Chapter 1

The Riemannian adiabatic limit

The purpose of this chapter is to study the adiabatic limit of the Levi-Civita connection on a fibred manifold. This study was initiated in [B86a], and continued in Bismut-Cheeger [BC89], Berline-Getzler-Vergne [BeGeV92], Berthomieu-Bismut [BerB94] and Bismut [B97].

This chapter is organized as follows. In Section 1.1, we introduce a smooth proper submersion $p : M \to S$.

In Section 1.2, we construct a family of Riemannian metrics g_ϵ^{TM}, and we study the limit as $\epsilon \to 0$ of the corresponding Levi-Civita connection and of related tensors.

Finally, in Section 1.3, we construct a trilinear form ρ_0 on the tangent bundle TM.

1.1 A smooth submersion

Let M, S be smooth manifolds. Let $p : M \to S$ be a smooth submersion with compact fibre X. Let $TX = TM/S$ denote the relative tangent bundle. We have the exact sequence of smooth vector bundles on M,

$$0 \to TX \longrightarrow TM \xrightarrow{p_*} p^*TS \to 0. \tag{1.1.1}$$

Let g^{TM} be a smooth Riemannian metric on TM, let g^{TX} be its restriction to TX. Let T^HM be the orthogonal bundle to TX in TM with respect to g^{TM}, so that TM splits orthogonally as

$$TM = TX \oplus T^HM. \tag{1.1.2}$$

Clearly p_* induces the isomorphism

$$T^HM = p^*TS. \tag{1.1.3}$$

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By (1.1.2), (1.1.3), we get

$$TM = TX \oplus p^*TS. \tag{1.1.4}$$

If $A \in TS$, let $A^H \in T^HM$ correspond to A via (1.1.3).

Let g^{T^HM} be the metric induced by g^{TM} on T^HM. Then g^{T^HM} can be viewed as a metric on p^*TS.

Let P^{TX}, P^{T^HM} be the projections from TM on TX, T^HM with respect to the splitting (1.1.2).

Let $\nabla^{TM,LC}$ be the Levi-Civita connection on $\left(TM, g^{TM}\right)$.

By the results of [B86a, section 1], $\left(T^HM, g^{TX}\right)$ uniquely determines a metric preserving connection $\nabla^{TX,LC}$ on TX. The connection $\nabla^{TX,LC}$ is the projection of $\nabla^{TM,LC}$ on TX with respect to the splitting (1.1.2), the crucial point being that it only depends on $\left(T^HM, g^{TX}\right)$. The restriction of $\nabla^{TX,LC}$ to one given fibre X is the Levi-Civita connection on the tangent bundle of the fibre.

If A is a smooth section of TS, let L_{A^H} be the Lie derivative operator associated with the vector field A^H. Then L_{A^H} acts on the tensor algebra of TX, and this action is a tensor also in $A \in TS$. By [B97, Theorem 1.1], if A, E are smooth sections of TS, TX, then

$$\nabla^{TX,LC}_{A^H} E = \left[A^H, E\right] + \frac{1}{2}\left(g^{TX}\right)^{-1}\left(L_{A^H} g^{TX}\right) E. \tag{1.1.5}$$

Let g^{TS} be a smooth Riemannian metric on TS. Let $\nabla^{TS,LC}$ be the Levi-Civita connection on $\left(TS, g^{TS}\right)$. We can write g^{T^HM} in the form

$$g^{T^HM} = g^{TS}k, \tag{1.1.6}$$

where k is a smooth section of $p^*\mathrm{End}\,(TS)$ over M which is self-adjoint and positive with respect to g^{TS}.

1.2 The limit of the Levi-Civita connection as $\epsilon \to 0$

For $\epsilon > 0$, let g^{TM}_ϵ be the metric on TM given by

$$g^{TM}_\epsilon = g^{TM} + \frac{1}{\epsilon} p^* g^{TS}. \tag{1.2.1}$$

Using (1.1.4), (1.1.6), we can rewrite (1.2.1) in the form

$$g^{TM}_\epsilon = g^{TX} \oplus g^{TS}\left(\frac{1}{\epsilon} + k\right). \tag{1.2.2}$$

Then g^{TM}_ϵ still induces the metric g^{TX} on TX, and T^HM is the orthogonal bundle to TX with respect to g^{TM}_ϵ. Let $\nabla^{TM,LC}_\epsilon$ be the Levi-Civita connection on TM with respect to the metric g^{TM}_ϵ. By the above, $\nabla^{TM,LC}_\epsilon$ projects on TX as the fixed connection $\nabla^{TX,LC}$.

