Chapter 2
Black Holes in String Theory

Iosif Bena, Sheer El-Showk and Bert Vercnocke

Abstract These lectures notes provide a fast-track introduction to modern developments in black hole physics within string theory, including microscopic computations of the black hole entropy as well as construction and quantization of microstates using supergravity. These notes are largely self-contained and should be accessible to students at an early PhD or Masters level. Topics covered include the black holes in supergravity, D-branes, Strominger-Vafa's computation of the black hole entropy via D-branes, AdS-CFT and its applications to black hole phyisics, multicenter solutions, and the geometric quantization of the latter.

2.1 Why Black Holes?

This is a lecture series about black holes, but that does not mean that every little detail about what a black hole is will be explained. Our purpose is not to give a comprehensive review of the subject, but rather to fast-track interested students and researchers to the "juicy" aspects of the field using as little sophistication as possible. Students who wish to devote the rest of their life to the study of black holes in string theory, while they may find this overview useful, are urged to follow the "classical" route of learning first all the gory details of string theory, then all the gory details of black holes in general relativity, then read ten or fifteen foundational articles from the glorious nineties, as well as a few more recent ones in their preferred sub-area of research.

I. Bena (✉) · S. El-Showk · B. Vercnocke
Institut de Physique Théorique, CEA Saclay, 91191 Gif sur Yvette, France
e-mail: iosif.bena@cea.fr

S. El-Showk
e-mail: sheer.el-showk@cea.fr

B. Vercnocke
e-mail: bert.vercnocke@cea.fr

S. Bellucci (ed.), *Black Objects in Supergravity*, Springer Proceedings in Physics 144,
DOI: 10.1007/978-3-319-00215-6_2,

We start these notes in this section by reviewing the main questions of black hole physics. For more details on the (GR) aspects of black holes, see for instance the Course de Physique at IPhT by Nathalie Deruelle in 2009 [1] and references therein.

Note: In these lecture notes, we choose for a pedagogical referencing style. We refer to useful books, lectures and reviews as much as possible, and we will give only the most relevant original papers when appropriate. More references can be found in the reviews and pedagogical papers we refer to.

2.1.1 Classical Black Holes

Black holes are *classical* solutions that appear naturally in GR. The first black hole metric was written down for the first time almost a century ago by Karl Schwarzschild (although at that point it was only used to model the geometry outside of a spherically symmetric object as the Sun or the Earth). It is a solution to the Einstein equations determined by one parameter, the mass.

Very crudely, we can picture such a black hole as a region of spacetime in which things can fall, or be thrown in, but nothing comes out, see Fig. 2.1 for a cartoon. The boundary from which no round-trip tickets are available any more, is called the event horizon. The name "black hole" fits very well: classically, a black hole does not emit anything, not even light.

We can say more than just drawing cartoons. In GR, there is a very well-defined picture one can make of a spacetime that showcases its causal properties, while it still fits on a page: the Penrose diagram. It can be obtained by performing a conformal transformation (scaling) on the metric. The Penrose diagram is then a two-dimensional picture of the conformal metric. The key feature is that time-like surfaces (light-rays) are still at 45° angles and we can therefore easily infer the causal structure of the spacetime. The Penrose diagram for the Schwarzschild black hole is shown in Fig. 2.2.

Any object travels on a causal curve: it has to stay within its future lightcone. We see that once something falls into the horizon, it can never get out again. From

Fig. 2.1 A *classical black hole* is the ultimate solution for those smelly diapers of your one-year-old daughter, nagging mother-in-laws or ageing national monuments: you can throw things in, but nothing comes out

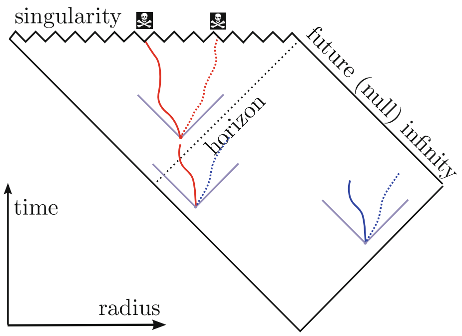


Fig. 2.2 The Penrose diagram for the Schwardschild metric. Some lightcones and particle trajectories are drawn outside and inside the *black hole* horizon. Note that the singularity (*sawtooth line*) is in the causal future of any object that falls behind the horizon

the Penrose diagram, we also see that anything that falls in will further collapse and eventually hit the singularity.

Two important observations where made by Carter, Hawking, Penrose…from the 1960s onwards:

- No memory in horizon region of what the black hole is made of, this region is smooth and has no special features:
  "Black holes have no hair"

- Black hole uniqueness theorems (1960s–1970s):
  A static black hole is fully characterized by its mass.[1]

A black hole of a certain mass could thus be made up out of anything: ipods, elephants, grad students…from the outside it will look the same.

2.1.2 *A Little Bit of Quantum Mechanics*

What happens if we add quantum mechanics to the game? The region of spacetime around the horizon of a black hole has a curvature and hence a certain energy density. We know that in QFT, energy can decay into a particle-antiparticle pair. This idea has led Hawking to perform a semiclassical analysis of QFT in a black hole background. Through the Hawking process, pairs will be created and once in a while one of the two falls into the black hole horizon, while the other escapes off to spatial infinity.

---

[1] The more general time-independent solution, a stationary black hole, is fully determined by its mass ánd angular momentum. When GR is coupled to an electromagnetic field, a black hole can have an electric and a magnetic charge as well. However, there is no additional memory of what formed the black hole: there are no higher multipole moments etc.

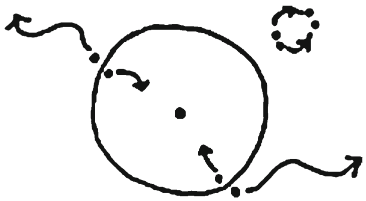


Fig. 2.3 A cartoon of the Hawking process. The *black hole* geometry is pictured as a point, the singularity, surrounded by a horizon. A QFT calculation in the *black hole* spacetime leads to pairwise particle creation such that close to the horizon, one of these particles can fall into the horizon, the other escaping to infinity

The net result is that the black hole mass is lowered and energy, under the form of thermal radiation, escapes to infinity, see Fig. 2.3.

The black hole behaves as a black body, with a temperature proportional to the strength of the gravitational field at the horizon. One finds this temperature is inversely proportional to the black hole mass:

$$T = \frac{\hbar c^3}{k_B} \frac{1}{8\pi G_4 M} \simeq 6 \times 10^{-8} \left(\frac{M}{M_\odot} \right) \text{Kelvin} \tag{2.1}$$

where $M_\odot \simeq 2 \times 10^{30}$ kg is the mass of the sun. The bigger the black hole is (more mass), the lower its gravitational field a the horizon and hence how lower its temperature. For a typical astrophysical black hole, ranging from several to several million solar masses, this is a very small temperature.

By the laws of black hole thermodynamics, a black hole also has an entropy. It was first conjectured by Bekenstein [2] and later proven by Hawking [3] that this entropy is proportional to the area of the black hole horizon:

$$S_{BH} = \frac{A_H}{4G_N}, \tag{2.2}$$

where G_N is Newton's constant, related to the Planck length as $G_N \sim l_P^2$. In Planck units, we thus have $S_{BH} = A_H/4l_P^2$ with $l_P \simeq 1.6 \times 10^{-35}$ m. The entropy of a typical black hole will thus be very large. For a Schwarzschild black hole, we find that the Bekenstein-Hawking entropy is proportional to the square of the black hole mass:

$$S_{Schw} \simeq 10^{76} \times \left(\frac{M}{M_\odot} \right)^2. \tag{2.3}$$

This is a huge entropy! For a solar mass black hole (which would have a radius of about 3 km) we find 10^{76}, for the black hole in the center of our galaxy of several million solar masses, we find about $S_{Gal} \simeq 10^{90}$.

How should we understand this entropy? Boltzmann has taught us that the entropy is related to a number N of microstates, microscopic configurations with the same macroscopic properties:

$$S = \log(N). \tag{2.4}$$

We would hence conclude that the quantum mechanics of black holes leads to an incredibly large amounts of microstates: $N_{QM} \sim e^{10^{76}}$. However, in the classical GR picture we do not understand this number, as there is only one stationary solution with the black hole mass (the macroscopic parameter of the configuration): $N_{GR} = 1$. This numerical discrepancy is the largest unexplained number in theoretical physics.[2]

2.1.3 Problems

- **Where are the microstates?** Maybe the $N = \exp(S_{BH})$ states live in the region of the singularity, and GR just does not see them? Recent arguments by Mathur and others point out that this would not solve the information paradox (second point), and black hole microstates should differ from the black hole significantly also at horizon scales. Such 'microstate geometries' do not exist within general relativity.
- **Information paradox**. The Hawking radiation process has positive feedback: as a black hole radiates, it loses mass, increasing its temperature, which increases the rate of radiation. If we wait long enough, by the Hawking process a black hole will continue radiating until all of its mass is radiated away and we are left with only thermal radiation. This leads to a problem: where has the information of the initial state gone? Once a black hole forms, the spacetime is completely determined by the mass. All other information of the initial state that went into the black hole seems gone: whether we make a black hole out of 2 seven-ton elephants, or 200 seventy-kilogram graduate students, the classical black hole geometry is indistinguishible. As the black hole evaporates, only the thermal radiation comes out, there is no information about the initial state in the Hawking radiation neither.

Note that a black hole we start from that goes to a universe without black hole, but filled with thermal radiation cannot be obtained by unitary evolution. People have come up with many ideas to solve this problem: maybe physics is not unitary, or the black hole does not evaporate completely and there is a remnant with high entropy, and other explanations. Not one has proven satisfactory. Currently, the most popular viewpoint among string theorists is that the physics is nuitary, the information paradox is just an artefact of semiclassical gravitational physics.

We would like to solve these problems. The solution is in the study of black holes in a quantum gravity theory, that can unify classical GR with quantum mechanics.

[2] For comparison, the famous cosmological constant problem is the large ratio $\Lambda_{QFT}/\Lambda_{obs} \sim 10^{120}$ between the "expected" value Λ_{QFT} and the observed value Λ_{obs}. This number is peanuts compared to the required number of black hole microstates!

String theory is a powerful mathematical framework that does exactly this. We do not have to believe that this theory describes the real world. As a quantum gravity theory, string theory can be tested by its answers to the issues related to black holes (information problem, entropy problem). If it does not pass this test, and cannot solve these problems, we throw it to the garbage as a quantum theory of gravity. If it does, we can start thinking about other tests and problems to attack—and maybe start believing it describes the real world after all.

2.2 Building Blocks

In this section, we provide the tools to construct black hole solutions of string theory. It is not our intention to give a lecture series on string theory: We will not tell you how to build the computer, but how to *programme* it. For further information on string theory basics, see the textbooks [4–9], and for supergravity, the low-energy limit of string theory, see [10].

2.2.1 Caught in the Web

String theory is a framework that has grown dynamically over the past thirty or so years. Various limits of this theory have been studied, see Fig. 2.4. Historically all the corners of this diagram were constructed as different theories and only about 15 years ago it was realized that they were all related through various dualities, and can be seen as limits of one theory. We reserve the term "string theory" for the encompassing framework.[3]

In these lectures, we will only consider M-theory, type IIA and type IIB string theory. The natural geometric interpretation of M-theory is 11-dimensional, while the type II strings live in ten dimensions. We will mainly study the low-energy limits of string theory. "Low energy" is relative. We mean that we stick to the zero mass

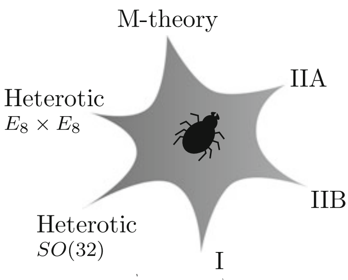


Fig. 2.4 We should view string theory as a web, of which we understand several corners, where perturbative and other techniques can be used. In these lectures, we will only consider M-theory, type IIA and type IIB string theory

[3] Often people refer to the entire framework as "M-theory". We like to view this eleven-dimensional theory as one of the corners of the string web instead.

Table 2.1 The theories we work in

Theory	Low-energy limit
M	11d supergravity
IIA	10d IIA supergravity
IIB	10d IIB supergravity

fields of the string spectrum. The low-energy limits of string theories are supergravity theories: gravity theories that are extensions of general relativity with other fields, whose couplings are fixed by the requirement of supersymmetry. See Table 2.1.

2.2.2 An Analogy for M Theory

To get a grip on the field content of these higher-dimensional beasts, we first make an analogy with Maxwell theory in four dimensions.

Maxwell Theory

The action for Maxwell theory coupled to gravity is:

$$S = \int d^4x \sqrt{-g}\,(R + \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}), \tag{2.5}$$

where $F_{\mu\nu}$ is the electromagnetic field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \equiv 2\partial_{[\mu} A_{\nu]}$.

What are the fundamental objects in this theory?

- **Electrons**. An electrically charged particle with electric charge e couples to the electric field as

$$S_{el} = e \int \left[A_\mu \frac{dx^\mu}{d\tau} \right] d\tau, \tag{2.6}$$

  where τ parameterizes the world-line of the particle and $x^\mu(\tau)$ describes the embedding of the particle's world-line in space-time (Fig. 2.5). A particle that is not moving in a certain reference frame, couples to the time component of the electric field as $e \int A_0 dx^0$ with $x^0 = \tau$. The electric field profile sourced by such a field is

$$A_0 = \frac{e}{r}, \qquad \boldsymbol{E} = \nabla A_0 = -\frac{e}{r^2} \boldsymbol{u}_r, \tag{2.7}$$

  where $\boldsymbol{u}_r$ is a unit vector in the radial direction. Note that a moving electron couples to magnetic components A_i of the gauge field as well.
  The electric field of a charged particle solves Maxwell's equation's (the equations of motion for the field A_μ) with a delta-function source:

$$\nabla^2 A_0 = e\delta(\boldsymbol{r}). \tag{2.8}$$

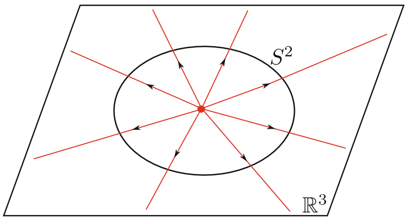


Fig. 2.5 Magnetic field lines from a magnetic monopole. The total charge is measured by integrating the flux over a surface (for instance a two-sphere) surrounding the source

- **Magnetic monopoles**. In theory, there can also be magnetically charged particles in four dimensions. These are monopole sources of the magnetic field. The charge of these particles can be measured by integrating the magnetic field lines over a two-sphere surrounding the charge (see Fig. 2.6a):

$$g_M = \frac{1}{4\pi}\int_{S^2} F_{\mu\nu} dx^\mu dx^\nu . \tag{2.9}$$

The magnetic monopole sources a profile for the magnetic field. For a flat metric $g_{\mu\nu} = \eta_{\mu\nu}$, we have:

$$F_{ij} = -\epsilon_{ijk} B^k . \tag{2.10}$$

The coupling to the electromagnetic field is found in an indirect way. Just as the electron couples to the gauge field, the magnetic monopole couples to the (Hodge) *dual* electric field:

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\sqrt{-g}\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} , \tag{2.11}$$

as

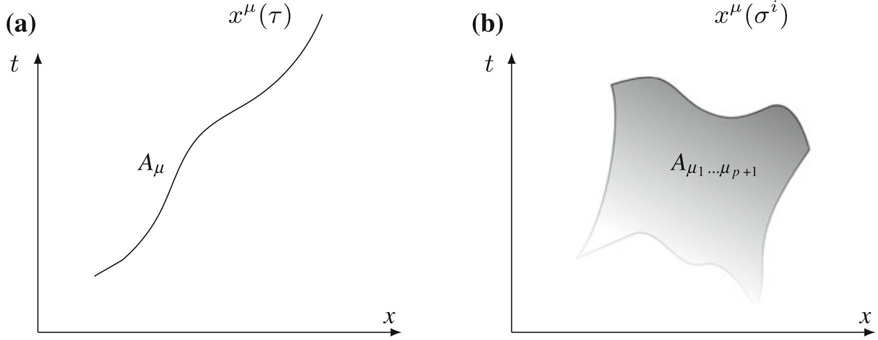


Fig. 2.6 A charged particle traces out a one-dimensional world line, its higher dimensional analogue (a p-brane) traces out a $(p+1)$-dimensional world volume, sourcing a $(p+1)$-form potential. For a p-brane, we parametrize the world volume in terms of σ^i ($i = 0 \ldots p$). **a** A charged particle. **b** A p-brane

$$S_{mag} = g_M \int \left[\tilde{A}_\mu \frac{dx^\mu}{d\tau} \right] d\tau. \tag{2.12}$$

The dual field sourced by a static magnetic monopole is then

$$\tilde{A}_0 = \frac{g_M}{r}. \tag{2.13}$$

In flat space with metric $ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$, this gives the magnetic field in polar coordinates (using 2.11)

$$F_{\theta\phi} = g_M \sin\theta\,, \qquad \text{or} \qquad \boldsymbol{B} = -\frac{g_M}{r^2}\boldsymbol{u}_r. \tag{2.14}$$

where $\boldsymbol{u}_r$ is a unit vector in the radial direction.

