Chapter 2
Stochastic Functional Differential Equations and Procedure of Constructing Lyapunov Functionals

2.1 Short Introduction to Stochastic Functional Differential Equations

Here the basic notation of the theory of stochastic differential equations [16, 17, 81, 84–87, 210] is considered.

Let $\{\Omega, F, \mathbf{P}\}$ be a probability space, $\{F_t, t \geq 0\}$ be a nondecreasing family of sub-σ-algebras of F, i.e., $F_{t_1} \subset F_{t_2}$ for $t_1 < t_2$, $\mathbf{P}\{\cdot\}$ be the probability of an event enclosed in the braces, and $\mathbf{E}$ be the mathematical expectation.

2.1.1 Wiener Process and Its Numerical Simulation

The Wiener process sometimes is also called the Brownian motion process. Originally, the Brownian motion process was posed by the English botanist Robert Brown as a model for the motion of a small particle immersed in a liquid and thus subject to molecular collisions. The Brownian motion assumes a central role in the theory of stochastic processes and statistics. It is basic to descriptions of financial markets, the construction of a large class of Markov processes called diffusions, approximations to many queuing models, and the calculation of asymptotic distributions in large sample statistical estimation problems.

Definition 2.1 A stochastic process $w(t)$ is called the standard Wiener process (relatively to the family $\{F_t, t \geq 0\}$) if it is F_t-measurable and

- $w(0) = 0$ ($\mathbf{P}$-a.s.);
- $w(t)$ is a process with stationary and mutually independent increments;
- the increments $w(t) - w(s)$ have the normal distribution with

$$\mathbf{E}\big(w(t) - w(s)\big) = 0, \quad \mathbf{E}\big(w(t) - w(s)\big)^2 = |t - s|;$$

- for almost all $\omega \in \Omega$, the functions $w(t) = w(t, \omega)$ are continuous on $t \geq 0$ [84].

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We will also consider an m-dimensional Wiener process

$$w(t) = \big(w_1(t), \dots, w_m(t)\big)' \in \mathbf{R}^m$$

where the components $w_i(t)$, $i = 1, \dots, m$, are mutually independent scalar Wiener processes, and

$$\mathbf{E}\big(w(t) - w(s)\big) = 0, \quad \mathbf{E}\big(w(t) - w(s)\big)\big(w(t) - w(s)\big)' = I|t - s|.$$

Here I is the $m \times m$ identity matrix, and the prime denotes the transposition.

The trajectories of the Wiener process are nondifferentiable functions, although formally the derivative of the Wiener process $\dot{w}(t)$ is called the white noise.

There are different ways to get numerical simulation of trajectories of a Wiener process. One of them is the following [244].

Let Y_i, $i = 1, \dots, n$, be independent random variables that are uniformly distributed on $[0, 1]$. Then $X_i = \sqrt{12}(Y_i - 0.5) = \sqrt{3}(2Y_i - 1)$, $i = 1, \dots, n$, are independent identically distributed random variables such that $\mathbf{E}X_i = 0$ and $\mathrm{Var}(X_i) = 1$. Define the random walk S_n, $n \ge 0$, by $S_0 = 0$ and $S_n = X_1 + \dots + X_n$ for $n > 0$. By the central limit theorem, $\frac{1}{\sqrt{n}} S_n$ converges in distribution to $N(0, 1)$, i.e., $\frac{1}{\sqrt{n}} S_n \to N(0, 1)$.

Define the continuous-time process $W_n(t) = \frac{1}{\sqrt{n}} S_{[nt]}$, $t \ge 0$, where $[t]$ is the integer part of t, i.e., the greatest integer less than or equal to t. Therefore, for any $t > 0$, we have

$$W_n(t) = \sqrt{\frac{[nt]}{n}} \frac{S_{[nt]}}{\sqrt{[nt]}} \to N(0, t).$$

Also, for $t > s$, we obtain

$$\begin{aligned} W_n(t) - W_n(s) &= \frac{S_{[nt]} - S_{[ns]}}{\sqrt{n}} = \frac{\sum_{j=[ns]+1}^{[nt]} X_j}{\sqrt{n}} = \frac{S_{[nt]-[ns]}}{\sqrt{n}} \\ &= \sqrt{\frac{[nt] - [ns]}{n}} \frac{S_{[nt]-[ns]}}{\sqrt{[nt] - [ns]}} \to N(0, t - s). \end{aligned}$$

Since the process $\{W_n(t), t \ge 0\}$ is not continuous, let us modify it in the following way:

$$W_n^{(c)}(t) = \frac{S_{[nt]}}{\sqrt{n}} + \big(nt - [nt]\big) \frac{X_{[nt]+1}}{\sqrt{n}}, \quad t \ge 0.$$

It is easy to see that $\mathbf{E}W_n^{(c)}(t) = 0$ and

$$\lim_{n \to \infty} \mathrm{Var}\big(W_n^{(c)}(t)\big) = \lim_{n \to \infty} \left(t - \frac{nt - [nt]}{n} + \frac{(nt - [nt])^2}{n} \right) = t.$$

So, as $n \to \infty$, $W_n^{(c)}(t)$ converges in distribution to the Wiener process $w(t)$. This means that for large enough n, the process $W_n^{(c)}(t)$ approximates the Wiener process

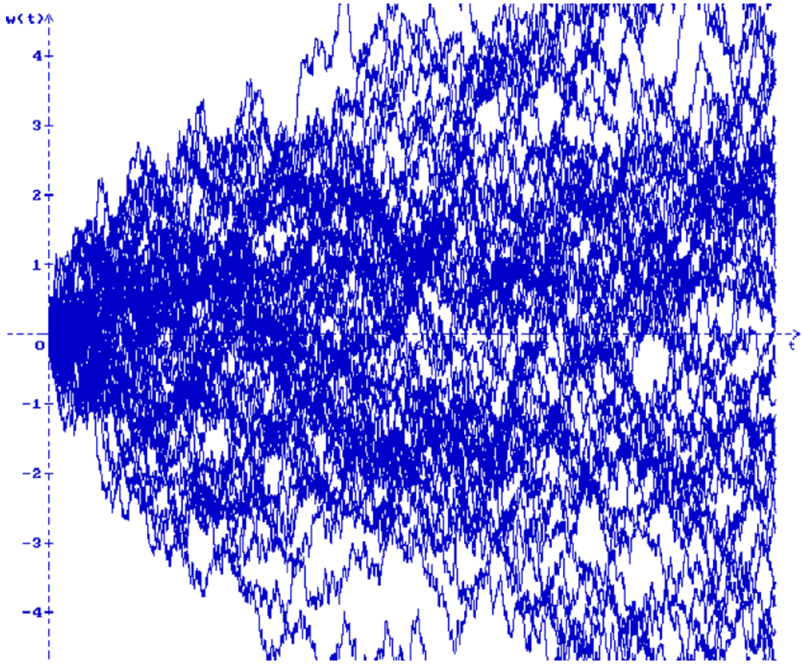

Fig. 2.1 50 trajectories of the Wiener process

well enough. 50 trajectories of the Wiener process obtained via this algorithm are shown in Fig. 2.1.

2.1.2 Itô Integral, Itô Stochastic Differential Equation, and Itô Formula

Let $H_2[0, T]$ be the space of random functions $f(t)$ that are defined and F_t-measurable for each $t \in [0, T]$ and for which

$$\int_0^t \mathbf{E} f^2(s)\, ds < \infty.$$

Then for all functions from $H_2[0, T]$, the Itô integral with respect to the Wiener process $w(t)$

$$\int_0^t f(s)\, dw(s)$$

is defined and has the following properties:

$$\mathbf{E}\int_0^t f(s)\,dw(s) = 0, \quad \mathbf{E}\left(\int_0^t f(s)\,dw(s)\right)^2 = \int_0^t \mathbf{E}f^2(s)\,ds.$$

Let H_p, $p > 0$, be the space of F_0-adapted stochastic processes $\varphi(\theta)$, $\theta \le 0$, with continuous trajectories that are independent on the σ-algebra $B_{[0,\infty)}(dw)$, where $B_{[t_1,t_2]}(dw)$ is the minimal σ-algebra generated by the random variables $w(s) - w(t)$ for arbitrary $s, t : t_1 \le t \le s \le t_2$. In the space H_p two norms are defined: $\|\varphi\|_0 = \sup_{s\le 0} |\varphi(s)|$ and $\|\varphi\|_1^p = \sup_{s\le 0} \mathbf{E}|\varphi(s)|^p$.