With respect to the splitting (1.1.4) of TM, $\nabla_\epsilon^{TM,LC}$ can be written in the form

$$\nabla_\epsilon^{TM,LC} = \begin{bmatrix} \nabla^{TX,LC} & S_\epsilon^{TX,LC} \\ -S_\epsilon^{TX,LC*} & \nabla_\epsilon^{TS} \end{bmatrix}. \tag{1.2.3}$$

Let $\nabla_{s,\epsilon}^{TS}$ be the connection on p^*TS,

$$\nabla_{s,\epsilon}^{TS} = \nabla^{TS,LC} + \frac{\epsilon}{2}\left(1+\epsilon k\right)^{-1} \nabla^{TS,LC} k. \tag{1.2.4}$$

Set

$$\nabla_{s,\epsilon}^{TM,LC} = \begin{bmatrix} \nabla^{TX,LC} & 0 \\ 0 & \nabla_{s,\epsilon}^{TS} \end{bmatrix}. \tag{1.2.5}$$

Then $\nabla_{s,\epsilon}^{TM,LC}$ is an Euclidean connection on $\left(TM, g_\epsilon^{TM}\right)$.

Set

$$\nabla_s^{TM,LC} = \begin{bmatrix} \nabla^{TX,LC} & 0 \\ 0 & \nabla^{TS,LC} \end{bmatrix}. \tag{1.2.6}$$

Then $\nabla_s^{TM,LC}$ is an Euclidean connection on $TX \oplus p^*TS$ equipped with the metric $g^{TX} \oplus p^* g^{TS}$.

By (1.2.4), (1.2.5), as $\epsilon \to 0$,

$$\nabla_{s,\epsilon}^{TM,LC} = \nabla_s^{TM,LC} + \mathcal{O}\left(\epsilon\right). \tag{1.2.7}$$

In (1.2.7), $\mathcal{O}\left(\epsilon\right)$ indicates that if K is a compact subset of M, for any $k \in \mathbf{N}$, the coefficients of the considered operator and its derivatives of order $\leq k$ can be dominated by $C_{K,k}\epsilon$. In the whole book, a similar notation will be used for other tensors.

Let $T_{s,\epsilon}, T_s$ be the torsions of $\nabla_{s,\epsilon}^{TM,LC}, \nabla_s^{TM,LC}$. Since $\nabla^{TX,LC}$ is fibrewise torsion free, $T_{s,\epsilon}, T$ both vanish on $TX \times TX$. By (1.2.7), we get

$$T_{s,\epsilon} = T_s + \mathcal{O}\left(\epsilon\right). \tag{1.2.8}$$

By (1.2.4), for $A, B \in TM$,

$$\frac{\partial}{\partial\epsilon} p_* T_{s,\epsilon}|_{\epsilon=0}\left(A,B\right) = \frac{1}{2}\left(\nabla_A^{TS} k p_* B - \nabla_B^{TS} k p_* A\right). \tag{1.2.9}$$

By (1.2.6), T_s takes its values in TX. Since ∇^{TS} is torsion free, if $A, B \in TS$,

$$T_s\left(A^H, B^H\right) = -P^{TX}\left[A^H, B^H\right]. \tag{1.2.10}$$

By [B97, Theorem 1.1] or by (1.1.5), if $A \in TS, E \in TX$,

$$T_s\left(A^H, E\right) = \frac{1}{2}\left(g^{TX}\right)^{-1}\left(L_{A^H} g^{TX}\right) E. \tag{1.2.11}$$

In particular if $A \in TS$, and if $E, F, \in TX$, then

$$\left\langle T_s\left(A^H, E\right), F\right\rangle_{g^{TX}} = \left\langle E, T_s\left(A^H, F\right)\right\rangle_{g^{TX}}. \tag{1.2.12}$$

By the above, we recover the known fact that the tensor T_s depends only on $\left(g^{TX}, T^H M\right)$.