Exercise 2.2.1 *Show that the magnetic monopole field solves the Bianchi identity up to a delta-function source:*

$$\partial_r F_{\theta\phi} + \partial_\theta F_{\phi r} + \partial_\phi F_{r\theta} = g_M \delta(\boldsymbol{r}). \tag{2.15}$$

Hint: integrate the equation on a ball of arbitrary radius R centered at $\boldsymbol{r} = 0$ (ball means a 'filled' two-sphere here). You can use the integral $\int_{r=0}^{r=R} \sqrt{g} dr\theta d\phi$ with the metric

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.16}$$

Eleven-Dimensional Supergravity

The features of eleven-dimensional supergravity (the low-energy limit of M-theory) are very similar to those of four-dimensional Einstein-Maxwell theory. The bosonic fields are again the metric and a gauge field, which is now a *three*-form potential $A_{\mu\nu\rho}$, instead of the one-form of Maxwell theory. These fields and their couplings are dictated by supersymmetry: supergravity theories are theories of gravity that are (locally) supersymmetric, and due to this extra symmetry, the possible fields and their couplings are constrained.

The three-form has a four-form field strength. We will often use form notation instead of writing everything out in components. The four-form field strength of M-theory is written as

$$F_4 = \frac{1}{4!} F_{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma. \tag{2.17}$$

where $F_{\mu\nu\rho\sigma}$ are the components of a four-form gauge field

$$F_{\mu\nu\rho\sigma} = 4!\partial_{[\mu} A_{\nu\rho\sigma]}. \tag{2.18}$$

The Lagrangian for eleven-dimensional supergravity is [11]

$$S = \int d^{11}x\, \sqrt{-g}\, \left(R + \tfrac{1}{2}\tfrac{1}{4!} F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma}\right) + \frac{1}{3}\int A_3 \wedge F_4 \wedge F_4, \qquad (2.19)$$

The last term does not contain the metric, it is topological. For single electric or magnetic sources this so-called Chern-Simons term does not contribute. We focus only on the other terms in the action, which are the straightforward generalization of Einstein-Maxwell theory.

What are the fundamental charged objects of this theory?

- **Electric object: M2-brane**. The counterpart of the electron (which couples to the gauge field component A_0) is an object that couples to the electric component of the three-form potential C_{0ij}. Because of the additional directions, this potential couples naturally to a two-dimensional extended object or membrane, with a three-dimensional world volume Σ (generalizing the particle with a one-dimensional world volume). This membrane of M-theory is also called M2-brane. For a membrane extended along the directions x^1, x^2 we have:

$$S_{M2} = Q_{M2} \int_{\Sigma} C_{012} dx^0 dx^1 dx^2. \qquad (2.20)$$

  where Q_{M2} is proportional to the charge of the M2-branes.
- **Magnetic object: M5-brane**. In analogy with the magnetic particle, we can also consider a magnetic monopole charge for the field strength $F_{\mu\nu\rho\sigma}$. To measure its charge, we have to integrate the field strength over a *four*-sphere, see Fig. 2.7:

$$Q_{M5} = \frac{1}{\mathrm{vol}(S^4)} \int_{S^4} F_{\mu\nu\rho\sigma} dx^\mu dx^\nu dx^\rho dx^\sigma, \qquad (2.21)$$

  From Fig. 2.7 we can also find the dimensionality of the magnetic monopole of M-theory. The field lines run in a five-dimensional transverse plane (directions

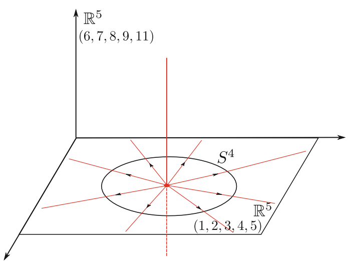


Fig. 2.7 Magnetic field lines from the M-theory magnetic are integrated over an S^4 in the transverse $\mathbb{R}^5(x^1 \ldots x^5)$. Hence, this magnetic monopole is a membrane extending in five space dimensions $(x^6 \ldots x^{10})$

1,2,3,4,5) and the magnetic monopole takes up the remaining five dimensions (6,7,8,9,11).[4] This object is called the M5-brane.

2.2.3 *Type II String Theory*

We relate ten-dimensional string theories and M-theory. See Chap. 8 of [8] for a more detailed account.

Type IIA Supergravity from Dimensional Reduction

Consider eleven-dimensional M-theory. We imagine making the direction x^{11} small and 'compactifying' it on a circle. See Fig. 2.8. What happens to the objects of M-theory? There are two distinct possibilities for each fundamental object: either the world-volume of the object is wrapped on x^{11}, meaning that one of its directions shrinks away, or the world-volume is completely inside the ten large dimensions of space-time. We summarize the possibilities for M-theory objects in Table 2.2.

An important new object is the momentum wave. Because we compactify on a circle, momentum along x^{11} is quantized and momentum waves excitations have a discrete mass spectrum:

$$m = \frac{1}{\ell_{11}}, \frac{2}{\ell_{11}}, \frac{3}{\ell_{11}} \dots \tag{2.22}$$

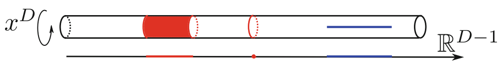


Fig. 2.8 Curling up one out of D dimensions makes a space-time look essentially $(D-1)$-dimensional. An object that is wrapped on the compact dimension has a world-volume of one dimension lower (a membrane becomes a string, a string becomes a point etc.), an unwrapped object remains of the same dimension

Table 2.2 Objects in IIA after compactifying M-theory on a circle

M-theory		IIA supergravity	
Object	Directions	Object	Directions
M2	0, 1, 11	String	0, 1
	0, 1, 2	Membrane	0, 1, 2
M5	0, 1, 2, 3, 4, 11	4d membrane	0, 1, 2, 3, 4
	0, 1, 2, 3, 4, 5	5d membrane	0, 1, 2, 3, 4, 5
Mom. wave	0, 11	Particle	0

[4] We choose to write the time directions as x^0 and space time directions $x^1, x^2, \dots$. However, we choose the 'eleventh' dimensions to be x^{11} and skip x^{10}.

where ℓ_{11} is the radius of the circle. Upon compactification, momentum waves have quantized excitations and become point particles.

We can now interpret all these new objects after compactification. The resulting ten-dimensional theory is called IIA string theory.[5] Its low-energy limit is IIA supergravity. It was found independently in the 1980s and only in the mid-1990s people realized its connection to eleven-dimensional supergravity and M-theory through compactification. The objects of IIA string theory, which were found earlier through quantization of the IIA string, correspond exactly to what we found above from compactifying M-theory (see [7] and references therein to guide you to the original works on the quantization of the IIA string). These are organized in two sectors[6]:

The NS–NS Sector

It contains the following objects:

- F1: the fundamental quantized string of IIA string theory. It comes from an M2-brane wrapped on x^{11}.
- NS5-brane: this is not a D-brane, but is in fact the 'magnetic monopole' associated to the 'electric' F1. It descends from the non-wrapped M5-brane.

The R–R Sector

These are Dirichlet branes, or D-branes for short. They arise from possible Dirichlet boundary conditions one can put on an open fundamental string. One finds that, depending on the type of string theory, only certain dimensionalities of submanifolds of space-time can provide such Dirichlet-boundary conditions while remaining stable objects. These are the allowed D-branes. In IIA one only finds stable D-branes of even dimensions (D0, D2, D4…). Surprisingly, one finds that these D-branes not only describe boundary conditions for strings, but they can also have a dynamics of their own. We will expand on this as we go on.

The relation of the D-branes to M-theory is:

- D0-brane: or D-particle, coming from a momentum wave along the compact eleventh dimension (eleven-dimensional metric degree of freedom).
- D2-brane: the D2-brane is an M2-brane that is not wrapped on the compact direction.

---

[5] Type IIA supergravity is one of the two possible ten-dimensional supergravity theories invariant under $\mathcal{N} = 2$ supersymmetry, namely the one for which the two supersymmetry generators (spinors) have opposite chirality. The other $\mathcal{N} = 2$ supergravity in ten dimensions is type IIB supergravity, the low-energy limit of IIB superstring theory, which has two supersymmetry generators with the same chirality.

[6] Different boundary conditions for the fermionic fields living on the world-volume of the type II string give different possible fields in the string spectrum. In the massless spectrum we observe that Neveu-Schwarz-boundary conditions (anti-periodic) give the NS-fields: metric $g_{\mu\nu}$, B-field $B_{\mu\nu}$, and dilaton ϕ. Ramond boundary conditions (periodic) give RR fields $C^{(0)}, C^{(2)}, C^{(4)}$.

- D4-brane: an M5-brane wrapped on x^{11}. It is the magnetic monopole associated to the D2-brane.
- D6-brane: yet another D-brane in the string spectrum. It descends from a certain smooth type of geometry in M-theory known as the Kaluza-Klein monopole, and is the magnetic equivalent of the D0-brane.

IIA Supergravity Action

We have seen what are the objects that appear in IIA string theory. Let us summarize the fields they couple to, and give the low-energy effective action of type IIA string theory. ('Low energy' is relative and means energies E well below the scale set by the string length $E \ll 1/l_s$. The energies reached in present-day accelerators are 'low' in this terminology.) In this limit, the only vibration modes of the string that are of relevance are the massless modes. They are described by type IIA supergravity. We are only concerned with the bosonic content of the theory, given by the following fields.

- The ten-dimensional *metric*, with components $g_{\mu\nu}$. Its excitations are gravitons.
- The *dilaton* ϕ. This is a scalar field. Its vacuum expectation value sets the value of the string coupling as $g_s = \langle e^{\phi} \rangle$. In eleven-dimensional M-theory, it is a metric component that sets the size of the eleventh direction. It plays an important role in string theory – it sets the value of the string coupling and determines the validity of perturbative string theory. When the eleventh dimension is small, we get weakly coupled IIA string theory and conversely, the strongly coupled limit of IIA theory opens up an extra space-time dimension giving M-theory. We will not consider the dilaton further.
- An *antisymmetric two-form* field with components B_{MN}. This is the gauge field three-form potential C of M-theory with one compactified direction:

$$B_{\mu\nu} \equiv C_{\mu\nu 11}. \tag{2.23}$$

  This field couples electrically to the F1 string and magnetically to the NS5-brane.
- *Higher-form gauge fields*. These are generalizations of the Maxwell field A_μ of four dimensions. We have a one-form potential with components C_μ and a three-form with components $C_{\mu\nu\rho}$.[7] The gauge field C_μ for the D0-brane is related to the eleven-dimensional metric $g^{(11)}_{\mu\nu}$ as $C_\mu = g^{(11)}_{\mu 11}$ (up to a factor involving the dilaton). Its magnetic monopole source is the D6-brane. In a similar fashion, the components of the three-form gauge field $C_{\mu\nu\rho} \equiv A_{\mu\nu\rho}$ in ten dimensions define the Ramond-Ramond three-form gauge field and they couple electrically to the D2 branes and magnetically to the D4-branes.

From now on, we use differential form notation and write B_2, C_1, C_3 (for instance $C_1 = C_M dx^M$ and $B_2 = \frac{1}{2} B_{MN} dx^M \wedge dx^N$) with associated field strengths

[7] We adopt common notation C for the Ramon-Ramond gauge field in ten dimensions that couple to D-branes, and A for the gauge field in elven dimensions that couples to M-branes.

$H_3 = dB_2$, $F_2 = dC_1$, $F_4 = dC_3$. All the fields above form the bosonic content of the type IIA supergravity action, which is, up to two derivatives, completely determined by supersymmetry to have the form [7]

$$S = \frac{1}{16\pi G_{10}} \int d^{10}x\, e^{-2\phi} \sqrt{-g} \left(R - \frac{1}{2}|H_{(3)}|^2 - \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}|F_{(2)})|^2 - \frac{1}{2}|\tilde{F}_{(4)}|^2 \right)$$
$$- \frac{1}{16\pi G_{10}} \int \frac{1}{2} B_{(2)} \wedge F_{(4)} \wedge F_{(4)}. \tag{2.24}$$

where G_{10} is Newton's constant in ten dimensions, we introduced $\tilde{F}_{(4)} \equiv F_{(4)} - C_{(1)} \wedge H_{(3)}$ and we have the notation $|F_{(n)}|^2 = \frac{1}{n!} F_{\mu_1 \ldots \mu_n} F^{\mu_1 \cdots \mu_n}$ and likewise for $|H_{(3)}|^2$.

Historically, all these higher-form gauge fields were first found in the spectrum of string theory, but people had at that point (the 1980s) no idea what objects they coupled to. It took until the mid-1990s ago before it was realized that the objects the R–R fields couple to are in fact the Dirichlet-branes.

In a similar way, IIB string theory has a plethora of higher-dimensional objects. The NS-sector (including the F1 string and the NS5 brane) also appears, but IIB has only stable branes of *uneven* dimensionality, versus the *even* branes of IIA. See Table 2.3.

Dualities

One may wonder how to relate IIB to IIA and M-theory, since at this point we wrote down the fields in a rather ad hoc way. The clue lies in several dualities of the string spectrum.

S-duality

We first focus on a symmetry of the spectrum of the IIB string. We observe that the spectrum can be organised in pairs of the same dimensions: F1–D1, NS5–D5 (we

Table 2.3 Coupling of branes to n-form potentials. In ten dimensions, an $n = (p+1)$-form potential couples to a p-brane through an electric coupling and to a $(6-p)$ through a magnetic coupling

Potential	IIA			IIB			
	B_2	C_1	C_3	B_2	C_0	C_2	C_4
Electric	F1	D0	D2	F1	D(−1)	D1	D3
Magnetic	NS5	D6	D4	NS5	D7	D5	D3

We give the brane couplings of the NS–NS sector (F1 stands for fundamental string, NS5 for the magnetically dual five-brane) and R–R sector of type IIA and type IIB string theory. (We do not consider the IIA (magnetic) D8-brane and its electric counterpart. The D(−1) brane should be seen as an instanton.)

also have NS7–D7, but that example is a little special so we ignore it further and refer the interested reader to the literature [12, 13] and [14–16]). This corresponds to the pairing of the B-field $B_{\mu\nu}$ with the RR two-form $C_{\mu\nu}$ and the same for their magnetic dual fields $\tilde{B}$ and $\tilde{C}$ (which are in fact 6-forms as Exercise 2.2.2 asks you to show).

Exercise 2.2.2 *Generalize the dualization rule (2.11) for two-forms in four space-time dimensions to arbitrary dimensions D and arbitrary p-forms (you need the inverse metric to raise indices). This operation is called Hodge duality (see for instance []). Use this to write down which form couples to which brane in both IIA and IIB theory.*

What about the D3 brane? What does it pair up with? The D3-brane couples electrically to a four-form potential, with a five-form field strength F_5. In fact, in IIB supergravity, F_5 obeys the property

$$F_{\mu_1\ldots\mu_5} = \tilde{F}_{\mu_1\ldots\mu_5} \equiv \frac{1}{5!}\sqrt{-g}\epsilon_{\mu_1\ldots\mu_5\,\mu_6\ldots\mu_{10}}F^{\mu_6\ldots\mu_{10}} \tag{2.25}$$

and therefore, using Exercise 2.2.2, the five-form field strength that couples to the D3 brane is self-dual $F_5 = \tilde{F}_5$. Hence the D3 brane 'pairs up with itself': the D3-brane is a dyon, it is both an electrically charged brane and a magnetic monopole! We will see below that this dyonic nature separates the D3-brane from the other branes.

There exists a clean symmetry interchanging the fields B_2 with C_2, while leaving F_5 unaltered. This transformations is called S-duality and it interchanges F1's with D1's, D5's with NS5's and leaves the D3 brane as it is. It is a very useful transformation in navigating through the zoo of brane solutions.[8]

T-duality

There is another symmetry that maps the string spectra of *different* string theories onto each other. Imagine wrapping the IIA string on a circle of radius R. A string wrapped on the compact dimension has a mass proportional to its tension T_{F1} times the radius R of the string. The string length ℓ_s is related to the string tension as $T_{F1} = 1/2\pi(\ell_s)^2$, so this mass comes in fundamental units of R/ℓ_s^2. The number of units is a topological number and describes how many times the string winds along the compactified dimensions. We call them (string) winding modes.

We can also put momentum modes on the string. These momentum modes should be viewed as oscillations travelling on the string. Again, these fundamental string excitations come in quanta, proportional to $1/R$; for larger radius R, the energy cost of a momentum mode goes down. We can play the same game for IIB string theory compactified on a circle of radius $\tilde{R}$. See Table 2.4 and Fig. 2.9.

[8] In the near-horizon geometry of a D3 brane, which is $AdS_5 \times S^5$ as we will see below, S-duality becomes the strong-weak coupling duality of $N = 4$ super Yang-mills, the theory dual to the $AdS_5 \times S^5$ background through the AdS/CFT duality.