We will consider the Itô stochastic functional differential equation [84]

$$\begin{aligned} &dx(t) = a_1(t, x_t)\,dt + a_2(t, x_t)\,dw(t), \\ &t \ge 0,\ x \in \mathbf{R}^n, \end{aligned} \tag{2.1}$$

with the initial condition

$$x_0 = \phi \in H_p. \tag{2.2}$$

Here $x \in \mathbf{R}^n$, $x_t = x(t+s)$, $s \le 0$, $w : [0,\infty) \to \mathbf{R}^m$ is the standard Wiener process, the continuous functionals $a_1(t,\varphi)$, $a_2(t,\varphi)$ are defined on $[0,\infty) \times H_p$, $a_1 \in \mathbf{R}^n$, a_2 is an $n \times m$-dimensional matrix. It is assumed also that the functionals a_i, $i = 1, 2$, satisfy the following conditions: $a_i(t, 0) \equiv 0$, and for arbitrary functions $\varphi_1(\theta)$, $\varphi_2(\theta)$ from H_p,

$$\begin{aligned} &\left|a_i(t,\varphi_1) - a_i(t,\varphi_2)\right|^2 \le \int_0^\infty \left|\varphi_1(-\theta) - \varphi_2(-\theta)\right|^2 dR_i(\theta), \\ &\int_0^\infty dR_i(\theta) < \infty, \quad i = 1, 2, \end{aligned} \tag{2.3}$$

where $R_i(\theta)$ are nondecreasing bounded functions.

A solution of problem (2.1)–(2.2) is a process $x(t)$ such that $x(\theta) = \phi(\theta)$ for $\theta \le 0$ and with probability 1

$$\begin{aligned} &x(t) = x(0) + \int_0^t a_1(s, x_s)\,ds + \int_0^t a_2(s, x_s)\,dw(s), \quad t \ge 0, \\ &x(0) = \phi(0). \end{aligned}$$

The last integral is understood in the Itô sense.

Sometimes, instead of $x(t)$, we will write $x(t,\phi)$ for the solution of (2.1) with the initial function (2.2). Existence and uniqueness theorems for problem (2.1)–(2.2) are considered in [84–87, 132, 133].

To calculate the stochastic differential of the process $\eta(t) = u(t, x(t))$, where $x(t)$ is a solution of problem (2.1)–(2.2), and the function $u : [0,\infty) \times \mathbf{R}^n \to \mathbf{R}$ has continuous partial derivatives

$$u_t = \frac{\partial u(t,x)}{\partial t}, \qquad \nabla u = \left(\frac{\partial u(t,x)}{\partial x_1}, \dots, \frac{\partial u(t,x)}{\partial x_n}\right),$$
$$\nabla^2 u = \left(\frac{\partial^2 u(t,x)}{\partial x_i \partial x_j}\right), \quad i, j = 1, \dots, n,$$

the following Itô formula [84] is used:

$$d\eta(t) = Lu\big(t, x(t)\big)\,dt + \nabla u'\big(t, x(t)\big) a_2(t, x_t)\,dw(t). \tag{2.4}$$

The operator L is called the generator of (2.1) and is defined in the following way:

$$\begin{aligned} Lu\big(t, x(t)\big) = u_t\big(t, x(t)\big) + \nabla u'\big(t, x(t)\big) a_1(t, x_t) \\ + \frac{1}{2}\operatorname{Tr}\big[a_2'(t, x_t)\nabla^2 u\big(t, x(t)\big) a_2(t, x_t)\big], \end{aligned} \tag{2.5}$$

where Tr denotes the trace of a matrix.

The generator L can be applied also for some functionals $V(t, \varphi) : [0, \infty) \times H_p \to \mathbf{R}_+$. Suppose that a functional $V(t, \varphi)$ can be represented in the form $V(t, \varphi) = V(t, \varphi(0), \varphi(\theta))$, $\theta < 0$, and for $\varphi = x_t$, put

$$\begin{aligned} &V_\varphi(t, x) = V(t, \varphi) = V(t, x_t) = V\big(t, x, x(t+\theta)\big), \\ &x = \varphi(0) = x(t), \ \theta < 0. \end{aligned} \tag{2.6}$$

Denote by D the set of the functionals for which the function $V_\varphi(t, x)$ defined by (2.6) has a continuous derivative with respect to t and two continuous derivatives with respect to x. For functionals from D, the generator L of (2.1) has the form

$$\begin{aligned} LV(t, x_t) = \frac{\partial V_\varphi(t, x(t))}{\partial t} + \nabla V_\varphi'\big(t, x(t)\big) a_1(t, x_t) \\ + \frac{1}{2}\operatorname{Tr}\big[a_2'(t, x_t)\nabla^2 V_\varphi\big(t, x(t)\big) a_2(t, x_t)\big]. \end{aligned} \tag{2.7}$$

From the Itô formula it follows that for functionals from D,

$$\mathbf{E}\big[V(t, x_t) - V(s, x_s)\big] = \int_s^t \mathbf{E}LV(\tau, x_\tau)\,d\tau, \quad t \geq s. \tag{2.8}$$

Together with (2.1), we will also consider the stochastic differential equation of neutral type [134]

$$\begin{aligned} &d\big(x(t) - G(t, x_t)\big) = a_1(t, x_t)\,dt + a_2(t, x_t)\,dw(t), \quad t \geq 0, \\ &x_0 = \phi \in H_p, \end{aligned} \tag{2.9}$$

with the additional conditions on the functional $G(t,\varphi)$: $G(t,0)=0$,

$$\big|G(t,\varphi)\big| \le \int_0^\infty \big|\varphi(-s)\big|\,dK(s), \qquad \int_0^\infty dK(s) < 1. \tag{2.10}$$

2.2 Stability of Stochastic Functional Differential Equations

2.2.1 Definitions of Stability and Basic Lyapunov-Type Theorems

Definition 2.2 The solution $x(t)$ of (2.1) with the initial function (2.2) for some $p>0$ is called:

- Uniformly p-bounded if $\sup_{t\ge 0}\mathbf{E}|x(t)|^p<\infty$.
- Asymptotically p-trivial if $\lim_{t\to\infty}\mathbf{E}|x(t)|^p=0$.
- p-integrable if $\int_0^\infty \mathbf{E}|x(t)|^p\,dt<\infty$.

Definition 2.3 The trivial solution of (2.1) for some $p>0$ is called:

- p-stable if for each $\varepsilon>0$, there exists $\delta>0$ such that $\mathbf{E}|x(t,\phi)|^p<\varepsilon$, $t\ge 0$, provided that $\|\phi\|_1^p<\delta$.
- Asymptotically p-stable if it is p-stable and for each initial function ϕ, the solution $x(t)$ of (2.1) is asymptotically p-trivial.
- Exponentially p-stable if it is p-stable and there exists $\lambda>0$ such that for each initial function ϕ, there exists $C>0$ (which may depend on ϕ) such that $\mathbf{E}|x(t,\phi)|^p\le Ce^{-\lambda t}$ for $t>0$.
- Stable in probability if for any $\varepsilon_1>0$ and $\varepsilon_2>0$, there exists $\delta>0$ such that the solution $x(t,\phi)$ of (2.1) satisfies the condition $\mathbf{P}\{\sup_{t\ge 0}|x(t,\phi)|>\varepsilon_1/F_0\}<\varepsilon_2$ for any initial function ϕ such that $\mathbf{P}\{\|\phi\|_0<\delta\}=1$.

In particular, if $p=2$, then the solution of (2.1) is called respectively mean-square bounded, mean-square stable, asymptotically mean-square stable, and so on.

Definition 2.4 A nonnegative functional $V(t,\varphi)$, defined on $[0,\infty)\times H_p$, such that $V(t,0)\equiv 0$ and $\lim_{t\to 0}\mathbf{E}V(t,x_t)=0$ if $\lim_{t\to 0}\mathbf{E}|x(t)|^p=0$, $p>0$, is called an F_p-functional.

Certain stability conditions for stochastic functional differential equations can be stated in terms of Lyapunov functionals. In the sequel, c_i are different positive numbers.