Set

$$S^{TM}_{s,\epsilon} = \nabla^{TM,LC}_{\epsilon} - \nabla^{TM,LC}_{s,\epsilon}. \tag{1.2.13}$$

By (1.2.3), (1.2.5), $S^{TM}_{s,\epsilon}$ is of the form

$$S^{TM}_{s,\epsilon} = \begin{bmatrix} 0 & S^{TX,LC}_{\epsilon} \\ -S^{TX,LC*}_{\epsilon} & S^{TS}_{\epsilon} \end{bmatrix}. \tag{1.2.14}$$

Since $\nabla^{TM,LC}_{\epsilon}$ is torsion free, if $A, B \in TM$, then

$$T_{s,\epsilon}(A,B) = -S^{TM}_{s,\epsilon}(A)B + S^{TM}_{s,\epsilon}(B)A. \tag{1.2.15}$$

Moreover, if $A, B, C \in TM$, we have the classical identity

$$\begin{aligned} 2\left\langle S^{TM}_{s,\epsilon}(A)B, C\right\rangle_{g^{TM}_{\epsilon}} &+ \left\langle T_{s,\epsilon}(A,B), C\right\rangle_{g^{TM}_{\epsilon}} \\ &+ \left\langle T_{s,\epsilon}(C,A), B\right\rangle_{g^{TM}_{\epsilon}} - \left\langle T_{s,\epsilon}(B,C), A\right\rangle_{g^{TM}_{\epsilon}} = 0. \end{aligned} \tag{1.2.16}$$

Using (1.2.8), (1.2.9), the fact that T_s takes its values in TX, and (1.2.16), we find that there is a smooth section $S^{TX,LC}$ of $T^*M \otimes \operatorname{Hom}(p^*TS, TX)$ such that as $\epsilon \to 0$,

$$S^{TX,LC}_{\epsilon} = S^{TX,LC} + \mathcal{O}(\epsilon), \qquad S^{TX,LC*}_{\epsilon} = \mathcal{O}(\epsilon), \qquad S^{TS}_{\epsilon} = \mathcal{O}(\epsilon). \tag{1.2.17}$$

In the sequel, we identify $S^{TX,LC}$ with the corresponding element of $T^*M \otimes \operatorname{End}(TM)$ that vanishes on TX. By (1.2.14), (1.2.17), as $\epsilon \to 0$,

$$S^{TM}_{s,\epsilon} = S^{TX,LC} + \mathcal{O}(\epsilon). \tag{1.2.18}$$

By (1.2.8), (1.2.15), and (1.2.17), if $A, B \in TM$,

$$T_s(A,B) = -S^{TX,LC}(A)P^{T^HM}B + S^{TX,LC}(B)P^{T^HM}A. \tag{1.2.19}$$

By (1.2.16), (1.2.18), if $A \in TM, B \in TS, C \in TX$, then

$$\begin{aligned} 2\left\langle S^{TX,LC}(A)B^H, C\right\rangle_{g^{TX}} &+ \left\langle T_s\left(A, B^H\right), C\right\rangle_{g^{TX}} + \left\langle \frac{\partial}{\partial\epsilon} T_{s,\epsilon}|_{\epsilon=0}(C,A), B^H \right\rangle_{g^{TS}} \\ - \left\langle T_s\left(B^H, C\right), P^{TX}A\right\rangle_{g^{TX}} &- \left\langle \frac{\partial}{\partial\epsilon} T_{s,\epsilon}|_{\epsilon=0}\left(B^H, C\right), P^{T^HM}A \right\rangle_{g^{TS}} = 0. \end{aligned} \tag{1.2.20}$$