Table 2.4 The mass m of winding and momentum modes of IIA string theory compactified on a circle of radius R and IIB theory compactified on a circle of radius $\tilde{R}$

IIA	Winding	Momentum	IIB	Winding	Momentum
$m =$	$\frac{R}{(\ell_s)^2}$	$\frac{1}{R}$	$m =$	$\frac{\tilde{R}}{(\ell_s)^2}$	$\frac{1}{\tilde{R}}$
	$\frac{2R}{(\ell_s)^2}$	$\frac{2}{R}$		$\frac{2\tilde{R}}{(\ell_s)^2}$	$\frac{2}{\tilde{R}}$
	$\frac{3R}{(\ell_s)^2}$	$\frac{3}{R}$		$\frac{3\tilde{R}}{(\ell_s)^2}$	$\frac{3}{\tilde{R}}$
	…	…		…	…

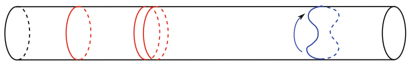


Fig. 2.9 *Left* (in *red*): a string winding one or several times around the compact dimensions, *right* (*blue*): a vibrational or momentum mode of the string

It turns out that the spectra of IIA and IIB compactified on circles of radius R and $\tilde{R} = (\ell_s)^2/R$ are *exactly* mapped into each other under T-duality: momentum modes map to winding modes and vice versa. See also Table 2.4. We reserve p for the units of momentum charge and F1 for the amount of string winding. Schematically, T-duality thus acts as:

$$\begin{array}{lcl} \textbf{IIA} & & \textbf{IIB} \\ \hline \mathrm{F1} & \longleftrightarrow & p \\ p & \longleftrightarrow & \mathrm{F1} \end{array}$$

The symmetry of the string spectra in these two different string theories opens up a huge portion of parameter space where we can actually have a geometric interpretation of string theory. Say we consider type IIA string theory. As long as R is large compared to the string scale, we have a pretty good control because string excitations behave as particles and we can use the supergravity approximation (action contains no more than two space-time derivatives). However, when the size of the circle is small compared to the string length scale, corrections due to the stringy nature are huge and we lose this control. Then T-duality makes it possible to go to type IIB theory with $\tilde{R} \gg \ell_s$. (Note that for circle radius $R \simeq \ell_s$, we still cannot say too much.)

Dualities for D-branes

Consider the setup of Fig. 2.10. We compactify string theory on a circle. A brane that is wrapped on this circle, will no longer extend along this direction after T-duality. Conversely, a D-brane that does not wrap the T-duality circle, will become a D-brane of one dimension higher wrapping the circle after T-duality.

To get the gist of it, we apply T-duality on the (supersymmetric) intersection of two species of D-branes. Let us start from a D3-D3 brane intersection in type IIB

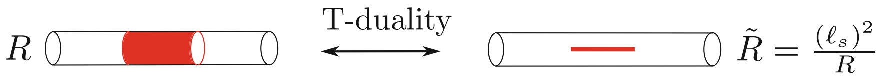


Fig. 2.10 Under T-duality, a D-brane wrapping the *circle* is mapped to a D-brane of one dimension lower and vice versa

$$\begin{array}{r|llllll} \text{IIB}: \text{D3} & 0 & 1 & 2 & 3 & & \\ \text{D3} & 0 & & & 3 & 4 & 5 \end{array}$$

Say we compactify the 3-direction. Under a T-duality to IIA, we get the branes:

$$\begin{array}{r|lllll} \text{IIA D2} & 0 & 1 & 2 & & \\ \text{D2} & 0 & & & 4 & 5 \end{array}$$

We can continue on this, see Exercise 2.2.3.

Exercise 2.2.3 *Show that three additional T-dualities on the two orthogonal D2-branes, along directions 1, 2 and 3 give the D1-D5 brane intersection:*

$$\begin{array}{r|llllll} \text{IIB}: \text{D1} & 0 & & & 3 & & \\ \text{D5} & 0 & 1 & 2 & 3 & 4 & 5 \end{array}$$

We will use this brane setup ('D1-D5 system') a lot in the study of black holes and their entropy.

Similarly, we can consider S-dualities. For instance, the D1-D5 setup of Exercise 2.2.3 becomes after S-duality:

$$\begin{array}{r|llllll} \text{IIB}: \ \text{F1} & 0 & & & 3 & & \\ \text{NS5} & 0 & 1 & 2 & 3 & 4 & 5 \end{array}$$

We see that the dualities give some insight in an entire zoo of complicated D-brane configurations. On the level of supergravity, they form a solution generating tool (see the next section). We can interpret all these two-brane intersections as really one solution, which takes on different forms in different 'duality frames'. We can get the supergravity solution in any frame in no time from the T-duality rules. This applies equally well to any other brane solution.

We will make extensive use of T- and S-dualities on black hole solutions. This will map to black holes which may look a bit different, but all have the same physical properties (entropy, temperature…). We will always work in the duality frame most adapted to the questions we are asking at that moment. In particular, we will often work in the D1-D5 duality frame of Exercise 2.2.3.

2.2.4 p-brane Supergravity Solutions

Let us consider some actual supergravity limits of Dp-brane solutions. For further references, see the complete, but extremely short account of [17], some more information for instance in [18, 19] or [20] for a more black hole oriented Dp-brane review (p runs over the allowed integers).

2-Brane Solution

For concreteness, we discuss the D2-brane solution of IIA supergravity, extending along directions 0,1,2 (time and two space directions). As in the analogy with electromagnetism, this brane sources a three-form potential C_{012}. It has a non-zero tension or mass density and hence it also couples to the metric. There is a third field it sources, the dilaton.

The exact way the D2 brane source affects those fields, is through one function of the space-time coordinates. We call that function Z. One finds the metric

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = Z^{-1/2}(-dx_0^2 + dx_1^2 + dx_2^2) + Z^{1/2}(dx_3^2 + \ldots + dx_9^2) \quad (2.26)$$

and the other non-zero fields are the three-form that couples to the 2-brane and the dilaton:

$$C_{012} = Z^{-1}, \qquad e^\phi = Z^{1/4}. \quad (2.27)$$

We will not consider the dilaton ϕ any further. Concentrating on the other fields, we see that the solution has Lorentz invariance along the D2 brane directions 0, 1, 2 and Euclidean symmetry in the transverse directions.

The D2 brane behaves as a point particle in the transverse $\mathbb{R}^7$. The function Z plays the role of the Maxwell potential in the transverse $\mathbb{R}^7$. From the supergravity equations of motion, one finds that it obeys the Laplace equation on $\mathbb{R}^7$:

$$\Delta_7 Z = 0. \quad (2.28)$$

In the presence of sources, this is modified to

$$\Delta_7 Z = \rho_{D2}. \quad (2.29)$$

For a stack of N_{D2} D2-branes sitting at the origin of our coordinate system, the source is a delta function $\rho_{D2} = N_{D2}\delta(\boldsymbol{r}_7)$ and we find the solution

$$Z = 1 + \frac{N_{D2}}{r^5}, \quad (2.30)$$

where r is the radius of the transverse space $r^2 \equiv x_3^2 + \dots x_9^2$. The integration constant can always be set to one by a constant rescaling of the coordinates.

As $r \to 0$, we approach the D2-brane source and $Z \to \infty$. From the expression for the metric, we see that the $\mathbb{R}^{1,2}$ factor shrinks, while the $\mathbb{R}^7$ blows up. This is not just a coordinate singularity, but $r = 0$ is a singular locus in space-time. This can be seen from the three-form potential C_{012}. It goes to zero at $r = 0$ but the energy of the C-field

$$E = \frac{1}{4!} F_{\mu\nu\rho\sigma} F_{\mu'\nu'\rho'\sigma'} g^{\mu\mu'} g^{\nu\nu'} g^{\rho\rho'} g^{\sigma\sigma'} \tag{2.31}$$

blows up as $r \to 0$ and we conclude that the D2 brane solution contains a singularity, which is not shielded by a horizon ('naked singularity').[9]

Note that the equations of motion are linear in the sense that we can add multiple (singular) D2-brane sources:

$$\Delta_7 Z = N_a \delta(\boldsymbol{r} - \boldsymbol{r}_a) + N_b \delta(\boldsymbol{r} - \boldsymbol{r}_b) + \dots . \tag{2.32}$$

We consider all D2-branes of the same 'species', with the world volume along the 0, 1, 2 directions.

Then the only thing that changes is that the function Z becomes a sum of harmonic functions, sourced at different locations:

$$Z = 1 + \frac{N_b}{|\boldsymbol{r} - \boldsymbol{r}_b|^5} + \frac{N_a}{|\boldsymbol{r} - \boldsymbol{r}_b|^5} + \dots . \tag{2.33}$$

We see that this solution can describe any density of D2-branes, even a continuous one.

D3-Brane from T-duality

Start from a continuous distribution of D2 branes along a line in the transverse space (this is also called 'smearing' the D-brane charge). Say that we put this smeared D2-branes along the x_7 direction, see Fig. 2.11.

Because the solution is now homogeneous in x_7, we can compactify this direction. Then the solution for such a continuous distribution of D2-brane charge on a finite line segment goes as:

$$Z = 1 + \frac{N_{D2}/L_7}{r^4}, \tag{2.34}$$

[9] Although there is a naked singularity in the supergravity solution, as a solution to string theory, a D-brane is well-defined. As $r \to 0$, the dilaton ϕ blows up. Since it sets the length of the eleven-dimensional compactification circle of M-theory, the eleventh dimension decompactifies near $r \to 0$. We hence get the near-M2-brane solution of eleven-dimensional M-theory, which is well-defined in all of space-time.

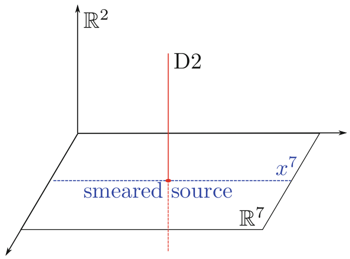


Fig. 2.11 A D2 brane smeared along x_7

with L_7 the length of the compactified direction. Next we perform a T-duality along x_7 to a D3-brane solution of type IIB string theory. What does this solution look like? Remember that the size of the compact circle is inverted after this duality transformation $\sqrt{g_{77}} \to (\ell_s)^2/\sqrt{g_{77}}$, and hence the metric of the resulting solution is (we set $\ell_s = 1$ for simplicity):

$$ds^2 = Z^{-1/2}(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_7^2) + Z^{1/2} ds_{6\text{d}}^2. \tag{2.35}$$

The three-form gets an additional leg to become the IIB four-form:

$$C_{0127} = Z^{-1}. \tag{2.36}$$

and the solution for the function Z is

$$Z = 1 + \frac{N_{D3}}{r^4}. \tag{2.37}$$

Near-Solution and Brane Throat

What does the geometry look like close to the D3-brane? We approach the D3-brane as we take $r \to 0$. This means that in the function Z, we can effectively drop the constant and write $Z = N_{D3}/r^4$ as $r \to 0$.

To reinstate the correct dimensions, we write $Z = R^4/r^4$, with R some reference radius. First write the transverse six-dimensional space in terms of polar coordinates as

$$ds_{6\text{d}}^2 = dr^2 + r^2 d\Omega_5^2, \tag{2.38}$$

Then the near-geometry of the D3-brane is

$$ds_{\text{near}}^2 = \frac{r^2}{R^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_7^2) + R^2 \frac{dr^2}{r^2} + R^2 d\Omega_5^2. \tag{2.39}$$

What has the D3-brane done? It has opened up a "throat": as we approach $r \to 0$ from infinity, the S^5 will get smaller and smaller. But near the D3 brane it attains a finite size, set by the radius R. Note that the metric distance to $r = 0$ from any

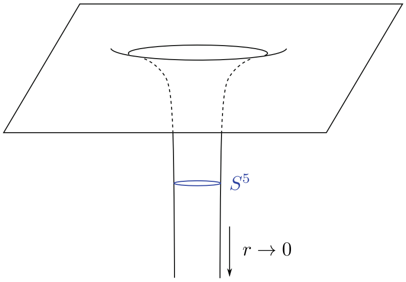


Fig. 2.12 A cartoon of the D3-brane geometry. As we approach the D3 brane, an infinity throat opens with constant transverse S^5 size

other point in space-time with $r > 0$ is actually infinite and the D3-brane throat is infinitely deep.

Physically, the D3-brane solution *forces* the $AdS_5 \times S^5$ geometry to appear.[10] This is a special feature of the D3-brane that the other D-branes do not possess (in fact, all the D0, D1...D6-branes have a naked singularity if we consider them in ten-dimensional supergravity). The origin lies in the dyonic nature of the D3 brane: it is both an electric and a magnetic charge for the four-form potential C_{0127} (Fig. 2.12).

The $AdS_5 \times S^5$ geometry is the riding horse of holography. Classical gravitational physics on this background is dual, through the AdS/CFT correspondence, to strongly coupled conformal field theory in $N = 4$ Super-Yang-Mills. We will come back to this later.

BPS Property: Mass = Charge

The charge of a D3 brane is given by integrating the gauge field that couples magnetically to it over a surface surrounding the brane (as for the magnetic monopole of electromagnetism)

$$Q_{D3} = \frac{1}{5!}\int_{S^5} F_{ijklm}\, dx^i dx^j dx^k dx^l dx^m, \tag{2.40}$$

where the field strength is $F_{ijklm} = 5!\partial_{[i}C_{jklm]}$. So far, we have only given the electric component of the gauge field C_{0123}. Exercise 2.2.4 asks you to derive the magnetic component of the field strength: since the five-form F_5 of IIB string theory is self-dual, it *must* have magnetic components as well.

Exercise 2.2.4 *Derive, using the duality*

$$F_{\nu_1\ldots\nu_5} = \tilde{F}_{\nu_1\ldots\nu_5} \equiv \frac{1}{5!}\sqrt{-g}\epsilon_{\nu_1\ldots\nu_5\,\mu_1\mu_2\mu_3\mu_4} g^{\mu_1\mu_1'} g^{\mu_2\mu_2'} g^{\mu_3\mu_3'} g^{\mu_4\mu_4'} g^{\mu_5\mu_5'} F_{\mu_1'\mu_2'\mu_3'\mu_4'\mu_5'}. \tag{2.41}$$

and the expression for the electric components of the field strength

[10] For the aficionados: this is the same mechanism that forces the extremal Reissner-Nordstrom black hole to have a near-horizon region of the form $AdS_2 \times S^2$.

$$F_{0123r} = \partial_r Z^{-1}. \tag{2.42}$$

the form of the magnetic components F_{45678}.

With this result, we find that from integrating over an S^5 at $r \to \infty$ to cover the entire flux emanating from the D3 brane, that the charge of the D-brane is

$$Q_{D3} = N_{D3}, \tag{2.43}$$

up to some numerical coefficient that we set to one for simplicity's sake.

The mass of the D3-brane can be derived from the component g_{tt} of the metric, following the prescription of Arnowit, Deser and Misner (ADM) (see [21] for more details on the ADM formalism in GR, and [20] for a discussion in p-brane space-times). In particular, when expanding this component for large r, in asymptotically flat D-dimensional space-time the leading terms for a point-like source are:

$$g_{tt} = -1 + \frac{16\pi G_N}{(D-2)\Omega_{D-2}} \frac{M}{r_{D-3}} \tag{2.44}$$

where G_N is Newton's constant and Ω_n is the area of the n-sphere of unit radius S^n. A D3-brane is effectively like a point in $D = 7$ and we see from (2.35) and (2.37) that M is proportional to the number N_{D3} of D-branes. We have not been too careful about prefactors in the expression for the metric, so we only state the dependence on g_s of the end result:

$$M_{D3} = \frac{N_{D3}}{g_s}, \tag{2.45}$$

where g_s ("g-string") is the string coupling constant. This is an interesting feature: the masses of all D-branes are inversely proportional to the string coupling constant. This should be contrasted with electromagnetism. The mass of the electron, the fundamental object, is independent of the coupling (let's call it g). On the other hand, the mass of a soliton in field theory goes as $1/g^2$. The magnetic monopole's mass has this behaviour. So we see that the D-brane is neither a fundamental object nor a soliton of string theory.

The mass of the fundamental string, the fundamental object of string theory, is independent of g_s (we have seen that the string tension, or mass density, is $T_{F1} = 1/2\pi\ell_s^2$). One finds that the mass of the NS5 brane goes as

$$M_{NS5} \sim \frac{1}{g_s^2}, \tag{2.46}$$

and the NS5 brane is really a soliton of string theory. The different dependence on g_s of the masses of all these objects shows up in the 'warp factor' Z of the supergravity solutions. We track the dependence on g_s and drop other proportionality factors, such as the string length ℓ_s. Newton's constant G_N goes as $G_N \sim g_s^2$ (this follows from the low-energy supergravity action of ten-dimensional string theories). In general we have

$$Z = 1 + \frac{G_N M}{r^{\#}}. \tag{2.47}$$

where # is the appropriate power. For a D-brane, this gives

$$Z_{\text{D-brane}} = 1 + \frac{N_{\text{D}} g_s}{r^{\#}}, \tag{2.48}$$

for an NS5 and a string we have

$$\begin{aligned} Z_{\text{NS5}} &= 1 + \frac{N_{\text{NS5}}}{r^{\#}}, \\ Z_{\text{F1}} &= 1 + \frac{N_{\text{F1}} g_s^2}{r^{\#}}. \end{aligned} \tag{2.49}$$

Going back to the D3 brane, we find in ‘dimensionless’ units that

$$Q_{D3} = M_{D3}. \tag{2.50}$$

We interpret this as: “the mass (density) of a D3-brane is equal to its charge (density)”.