Theorem 2.1 *Let $V:[t_0,\infty)\times H_p\to\mathbf{R}_+$ be a continuous functional such that for any solution $x(t)$ of problem* (2.1)–(2.2) *and $p\ge 2$, the following inequalities hold*:

$$\mathbf{E}V(t,x_t)\ge c_1\mathbf{E}\big|x(t)\big|^p, \qquad t\ge 0, \tag{2.11}$$

$$\mathbf{E}V(0,\phi) \le c_2\|\phi\|_1^p, \tag{2.12}$$

$$\mathbf{E}\big[V(t,x_t) - V(0,\phi)\big] \le -c_3\int_0^t \mathbf{E}\big|x(s)\big|^p\,ds, \qquad t \ge 0. \tag{2.13}$$

Then the trivial solution of (2.1) *is asymptotically* p*-stable.*

Proof From (2.11)–(2.13) we have

$$c_1\mathbf{E}\big|x(t)\big|^p \le \mathbf{E}V(t,x_t) \le \mathbf{E}V(0,\phi) \le c_2\|\phi\|_1^p. \tag{2.14}$$

This proves the p-stability.

To prove the asymptotic p-stability, let us show that the solution of (2.1) is asymptotically p-trivial for any initial function ϕ. Note that from (2.14) we obtain

$$\sup_{t\ge 0}\mathbf{E}\big|x(t)\big|^p \le \frac{c_2}{c_1}\|\phi\|_1^p. \tag{2.15}$$

From (2.13) and (2.12) it follows that

$$\int_0^\infty \mathbf{E}\big|x(s)\big|^p\,ds \le \frac{1}{c_3}\mathbf{E}V(0,\phi) \le \frac{c_2}{c_3}\|\phi\|_1^p < \infty, \tag{2.16}$$

i.e., the solution of (2.1) is p-integrable. Applying the generator L to the function $|x(t)|^p$ via (2.5), we obtain

$$\begin{aligned}
\mathbf{E}L\big|x(t)\big|^p = \frac{p}{2}\Big[&2\mathbf{E}\big|x(t)\big|^{p-2}x'(t)a_1(t,x_t)\\
&+\mathbf{E}\big|x(t)\big|^{p-2}\operatorname{Tr}\big[a_2'(t,x_t)a_2(t,x_t)\big]\\
&+(p-2)\mathbf{E}\big|x(t)\big|^{p-4}\big|x'(t)a_2(t,x_t)\big|^2\Big].
\end{aligned}$$

By the Hölder inequality, (2.3), and (2.15), there is a constant c_4 such that

$$\begin{aligned}
&2\mathbf{E}\big|x(t)\big|^{p-2}\big|x'(t)a_1(t,x_t)\big|\\
&\quad\le \mathbf{E}\big|x(t)\big|^{p-2}\Big[\big|x(t)\big|^2+\big|a_1(t,x_t)\big|^2\Big]\\
&\quad\le \mathbf{E}\big|x(t)\big|^p + \int_0^\infty \mathbf{E}\big|x(t)\big|^{p-2}\big|x(t-\theta)\big|^2\,dR_1(\theta)\\
&\quad\le \mathbf{E}\big|x(t)\big|^p + \int_0^\infty \big(\mathbf{E}\big|x(t)\big|^p\big)^{\frac{p-2}{p}}\big(\mathbf{E}\big|x(t-\theta)\big|^p\big)^{\frac{2}{p}}\,dR_1(\theta)\\
&\quad\le c_4.
\end{aligned}$$

Analogously,

$$\mathbf{E}\big|x(t)\big|^{p-2}\big|\operatorname{Tr}\big[a_2'(t,x_t)a_2(t,x_t)\big]\big| \le c_4,$$

$$\mathbf{E}\big|x(t)\big|^{p-4}\big|x'(t)a_2(t,x_t)\big|^2 \le c_4.$$

Hence, there exists a constant c_5 such that $|\mathbf{E}L|x(t)|^p| \le c_5$, and using (2.8) for $t_2 \ge t_1 \ge 0$, we obtain

$$\Big|\mathbf{E}\big|x(t_2)\big|^p - \mathbf{E}\big|x(t_1)\big|^p\Big| \le c_5(t_2 - t_1),$$

i.e., the function $\mathbf{E}|x(t)|^p$ satisfies the Lipschitz condition. From this, (2.15), and (2.16) it follows that $\lim_{t\to\infty}\mathbf{E}|x(t)|^p = 0$. The proof is completed. □

Remark 2.1 From (2.8) it follows that for the functional $V \in D$, condition (2.13) in Theorem 2.1 follows from the inequality

$$\mathbf{E}LV(t,x_t) \le -c_3\mathbf{E}\big|x(t)\big|^p, \quad t \ge 0. \tag{2.17}$$

Theorem 2.2 *Let there exist a functional $V(t,\varphi) \in D$ such that for any solution $x(t)$ of problem (2.1)–(2.2) and $p \ge 2$, the following inequalities hold*:

$$V(t,x_t) \ge c_1\big|x(t)\big|^p, \tag{2.18}$$

$$V(0,\phi) \le c_2\|\phi\|_0^p, \tag{2.19}$$

$$LV(t,x_t) \le 0, \quad t \ge 0, \tag{2.20}$$

$c_i > 0$, *for any initial function ϕ such that $\mathbf{P}\{\|\phi\|_0 \le \delta\} = 1$, where $\delta > 0$ is small enough. Then the zero solution of* (2.1) *is stable in probability*.

Proof Let us suppose that $\mathbf{P}\{\|\phi\|_0 \le \delta\} = 1$. From (2.20) it follows that the process $V(t,x_t)$ is a supermartingale. By (2.18), (2.19), and the inequality for supermartingales [84–87] we have

$$\mathbf{P}\Big\{\sup_{t\ge t_0}\big|x(t,\phi)\big| > \varepsilon_1/F_0\Big\} \le \mathbf{P}\Big\{\sup_{t\ge t_0}V(t,x_t) > c_1\varepsilon_1^p/F_0\Big\} \le \frac{V(t_0,\phi)}{c_1\varepsilon_1^p} \le \frac{c_2\delta^p}{c_1\varepsilon_1^p} < \varepsilon_2$$

for $\delta < \varepsilon_1(c_1\varepsilon_2/c_2)^{1/p}$. The theorem is proven. □

Theorem 2.3 *Let there exist a functional $W : [0,\infty) \times H_2 \to \mathbf{R}_+$ satisfying the condition $\mathbf{E}W(t,\varphi) \le c_1\|\varphi\|_1^2$ and such that for the functional*

$$V(t,\varphi) = W(t,\varphi) + \big|\varphi(0) - G(t,\varphi)\big|^2, \tag{2.21}$$

where $G(t,\varphi)$ satisfies condition (2.10), *the following estimates are valid*:

$$\begin{aligned} &\mathbf{E}V(0,\phi) \le c_2\|\phi\|_1^2, \\ &\mathbf{E}V(t,x_t) - \mathbf{E}V(0,\phi) \le -c_3\int_0^t \mathbf{E}\big|x(s)\big|^2\,ds, \quad t \ge 0, \end{aligned} \tag{2.22}$$

where c_i, $i = 1,2,3$, are some positive constants. Then the zero solution of (2.9) *is asymptotically mean-square stable*.

The proof of Theorem 2.3 is similar to Theorem 2.1 and can be found in [132–135].

Theorem 2.4 *Let there exist a functional $V(t,\varphi)\in D$ such that for some $p>0$ and $\lambda>0$, the following conditions hold*:

$$\mathbf{E}V(t,x_t)\geq c_1 e^{\lambda t}\mathbf{E}\big|x(t)\big|^p,\quad t\geq 0, \tag{2.23}$$

$$\mathbf{E}V(0,\phi)\leq c_2\|\phi\|_1^p, \tag{2.24}$$

$$\mathbf{E}LV(t,x_t)\leq 0,\quad t\geq 0. \tag{2.25}$$

Then the trivial solution of (2.1) is exponentially p-stable.

Proof Integrating (2.25) via (2.8), we obtain $\mathbf{E}V(t,x_t)\leq \mathbf{E}V(0,\phi)$. From this and from (2.23)–(2.24) it follows that

$$c_1\mathbf{E}\big|x(t)\big|^p\leq e^{-\lambda t}\mathbf{E}V(0,\phi)\leq c_2\|\phi\|_1^p.$$

The inequality $c_1\mathbf{E}|x(t)|^2\leq c_2\|\phi\|_1^p$ means that the trivial solution of (2.1) is p-stable. Besides, from the inequality $c_1\mathbf{E}|x(t)|^2\leq e^{-\lambda t}\mathbf{E}V(0,\phi)$ it follows that the trivial solution of (2.1) is exponentially p-stable. The proof is completed. □

Corollary 2.1 *Let there exist a functional $V_0(t,\varphi)\in D$ such that for some $p>0$, the following conditions hold*:

$$\begin{aligned} c_1\mathbf{E}\big|x(t)\big|^p &\leq \mathbf{E}V_0(t,x_t)\\ &\leq c_2\mathbf{E}\big|x(t)\big|^p+\sum_{i=0}^m\int_0^\infty\int_{t-\theta}^t(s-t+\theta)^i\mathbf{E}\big|x(s)\big|^p\,ds\,dK_i(\theta), \end{aligned} \tag{2.26}$$

$$\mathbf{E}LV_0(t,x_t)\leq -c_3\mathbf{E}\big|x(t)\big|^p, \tag{2.27}$$

where $m\geq 0$, $K_i(\theta)$, $i=0,1,\dots,m$, are nondecreasing functions such that, for some small enough $\lambda>0$,

$$\sum_{i=0}^m\eta_i(\lambda)<\infty,\qquad \eta_i(\lambda)=\frac{1}{i+1}\int_0^\infty e^{\lambda\theta}\theta^{i+1}\,dK_i(\theta). \tag{2.28}$$

Then the trivial solution of (2.1) is exponentially p-stable.

