiv \mathcal{N}. \end{aligned}$$

Note, that the centres of the $\varepsilon/4$-covering thus obtained do not necessarily belong to K but we evidently can construct the $\varepsilon/2$-covering with centres in K and with the same number of balls.

Thus, having the initial ε-covering of K in H with the number of balls N_ε we have constructed the $\varepsilon/2$-covering with the number of balls $N_{\varepsilon/2} = \mathcal{N} N_\varepsilon$.

Consequently, the ε-entropy of the set K possesses the following estimate.

In fact the assertion of the theorem is a corollary of this recurrent estimate. Indeed, since $K \Subset H$ then there exists ε_0 such that $K \subset B(\varepsilon_0, k_0, H)$ and consequently

$$\mathbb{H}_{\varepsilon_0}(K, N) = 0. \tag{2.5}$$

Iterating the estimate (2.5) n-times we obtain that

$$\mathbb{H}_{\varepsilon_0/2^n}(k, H) \le n \log_2 \mathcal{N}.$$

Fix now an arbitrary $\varepsilon > 0$ and choose $n = n(\varepsilon)$ in such a way that

$$\frac{\varepsilon_0}{2^n} \le \varepsilon \le \frac{\varepsilon_0}{2^n - 1}.$$

Then

$$\mathbb{H}_{\varepsilon}(K) \le \mathbb{H}_{\varepsilon/2^n}(K) \le n \log_2 \mathcal{N} \le \log_2\left(\frac{2\varepsilon_0}{\varepsilon}\right) \log_2 \mathcal{N}. \tag{2.6}$$

Thus (2.4) is an immediate consequence of (2.6). Theorem 2.3 is proved. □

http://www.springer.com/978-3-0348-0614-5

Evolution Equations Arising in the Modelling of Life Sciences
Efendiev, M.
2013, XII, 217 p., Hardcover
ISBN: 978-3-0348-0614-5
A product of Birkhäuser Basel