What does this mean physically? The gravitational attraction and the electric repulsion are exactly balanced, even though both forces are huge. This is why we can have D-brane solutions with sources at many points and still remain stable. This is different in electromagnetism, where two electrons would fly apart; the electric repulsion always takes the upper hand and we cannot build multi-center electron-solutions.

Note that there is an underlying physical bound $M \geq Q$ for any charged object. When the mass is smaller than the charge, then the solution is unphysical (more on this in the next section). This bound is called BPS bound after Bogomol’nyi, Prasad and Sommerfield. Note that this bound typically appears in supersymmetric theories. In the real world, supersymmetry is not manifest and elementary particles such as electrons do not satisfy the BPS bound: $e > m_e$ in the units we are using here.

We call the equal mass and charge of the D3 brane a BPS property. The BPS-ness of the D3-brane and all the other D-branes is a consequence of supersymmetry. All D-brane solutions (and the F1 and NS5) are invariant under a set of supersymmetry transformations, and the mass of any supersymmetric object is equal to its charge in natural units.

General D*p*-Brane Solution

For completeness, we give the solution for a Dp-brane to the IIA action given in (2.24) with general p. It has the non-zero fields:

$$
\begin{aligned}
ds^2 &= H^{-1/2}(-dt^2 + dx^m dx^m) + H^{1/2}(dr^2 + r^2 d\Omega^2_{(q-1)}) \\
e^{2\phi} &= H^{-\frac{1}{2}(p-3)} \\
C_{t\,\mu_1\ldots\mu_p r} &= Z^{-1}
\end{aligned}
\tag{2.51}
$$

where H is a harmonic function

$$
Z(r) = 1 + \frac{Q_p}{r^{7-p}}. \tag{2.52}
$$

Also the Dp-branes of type IIB supergravity have this form for uneven p.

2.3 Black Hole Solutions

We discuss how to obtain black hole solutions from D-branes that are wrapped on compact spaces. For completeness, we first show how to make a black D-brane. We will later focus on supersymmetric black holes, because these are easier to construct and understand microscopically.

2.3.1 Non-extremal Black Holes

Let us forget about supersymmetry for a moment, and see if we can make a black hole with $M > Q$. We do not try to make a multi-D-brane solution or anything like that, but just want to make a black hole, or a black object, with more mass than charge. An easy such solution is a black D3-brane. Its metric is given by

$$
ds^2 = -Z^{-1/2}\left(-f(r)dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + Z^{1/2}\left(\frac{dr^2}{f(r)} + r^2 d\Omega_5^2\right). \tag{2.53}
$$

and the gauge field takes the same form as for the ordinary D3-brane

$$
C_{0123} = Z^{-1}. \tag{2.54}
$$

When the function $f(r) = 1$, this is just the supersymmetric D3 brane we have encountered before. By adding the function $f(r)$, the D3-brane is turned into a "black brane". The function f obeys the same Laplace equation in transverse space as Z:

$$
\Delta f = 0. \tag{2.55}
$$

Typically, one considers the solution

$$
f(r) = 1 - \frac{\Delta M}{r^4}. \tag{2.56}
$$

Table 2.5 Near-horizon geometry of black D3-brane and thermal physics in field theory

String theory on $AdS_5 \times S^5$	$N = 4$ Super-Yang-Mills (4d)
Weak coupling	Strong coupling
Black hole in $AdS_5 \times S^5$ at temperature T	$N = 4$ SYM at temperature T

The warp factor $Z = c + N/r^4$, where c is a constant. When $c = 1$ the metric describes a black *membrane* in ten dimensions, with flat asymptotics. When $c = 0$, the metric describes a black *hole* in the AdS_5 factor of the $AdS_5 \times S^5$ near-horizon geometry of D3-branes (2.39). We consider the former. The charge for this solution is still given as for the normal D3 brane

$$Q = \int F_5 = N. \tag{2.57}$$

The mass (obtained from g_{tt} as before) is now

$$M = Q + \Delta M \tag{2.58}$$

We make two remarks. First note that when $\Delta M < 0$, this describes a singular solution with a naked singularity. Hence we consider $\Delta M > 0$ for physical reasons. Also, we see that unlike the supersymmetric D3 brane, two (or more) of these objects are not in equilibrium any more. Two black branes will attract and eventually collapse to a single black object, because the gravitational attraction is larger than the electrostatic repulsion.

A black hole, or black brane, that saturates the BPS bound $M = Q$ is also called extremal. Such a black object has zero Hawking temperature and does not emit radiation. When $M > Q$, the black object has a non-zero temperature and is called non-extremal. For small ΔM, the temperature is proportional to the mass excess:

$$T \sim \Delta M. \tag{2.59}$$

We see that by the $f(r)$ "black deformation", we can create a solution with non-trivial mass, charge and temperature.

This solution is very useful for holography. In the near-brane region $r \to 0$, we have $Z \sim 1/r^4$ and the black brane metric describes a black hole in $AdS_5 \times S^5$. Following the AdS/CFT correspondence, this maps to turning on a temperature in $N = 4$ Super-Yang Mills theory in four dimensions, see Table 2.5. So a black hole corresponds to warming up the field theory. Conversely, a temperature in field theory gives a black hole in AdS_5.

In conclusion, we see that by warming up the D3 brane with $f(r)$, we can study the dual field theory and its properties (conductivity, transport coefficients ...) from weakly coupled strings in the $AdS_5 \times S^5$ black hole background. We could call this field "applied string theory". A lot of people nowadays use string theory no longer as a theory that describes the real world, but as a sort of calculator that we can use to teach us valuable information in other, strongly coupled, systems.

- What about quantum effects? Quantum effects are controlled by g_s, the string coupling. In the limit we consider (horizon area A_H and charge Q very large in Planck units, for instance $A_H \gg \ell_P^2$ with ℓ_P the Planck length) such that supergravity is a good description, we expect quantum effects to not destroy the geometry. Of course, when we only consider one D-brane, this limit does not hold and the question of quantum corrections becomes really important. More on this in Sect. 2.4.1.

Note that other D-branes also have such a non-extremal version. We can get for instance a black D2-brane very easily by T-duality. See Exercise 2.3.5. Black p-branes all have

$$M > Q \qquad T > 0, \tag{2.60}$$

and they are found from a deformation of the metric by one function $f(r)$ determined by

$$\Delta_d f = 0, \tag{2.61}$$

where d is the number of transverse dimensions.

Exercise 2.3.5 *T-dualize the metric (2.53) and four-form potential (2.54) of the black D3 brane along direction x^3. Show that this becomes a smeared black 2-brane. In particular, repeat the mass calculation from the g_{tt} metric component and show that $M = Q + \Delta M$.*

2.3.2 Supersymmetric Black Holes in Four Dimensions

For the largest part of these lectures, we concentrate on supersymmetric black holes. The reason is that when a black hole solution preserves supersymmetry, it can be constructed more easily and even be understood microscopically.

Consider again the orthogonal D2-brane system

$$\begin{array}{llllll} \text{IIA}: & \text{D2} & 0 & 1 & 2 & & \\ & \text{D2} & 0 & & & 3 & 4 \end{array}$$

Normally any two objects that we put together would either attract or repel. However, this combination of D2-branes is supersymmetric and feels a flat potential: supersymmetry exactly amounts to canceling forces and we can put the branes together in a stable fashion.

We can even do more. It turns out to be possible to add an extra D2-brane and even a D6 brane, while still preserving supersymmetry, in the following configuration:

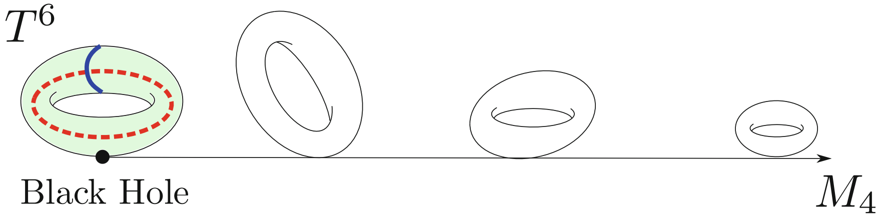


Fig. 2.13 Four-dimensional *black hole* from compactification on a six-torus. The T^6 has a different size and shape at each point of four-dimensional spacetime M_4. At the position of the *black hole*, there are branes present that are wrapped on T^6

$$
\begin{array}{lllllllll}
\mathrm{IIA}: & \mathrm{D2}_1 & 0 & 1 & 2 & - & - & - & - \\
 & \mathrm{D2}_2 & 0 & - & - & 3 & 4 & - & - \\
 & \mathrm{D2}_3 & 0 & - & - & - & - & 5 & 6 \\
 & \mathrm{D6}_4 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
$$

This combination of branes experiences a flat potential and is stable. This follows from the supersymmetry it preserves. To show this, we would need to check the supersymmetry algebra; we will not do this in these lectures. For the black hole discussion, we smear the D2 branes on their transverse directions inside $x^1 \dots x^6$, which we denote by "–" and we number the branes from 1 to 4 for practical reasons.

The solution for the D2-D6 system can actually be written down! The metric takes a very intuitive form:

$$
\begin{aligned}
ds^2 = &-(Z_1 Z_2 Z_3 Z_4)^{-1/2} dt^2 + (Z_1 Z_2 Z_3 Z_4)^{1/2}(dx_7^2 + dx_8^2 + dx_9^2) \\
&+ \frac{(Z_2 Z_3)^{1/2}}{(Z_1 Z_4)^{1/2}}(dx_1^2 + dx_2^2) + \frac{(Z_1 Z_3)^{1/2}}{(Z_2 Z_4)^{1/2}}(dx_3^2 + dx_4^2) + \frac{(Z_1 Z_2)^{1/2}}{(Z_3 Z_4)^{1/2}}(dx_5^2 + dx_6^2).
\end{aligned}
\tag{2.62}
$$

This solution reduces to any single brane solution when only one of the four branes is present (for say Z_1 non-trivial, and the other ones constant, $Z_2 = Z_3 = Z_4 = 1$, we retrieve the metric of the D2-brane). Amazingly, this D2-D2-D2-D6 solution, which is constructed from the same "harmonic function rule" we had for single Dp-branes ($Z^{-1/2}$ metric component when the brane worldvolume is along that direction, $Z^{1/2}$ when it is orthogonal) applies to all of the Z_i individually, regardless of the presence of the other branes. This is a very non-trivial feature and would not happen for a generic solution; it is only for such a special class of supersymmetric solutions, that we get such a nice structure at the end of the day. For more information, see the original references [22].

Exercise 2.3.6 *Consider the directions $x^1 \dots x^6$ to be compact and to describe a six-torus, or T^6. T-dualize the D2-D2-D2-D6 metric 6 times along $x^1, \dots, x^6$. Write down the resulting D4-D4-D4-D0 metric.*

In order to write down the explicit solution, we need to smear the D2 branes uniformly along the transverse directions in T^6 (the compact directions $x^1 \dots x^6$).[11] This means we have to smear $D2_1$ along directions 3456, $D2_2$ along 1256 and $D2_3$ along 1234. Then all branes are points in three-dimensional space spanned by x^7, x^8, x^9. Therefore the warp factors Z_i obey (Fig. 2.13):

$$\Delta_3 Z_i = 0 \qquad \to \qquad Z_i = 1 + \frac{Q_i}{r}. \tag{2.63}$$

We will show in the next section how the dimensionful charges Q_i are related to the integers N_i counting the number of D-branes.

Exercise 2.3.7 *Convince yourself that smearing a D-brane along a spacelike direction changes the radial dependence of Z in the correct way. For example, take a D2 brane along directions x^1, x^2 and smear it along the circular dimension x^3 with a homogeneous density $\rho_{\text{smear}} \sim 1/R_3$, where R_3 is the radius of the 3-circle. Show that in this process, the solution to the Laplace equation becomes $Z = 1 + \tilde{Q}_{D2}/r^4$ rather than $Z = 1 + Q_{D2}/r^5$, with $\tilde{Q}_{D2} = Q_{D2}/R_3$.*

What is our solution? We study the asymptotic limits.

- At $r \to \infty$: The metric becomes that of four-dimensional Minkowski spacetime times a flat torus with fixed radii:

$$ds^2_{r\to\infty} = -dt^2 + ds^2(\mathbb{R}^3) + ds^2(T^6) \tag{2.64}$$

  This means we have compactified string theory on a six-torus to a four-dimensional theory. The leading terms in the $1/r$ expansion of the g_{tt} metric component are

$$g_{tt} = 1 - \frac{1}{2}\frac{Q_1 + Q_2 + Q_3 + Q_4}{r}, \tag{2.65}$$

  and we see that the mass of this solution is (up to factors we do not care about at this point)

$$M = Q_1 + Q_2 + Q_3 + Q_4. \tag{2.66}$$

  Again, this solution saturates the BPS bound and is extremal (the minimal amount of gravitational mass for given charges and angular momenta): its mass is the sum of the charges of the individual branes; there is no binding energy. This is a consequence of supersymmetry.
- At $r \to 0$: all the 1's drop out of the warp factors Z_i and the metric becomes

[11] If we did not smear the individual D-branes making up the black hole solution, then the metric would depend on some of the internal coordinates as well. We only want dependence on four-dimensionsional space-time. In addition, if we T-dualize one D-brane, then the result becomes smeared along the dualization direction. To get a four-dimensional black hole that looks the same in all duality frames, we need to work in a duality frame where the branes are smeared on orthogonal compact directions.

$$ds^2_{r\to 0} = -\frac{r^2}{R^2}dt^2 + \frac{R^2}{r^2}(dx_7^2 + dx_8^2 + dx_9^2) + \left(\frac{Q_2 Q_3}{Q_1 Q_4}\right)^{1/2}(dx_1^2 + dx_2^2)$$
$$+ \left(\frac{Q_1 Q_3}{Q_2 Q_4}\right)^{1/2}(dx_3^2 + dx_4^2) + \left(\frac{Q_1 Q_2}{Q_3 Q_4}\right)^{1/2}(dx_5^2 + dx_6^2). \quad (2.67)$$

The six-torus spanned by the directions $x_1 \dots x_6$ has constant radii. If we go to polar coordinates in $\mathbb{R}^3$ spanned by x_7, x_8, x_9, then the metric is

$$ds^2 = -\frac{r^2}{R^2}dt^2 + R^2\frac{dr^2}{r^2} + R^2 d\Omega_2^2 + ds^2(T^6). \quad (2.68)$$

The first two terms describe AdS_2. The other terms describe an S^2 and a T^6 of constant radii. Therefore, the $r \to 0$ limit of the D2-D2-D2-D6 spacetime describes a compactification of string theory on T^6 to the four-dimensional $AdS_2 \times S^2$ geometry.
We also observe that g_{tt} vanishes as $r \to 0$ and hence $r \to 0$ describes an event horizon.

From these facts we conclude that the metric of this D2-D2-D2-D6 brane system describes a real black hole in four dimensions. We will refer to this four-dimensional black hole as the "four-charge black hole".

Note that all the Q_i appearing in the metric are positive. Only the gauge fields (which we have not given) are sensitive to the sign of the charges. The gravitational field sourced by a positive or a negative charge is exactly the same. An anti-D2 brane would have the same metric and mass as a D2-brane, but opposite electric field.

To understand the full spacetime, we make our lives a bit easier and restrict to all charges equal:

$$Q_i \equiv Q\,, \qquad Z_i \equiv Z = 1 + \frac{Q}{r}. \quad (2.69)$$

The black hole metric (2.62) becomes

$$ds^2 = -\left(1 + \frac{Q}{r}\right)^{-2} dt^2 + \left(1 + \frac{Q}{r}\right)^2 (dr^2 + r^2 d\Omega_2^2) + ds^2(T^6). \quad (2.70)$$

The T^6 has a constant metric and does not play a role in the physics. We further concentrate only on the four-dimensional part of the geometry.

Our claim is that this metric represents a black hole. A very special one even, with $M = Q$. Let us go over the textbook black hole teachings to see if our claim is valid.

1. The first black hole you learn about in classical GR, is the Schwarzschild black hole. It is a solution to vacuum gravity, described by the metric

$$ds_S^2 = -\left(1 - \frac{2M}{\rho}\right) dt^2 + \left(1 - \frac{2M}{\rho}\right)^{-1} d\rho^2 + \rho^2 d\Omega_2^2. \quad (2.71)$$

This is clearly not the same as our solution. We need to include a charge for the black hole.

2. Luckily there is also the second black hole you encounter in your favourite classical GR course. It is the Reissner-Nordström black hole. This black hole is a solution to Einstein-Maxwell theory (the Lagrangian (2.5)). It is given by

$$ds^2_{\mathrm{RN}} = -\left(1 - \frac{2M}{\rho} + \frac{Q}{\rho^2}\right) dt^2 + \left(1 - \frac{2M}{\rho} + \frac{Q}{\rho^2}\right)^{-1} d\rho^2 + \rho^2 d\Omega^2_2. \quad (2.72)$$

It has a very interesting limit

$$M = Q. \quad (2.73)$$

Then the metric becomes

$$ds^2_{\mathrm{RN}} = -\left(1 - \frac{Q}{\rho}\right)^2 dt^2 + \left(1 - \frac{Q}{\rho}\right)^{-2} d\rho^2 + \rho^2 d\Omega^2_2. \quad (2.74)$$

What does this have to do with our black hole metric, which has $g_{tt} = Z^{-2} = 1/(1 + Q/r)^2$? If we redefine

$$r = \rho - Q, \quad (2.75)$$

then we find the D-brane black hole solution on the nose!