Proof It is enough to show that by the conditions (2.26)–(2.28) there exists a functional $V(t,\varphi)$ that satisfies the conditions of Theorem 2.4. Indeed, put $V_1(t,\varphi)=e^{\lambda t}V_0(t,\varphi)$. By (2.26) and (2.27) we have

$$
\begin{aligned}
\mathbf{E}LV_1(t,x_t) &= \lambda e^{\lambda t}\mathbf{E}V_0(t,x_t) + e^{\lambda t}\mathbf{E}LV_0(t,x_t) \\
&\le e^{\lambda t}\left(\lambda\left(c_2\mathbf{E}\big|x(t)\big|^p + \sum_{i=0}^m \int_0^\infty \int_{t-\theta}^t (s-t+\theta)^i \mathbf{E}\big|x(s)\big|^p\,ds\,dK_i(\theta)\right)\right. \\
&\qquad \left. - c_3\mathbf{E}\big|x(t)\big|^p\right) \\
&= e^{\lambda t}\left((\lambda c_2 - c_3)\mathbf{E}\big|x(t)\big|^p\right. \\
&\qquad \left. + \lambda\sum_{i=0}^m \int_0^\infty \int_{t-\theta}^t (s-t+\theta)^i \mathbf{E}\big|x(s)\big|^p\,ds\,dK_i(\theta)\right).
\end{aligned}
$$

Now put

$$
V_2(t,x_t) = \lambda\sum_{i=0}^m \frac{1}{i+1}\int_0^\infty \int_{t-\theta}^t e^{\lambda(s+\theta)}(s-t+\theta)^{i+1}\big|x(s)\big|^p\,ds\,dK_i(\theta).
$$

Then

$$
\begin{aligned}
LV_2(t,x_t) &= \lambda\sum_{i=0}^m\left(e^{\lambda t}\eta_i(\lambda)\big|x(t)\big|^p\right. \\
&\qquad \left. - \int_0^\infty \int_{t-\theta}^t e^{\lambda(s+\theta)}(s-t+\theta)^i\big|x(s)\big|^p\,ds\,dK_i(\theta)\right) \\
&= \lambda e^{\lambda t}\sum_{i=0}^m\left(\eta_i(\lambda)\big|x(t)\big|^p\right. \\
&\qquad \left. - \int_0^\infty \int_{t-\theta}^t e^{\lambda(s-t+\theta)}(s-t+\theta)^i\big|x(s)\big|^p\,ds\,dK_i(\theta)\right),
\end{aligned}
$$

and via $s \ge t-\theta$, for the functional $V = V_1 + V_2$, it follows that

$$
\begin{aligned}
\mathbf{E}LV(t,x_t) &\le e^{\lambda t}\left(\left(\lambda\left(c_2 + \sum_{i=0}^m \eta_i(\lambda)\right) - c_3\right)\mathbf{E}\big|x(t)\big|^p\right. \\
&\qquad \left. - \lambda\sum_{i=0}^m \int_0^\infty \int_{t-\theta}^t \left(e^{\lambda(s-t+\theta)} - 1\right)\mathbf{E}\big|x(s)\big|^p\,ds\,dK_i\theta\right) \\
&\le e^{\lambda t}\left(\lambda\left(c_1 + \sum_{i=0}^m \eta_i(\lambda)\right) - c_3\right)\mathbf{E}\big|x(t)\big|^p.
\end{aligned}
$$

From this and from (2.28), for small enough $\lambda > 0$, we obtain (2.25). It is easy to check that conditions (2.23)–(2.24) hold too. The proof is completed. □

Example 2.1 Consider the linear stochastic differential equation

$$\dot{x}(t) = ax(t) + bx(t-h) + \sigma x(t-\tau)\dot{w}(t). \tag{2.29}$$

For the functional

$$V(t, x_t) = x^2(t) + |b| \int_{t-h}^{t} x^2(s)\,ds + \sigma^2 \int_{t-\tau}^{t} x^2(s)\,ds, \tag{2.30}$$

we have

$$\begin{aligned} LV(t, x_t) &= 2x(t)\big(ax(t) + bx(t-h)\big) + \sigma^2 x^2(t-\tau) \\ &\quad + |b|\big(x^2(t) - x^2(t-h)\big) + \sigma^2\big(x^2(t) - x^2(t-\tau)\big) \\ &\le \big(2\big(a + |b|\big) + \sigma^2\big)x^2(t). \end{aligned} \tag{2.31}$$

So, by the condition

$$a + |b| + \frac{1}{2}\sigma^2 < 0 \tag{2.32}$$

the functional (2.30) satisfies the conditions of Theorem 2.1 with $p = 2$, and therefore the trivial solution of (2.29) is asymptotically mean-square stable.

On the other hand, by (2.31)–(2.32) the functional (2.30) satisfies the conditions of Corollary 2.1 with $p = 2$, $m = 0$, and $dK_0(s) = (|b|\delta(s-h) + \sigma^2\delta(s-\tau))\,ds$. Therefore, the trivial solution of (2.29) is exponentially mean-square stable. By Theorems 2.1–2.4 the construction of stability conditions is reduced to the construction of some Lyapunov functionals satisfying the assumptions of these theorems. Below we will use the general method of construction of Lyapunov functionals [136–139, 146–148, 150, 265, 269–272, 278] allowing one to obtain stability conditions immediately in terms of parameters of systems under consideration. This method was successfully used for functional differential equations, for difference equations with discrete and continuous time [278], and for partial differential equations [48].

2.2.2 *Formal Procedure of Constructing Lyapunov Functionals*

The formal procedure of constructing Lyapunov functionals consists of four steps.

Step 1 Let us represent (2.9) in the form

$$dz(t, x_t) = \big(b_1\big(t, x(t)\big) + c_1(t, x_t)\big)\,dt + \big(b_2\big(t, x(t)\big) + c_2(t, x_t)\big)\,dw(t), \tag{2.33}$$

where $z(t, x_t)$ is some functional of x_t, $z(t, 0) = 0$, the functionals $b_i(t, x(t))$, $i = 1, 2$, depend on t and $x(t)$ only and do not depend on the previous values $x(t + s)$, $s < 0$, of the solution, and $b_i(t, 0) = 0$.

Step 2 Consider the auxiliary differential equation without memory

$$dy(t) = b_1\big(t, y(t)\big)\,dt + b_2\big(t, y(t)\big)\,dw(t). \tag{2.34}$$

Let us assume that the zero solution of (2.34) is asymptotically mean-square stable and therefore there exists a Lyapunov function $v(t, y)$ such that $c_1|y|^2 \le v(t, y) \le c_2|y|^2$ and $L_0 v(t, y) \le -c_3|y|^2$. Here L_0 is the generator of (2.34), $c_i > 0$, $i = 1, 2, 3$.

Step 3 Replacing the second argument y of the function $v(t, y)$ by the functional $z(t, x_t)$ from left-hand part of (2.33), we obtain the main component $V_1(t, x_t) = v(t, z(t, x_t))$ of the functional $V(t, x_t)$. Then it is necessary to calculate LV_1, where L is the generator of (2.33), and in a reasonable way to estimate it.

Step 4 Usually, the functional V_1 almost satisfies the requirements of stability theorems. In order to satisfy these conditions completely, an auxiliary component V_2 can be easily chosen by some standard way. As a result, the desired functional $V(t, x_t)$ takes the form $V = V_1 + V_2$.

Let us make remarks on some peculiarities of this procedure.