By (1.2.9), we can rewrite (1.2.20) in the form

$$\begin{aligned} 2\left\langle S^{TX,LC}(A)B^H, C\right\rangle_{g^{TX}} &+ \left\langle T_s\left(A, B^H\right), C\right\rangle_{g^{TX}} \\ - \left\langle T_s\left(B^H, C\right), P^{TX}A\right\rangle_{g^{TX}} &+ \left\langle \nabla^{TS}_C kB, p_*A\right\rangle_{g^{TS}} = 0. \end{aligned} \tag{1.2.21}$$

Set

$$\nabla_0^{TM,LC} = \nabla_s^{TM,LC} + S^{TX,LC}. \tag{1.2.22}$$

Equivalently,

$$\nabla_0^{TM,LC} = \begin{bmatrix} \nabla^{TX,LC} & S^{TX,LC} \\ 0 & \nabla^{TS,LC} \end{bmatrix}. \tag{1.2.23}$$

By (1.2.7), (1.2.13), and (1.2.18), as $\epsilon \to 0$,

$$\nabla_\epsilon^{TM,LC} = \nabla_0^{TM,LC} + \mathcal{O}(\epsilon). \tag{1.2.24}$$

Since $\nabla_\epsilon^{TM,LC}$ is torsion free, $\nabla_0^{TM,LC}$ is also torsion free. Equation (1.2.19) is a reformulation of this fact.

1.3 The trilinear form ρ_0

Definition 1.3.1. For $A, B, C \in TM$, set

$$\rho_\epsilon(A, B, C) = \left\langle S_{s,\epsilon}^{TM}(A) B, C \right\rangle_{g_\epsilon^{TM}}. \tag{1.3.1}$$

If $A, B, C \in TM$, let $\rho_0(A, B, C) \in \mathbf{R}$ be defined by

$$\begin{aligned} &2\rho_0(A, B, C) + \left\langle T_s(A, B), P^{TX} C \right\rangle_{g^{TX}} + \left\langle T_s(C, A), P^{TX} B \right\rangle_{g^{TX}} \\ &- \left\langle T_s(B, C), P^{TX} A \right\rangle_{g^{TX}} - \left\langle \nabla_B^{TS} k p_* A, p_* C \right\rangle_{g^{TS}} + \left\langle \nabla_C^{TS} k p_* A, p_* B \right\rangle_{g^{TS}} = 0. \end{aligned} \tag{1.3.2}$$

Proposition 1.3.2. *As $\epsilon \to 0$,*

$$\rho_\epsilon = \rho_0 + \mathcal{O}(\epsilon). \tag{1.3.3}$$

Moreover, if $A \in TX, B, C \in TM$, $\rho_0(A, B, C)$ does not depend on g^{TS}, and is given by

$$\begin{aligned} 2\rho_0(A, B, C) + \left\langle T_s(A, B), P^{TX} C \right\rangle_{g^{TX}} &+ \left\langle T_s(C, A), P^{TX} B \right\rangle_{g^{TX}} \\ &- \left\langle T_s(B, C), P^{TX} A \right\rangle_{g^{TX}} = 0. \end{aligned} \tag{1.3.4}$$

Proof. Equation (1.3.3) follows from (1.2.9), (1.2.16), and (1.3.1). When $A \in TX$, $p_* A = 0$, and the last two terms in the left-hand side of (1.3.2) do vanish, so that we get (1.3.4). This shows that $\rho_0(A, B, C)$ does not depend on g^{TS}. The proof of our proposition is completed. □

Definition 1.3.3. Let $S_0^{TM} \in T^*M \otimes \operatorname{End}(TM)$ be such that if $A, B, C \in TM$, then

$$\left\langle S_0^{TM}(A) B, C \right\rangle_{g^{TX} \oplus g^{TS}} = \rho_0(A, B, C). \tag{1.3.5}$$

By (1.2.21), (1.3.2), S_0^{TM} can be written in the form

$$S_0^{TM} = \begin{bmatrix} 0 & S^{TX,LC} \\ -S^{TX,LC*} & S^{TS} \end{bmatrix}. \tag{1.3.6}$$

As the notation indicates, $S^{TX,LC*}$ is the adjoint of $S^{TX,LC}$.

Remark 1.3.4. The trilinear form ρ_0 was already obtained in [B86a, section 1 c)] when the metric g^{TM} defines a Riemannian submersion.

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