These "$M = Q$" black holes are the ones we can construct most easily in string theory. They are extremal and are frozen at zero temperature:

$$T_{BH} = 0, \quad (2.76)$$

but have a non-zero mass M and entropy S_{BH}. The Bekenstein-Hawking entropy is given by the horizon area (we ignore numerical factors)

$$S_{BH} = \frac{A_H}{4G_N} = \pi R^4 = 2\pi\sqrt{Q_1 Q_2 Q_3 Q_4}, \quad (2.77)$$

or

$$S_{BH} = 2\pi Q^2 \quad (2.78)$$

when all $Q_i = Q$. This entropy comes from some microscopic states. Who makes this entropy? We will answer this in the sections to come.

In ρ-coordinates, this is clearly a black hole. The horizon is at $\rho = Q$, where $g_{tt} = 0$. The coordinate r we used for the string theory metric is only well-defined outside the horizon ($r > 0$).[12] Note that in the single D2-brane solution, the coordinate r is a measure for the distance from the brane at $r = 0$ (the same goes for D0,

[12] At the position of the horizon, we have a degenerate coordinate system, but there is no physical singularity at $r = 0$.

D1, D4, D5 and D6 brane solutions). For the supersymmetric black hole, space is "created" behind the $r = 0$ point and a large $AdS_2 \times S^2$ throat develops. The way to see this is by passing through a set of coordinates for which the metric extends beyond the horizon to the black hole singularity.

We come back to the regime of validity of supergravity. One can perform a calculation to show that the curvature of a D-brane goes as $1/Q$. In terms of the number N of D-branes, this gives a curvature proportional to $1/g_s N$. Therefore the solutions we have considered are only valid when $g_s N \gg 1$ (small curvature, we can trust classical physics). When the number of branes is too small and $g_s N \ll 1$, supergravity can no longer be used to describe the solution. The large curvature of spacetime takes us out of the low-energy description and higher energy modes should be taken into account. Note that this does *not* mean there is no D-brane any more. Think of classical electromagnetism. The electron is also a singular solution, but this gets resolved in the quantum theory. Similarly, string theory takes over for the quantum description of D-branes when $g_s N \sim 1$: string loops are suppressed by $g_s N$, rather than g_s. We discuss this in Sect. 2.4.

2.3.3 Supersymmetric Black Holes in Five Dimensions

For phenomenological and other existential reasons, we like four dimensions. Nonetheless, we make the switch to five dimensions, because five-dimensional black holes are easier to construct and analyze. Using dualities and dimensional reduction, a lot of what we do can be connected to four-dimensional physics.

Consider eleven-dimensional supergravity, with three orthogonal M2 branes as:

$$\begin{array}{llllllll}
\mathrm{M2}_1 & 0 & 1 & 2 & - & - & - & - \\
\mathrm{M2}_2 & 0 & - & - & 3 & 4 & - & - \\
\mathrm{M2}_3 & 0 & - & - & - & - & 5 & 6
\end{array}$$

We consider the directions $x^1 \dots x^6$ to be compactified such that they again form a T^6. As for the four-dimensional black hole, the branes are smeared on their transverse directions on T^6, denoted by "–" in the table. Therefore the M2-branes are all pointlike in the transverse $\mathbb{R}^4$ spanned by x^7, x^8, x^9, x^{10}. The solution is determined by three functions:

$$Z_1 = 1 + \frac{Q_1}{\rho^2}, \qquad Z_2 = 1 + \frac{Q_2}{\rho^2}, \qquad Z_3 = 1 + \frac{Q_3}{\rho^2}. \tag{2.79}$$

We will use ρ for the radius for black holes in five-dimensional space-time, to distinguish from r for four-dimensional black holes. Note the power $1/\rho^2$ for harmonic functions on in five dimensional space-time.

It turns out that for an eleven-dimensional supergravity solution, we can play the same game with the harmonic functions. The only difference is that different powers appear in the metric. When an M2 brane is wrapped along a direction, we multiply the corresponding metric component with an extra factor $Z^{-2/3}$, when the brane is transverse, we multiply it with $Z^{1/3}$. In particular, the supergravity solution for the (supersymmetric) M2-M2-M2 brane system is

$$\begin{aligned} ds^2 = &-(Z_1 Z_2 Z_3)^{-2/3} dt^2 + (Z_1 Z_2 Z_3)^{1/3} (d\rho^2 + \rho^2 d\Omega_3^2) \\ &+ \left(\frac{Z_2 Z_3}{Z_1^2}\right)^{1/3} (dx_1^2 + dx_2^2) + \left(\frac{Z_1 Z_3}{Z_2^2}\right)^{1/3} (dx_3^2 + dx_4^2) \\ &+ \left(\frac{Z_1 Z_2}{Z_3^2}\right)^{1/3} (dx_5^2 + dx_6^2). \end{aligned} \tag{2.80}$$

This solution describes a black hole in five spacetime dimensions. This can be seen from the limits

- $r \to \infty$: The metric describes a direct product of five-dimensional flat Minkowski spacetime with a six-torus with constant radii. This is a compactification of flat eleven-dimension spacetime to five dimensions.
- $r \to 0$: This is the horizon of the black hole. Write the transverse $\mathbb{R}^4$ metric in polar coordinates $dx_{78910}^2 = dr^2 + r^2 d\Omega_3^2$. Then for $r \to 0$, the metric looks like

$$ds^2 = -\frac{r^4}{R^4} dt^2 + R^2 \frac{dr^2}{r^2} + R^2 d\Omega_3^2 + ds^2(T^6), \tag{2.81}$$

where $R^2 = (Q_1 Q_2 Q_3)^{1/3}$ and the last term describes a torus of constant radii. By the coordinate redefinition $\rho = r^2$, we see that the first two terms form the metric of AdS_2 ($g_{tt} \to 0$ and $g_{rr} \to \infty$ in such a way to give AdS_2) and the S^3 has constant radius. Hence the near-horizon geometry is $AdS_2 \times S^3 \times T^6$.

We have encountered many examples of $AdS_p \times S^q$ geometries from D-branes: $AdS_5 \times S^5$ from the D3 brane, $AdS_2 \times S^2 \times T^6$ from D2-D2-D2-D6, $AdS_2 \times S^3 \times T^6$ from M2-M2-M2. We will later also encounter $AdS_3 \times S^3 \times T^4$.

Entropy in Gory Detail

We want to give the exact expression for the Bekenstein-Hawking entropy of the black hole. This entropy is proportional to the horizon area in Planck units, or more precisely

$$S_{BH} = \frac{A_H}{4G_N}. \tag{2.82}$$

Note that this looks independent of the dimension. However, if we use the horizon area in D dimensions, we should also use Newton's constant in D dimensions.

Let us get our hands dirty and derive this beast. The horizon area of the eleven-dimensional metric is really the area of $S^3 \times T^6$ at $r = 0$:

$$A_H = \int_{S^3\times T^6}\sqrt{g} = \int_{S^3}\sqrt{g_{S^3}}\int_{T^6}\sqrt{g_{T^6}}\bigg|_{r\to 0}. \tag{2.83}$$

The area of the S^3 in the metric (2.81) is:

$$\int_{S^3}\sqrt{g_{S^3}} = \sqrt{R^6}\,\Omega_3 = 2\pi^2\sqrt{Q_1Q_2Q_3}, \tag{2.84}$$

where $\Omega_3 = 2\pi^2$ is the area of a three-sphere with unit radius. The area of the T^6 for the metric (2.81) is

$$\begin{aligned}\int_{T^6}\sqrt{g_{T^6}} &= \int dx_1\dots dx_6\sqrt{\left(\frac{Z_2Z_3}{Z_1^2}\right)^{1/3}\left(\frac{Z_1Z_3}{Z_2^2}\right)^{1/3}\left(\frac{Z_1Z_2}{Z_3^2}\right)^{1/3}}\\ &= \prod_{i=1}^{6}(2\pi L_i),\end{aligned} \tag{2.85}$$

where L_i are the radii of the x_i circles.[13]

We want to express the entropy in terms of a dimensionless number that can be related to a number of microstates. Before we can continue, we have to find the exact relation of the supergravity charges Q_i to the actual integer numbers that count the number of M2 branes of type i that source the supergravity solution. So far, we have been sloppy with the distinction between the supergravity charges Q_i (with dimensions of length squared and appearing in the functions Z_i) and the actual brane numbers N_i. All numerical factors in the exact relation $Q_i = (\dots)N_i$ are extremely important: these will become prefactors in the entropy, which is exponentiated to get the number of black hole microstates. A mistake of a factor of 2 in a number as e^N or e^{2N} has huge consequences!

To find the relation between Q_i and N_i, we first consider the gauge fields of the solution. These are given by

$$C_{012} = Z_1^{-1}, \qquad C_{034} = Z_2^{-1}, \qquad C_{056} = Z_3^{-1}. \tag{2.86}$$

Remember that Q_i represent *densities* of M2-branes, smeared on some directions. For instance, Q_1 describes the density of N_1 M2-branes smeared on the directions x^3, x^4, x^5, x^6. Hence on general grounds, we expect that such a density should scale as

$$Q_1 = \frac{N_1}{L_3L_4L_5L_6}(\dots). \tag{2.87}$$

The exact coefficient $(\dots)$ is left to as a homework problem in Exercise 2.3.8.

[13] The T^6 radii L_i are defined by identifying the x_i periodically as $x_i = x_i + 2\pi L_i$.

Exercise 2.3.8 *The number of M2-branes can be read off by integrating the magnetic gauge field strength over a surface that surrounds the M2-branes as*[14]*:*

$$(2\pi\ell_P)^6 N_{M2} = \int_{\Sigma_7} F_7, \tag{2.88}$$

where ℓ_P is Planck's constant (in eleven dimensions) and Σ_7 is the surface surrounding the M2-branes. For the two-torus T^2_{12} spanned by x_1, x_2, this is:

$$\Sigma^7 = T^2_{34} \times T^2_{56} \times S^3, \tag{2.89}$$

The magnetic seven-form gauge field is found from the Hodge dualization relation $F_7 = \star_{11} F_4$, which is written out as

$$F_{i_1 \ldots i_7} = \frac{1}{4!}\sqrt{-g}\epsilon_{i_1 \ldots i_7; j_8 j_9 j_{10} j_{11}} g^{j_8 j_8'} g^{j_9 j_9'} g^{j_{10} j_{10}'} g^{j_{11} j_{11}'} F_{j_8 j_9 j_{10} j_{11}}. \tag{2.90}$$

Take the metric (2.81) and the four-form field strength $F_4 = dC$ with components

$$F_{012r} = \partial_r C_{012} = \partial_r (Z_1^{-1}), \tag{2.91}$$

and analogously for F_{034r} and F_{056r}. Calculate the dual seven-form and use (2.88) to express the charges Q_i in terms of the integers N_i that count the number of branes. Show that the exact relation is

$$Q_1 = \frac{N_1(\ell_P)^6}{L_3 L_4 L_5 L_6}, \qquad Q_2 = \frac{N_2(\ell_P)^6}{L_1 L_2 L_5 L_6}, \qquad Q_3 = \frac{N_3(\ell_P)^6}{L_1 L_2 L_3 L_4}, \tag{2.92}$$

where L_i are the radii of the circles at infinity.

We continue with the horizon area (2.83). It is given in terms of the charges as

$$A_H = 2\pi^2 \sqrt{Q_1 Q_2 Q_3} \prod_{i=1}^{6} (2\pi L_i). \tag{2.93}$$

By substituting the result from Exercise 2.3.8, Eq. (2.92), we get

$$A_H = 2\pi^2 (2\pi)^6 (\ell_P)^9 \sqrt{N_1 N_2 N_3}. \tag{2.94}$$

We want to we evaluate the entropy $S_{BH} = A_H / 4G_N$. The definition of the Planck length in terms of Newton's constant for any dimension D is:

[14] Remember the analogy with electromagnetism, for a magnetic monopole we find the quantized monopole charge N is $N \propto \int_{S^2} F_2$.

$$16\pi G_N = (2\pi)^{D-3}(\ell_P)^{D-2}. \tag{2.95}$$

Plugging this into $S_{BH} = A_H/4G_N$ for $D = 11$, we conclude that the Bekenstein-Hawking entropy of the black hole is

$$\boxed{S_{BH} = 2\pi\sqrt{N_1 N_2 N_3}}\,. \tag{2.96}$$

A few remarks are in order. Note that we began with D-branes on a torus. As the torus gets smaller or larger (by changing L_i), the solution changes drastically. We get different black holes because the Q_i's change. But: the entropy does not care whether the torus is of diameter 1 mm or 1 Mpc. This is a very interesting fact: S_{BH} does not change as you change the torus radii. For a non-supersymmetric solution, you would expect that the entropy depends on the parameters of the torus; the invariance of S_{BH} under variations of the torus radii is a feature due to supersymmetry.

We can use this feature to do dualities on the internal torus. The five-dimensional black hole will have the same entropy, but the black hole can be made up out of different branes in some other string theory. Take for instance the duality chain: (1) reduce along x_6 to IIA, (2) two T-dualities along x_1, x_2, (3) a T-duality along x_5:

				IIA:								**IIA:**								**IIB:**		
D2	0	1	2	–	–	–	T	D0	0	–	–	–	–	–	T	D1	0	–	–	–	–	5
D2	0	–	–	3	4	–	$\rightarrow$	D4	0	1	2	3	4	–	$\rightarrow$	D5	0	1	2	3	4	5
F1	0	–	–	–	–	5	$x_{1,2}$	F1	0	–	–	–	–	5	x_5	p	0	–	–	–	–	5

For this T-dualization, you need to know that for F1's that do not wrap the T-duality circle, nothing happens at all: they remain F1's.) The end result of this little exercise is an intersection of D5 branes with D1 branes and momentum along the common direction. This is the celebrated D1-D5-P system. We will mainly study the three-charge black hole in five-dimensions in this duality frame.

Exercise 2.3.9 *Use dimensional reduction and the T-duality chain from the M2-M2-M2 system to the D1-D5-P frame to show that the metric becomes*

$$\begin{aligned} ds^2 = &-(Z_1 Z_5)^{-1/2}(dt^2 + dz^2) + (Z_1 Z_5)^{-1/2}(Z_p^{-1} - 1)(dz - dt)^2 \\ &+ (Z_1 Z_5)^{1/2} dx_{78910}^2 + (Z_1 Z_5)^{-1/2} dx_{1234}^2. \end{aligned} \tag{2.97}$$

You will need to perform a minor change of coordinates and use that the reduction ansatz from M-theory to IIA supergravity is

$$ds_{11}^2 = e^{2\phi/3} ds_{10}^2 + e^{-4\phi/3} dx_{10}^2, \tag{2.98}$$

with ϕ the dilaton and ds_{10}^2 the ten-dimensional metric.

When $p = 0$, there is no momentum charge. The metric only depends on the functions

$$Z_{1,5} = 1 + \frac{g_s N_{1,5}}{r^2}, \qquad Z_p = 1 + \frac{(g_s)^2 N_p}{r^2}. \tag{2.99}$$

In the limit $r \to 0$, we can drop the 1's in the harmonic functions and the metric becomes $AdS_3 \times S^3 \times T^4$, see Exercise 2.3.10. Also this geometry is very useful for holography. String theory on this background is dual to a $(1+1)$-dimensional CFT.

Exercise 2.3.10 *Show that for $r \to 0$, the metric (2.97) with $p = 0$ ($Z_p = 1$), the metric becomes*

$$ds^2 = r^2(-dt^2 + dx_5^2) + \frac{dr^2}{r^2} + d\Omega_3^2 + ds^2(T^4), \tag{2.100}$$

where the last term describes the metric on a T^4 with constant radii.

But …there is a "but": for $Z_p = 1$ (no momentum charge) the horizon area is zero. This can be seen from the metric. At $r = 0$ it is singular and one can show that the Ricci scalar in five-dimensional spacetime blows up at $r = 0$ and the horizon coincides with a curvature singularity.

When $p \neq 0$, the entropy is

$$S_{BH} = 2\pi\sqrt{N_1 N_5 N_p}. \tag{2.101}$$

String theory on the near-horizon region is dual to the same $(1+1)$-dimensional CFT as for the $p = 0$ solution. Now there is a non-trivial momentum in the game, which translates into an extra charge of that the CFT states that are dual to the black hole can have.

The D1-D5-P black hole entropy comes from the many ways in the which the CFT can carry this momentum p. This result is proven in the next section. It is *most amazing*: the entropy of a black hole is recovered from counting states in a $(1+1)$-dimensional CFT!

2.4 Black Hole Microscopics

To properly account for the entropy of the black hole we first have to learn some very basic string theory. In the spirit of the rest of the lectures we'll eschew any details we don't need and ask the reader to trust us since we're supposed to be experts.