(1) It is clear that the representation (2.33) in the first step of the procedure is not unique. Hence, for different representations, one can construct different Lyapunov functionals, and, as a result, one can get different stability conditions.
(2) In the second step, for the auxiliary equation (2.34), one can choose different Lyapunov functions $v(t, y)$. So, for the initial equation (2.9), different Lyapunov functionals can be constructed, and, as a result, different stability conditions can be obtained.
(3) It is necessary to emphasize also that by choosing different ways of estimation of LV_1 one can construct different Lyapunov functionals and, as a result, one can get different stability conditions.
(4) At last, some standard way of the construction of the additional functional V_2 sometimes allows one to simplify the fourth step and do not use the functional V_2 at all. Below the corresponding auxiliary Lyapunov-type theorems are considered.

2.2.3 Auxiliary Lyapunov-Type Theorem

The following theorem in some cases allows one to use the procedure of constructing Lyapunov functionals without an additional functional V_2.

Theorem 2.5 *Let there exist a functional $V_1(t,x_t)\in D$ of type (2.21) such that*

$$
\begin{aligned}
\mathbf{E}LV_1(t,x_t) \le{}& \mathbf{E}x'(t)D(t)x(t)+\sum_{i=1}^{l}\int_0^\infty \mathbf{E}x'(t-s)S_i(t-s)x(t-s)\,d\nu_i(s)\\
&+\sum_{i=1}^{n}\int_0^\infty \mathbf{E}x_i^2(t-s)\,dK_i(s)\\
&+\sum_{i=1}^{k}\mathbf{E}x'\big(t-\tau_i(t)\big)Q_i\big(t-\tau_i(t)\big)x\big(t-\tau_i(t)\big)\\
&+\sum_{j=0}^{m}\int_0^\infty d\mu_j(s)\int_{t-s}^{t}(\theta-t+s)^j\mathbf{E}x'(\theta)R_j(\theta)x(\theta)\,d\theta\\
&+\int_0^\infty d\mu(s)\int_{t-s}^{t}\mathbf{E}x'(\tau)R(\tau+s,t)x(\tau)\,d\tau, \qquad (2.35)
\end{aligned}
$$

where L is the generator of (2.9), $D(t)$ is a negative definite matrix, $S_i(t)$, $i=1,\dots,l$, $Q_i(t)$, $i=1,\dots,k$, $R_j(t)$, $j=0,\dots,m$, $R(s,t)$, $s\ge t\ge 0$, are nonnegative definite matrices, $\tau_i(t)$, $i=1,\dots,k$, $t\ge 0$, are differentiable nonnegative functions with $\dot\tau_i(t)\le\hat\tau_i<1$, $K_i(s)$, $i=1,\dots,n$, $\nu_i(s)$, $i=0,\dots,l$, $\mu_j(s)$, $j=0,\dots,m$, and $\mu(s)$, $s\ge 0$, are nondecreasing functions of bounded variation such that

$$
\begin{aligned}
&k_i=\int_0^\infty dK_i(s)<\infty, \qquad q_i=\int_0^\infty d\nu_i(s)<\infty,\\
&r_j=\int_0^\infty \frac{s^{j+1}}{j+1}\,d\mu_j(s)<\infty,
\end{aligned} \qquad (2.36)
$$

and the matrix

$$
\begin{aligned}
G(t)={}&D(t)+K+\sum_{i=1}^{l}q_iS_i(t)+\sum_{i=1}^{k}\frac{1}{1-\hat\tau_i}Q_i(t)\\
&+\sum_{j=0}^{m}r_jR_j(t)+\int_0^\infty d\mu(s)\int_t^{t+s}R(t+s,\theta)\,d\theta, \qquad (2.37)
\end{aligned}
$$

where K is the diagonal matrix with elements k_i, $i=1,\dots,n$, is uniformly negative definite matrix with respect to $t\ge 0$, i.e.,

$$
x'G(t)x\le -c|x|^2, \quad c>0,\ x\in\mathbf{R}^n. \qquad (2.38)
$$

Then the zero solution of (2.9) is asymptotically mean-square stable.

Proof Put

$$
\begin{aligned}
V_2(t,x_t) &= \sum_{i=1}^{l}\int_0^\infty d\nu_i(s)\int_{t-s}^{t} x'(\theta)S_i(\theta)x(\theta)\,d\theta \\
&\quad + \sum_{i=1}^{n}\int_0^\infty dK_i(s)\int_{t-s}^{t} x_i^2(\theta)\,d\theta + \sum_{i=1}^{k}\frac{1}{1-\hat{\tau}_i}\int_{t-\tau_i(t)}^{t} x'(s)Q_i(s)x(s)\,ds \\
&\quad + \sum_{j=0}^{m}\int_0^\infty d\mu_j(s)\int_{t-s}^{t}\frac{(\theta-t+s)^{j+1}}{j+1}x'(\theta)R_j(\theta)x(\theta)\,d\theta \\
&\quad + \int_0^\infty d\mu(s)\int_{t-s}^{t}\int_t^{\tau+s} x'(\tau)R(\tau+s,\theta)x(\tau)\,d\theta\,d\tau.
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathbf{E}LV_2(t,x_t) &= \sum_{i=1}^{l} q_i\mathbf{E}x'(t)S_i(t)x(t) - \int_0^\infty \mathbf{E}x'(t-s)S(t-s)x(t-s)\,d\nu(s) \\
&\quad + \sum_{i=1}^{n} k_i\mathbf{E}x_i^2(t) - \sum_{l=1}^{n}\int_0^\infty \mathbf{E}x_l^2(t-s)\,dK_l(s) \\
&\quad + \sum_{i=1}^{k}\frac{1}{1-\hat{\tau}_i}\mathbf{E}x'(t)Q_i(t)x(t) \\
&\quad - \sum_{i=1}^{k}\frac{1-\dot{\tau}_i(t)}{1-\hat{\tau}_i}\mathbf{E}x'\big(t-\tau_i(t)\big)Q_i\big(t-\tau_i(t)\big)x\big(t-\tau_i(t)\big) \\
&\quad + \sum_{j=0}^{m} r_j\mathbf{E}x'(t)R_j(t)x(t) \\
&\quad - \sum_{j=0}^{m}\int_0^\infty d\mu_j(s)\int_{t-s}^{t}(\theta-t+s)^j\mathbf{E}x'(\theta)R_j(\theta)x(\theta)\,d\theta \\
&\quad + \int_0^\infty d\mu(s)\int_t^{t+s}\mathbf{E}x'(t)R(t+s,\theta)x(t)\,d\theta \\
&\quad - \int_0^\infty d\mu(s)\int_{t-s}^{t}\mathbf{E}x'(\tau)R(\tau+s,t)x(\tau)\,d\tau. \qquad (2.39)
\end{aligned}
$$

From (2.35), (2.37), and (2.39) for the functional $V(t,x_t)=V_1(t,x_t)+V_2(t,x_t)$ it follows that

$$
\mathbf{E}LV(t,x_t) \le \mathbf{E}x'(t)G(t)x(t). \qquad (2.40)
$$

By (2.38) and Remark 2.1 this means that the functional $V(t, x_t)$ satisfies conditions (2.22) of Theorem 2.3, and therefore the zero solution of (2.9) is asymptotically mean-square stable. The proof is completed. □

Corollary 2.2 *Let in Theorem 2.5 inequality (2.35) be the exact equality,* $G(t) = G = \text{const}$, *and the functional* $V = V_1 + V_2$ *be* F_2*-functional. Then in the scalar case the condition* $G < 0$ *is a necessary and sufficient condition for asymptotic mean-square stability of the zero solution of* (2.9).

Proof In the considered case, for the functional $V = V_1 + V_2$, we have $\mathbf{E}LV(t, x_t) = G\mathbf{E}x^2(t)$. If $G \geq 0$, then from this and from (2.8) it follows that

$$\mathbf{E}V(t, x_t) = \mathbf{E}V(0, \phi) + G\int_0^t \mathbf{E}x^2(\tau)\,d\tau \geq \mathbf{E}V(0, \phi) > 0.$$

This means that $\lim_{t\to\infty} \mathbf{E}V(t, x_t) \neq 0$ and therefore $\lim_{t\to\infty} \mathbf{E}|x(t)|^2 \neq 0$. The proof is completed. □

Remark 2.2 Since the functional $V(t, x_t)$ constructed in Theorem 2.5 satisfies the conditions (2.22) and $V(t, x_t) \geq 0$, we have that $c_3 \int_0^t \mathbf{E}|x(s)|^2\,ds \leq \mathbf{E}V(0, \phi) < \infty$. This means that by conditions (2.35) and (2.38) the solution of (2.9) is also mean-square integrable.