We explain how to derive the black hole entropy from a microscopic counting of states for a:

1. **D1-D5-P black hole** (also "three-charge black hole") with entropy:

$$S_{BH} = 2\pi\sqrt{N_1 N_5 N_p}. \tag{2.102}$$

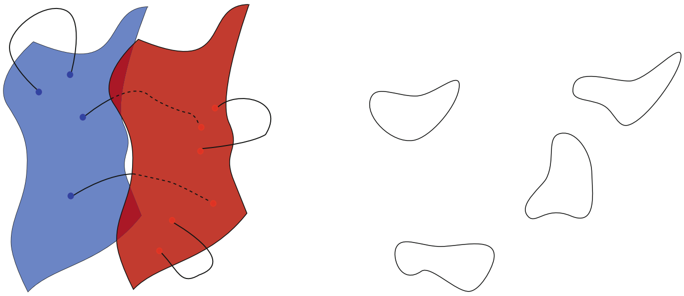


Fig. 2.14 Strings can be of two types, depending on the boundary conditions we put on the string: closed or open strings. The end-point of open strings are confined to D-branes. **a** Open strings end on D-branes. **b** Closed strings propagating in spacetime

2. **D6-D2-D2-D2 black hole** (also: "four-charge black hole") with entropy:

$$S_{BH} \sim \sqrt{N_{\mathrm{D2}} N_{\mathrm{D2}} N_{\mathrm{D2}} N_{\mathrm{D6}}}. \tag{2.103}$$

When all the charges above are equal this black hole has a very nice interpretation as the extremal Reissner-Nordtström black hole in four dimensions.

We will discuss the three-charge black hole first. Historically, this was the first black hole for which a microscopic counting was done that could explain the entropy (by Strominger and Vafa [23]). We will treat the four-charge black hole in four dimensions afterwards. It is the latter one which may have more appeal, as it describes the extremal black hole of Einstein-Maxwell theory in four dimensions ('extremal Reissner-Nordström black hole').[15]

2.4.1 A Brief Review of Open and Closed String Theory

String theory is a theory of (surprise, surprise:) strings. Strings come in two types: closed strings form closed loops in spacetime with no end-points (imagine rubber bands floating around in spacetime), while open strings have two ends (imagine a strand of rope stretched between…between what?), see Fig. 2.14.

In general the ends of open strings are not free to move in all directions of spacetime but are constrained to lie along higher dimensional "membranes". It turns out that these membranes are nothing other than the D-branes we found before as solu-

[15] This does not mean that it is a realistic astrophysical black hole. In nature, black holes will shed (almost) all their charge and be charge neutral. Supersymmetric black holes are extremal; they have the maximum amount of charge allowed for their mass and are hence not the black holes we observe in the sky.

tions to supergravity! Although it is hard to see why this is so, we will try to argue it briefly later.

Scales and Limits

One of the nice features of string theory is that it very naturally introduces a new length scale, ℓ_s, the string length. This is because fundamental strings (like all strings) have a tension, τ_{F1}, and this can be defined in terms of the string length, ℓ_s, a new fundemental length scale defined by this tension,

$$\tau_{F1} = \frac{1}{\ell_s^2}. \tag{2.104}$$

Note that the length dimension of τ_{F1} is defined so that integrating the tension over a one-dimensional volume yields a unit of mass, namely the mass of a string.

Oscillations on a world-volume of a string have an energy cost dependent on the string tension just like a regular guitar string. The mass of the harmonic modes is quantized in units of the string mass

$$M_s \propto \frac{1}{\ell_s}. \tag{2.105}$$

When this value is large then stringy modes are very massive and we can, to a good approximation, restrict ourselves to only the lowest lying sector corresponding to massless strings, see Fig. 2.15. In this limit when the string mass is very large and only a few modes remain strings essentially look like point particles and (owing to the various possible massless oscillations possible) generate a spectrum of fields in spacetime, see Table 2.6.

Even though we will not generally need the details of this spectrum, it is important to realize that the closed string spectrum generates supergravity with the associated fields. Open strings, on the other hand, are described at low energy by a gauge theory since A_μ has the degrees of freedom to be a gauge field (coupled to matter and

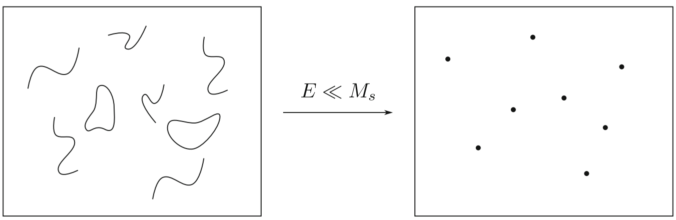


Fig. 2.15 When we probe strings at energy scales far below the string scale, $E \ll M_s$, then we can't excite oscillators on the strings so they look and act like point particles

Table 2.6 The massless spectrum of closed and open strings

String type	Spacetime fields generated
Closed	$g_{\mu\nu}, B_{\mu\nu}, C^{(3)}_{\mu_1\mu_2\mu_3}, \psi^{\alpha}_{\mu}, \ldots$
Open	$A_{\mu}, \phi, \psi_{\alpha}, \ldots$

fermions). This theory however does not live on all of spacetime but only on the D-branes on which the open strings are restricted to end.

In any gravity theory, including string theory, there is a fundamental length scale related to Newton's constant and the strength of gravitational interactions: the Planck scale. This is set by the Planck length ℓ_P, through the relation with Newton's constant (in D dimensions of spacetime):

$$G_N = (2\pi)^{D-3}(\ell_P)^{D-2}. \tag{2.106}$$

The introduction of a second fundamental length scale in string theory, ℓ_s, means that string theory has an associated dimensionless constant, the string coupling or g-string

$$g_s = f\left(\frac{\ell_s}{\ell_P}\right). \tag{2.107}$$

The exact dependence can be derived using the fact that the low-energy limit of closed string scattering (which depends on g_s and ℓ_s, as explained below) can be related to graviton scattering (which depends on G_N). A graviton propagator is controlled by Newton's constant. If we interpret this as a string exchange, we get two factors of g_s, one for emitting and one for absorbing a closed string. This gives

$$G_N \propto \ell_P^{D-2} \propto g_s^2\, \ell_s^{D-2}, \tag{2.108}$$

and we find

$$g_s \propto \frac{\ell_s}{\ell_P}. \tag{2.109}$$

From this we see that g_s controls the hiarachy of scales in string theory. When $g_s \ll 1$ we have $\ell_s \ll \ell_P$ so stringy excitations are much less massive than the Planck scale and we can do "classical string theory". On the other hand when $g_s \gg 1$ then any stringy excitations is more massive than the Planck scale and thus highly quantum. Therefore g_s acts as a *dimensionless* coupling in string theory telling us when the theory can be treated classically versus when it is necessarily strongly coupled.

While tuning g_s puts us in a theory with a certain string and Planck scale we have a further freedom to choose the energy scale at which we probe this theory. In any given physical process there is an associated dimensionalful energy scale such as e.g. the mass of the heaviest particle we consider, the energy of a scattering process, etc…. Thus even if we choose $g_s \ll 1$ we have the further freedom to consider only processes with E restricted to

$$E \ll M_s \ll M_P, \tag{2.110}$$

which means that the scale of our physics is smaller than the string scale (which in turn is lower than the Planck scale). The limit $M_s \ll M_P$ (or $g_s \ll 1$) means that string perturbation theory is valid and that we can look at classical string theory. The limit $E \ll M_s$ means that strings effectively look like point particles (we only look at those excitations that have very low energy compared to the scale set by the string length and we cannot distinguish the stringy nature of the string). This limit, in which semi-classical particle physics is a good approximation, is one in which we will often find ourselves.

String Perturbation Theory

Above we motivated g_s as a dimensionless coupling emerging from comparing the dimensionful ℓ_s and ℓ_P but within string theory this can actually be derived. String perturbation theory is described in terms of the mathematical "genus" of the string world-sheet (the two dimensional submanifold describing the strings path in space-time). Let's take a look at the loop expansion of a string process. As a Feynman diagram represents the worldlines of in- and outgoing particles and intermediate processes (propagators, loops), a string diagram represents the worldvolume of a string.

We represent perturbation theory for an ingoing closed string to an outgoing closed string in Fig. 2.16, which explaines visually the genus expansion. For every number of loops, there is exactly one type (topology) of string worldvolume.

Every loop in a closed string diagram introduces an extra factor of $(g_s)^2$. The limit where $g_s \to 0$ suppressed the loops and hence also quantum effects: this is the classical limit. If we further impose the extra "low-energy limit" $E \ll M_s$, such that the strings look like particles then the string diagrams reduce to standard Feynman diagrams because in this limit we send $\ell_s \to 0$ so the worldsheet compresses down to a world-line (see also Fig. 2.16).

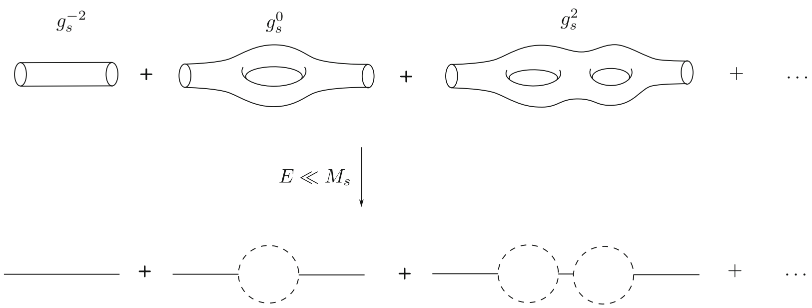


Fig. 2.16 String perturbation theory is a genus (# holes) expansion of string world-sheets. For closed strings, every hole introduces a factor of $(g_s)^2$ in the expansion. For excitations well below the string scale, strings behave like particles and we recover ordinary Feynman diagrams

We will generally work in this regime and keep only the zero-mass excitations of the string.[16] Then the closed string gives exactly the fields of supergravity, see Table 2.6: the metric, dilaton and B-field and the gauge potentials that we have seen when discussing D-branes (Ramond-Ramond fields). Thus one can think of supergravity as the *low energy limit* of *weakly coupled* string theory (and indeed this is where we will mostly be working).

If we consider Fig. 2.16 with in- and out-going graviton states (in the $E \ll M_s \ll M_P$ limit) then the prefactor for the first loop diagram (in the bottom row) is G_N. Computing this same diagram in string theory one finds a pre-factor $g_s^2\, \ell_s^{D-2}$ where the g_s factors come from the genus-counting and the ℓ_s dependence must follow from dimensional analysis (ℓ_s is the only length scale in string perturbation theory). This is the origin of Eq. (2.108).

What about open string perturbation theory? Open strings strech between D-branes and their end-points are labelled by the branes they end on. Thus open string perturbation theory gains an additional factor, N, the number of D-branes, from the degeneracy of open string considered in any scattering process (see Fig. 2.17). Thus the perturbative series is a power series in $g_s N$. This is similar to the expansion in a gauge theory with N_c colors, where we get an expansion in powers of gN_c, and indeed as we will see below this resemblance is no accident.

The low lying (massless) sector of the open strings are a vector field A_μ, a number of spinors ψ^α (fermions) and scalar fields ϕ^i, see Table 2.6. These fields are bound to the brane, because the open string endpoints are. The gauge fields can be interpreted as describing the D-brane dynamics: the scalars describe the transverse motion of the brane (there is one scalar for every direction transverse to the brane worldvolume), the vector (which has only directions on the worldvolume) describes a gauge theory living on the brane and the fermions are needed for supersymmetry. Note that if we only consider open strings, we cannot get a metric: a metric (gravitons) sits only in the closed string spectrum.

Questions from the audience:

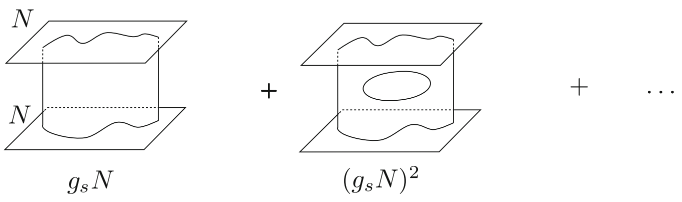


Fig. 2.17 Open string perturbation theory is an expansion in $g_s N$ where N is the number of branes. This is because each loop increases the genus by one (another factor of g_s) and also generates an additional trace over the N gauge factors (another factor of N). Note there is an additional overall factor of g_s above; we show only the relative g_s factors

[16] These are not the lowest-energy modes of the string. Those are tachyonic (negative energy) modes, that can be consistently projected out of the spectrum of string theory.

- *Have we not introduced a cut-off E by restricting our energies to $E \ll M_s$.* No because what we mean by $E \ll M_s$ is that we consider only massless excitations so the cuttoff is actually $E \sim 0$. Or said better we are sending $M_s/E \to \infty$ so we decouple stringy excitations. We assume that any dynamics or additional scales we introduce will be small with respect to M_s unless we explicitly state otherwise. Note that the number of massless excitations can be very large: for open strings on N D-branes, we get a $U(N)$ gauge theory, which has many (massless) fields.
- *Why and how do open strings leave a D-brane?* We have not yet said what closed strings do with a D-brane. Figure 2.19 shows the process by which a closed string leaves a D-brane.

The gauge/gravity duality we mentioned before, is really an open/closed string duality. The theory living on the worldvolume of a string (the so-called worldsheet theory) which describes the propagation of a string in spacetime has a symmetry allowing us to interchange proper time (τ) and proper length (σ) on the worldsheet (it is a symmetry of the string itself). Then a loop diagram in open string theory, looks like a tree level diagram describing the exchange of closed strings between two D-branes, see Fig. 2.18. We will return to this later.

From Fig. 2.19, we see that a process of a closed string interacting with a D-brane has a factor of $(g_s)^2$: we can see this as a graviton exchange. This is another way to see why $G_N \sim (g_s)^2$.

For the discussion of the black hole entropy, we will take the limit $g_s \to 0$. In this limit, open and closed strings naively decouple, since their interaction (Fig. 2.19) is suppressed. Note however that the open-closed diagram receives an enhancement from the degeneracy of open strings so the final effective coupling controlling the interaction of closed and open strings will be $g_s N$, the same coupling that governs interactions between open strings. Thus by taking $g_s \to 0$ but with $g_s N$ fixed we can supress quantum gravity effects but still allow open-strings, or D-branes, to source closed strings (yielding the supergravity solutions described in previous sections). We will return to this later.

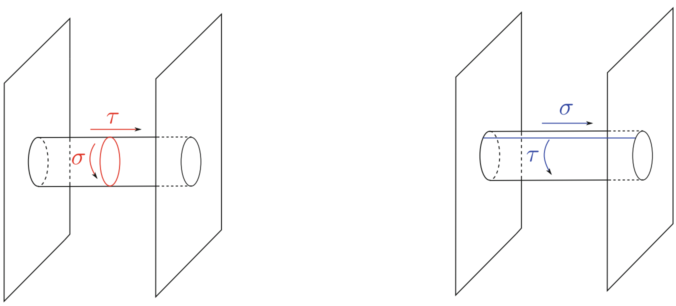


Fig. 2.18 By exchanging the role of string time (τ) and length σ, we can interpret this diagram as an exchange of closed string between D-branes (*left*), or a loop diagram in open string theory (*right*)

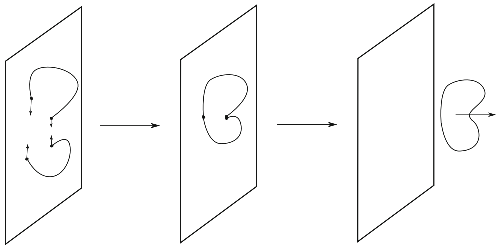


Fig. 2.19 Interpretation of a closed string leaving from a D-brane from open string interaction. Note that each interaction (each pair of end points joining) introduces a factor of g_s in the amplitude of this process

The Stringy D1-D5-P Black Hole

We consider the D1-D5-P system along the following directions. The D5 branes are on compact directions in spacetime, the D1 and the momentum are along one of the directions of the D5:

		T^4				S^1	$\mathbb{R}^4$			
	0	1	2	3	4	5	6	7	8	9
$D5$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$				
$D1$	$\times$					$\times$				
P	$\times$					$\times$				

We can picture this as in Fig. 2.20.

Question from the audience:

- *What is "P", the momentum, exactly?* This can be thought of as a gravitational wave propagating along the S^1 direction. We can see this by a manipulation of the metric (2.97). By changing coordinates, $x_- \to x_5 - t$, the metric looks like

$$\begin{aligned} ds^2 &= -(Z_1 Z_5)^{-1/2}\mathbf{dt}^2 + (Z_1 Z_5)^{1/2} dx_-^2 + Z_p^{-1} dt dx_- \\ &\quad + (Z_1 Z_5)^{1/2}(\mathbf{dr}^2 + r^2 d\Omega_3^2) + ds^2(T^4), \qquad Z_i = 1 + \frac{Q_i}{r^2}. \end{aligned} \tag{2.111}$$

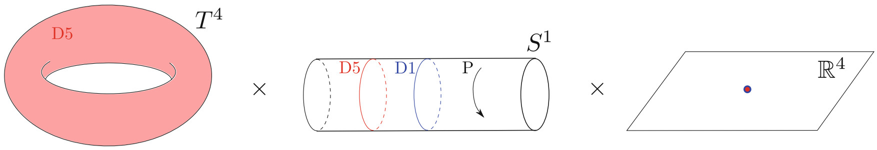


Fig. 2.20 The D1-D5-P system. The D5's are wrapped on $T^4 \times S^1$, along the S^1 we also wrap D1's and we put gravitational waves (momentum), denoted P

The angular momentum of this solution is related to the mixed time-space components of the metric: in this case $p \sim \partial/\partial x^5$ is given by the $1/r^2$ term $Z_p^{-1} = 1 - Q_p/r^2 + \ldots$, so Q_p is indeed the momentum charge.