Remark 2.3 In the scalar case, from Remark 2.2 it follows that if by condition (2.35) the solution of (2.9) is mean-square nonintegrable, i.e., $\int_0^\infty \mathbf{E}x^2(t)\,dt = \infty$, then $\sup_{t\geq 0} G(t) \geq 0$.

Remark 2.4 Theorem 2.5 is a useful development and improvement of the general method of construction of Lyapunov functionals. It allows one not to use Step 4 of the procedure and get good stability conditions using much more simple Lyapunov functional than via Theorem 2.3. It can be used in different applications.

2.3 Some Useful Statements

2.3.1 Linear Stochastic Differential Equation

Consider now conditions for asymptotic mean-square stability of the trivial solution of the linear Itô stochastic differential equation

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + \sigma x(t-h)\dot{w}(t), \tag{2.41}$$

where $A, B, \sigma, \tau \geq 0, h \geq 0$ are known constants.

Lemma 2.1 *A necessary and sufficient condition for asymptotic mean-square stability of the trivial solution of* (2.41) *is*

$$A + B < 0, \qquad G^{-1} > \frac{1}{2}\sigma^2, \tag{2.42}$$

where

$$G = \begin{cases} \frac{Bq^{-1}\sin(q\tau)-1}{A+B\cos(q\tau)}, & B + |A| < 0,\ q = \sqrt{B^2 - A^2}, \\ \frac{1+|A|\tau}{2|A|}, & B = A < 0, \\ \frac{Bq^{-1}\sinh(q\tau)-1}{A+B\cosh(q\tau)}, & A + |B| < 0,\ q = \sqrt{A^2 - B^2}. \end{cases} \tag{2.43}$$

Remark 2.5 If $A = -a$ and $B = 0$, then the necessary and sufficient stability condition (2.42)–(2.43) takes the form $a > \frac{1}{2}\sigma^2$.

Note that the proof of Lemma 2.1 is based on two old enough papers [243, 290] as it was shown briefly in the author recent book [278]. Following to advices and requests of some readers of the book [278], the author took the decision to write here the proof of this lemma in more detail.

Proof of the Lemma 2.1 A necessary and sufficient stability condition (2.42) with

$$G = 2\int_0^\infty x^2(s)\,ds, \tag{2.44}$$

where $x(t)$ is a solution of (2.41) in the deterministic case, i.e., with $\sigma = 0$, was obtained in [243]. By the Plancherel theorem the integral (2.44) coincides [243, 290] with

$$G = \frac{2}{\pi}\int_0^\infty \frac{dt}{(A + B\cos\tau t)^2 + (t + B\sin\tau t)^2}. \tag{2.45}$$

Let us obtain for this integral the representation (2.43) in elementary functions. Following [243], consider the functional

$$\begin{aligned} V(x_t) = {} & \frac{1}{2}Gx^2(t) + \int_{t-\tau}^t \beta(s-t)x(s)x(t)\,ds \\ & + \int_{t-\tau}^t \int_s^t \delta(s-t, \theta-t)x(\theta)x(s)\,d\theta\,ds, \end{aligned} \tag{2.46}$$

where G is a constant, and $\beta(s)$ and $\delta(s,\theta)$ are continuously differentiable functions. By (2.46) and (2.41) with $\sigma = 0$ we obtain

$$\begin{aligned} \frac{dV(x_t)}{dt} = {} & \left(Gx(t) + \int_{t-\tau}^t \beta(s-t)x(s)\,ds\right)\big(Ax(t) + Bx(t-\tau)\big) \\ & + \beta(0)x^2(t) - \beta(-\tau)x(t-\tau)x(t) - \int_{t-\tau}^t \frac{d\beta(s-t)}{ds}x(s)x(t)\,ds \end{aligned}$$

$$
\begin{aligned}
&+\int_{t-\tau}^{t}\delta(s-t,0)x(t)x(s)\,ds-\int_{t-\tau}^{t}\delta(-\tau,\theta-t)x(\theta)x(t-\tau)\,d\theta\\
&-\int_{t-\tau}^{t}\int_{s}^{t}\left(\frac{\partial\delta(s-t,\theta-t)}{\partial s}+\frac{\partial\delta(s-t,\theta-t)}{\partial\theta}\right)x(\theta)x(s)\,d\theta\,ds\\
=&\big(GA+\beta(0)\big)x^2(t)+\big(GB-\beta(-\tau)\big)x(t)x(t-\tau)\\
&+\int_{t-\tau}^{t}\left(A\beta(s-t)-\frac{d\beta(s-t)}{ds}+\delta(0,s-t)\right)x(s)x(t)\,ds\\
&+\int_{t-\tau}^{t}\big(B\beta(s-t)-\delta(-\tau,s-t)\big)x(s)x(t-\tau)\,ds\\
&-\int_{t-\tau}^{t}\int_{s}^{t}\left(\frac{\partial\delta(s-t,\theta-t)}{\partial s}+\frac{\partial\delta(s-t,\theta-t)}{\partial\theta}\right)x(\theta)x(s)\,d\theta\,ds.
\end{aligned}
\tag{2.47}
$$

Let us suppose that the functions $\beta(s)$ and $\delta(s,\theta)$ satisfy the conditions

$$
\begin{gathered}
GA+\beta(0)=-1,\qquad GB-\beta(-\tau)=0,\\
A\beta(s)-\frac{d\beta(s)}{ds}+\delta(s,0)=0,\qquad B\beta(s)-\delta(-\tau,s)=0,\\
\frac{\partial\delta(s,\theta)}{\partial s}+\frac{\partial\delta(s,\theta)}{\partial\theta}=0.
\end{gathered}
\tag{2.48}
$$

Then from (2.47) and (2.48) it follows that $\frac{dV(x_t)}{dt}=-x^2(t)$, and therefore (if the condition for asymptotic stability of the trivial solution of the considered equation holds, i.e., $\lim_{t\to\infty}V(x_t)=0$),

$$
\int_0^\infty x^2(t)\,dt=-\int_0^\infty\frac{dV(x_t)}{dt}\,dt=V(x_0).
\tag{2.49}
$$

Using the initial function

$$
x_\varepsilon(s)=\begin{cases}0 & \text{if } -\tau\le s\le-\varepsilon,\\ 1+\frac{s}{\varepsilon} & \text{if } -\varepsilon\le s\le 0,\end{cases}
$$

and the limit $\varepsilon\to 0$, from (2.46) we obtain $V(x_0)=\frac{1}{2}G$. From this and from (2.49) it follows that G in the functional (2.46) indeed coincides with (2.44) and has the representation (2.45).

To get for G the representation (2.43), let us solve system (2.48). From the last equation of (2.48) it follows that $\delta(s,\theta)=\varphi(s-\theta)$ and, by the forth equation of (2.48), $\varphi(s)=B\beta(-\tau-s)$. Substituting this $\varphi(s)$ into the third equation of (2.48), we obtain that the function $\beta(t)$ is defined by the differential equation

$$
\dot\beta(t)=A\beta(t)+B\beta(-t-\tau)
\tag{2.50}
$$

with the conditions

$$GA + \beta(0) + 1 = 0, \qquad GB - \beta(-\tau) = 0. \tag{2.51}$$

Suppose that $q^2 = B^2 - A^2 > 0$. Then, by (2.50),

$$\begin{aligned}\ddot{\beta}(t) &= A\dot{\beta}(t) - B\dot{\beta}(-t-\tau)\\ &= A\big(A\beta(t) + B\beta(-t-\tau)\big) - B\big(A\beta(-t-\tau) + B\beta(t+\tau-\tau)\big)\\ &= -q^2\beta(t)\end{aligned}$$

or

$$\ddot{\beta}(t) + q^2\beta(t) = 0. \tag{2.52}$$

Substituting the general solution $\beta(t) = C_1 \cos qt + C_2 \sin qt$ of (2.52) into (2.50) and (2.51), we obtain two equations for G, C_1, and C_2

$$GA + C_1 = -1, \qquad GB = C_1 \cos q\tau - C_2 \sin q\tau, \tag{2.53}$$

and two homogeneous linear dependent equations for C_1 and C_2

$$\begin{aligned}C_1(A + B\cos q\tau) - C_2(q + B\sin q\tau) = 0,\\ C_1(q - B\sin q\tau) + C_2(A - B\cos q\tau) = 0.\end{aligned} \tag{2.54}$$

By (2.54) we have

$$C_2 = C_1 \frac{A + B\cos q\tau}{q + B\sin q\tau} = -C_1 \frac{q - B\sin q\tau}{A - B\cos q\tau}. \tag{2.55}$$

Substituting the first equality (2.55) into (2.53) and excluding C_1, we obtain

$$G = \frac{A\sin q\tau - q\cos q\tau}{q(A\cos q\tau + q\sin q\tau + B)}. \tag{2.56}$$

Multiplying the numerator and the denominator of the obtained fraction by $B\sin q\tau - q$, one can convert (2.56) to the form of the first line in (2.43). Note that the same result can be obtained using the second equality (2.55).

Suppose now that $q^2 = A^2 - B^2 > 0$. Then, similarly to (2.52), we obtain the equation $\ddot{\beta}(t) - q^2\beta(t) = 0$ with the general solution $\beta(t) = C_1 e^{qt} + C_2 e^{-qt}$. Substituting this solution into (2.50) and (2.51), similarly to (2.53) and (2.54), we have

$$\begin{aligned}GA + C_1 + C_2 = -1, \qquad GB = C_1 e^{-q\tau} + C_2 e^{q\tau},\\ C_1(q - A) - C_2 B e^{q\tau} = 0, \qquad C_1 B + C_2(q + A)e^{q\tau} = 0.\end{aligned} \tag{2.57}$$

By the two last equations of (2.57),

$$C_2 = C_1 \frac{q - A}{B} e^{-q\tau} = -C_1 \frac{B}{q + A} e^{-q\tau}. \tag{2.58}$$

From the first equality of (2.58) and the two first equations of (2.57) we obtain

$$G = \frac{q - A + Be^{-q\tau}}{q(q - A - Be^{-q\tau})}. \tag{2.59}$$

Put now $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$ (respectively, hyperbolic sine and hyperbolic cosine). Multiplying the numerator and the denominator of (2.59) by $B \sinh q\tau - q$ in the denominator, we have

$$\begin{aligned}
&\big(q - A - Be^{-q\tau}\big)(B \sinh q\tau - q) \\
&\quad = \big(q - A - Be^{-q\tau}\big)\big(B \cosh q\tau - Be^{-q\tau} - q\big) \\
&\quad = qB \cosh q\tau - AB \cosh q\tau - B^2 e^{-q\tau} \cosh q\tau \\
&\qquad - Bqe^{-q\tau} + ABe^{-q\tau} + B^2 e^{-2q\tau} - A^2 + B^2 + Aq + Bqe^{-q\tau} \\
&\quad = (q - A)(A + B \cosh q\tau) + Be^{-q\tau}\big(A + B\big(e^{q\tau} + e^{-q\tau} - \cosh q\tau\big)\big) \\
&\quad = \big(q - A + Be^{-q\tau}\big)(A + B \cosh q\tau).
\end{aligned}$$

As a result, we obtain (2.59) in the form of the third line in (2.43). Note also that the same result can be obtained using the second equality of (2.58).

The second line of (2.43) can be obtained from the first (or the third) line in the limit as $q \to 0$. The proof is completed. □

2.3.2 System of Two Linear Stochastic Differential Equations

Consider the system of two stochastic differential equations without delays

$$\begin{aligned}
\dot{x}_1(t) &= a_{11}x_1(t) + a_{12}x_2(t) + \sigma_1 x_1(t)\dot{w}_1(t), \\
\dot{x}_2(t) &= a_{21}x_1(t) + a_{22}x_2(t) + \sigma_2 x_2(t)\dot{w}_2(t),
\end{aligned} \tag{2.60}$$

where a_{ij}, σ_i, $i, j = 1, 2$, are constants, and $w_1(t)$ and $w_2(t)$ are mutually independent standard Wiener processes.

Put $A = \|a_{ij}\|$, $i, j = 1, 2$, and

$$\delta_i = \frac{1}{2}\sigma_i^2, \quad i = 1, 2. \tag{2.61}$$

Remark 2.6 If $\sigma_1 = \sigma_2 = 0$, then (by Corollary 1.1) the trivial solution of (2.60) is asymptotically stable if and only if

$$\operatorname{Tr}(A) = a_{11} + a_{22} < 0, \qquad \det(A) = a_{11}a_{22} - a_{12}a_{21} > 0. \tag{2.62}$$

If $a_{12} = a_{21} = 0$, then (by Remark 2.5) the necessary and sufficient conditions for asymptotic mean-square stability of the trivial solution of (2.60) are

$$a_{11} + \delta_1 < 0, \qquad a_{22} + \delta_2 < 0. \tag{2.63}$$

Lemma 2.2 *Let for some positive definite matrix $P = \|p_{ij}\|$, $i, j = 1, 2$, the parameters of system (2.60) satisfy the conditions*

$$\begin{aligned}
&p_{12}a_{21} + p_{11}(a_{11} + \delta_1) < 0,\\
&p_{12}a_{12} + p_{22}(a_{22} + \delta_2) < 0,\\
&4\big(p_{12}a_{21} + p_{11}(a_{11} + \delta_1)\big)\big(p_{12}a_{12} + p_{22}(a_{22} + \delta_2)\big)\\
&\quad > \big(p_{11}a_{12} + p_{22}a_{21} + p_{12}\operatorname{Tr}(A)\big)^2.
\end{aligned} \tag{2.64}$$

Then the trivial solution of system (2.60) is asymptotically mean-square stable.

Proof Let L_0 be the generator of system (2.60). Using the Lyapunov function

$$v(t) = p_{11}x_1^2(t) + 2p_{12}x_1(t)x_2(t) + p_{22}x_2^2(t) \tag{2.65}$$

and (2.61)–(2.62) for system (2.60), we have

$$\begin{aligned}
L_0 v(t) &= 2\big(p_{11}x_1(t) + p_{12}x_2(t)\big)\big(a_{11}x_1(t) + a_{12}x_2(t)\big) + p_{11}\sigma_1^2 x_1^2(t)\\
&\quad + 2\big(p_{12}x_1(t) + p_{22}x_2(t)\big)\big(a_{21}x_1(t) + a_{22}x_2(t)\big) + p_{22}\sigma_2^2 x_2^2(t)\\
&= 2\big(p_{12}a_{21} + p_{11}(a_{11} + \delta_1)\big)x_1^2(t) + 2\big(p_{12}a_{12} + p_{22}(a_{22} + \delta_2)\big)x_2^2(t)\\
&\quad + 2\big(p_{11}a_{12} + p_{22}a_{21} + p_{12}\operatorname{Tr}(A)\big)x_1(t)x_2(t).
\end{aligned}$$

By (2.64) $L_0 v(t)$ is a negative definite square form, i.e., the function $v(t)$ satisfies (2.17) with $p = 2$. So, the trivial solution of system (2.60) is asymptotically mean-square stable. The proof is completed. □

Corollary 2.3 *Suppose that conditions (2.62) hold, $a_{12} \neq 0$, and*

$$\delta_1 < \frac{|\operatorname{Tr}(A)|\det(A)}{A_2}, \qquad \delta_2 < \frac{|\operatorname{Tr}(A)|\det(A) - A_2\delta_1}{A_1 - |\operatorname{Tr}(A)|\delta_1}, \tag{2.66}$$

where

$$A_1 = \det(A) + a_{11}^2, \qquad A_2 = \det(A) + a_{22}^2. \tag{2.67}$$

Then the trivial solution of system (2.60) is asymptotically mean-square stable.

Proof By Remark 1.1, from (1.29) and (2.64) it follows that if, for some $q > 0$,

$$-q + 2p_{11}\delta_1 < 0, \qquad -1 + 2p_{22}\delta_2 < 0, \tag{2.68}$$

then the trivial solution of (2.60) is asymptotically mean-square stable.

Using (1.29), we can represent (2.68) in the form

$$\frac{(A_2 q + a_{21}^2)\delta_1}{|\operatorname{Tr}(A)|\det(A)} < q, \qquad \frac{(A_1 + a_{12}^2 q)\delta_2}{|\operatorname{Tr}(A)|\det(A)} < 1.$$

From this we have

$$\frac{a_{21}^2\delta_1}{|\operatorname{Tr}(A)|\det(A) - A_2\delta_1} < q < \frac{|\operatorname{Tr}(A)|\det(A) - A_1\delta_2}{a_{12}^2\delta_2}. \tag{2.69}$$

So, if

$$\frac{a_{21}^2\delta_1}{|\operatorname{Tr}(A)|\det(A) - A_2\delta_1} < \frac{|\operatorname{Tr}(A)|\det(A) - A_1\delta_2}{a_{12}^2\delta_2}, \tag{2.70}$$

then there exists $q > 0$ such that (2.69), and therefore (2.68) holds.