Remember that the supergravity charges are actually charge densities (we omit numerical factors):

$$Q_1 \sim g_s(\ell_s)^2 N_1 \qquad Q_5 \sim g_s(\ell_s)^2 N_5\,, \qquad Q_p \sim g_s^2 N_p. \tag{2.112}$$

The horizon area depends on the string length and the string coupling:

$$A_H \sim \sqrt{Q_1 Q_5 Q_p} \sim g_s^2(\ell_s)^3 \sqrt{N_1 N_5 N_p}, \tag{2.113}$$

but the Bekenstein-Hawking entropy is independent of the coupling and length scales:

$$S_{BH} = \frac{A_H}{4G_N} = 2\pi\sqrt{N_1 N_5 N_p}. \tag{2.114}$$

From D-branes to Black Holes

Let us now use the observation that S_{BH} is independent of g_s to our advantage. Namely we will argue that by tuning g_s we can interpolate between a regime where the system is described by open strings ending on D-branes to a regime where the system is a black hole with a horizon area which is large in string units $A_H/\ell_s^3 \gg 1$ (i.e. a regular looking supergravity black hole). To do this let us recall:

- g_s is the perturbative parameter in both string theory and gravity. $g_s \ll 1$ is the (semi)classical regime while $g_s \sim 1$ is the quantum regime.
- The coupling between closed and open strings, on the other hand, is controlled by $g_s N$ so if we fix $g_s N$ to be large then D-branes back-react on closed strings (giving geometry) even if we send $g_s \to 0$. This is analogous to saying we can send $G_N \to 0$ (the exactly classical limit of GR) while keeping $G_N M$ fixed for some source so we have a reasonable non-trivial limit giving classical GR solutions.
- Thus our approach will be to fix the entropy by fixing the N's (number of branes) to some very large value but then tune g_s such that we vary from $g_s N \sim 0$ to $g_s N \gg 1$. At $g_s N \sim 0$ we can describe the system in terms of weakly coupled *open* strings on stacks of N D-branes. Closed and open strings decouple in this regime and we can neglect gravity. At $g_s N \gg 1$, on the other hand, the D-branes back react and form a large black hole

Let us see this all in more detail.

Black Holes at $g_s N \gg 1$

The scale of the supergravity solution is set by the charges Q_i in the warp factors. Remember that the supergravity charges appear as $Z = 1 + Q_i/r^2$ and they determine

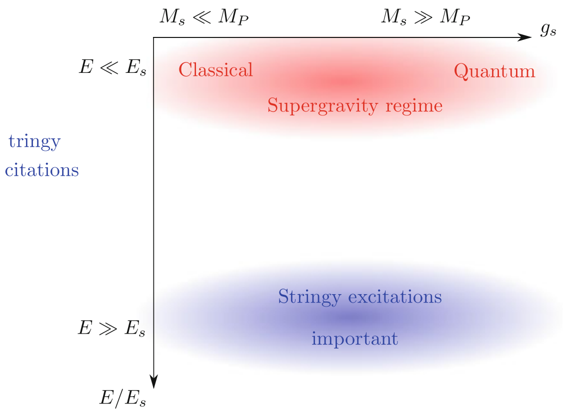


Fig. 2.21 Tuning the coupling in closed string perturbation theory. Kepping only the low energy (zero mass) modes, we have a theory of particles, supergravity. We restrict to small g_s, and only consider *classical* supergravity

the size of the solution. In general, we have $Q_i = G_N M_i$, see Eqs. (2.47–2.49). For a D-brane, we have $M_D = N/g_s$ and hence $Z \sim g_s N_D$, while for the momentum excitations, we have $M_p = N_p$ (just an excitation) and hence $Z \sim g_s^2 N_p$. If $g_s N$ is small (order 1), the area of the black hole is small in string units. Hence we cannot use supergravity to describe it: massive string modes become important, and supergravity only describes the *massless* modes. This violates our earlier physical requirement $E \ll M_s$ (put another way such black holes would involve curvature of the order of the inverse of the string length and thus probing them would involve energies at this scale). We see that we need the horizon to be large in string units to describe (super)gravity black holes and thus we consider instead the regime:

$$g_s N \gg 1. \tag{2.115}$$

We further impose $g_s \to 0$. Closed string theory is non-interacting in this regime as this limit suppresses quantum gravity corrections. This is true whether you are in string theory or in gravity since at low energy a closed string loop looks like a graviton loop, see Fig. 2.21. Thus in the limit $g_s N \gg 1$ with $g_s \to 0$ the D1-D5-P system resembles a large supergravity black hole.

Open Strings at $g_s N \ll 1$

For a large (semi-classical) black hole, we need A_H to be large both in string units, $A_H \gg (\ell_s)^3$, and in Plank units so, via (2.113–2.114), we must take $N_1 N_5 N_p$ to be very large; thus we take the "$N \to \infty$" where this is understood to apply to all the N's.

But as we may still vary g_s we can dial the coupling $g_s N$ allowing us to interpolate between the large black hole like description above (at large $g_s N$) and a weakly-coupled open string description where the open strings end on the branes and don't

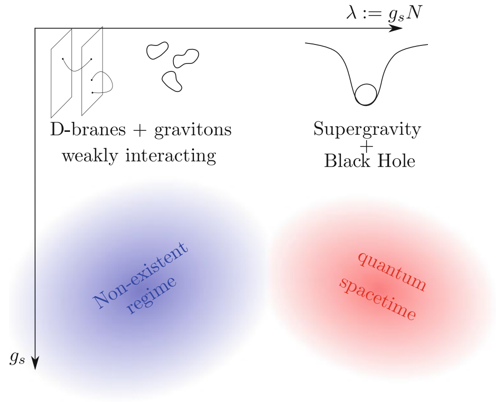


Fig. 2.22 By dialling the coupling $g_s N$ (while keeping g_s small), we can interpret the D1-D5-P system as a *black hole* or as open strings stretching between D-branes. Since the torus volume goes as $V_{T^4} \sim Q_1/Q_5$ in string units, it disappears from the picture and we only retain the five-dimensional geometry. Note that the lower *left* region is non-existent (since we always have that $g_s N > g_s$)

interact with closed strings and gravity (and open string perturbation theory is valid since $g_s N \ll 1$). This tuning is depicted in Fig. 2.22.

Because the entropy is independent of the coupling, g_s, we expect to be able to reproduce the entropy from a counting of supersymmetric states in the weakly coupled open string picture. Note that we take $g_s \to 0$ throughout this diagram so closed strings and gravity are always semi-classical but the open string coupling is $g_s N$ so if we also take $g_s N \to 0$ open strings become weakly coupled and furthermore there is no interaction between the closed and open string sector (even though closed string perturbation theory goes with powers of g_s the couplings to N D-branes goes as $g_s N$ so only in this limit do D-branes not source gravitons). Thus the limit $g_s \to 0$ with $g_s N \to 0$ gives weakly coupled open strings on D-branes in flat spacetime.

We summarize:

- If $g_s \to 0$, you always suppress closed string loop effects (quantum gravity effects)
- $g_s N$ tells you how much closed strings (and gravitons) feel the source. From an open string perspective tuning $g_s N$ is turning open string loop effects on/off.
- If $g_S N \ll 1$, you can count the number of states of these strings stretching between the D-branes, because essentially we get a free (open string) theory (loop effects suppressed). This is reminiscent of holography, where we have $g_s N \ll 1$ giving Yang-Mills weakly coupled, no gravity, and $g_s N \gg 1$ giving Yang-Mills strongly coupled, or AdS_5 gravity.

Note that if the entropy did depend on g_s, then none of this would make sense. A toy model will follow with a rigourous proof that it is g_s is independent.[17]

A question from the audience:

- *Can we get the gravity solution from open string calculations?* Yes you can, but it's a pain. Say we want to find the metric. You can expand the gravitational solution

[17] Extrapolating from toy models is many a string theoriest's idea of a mathematical proof of complicated string theory effects.

in the open string coupling $g_s N$

$$g_{tt} = (Z_1 Z_5)^{-1/2} \sim 1 + g_s N + (g_s N)^2 + g_s^3 + \ldots \tag{2.116}$$

One can then try to match this to an open string loop expansion. The one-loop computation is doable and has been done (Stefano Giusto, a former postdoc at IPhT is doing this). Higher loops are extremely tough; solving supergravity equations of motion is much simpler.

2.4.2 Supersymmetric Indices

We have just argued that the D1-D5-P system looks like a black hole for $g_s N \gg 1$, and like a system of very weakly coupled strings for $g_s N \ll 1$. We want to count the states that make up the entropy in the weakly coupled theory. Why can we trust such a computation? The answer is that in supersymmetric theories certain quantities are protected and cannot depend on continuous parameters such as g_s. Although we will not give a proof of this for the D1-D5-P system we illustrate the idea with a simpler toy model.

Consider supersymmetric quantum mechanics. It is defined by the Hamiltonian

$$H = \{Q, Q^\dagger\} \equiv Q^\dagger Q + Q Q^\dagger. \tag{2.117}$$

The operator Q is fermionic, and anticommutes with itself:

$$\{Q, Q\} = 2Q^2 = 0 \tag{2.118}$$

We define BPS states (or "supersymmetric states") as states that are annihilated by Q, but are not given by acting with Q on another state (Q-closed but not Q-exact):

$$|\psi\rangle_{\text{BPS}} : \qquad Q|\psi\rangle_{\text{BPS}} = 0\,, \quad |\psi\rangle_{\text{BPS}} \neq Q|\psi'\rangle. \tag{2.119}$$

Exercise 2.4.11 *Prove the following properties:*

1. *The Hamiltonian H has only positive eigenvalues. Show that BPS states are states of minimal (zero) energy:*
$$H|\psi\rangle_{\text{BPS}} = 0. \tag{2.120}$$

2. *Let $|\phi\rangle$ be a non-BPS state. Prove that ϕ is degenerate to*

$$|\phi'\rangle = Q|\phi\rangle, \qquad E_\phi = E_{\phi'}. \tag{2.121}$$

Introduce the operator $(-1)^F$, defined through its action on bosonic and fermionic states as:

$$(-1)^F|\text{boson}\rangle = |\text{boson}\rangle, \qquad (-1)^F|\text{fermion}\rangle = -|\text{fermion}\rangle. \tag{2.122}$$

This operator $\mathbb{Z}_2$-grades the Hilbert space. Note that it anticommutes with the operator Q:

$$\{(-1)^F, Q\} = 0. \tag{2.123}$$

Define the Witten index

$$Z = \text{Tr}\,[(-1)^F e^{-\beta H}], \tag{2.124}$$

where β is a number.

3. *Show that*

$$Z = (\text{\# bosonic BPS states}) - (\text{\# fermionic BPS states}) \tag{2.125}$$

4. *Show that*

$$\frac{\partial Z}{\partial \beta} = 0. \tag{2.126}$$

5. *Redo the calculation with the Hamilatonian*

$$H = H_0 + gH_1, \tag{2.127}$$

where both the original Hamiltonian H_0 and the perturbed Hamiltonian H obey the supersymmetry property

$$H_0 = \{Q_0, Q_0^\dagger\}, \qquad H = \{Q, Q^\dagger\} \tag{2.128}$$

for two different fermionic operators Q_0, Q. Show that the function Z is independent of g.

In this exercise, you have proven that the Witten index, which counts the difference in the number of bosonic and fermionic ground states, is independent of the coupling g. The key thing to note is that at strong coupling, the total number of ground states is equal to the Witten index. By its independence on the coupling g, we can calculate the Witten index at weak coupling to count the number of ground states at strong coupling.

We rephrase that in a more mathematical language. Define the trace over the BPS Hilbert space:

$$Z_{\text{BPS}} = \text{Tr}_{\text{BPS}}\,(e^{-\beta H}) = \text{Tr)BPS1} = \text{\#bosons} + \text{\#fermions}. \tag{2.129}$$

This counts the total number of ground states (in the exercise you have proven that the BPS states are exactly the ground states of the Hamiltonian). Note that this is always larger than the Witten index:

$$Z_{\text{BPS}} > Z_{\text{Witten}}. \tag{2.130}$$

At weak coupling, we expect that this is much larger. But at large values of the coupling, you expect that the number of BPS states will match the index because "Anything that can lift, will lift"; i.e. perturbing the system enough will lift degenerate boson/fermions pairs until we have only one species or the other left (i.e. the minimum necessary to preserve the Witten index which cannot vary as we mess around with the couplings). Thus at strong coupling we expect the number of states to match the Witten index. Since the latter is independent of the value of the coupling, we can calculate it at weak coupling and use it to know the number of BPS states at strong coupling.

Question:

- *Are there any restrictions on the validity of the extrapolation to strong coupling?* One way it could break down, is because of a phase transition or discontinuity. There are no walls of marginal stability for this index, so that does not pose a problem. However for extended-supersymmetry theories, where you have several operators Q_i:

$$H = \sum_{i,j=1}^{n} \epsilon^{ij} \{Q_i, Q_j^\dagger\}, \tag{2.131}$$

  the counting of $1/n$ BPS states (that are only annihilated by 1 of the n operators Q_i), is a lot more subtle. And the black hole states are exactly of this form—but we will not go into the details.

2.4.3 *Counting States for the Three-Charge Black Hole*

We study the D1-D5-P system of Fig. 2.20 in the limit $R^4 \gg V_{T^4}$, which means in terms of the charges

$$\frac{Q_p}{Q_1 Q_5} \gg 1. \tag{2.132}$$

In this regime the S^1 is much larger than the other compact directions on which the branes are wrapped so the theories on the D1 and D5 reduce to a theory living on the S^1 with radius R as depicted in Fig. 2.23. The rotation (momentum along x^5) of the D1 and D5 will translate into rotation of the open strings, so we put momentum on the strings to account for Q_p.

We motivate everything from the open string picture. It is not easy to show that D1/D5 momentum follows from F1 with momentum, so you will have to take our word for it. In principle, we can divide momentum over all possibilities: open F1, closed F1, D1's, D5's one or several branes, combinations, single wrapping, multiple wrapping etc: everything can carry momentum. We are interested in the typical, dominant contributions. We will find that we get the most entropy by putting all of

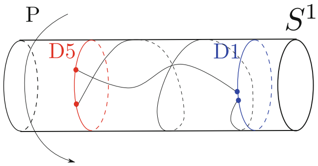


Fig. 2.23 In the limit $R^4 \gg V_{T^4}$ we can largely ignore excitation on the torus and the physics is effectively described by an open string stretched between the D1- and D-5 branes which wrap the *circle*. The open string also carries momentum along the S^1

the momentum in the open string sector because of fractional momentum quantization described on the next page.

To arrive at this picture of the black whole we have to go to weak coupling by tuning $g_s \to 0$ such that $g_s N \ll 1$; in this regime the D-branes are heavy static objects (their mass goes as N/g_s) but they decouple from gravity and are entirely described weakly interacting open strings ending on them. Moreover because we are interested in supersymmetric configurations (as our black hole is supersymmetric) it suffices to restrict to the ground states of the open strings as excited modes break more supersymmetry (recall from the exercises above that supersymmetry tends to require minimal energy). Thus the open strings essentially become point particles connecting two coincident branes. Moreover, at very small $g_s N$ the open strings are essentially free so their wavefunctions are momentum eigenstates on the S^1

$$\psi(x_5) = \sum_n e^{-\frac{2\pi n}{R} x_5}. \tag{2.133}$$

The wave function of a particle normally has to be single valued as we go around a circle but, because these particles carry additional labels, corresponding to the D-brane they're ending on, this is no longer the case. For instance a string ending on a D1 that wraps twice around the circle carries a coordinate, $x^{(1)}$, its location on the D1 and this coordinate itself is not single-valued on the S^1 (i.e. the coordinate length is 4π). This lack of single-valuedness may be familiar from fermions which need not be periodic on a circle because they carry internal (spinorial) indices. Here the additional internal data is just the coordinate on the brane the string endpoint is attached to.

Let us now consider a string with two endpoints going around the circle several times. Take for example a string stretched between a D1 brane that wraps the circle twice, and a D5 brane that wrap the circle three times.[18] If we unwrap the circle, this configuration looks like Fig. 2.23.

The open string wave function depends on the string coordinate x_5 and has two labels, coordinates on the D1-branes and D5-branes (Fig. 2.24):

[18] Note that for "2 D-branes" on a compact circle, we have either 2 distinct D-branes or a D-brane wrapping the circle 2 times.

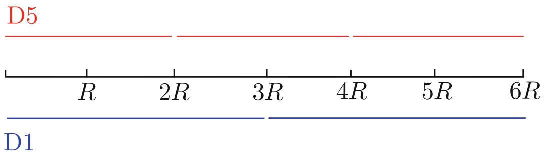


Fig. 2.24 A D1 brane wrapping the *circle* twice and a D5 brane wrapping the *circle* three times. We need to go six times around the *circle* before we reach the same point again

$$\psi(x^{(1)}, x^{(5)}). \tag{2.134}$$

Depending on the label, we have the periodicities:

$$x^{(1)} \sim x^{(1)} + 2R\,, \qquad x^{(5)} \sim x^{(5)} + 3R. \tag{2.135}$$

The wave function of the string then, depending on both $x^{(1)}$ and $x^{(5)}$ is not periodic in R, but rather has a periodicity of $6R$:

$$\psi(x^{(1)}, x^{(5)}) = \psi(x^{(1)} + 6R, x^{(5)} + 6R). \tag{2.136}$$

For a general number of branes, we conclude that the string wave function is periodic in $N_1 N_5 R$ (at least if N_1 and N_5 are coprime). Thus we can expand any such wavefunction in a set of modes with this periodicity:

$$\psi(x_5) \sim e^{-2\pi \frac{n}{N_1 N_5 R} x_5} \tag{2.137}$$

The number n denotes the number of momentum units; momentum on such D1-D5-string is quantized in units of $1/N_1 N_5 R$ rather than $1/R$. This phenomena is referred to as momentum *fractionalization* because momenta can now come in fractional units.