Let us show that (2.70) holds. Indeed, by the first condition (2.66) we can rewrite (2.70) in the form

$$\begin{aligned} a_{12}^2a_{21}^2\delta_1\delta_2 &< \left(\left|\operatorname{Tr}(A)\right|\det(A) - A_2\delta_1\right)\left(\left|\operatorname{Tr}(A)\right|\det(A) - A_1\delta_2\right) \\ &= \left(\left|\operatorname{Tr}(A)\right|\det(A)\right)^2 - \left|\operatorname{Tr}(A)\right|\det(A)(A_1\delta_2 + A_2\delta_1) + A_1A_2\delta_1\delta_2. \end{aligned} \tag{2.71}$$

By (2.67) we have

$$\begin{aligned} A_1A_2 &= \left(\det(A) + a_{11}^2\right)\left(\det(A) + a_{22}^2\right) \\ &= \left(\det(A) + a_{11}^2 + a_{22}^2\right)\det(A) + a_{11}^2a_{22}^2 \\ &= \left(\left|\operatorname{Tr}(A)\right|^2 - (a_{11}a_{22} + a_{12}a_{21})\right)\det(A) + a_{11}^2a_{22}^2 \\ &= \left|\operatorname{Tr}(A)\right|^2\det(A) + a_{12}^2a_{21}^2 \\ &\geq \left|\operatorname{Tr}(A)\right|^2\det(A). \end{aligned} \tag{2.72}$$

So, by (2.71) and (2.72) it is enough to show that

$$0 < \left|\operatorname{Tr}(A)\right|\det(A) - A_2\delta_1 - \left(A_1 - \left|\operatorname{Tr}(A)\right|\delta_1\right)\delta_2. \tag{2.73}$$

Note that from (2.66) and (2.72) it follows that

$$\delta_1 < \frac{|\operatorname{Tr}(A)|\det(A)}{A_2} \leq \frac{A_1}{|\operatorname{Tr}(A)|}.$$

So, (2.73) is equivalent to (2.66). The proof is completed. □

Remark 2.7 If $a_{12} = 0$, then conditions (2.66) coincide with (2.63). Indeed, by (2.62) from (2.66) we obtain

$$\delta_1 < \frac{|a_{11} + a_{22}|a_{11}a_{22}}{(a_{11} + a_{22})a_{22}} = -a_{11}, \qquad \delta_2 < \frac{|a_{11} + a_{22}|a_{22}(a_{11} + \delta_1)}{(a_{11} + a_{22})(a_{11} + \delta_1)} = -a_{22}.$$

Remark 2.8 From the conditions (2.63) and $a_{12}a_{21} \le 0$ it follows that

$$|a_{11}| \le \frac{|\operatorname{Tr}(A)| \det(A)}{A_2}, \qquad |a_{22}| \le \frac{|\operatorname{Tr}(A)| \det(A)}{A_1}.$$

So, from the conditions (2.63) and $a_{12}a_{21} \le 0$ it follows that

$$\delta_1 < \frac{|\operatorname{Tr}(A)| \det(A)}{A_2}, \qquad \delta_2 < \frac{|\operatorname{Tr}(A)| \det(A)}{A_1}.$$

Corollary 2.4 *Suppose that the parameters of system* (2.60) *satisfy the conditions* (2.62),

$$a_{21} > 0, \qquad A_2 > \big|\operatorname{Tr}(A)\big|\delta_2, \tag{2.74}$$

and the intervals

$$I_1 = \left(-\frac{a_{12}(a_{22}+\delta_2)}{A_2 - |\operatorname{Tr}(A)|\delta_2}, -\frac{a_{11}+\delta_1}{a_{21}}\right) \tag{2.75}$$

and

$$I_2 = \left(\frac{|\operatorname{Tr}(A)| - \sqrt{(a_{11}-a_{22})^2 + 4\det(A)}}{2a_{21}}, \frac{|\operatorname{Tr}(A)| + \sqrt{(a_{11}-a_{22})^2 + 4\det(A)}}{2a_{21}}\right) \tag{2.76}$$

have common points. Then the trivial solution of system (2.60) *is asymptotically mean-square stable.*

Proof Consider the function $v(t)$ given by (2.65) with $p_{11} = 1$, $p_{12} = \mu$, $p_{22} = \gamma$, where $\gamma = a_{21}^{-1}(\mu|\operatorname{Tr}(A)| - a_{12})$. From (2.64) it follows that $\mu \in I_1$. On the other hand, the function $v(t)$ is positive definite if and only if $\gamma > \mu^2$, which is equivalent to $\mu \in I_2$. So, the appropriate μ exists if and only if the intervals I_1 and I_2 have common points. The proof is completed. □

2.3.3 Some Useful Inequalities

Lemma 2.3 *For arbitrary vectors* $a \in \mathbf{R}^n$, $b \in \mathbf{R}^n$ *and an* $n \times n$ *matrix* $R > 0$, *it follows that*

$$a'b + b'a \le a'Ra + b'R^{-1}b.$$

Proof The proof follows from the simple equality

$$0 \le \big(a - R^{-1}b\big)'R\big(a - R^{-1}b\big) = a'Ra + b'R^{-1}b - a'b - b'a.$$ □

Lemma 2.4 *For positive* P_2, x *and nonnegative* P_1, Q *such that* $P_2 > Qx$, *the following inequality holds*:

$$\frac{P_1 + Qx^{-1}}{P_2 - Qx} \geq \left(\frac{\sqrt{Q^2 + P_1P_2} + Q}{P_2}\right)^2.$$

Proof It is enough to check that the function

$$f(x) = \frac{P_1 + Qx^{-1}}{P_2 - Qx}$$

reaches its minimum at the point

$$x_0 = \frac{P_2}{\sqrt{Q^2 + P_1P_2} + Q}$$

and this minimum equals x_0^{-2}. The proof is completed. □

2.4 Some Unsolved Problems

In spite of the fact that the theory of stability for stochastic hereditary systems is very popular in researches, there are simply and clearly formulated problems with unknown decisions. In order to attract attention to such problems, one of them for stochastic difference equation with continuous time is represented in [277], and two unsolved stability problems for stochastic differential equations with delay are described below.

2.4.1 Problem 1

Consider the linear stochastic differential equation with delays

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} B_i x(t - \tau_i) + \sigma x(t - h)\dot{w}(t), \tag{2.77}$$

where A, B_i, σ, $\tau_i > 0$, $h \geq 0$ are known constants, and $w(t)$ is the standard Wiener process.

It is known [290] that a necessary and sufficient condition for asymptotic mean-square stability of the zero solution of (2.77) can be represented in the form

$$G^{-1} > \frac{\sigma^2}{2}, \quad G = \frac{2}{\pi}\int_0^\infty \frac{dt}{(A + \sum_{i=1}^m B_i \cos \tau_i t)^2 + (t + \sum_{i=1}^m B_i \sin \tau_i t)^2}. \tag{2.78}$$

By Lemma 2.1, in the particular case $m = 1$, $B_1 = B$, $\tau_1 = \tau$ the integral (2.78) can be calculated in elementary functions, and the stability conditions take the form (2.42)–(2.43).

The problem is: *to calculate the integral* (2.78) *in elementary functions for* $m \geq 2$*, in particular, for* $m = 2$.

2.4.2 Problem 2

From (2.42) and (2.43) it follows that the zero solution of the differential equation with a constant delay $\dot{x}(t) = -bx(t-h)$ is asymptotically stable if and only if

$$0 < bh < \frac{\pi}{2}. \tag{2.79}$$

It is also known [221, 318] that the zero solution of the differential equation with a varying delay $\dot{x}(t) = -bx(t-\tau(t))$ is asymptotically stable for an arbitrary delay $\tau(t)$ such that $\tau(t) \in [0, h]$ if and only if

$$0 < bh < \frac{3}{2}. \tag{2.80}$$

Consider the stochastic differential equation with a constant delay

$$\dot{x}(t) = -bx(t-h) + \sigma x(t)\dot{w}(t). \tag{2.81}$$

From (2.42) and (2.43) it follows that the zero solution of (2.81) is asymptotically mean-square stable if and only if

$$0 < bh < \arcsin\frac{b^2 - p^2}{b^2 + p^2}, \quad p = \frac{\sigma^2}{2}. \tag{2.82}$$

In the deterministic case ($\sigma = 0$) condition (2.82) coincides with (2.79).

Consider the stochastic differential equation

$$\dot{x}(t) = -bx\big(t - \tau(t)\big) + \sigma x(t)\dot{w}(t) \tag{2.83}$$

with a varying delay $\tau(t)$ such that $\tau(t) \in [0, h]$.

The problem is: *to generalize condition* (2.80) *for* (2.83).

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