Note that the total spacetime momentum, N_p, as measured e.g. at infinity in black hole solution, is still quantized in units of $1/R$ because metric modes (which carry the momentum) are single valued around the S^1. But the individual open strings carrying the momentum can carry fractional momentum – it is only the sum of all the momenta that must be integrally quantized (in units of $1/R$).

What about non-coprime N_1, N_5? We can always consider the nearest-coprime number by subtracting a small number $m \ll N_{1,5}$ such that $N_1 - m$ and N_5 are coprime. Then the leading contribution to the entropy is still $N_1 N_5 N_p$ as any difference will be suppressed by powers of m/N_1. As we will explain below it is always entropically favourable to be in the configuration with maximal fractionalization so this configuration will dominate.

We want to put N_p units of momentum on the D1-D5-string system but there are many ways of doing this by putting different amounts of momenta on different open strings. Thus the entropy of the system is given by considering:

In how many ways can we get the momentum $p = N_p/R$ from partitioning the momentum over the D1-D5 open strings (with wave function (2.137)?

We can translate this to counting the number of partitions

$$\sum_{m=1}^{\infty} \frac{n_m m}{N_1 N_5 R} = \frac{N_p}{R}. \tag{2.138}$$

The number m counts the momentum in units of $1/N_1 N_5 R$ added by n_m strings of this type. For instance, the easiest (but not most entropic) way to get such a partitioning is to take one string with $m = N_p N_1 N_5$ units of momenta.

We count the number of different ways to form free strings (free excitations)

$$M \equiv N_1 N_5 N_p = \sum_{m=1}^{\infty} n_m m. \tag{2.139}$$

This is a counting of partitions of integers. We claim that this is counted by the partition function

$$Z = (1 + q + q^2 + \ldots)(1 + q^2 + q^4 + \ldots)(1 + q^3 + q^6 + \ldots)(\ldots). \tag{2.140}$$

The first contributions are

$$Z = 1 + q + 2q^2 + 3q^3 + \ldots \tag{2.141}$$

and the coefficients of q^n indeed count the partitions of the numbers n: one partitioning of 1, two of the number 2 ($1+1$ and 2), three for 3 ($1+2$, $2+1$ and 3) and so on. If we write the partition function as

$$Z = \sum_{n=0}^{\infty} d_n q^n, \tag{2.142}$$

then d_n counts the number of partitions of the integer n.

Using our knowledge of a geometric series for $q < 1$:

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}. \tag{2.143}$$

we see that the partition function can be written as the product

$$Z = \prod_{n=1}^{\infty} \frac{1}{1-q^n}. \tag{2.144}$$

How can we evaluate this partition function? We perform a calculation in the canonical ensemble: rather than fixing M we fix a dual "effective inverse temperature" β and we will go to a "high temperature"-limit. First we write q as

$$q = e^{-\beta}. \tag{2.145}$$

We calculate the average occupation number

$$\langle n \rangle = \frac{1}{Z} \sum_n n\, d_n e^{-\beta n} = \frac{\partial}{\partial \beta} \log Z. \tag{2.146}$$

This number will give us the leading contribution to the entropy.

First we evaluate the logarithm of the partition function:

$$\begin{aligned} \log Z &= -\sum_{n=1}^{\infty} \log(1 - q^n) \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(q^n)^m}{m} \\ &= \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=1}^{\infty} (q^m)^n \\ &= \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=1}^{\infty} \left(\frac{1}{1 - q^m} - 1 \right) \\ &= \sum_{m=1}^{\infty} \frac{1}{m} \frac{q^m}{1 - q^m}. \end{aligned} \tag{2.147}$$

In the second to last line we used (2.143), and compensated for the over counting for $n = 0$.

Now we take a "high temperature"-limit, by taking $\beta \ll 1$:

$$q \lesssim 1. \tag{2.148}$$

Then $\langle n \rangle$ will be large because we get the large n contributions of the sum $Z = \sum_n d_n q^n$. The leading terms in this limit are

$$q = 1 - \beta + \mathcal{O}(\beta^2)\,, \qquad q^m = 1 - m\beta + \mathcal{O}(\beta^2). \tag{2.149}$$

The logarithm of the partition function becomes

$$\log Z = \frac{1}{\beta} \sum_m m^{-2} + \mathcal{O}(\beta^0). \tag{2.150}$$

We can rewrite this in terms of $\zeta(n)$, Riemann's ζ function, which gives for n an integer:

$$\zeta(n) \equiv \sum_{m=1}^{\infty} \frac{1}{m^n}. \tag{2.151}$$

Then the average particle number is

$$\langle n \rangle = \frac{\zeta(2)}{\beta^2}. \tag{2.152}$$

Standard thermodynamics gives us that the entropy in the canonical ensemble is

$$S = \log Z + \beta \langle n \rangle, \tag{2.153}$$

and this gives

$$S = \frac{2}{\beta}\zeta(2). \tag{2.154}$$

To express the entropy in terms of the number $M \equiv \langle n \rangle$, we invert the relation (2.152), $\beta = \sqrt{\zeta(2)/M}$, and we use that $\zeta(z) = \pi^2/6$. This gives the entropy:

$$S = 2\pi\sqrt{\frac{M}{6}} = 2\pi\sqrt{\frac{N_1 N_5 N_p}{6}}. \tag{2.155}$$

There is a factor of 6 off in the square root compared to the supergravity result! Did we make a counting mistake?

Some remarks:

- Why do we count in canonical ensemble in terms of $\langle n \rangle$ instead of counting the d_n directly for $d = M$ (i.e. working in the microcanonical ensemble)? Recall that for large occupation numbers the canonical and microconincal ensemble are equivalent and we are interested in the large M asymptotics. This is exactly what we do in standard statistical mechanics: E in the canonical ensemble is replaced by $\langle H \rangle$, the expectation value of the Hamiltonian.
- We assumed $\beta \to 0$. We need to check this was a valid assumption. By

$$\beta = \sqrt{\frac{\zeta(2)}{M}}, \tag{2.156}$$

this gives $M \to \infty$: this is exactly the regime we are interested in from the validity of the supergravity solution.

Let us get back to this factor of 6. With the results of Exercise 2.4.12 , we find that the entropy for a "supersymmetric system" (equal number of fermionic and bosonic excitations) is

$$S = 2\pi\sqrt{\frac{cM}{4}}, \tag{2.157}$$

with c the number of bosons. We count the number of massless modes on S^1, but the entropically dominant strings are those stretching between the D1 and the D5-branes. Those 1–5 strings have four bosonic degrees of freedom from their freedom of moving around in T^4 (and these modes have 4 fermionic superpartners justifying the use of the supersymmetric counting formula).[19]

Therefore, the D1-D5-P system has $c = 4$ and we reproduce the black hole entropy on the nose:

$$S = 2\pi\sqrt{N_1 N_5 N_p}. \tag{2.158}$$

Hooray to string theory!

Exercise 2.4.12 *Prove the following statements:*

- *For the partition function*

$$Z_c = \left(\prod_{n=1}^{\infty} \frac{1}{1-q^n}\right)^c, \qquad c \in \mathbb{N}, \tag{2.159}$$

  the entropy in the large temperature limit is

$$S = 2\pi\sqrt{\frac{cN}{6}}. \tag{2.160}$$

  In a free theory, this formula is easy to show. This partition function is nothing but the partition function of c free bosonic oscillators.
- *Z_c was the partition function for c bosons. For fermions, which have either occupation number 0 or 1, we need to put in something extra. Using similar reasoning as for c* $= 1$ *boson partition function, show that the partition function for fermions is*

$$Z_{\text{fermions}} = \prod_{n=1} (1+q^n). \tag{2.161}$$

  Show that for the partition function for c bosonic and fermionic string excitations is

$$Z = \left[\prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n}\right)\right] \tag{2.162}$$

  and that in the high temperature limit, this gives the entropy

[19] There are also contributions from 1-1 and 5-5 strings, but these have momentum quantized in units of $p \sim 1/N_1$ and $- \sim 1/N_5$ and are hence subleading.

$$S = 2\pi\sqrt{\frac{cN}{4}}. \tag{2.163}$$

2.5 AdS/CFT

In this section we will "formalize" the counting arguments of the previous section by putting it in the much larger context of AdS/CFT, a very deep duality between gauge theory and gravity (or between open and closed strings), discussed first in [24–26].

We have seen that g_s, the string coupling, and the number of D-branes N allow us to interpolate between different regimes, see Figs. 2.21 and 2.22. The coupling g_s sets the "quantum" nature of closed string interactions. When $g_s \ll 1$: we have $M_s \ll M_P$ and string theory is classical. Low-lying string excitations are not so massive as to require quantum gravity to understand them. When $g_s \gg 1$ on the other hand, any massive stringy excitation (except the point-like ground states) are in the quantum gravity regime and there is no such thing as classical string theory.

Recall that in the previous section we very heuristically suggested that there is a general duality between open and closed strings: in the presence of a D-brane tree-level closed string diagrams can alternately be interpreted as an open string loop diagrams (see e.g. Fig. 2.18). While we believe this duality holds in general it is quite hard to study because its rather difficult to study excited stringy states. What has been studied and demonstrated in great detail however is a very particular low-energy limit of this duality: AdS/CFT.

In this section, we wish to motivate and study this particular limit and the associated duality. We consider string theory with N D-branes and take a low-energy limit by fixing the energy *at asymptotic infinity* such that $E \ll M_s$ (in a sense we will describe in more detail below). In this low-energy limit we want to consider the regimes:

- $g_s N \ll 1$: Open string theory reduces to a weakly coupled gauge theory describing the system. As we will explain below the description in terms of closed strings is not very tractable in this regime because the near-brane geometry has string-scale curvature and would require the full complex machinery of closed string theory to describe it (i.e. a reduction to massless supergravity modes is not sufficient).
- $g_s N \gg 1$: The same gauge theory above is now strongly coupled and while we can still think of it in terms of open strings this description is not very traceable. Rather a more tractable description is the dual closed string or supergravity picture with D-branes back-reacting and giving a near-brane geometry with a low curvature scale.

The main point is that open-closed duality implies either picture is valid but one may be more computationally tractable in a certain regime than the other. Here we will motivate this duality primarily in its low-energy limit where it becomes the AdS/CFT correspondence.

2.5.1 'Deriving' AdS/CFT

For simplicity in the exposition below we will take $N_1 = N_5 = N$. We further define

$$\lambda = g_s N. \tag{2.164}$$

We represent the small λ and large λ system in Fig. 2.22.

An Open String Perspective ($\lambda \ll 1$)

Let us start by considering the weak coupling picture, $g_s N \ll 1$, where we have a description in terms of perturbative closed and open strings with the latter ending on infinitely heavy, static D-branes. We will restrict ourselves to low energy excitations in this regime as we explain in more detail below.

The spacetime geometry at $\lambda = g_s N \ll 1$ is:

$$M_{1,10} = \mathbb{R}^{1,4} \times S^1 \times T^4. \tag{2.165}$$

In flat space there is a globally defined notion of energy which is the same for an observer near the brane as for an asymptotic observer:

$$E_0 = E_\infty. \tag{2.166}$$

Here E_0 is the energy of an observer in the bulk, or near the brane (this distinction will become important at strong coupling where warp factors shift energies measured at infinity with respect to those near the brane).

How does a process where open strings interact with closed strings depend on the this characteristic energy scale? Such a process was depicted in Fig. 2.19. At low energy, we have gravitons leaving the brane. The amplitude for such a process is proportional to:

$$g_s^2 \ell_s^{D-2} N^2 = G_N N^2. \tag{2.167}$$

From a closed string perspective this is just a gravitational interaction that must be proportional to the masses and G_N. From an open perspective there is one factor of $g_s N$ for each open string endpoint on N D-branes. For instance, for the D1-D5 system, we have to sum over all the ways we can get this process, and there are $N_1 N_5$ possible ways of making 1–5 strings, see Fig. 2.25. The g_s factors follow because as evident from Fig. 2.18 closed emission from a brane looks like an open loop diagram.

Let us take a six-dimensional emission perspective, focusing on $\mathbb{R}^{1,4} \times S^1$ (dropping the T^4 part of the geometry). The dimensionless rate is fixed, on dimensional grounds to depend on the energy E of the process as

$$E^4 G_N = (g_s N)^2 \ell_s^{D-2} E^4 = \lambda^2 (E \ell_s)^4. \tag{2.168}$$

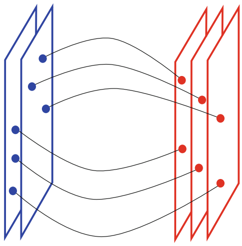


Fig. 2.25 Strings stretching between N_1 D1 branes and N_5 D5 branes. There are $N_1 N_5$ ways of making D1-D5 strings

We would like to work at low enough energies so this process is highly suppressed and the physics of the brane effectively decouples from that of the rest of spacetime. Thus we need to consider energies such that

$$E\ell_s \ll 1/\sqrt{\lambda}$$

In the limit above the open string physics on the brane decouples from interactions with bulk closed strings but open string theory is still rather complicated so let us consider a further limit $E\ell_s \ll 1$. In this limit stringy excitations are very massive and can be integrated out and open string theory on the brane reduces to gauge theory. Thus the limit we really want to consider is

$$E\ell_s = E^{\infty}\ell_s \ll \min(1, 1/\sqrt{\lambda}). \tag{2.169}$$

In this limit the physics of the D-brane "decouples" from that of the bulk and gives, at $\lambda = g_s N \ll 1$, a weakly coupled gauge theory living on the brane. The gauge theory is weakly coupled because λ is nothing other than the 't Hooft parameter—the natural coupling constant of a large N gauge theory (see [27] for a pedagogical exposition of large N gauge theories). But notice that we could also have taken the same limit at large λ and this should in principle describe the strongly coupled version of this gauge theory. Because we restrict to energies satisfying both $E\ell_s \ll 1$ and $E\ell_s \ll 1/\sqrt{\lambda}$ for any value of λ the decoupling of the brane from the rest of the geometry should remain valid as should the "gauge theory limit" of the open strings. The only thing that changes is that the gauge theory becomes strongly coupled. Can we understand this from the closed string perspective?

A Closed String Perspective ($\lambda \gg 1$)

Let's now move to the closed string perspective at $\lambda \gg 1$. Take the metric of the D1-D5-P system

$$ds^2 = \frac{1}{\sqrt{Z_1 Z_5}}(-dt^2 + dx_5^2 + Z_p dx_-^2) + \sqrt{Z_1 Z_5}\, d\boldsymbol{x}_4^2, \tag{2.170}$$

with the light-cone coordinate

$$x_- = t - x_5. \tag{2.171}$$

This metric describes a momentum excitation along one direction, because the light-cone coordinate x_+ is absent.

Remember that this metric has the following regions:

- Asymptotically flat $\mathbb{R}^{1,4} \times S^1 \times T^4$.
- Near horizon region. There is an $AdS_3 \times S^3 \times T^4$ throat and the black hole horizon sits at the bottom of this throat. The quick way to get this decoupled region is to drop the constants in the Z_1 and Z_5 harmonic function but keeping the constant in the Z_p harmonic function. See e.g. [28] for a more detailed exposition of this limit.

so the metric and spacetime at infinity look the same as in the weak coupling limit; only the region near the branes changes.

From the metric, we know that the charges Q in the harmonic functions $Z = 1 + Q/r^2$ go as $Q \sim g_s N (\ell_s)^2 =: \lambda (\ell_s)^2$. Therefore the scale of the throat is set by

$$L \sim \sqrt{\lambda} \ell_s. \tag{2.172}$$

Low Energy Excitations

As above we want to work with "low energy excitations". But: what is energy in this setup? Let us start with the energy E^∞ measured by an observer at infinity in the black hole spacetime and let us restrict, once more, to[20]

$$E^\infty \ell_s \ll 1. \tag{2.173}$$

This means that no strings are excited and we only see gravity modes. Asymptotically, string theory reduces to just (super)gravity.

On the other hand, the throat also has a characteristic energy scale set by $E_{\text{throat}} = 1/L \sim 1/\sqrt{\lambda}\ell_s$. Any asymptotic excitation with an energy lower than

$$E^\infty \ell_s < \frac{1}{\sqrt{\lambda}} \tag{2.174}$$

decouples from the throat: its wave length is larger than the scale of the throat and any such mode shot in from infinity will fly by and not be absorbed by the throat.

Thus at this scale asymptotic excitations decouple from excitations in the throat just as they did in the open string analysis at $g_s N \ll 1$. The low energy limit (2.169) thus has the effect of isolating the "near-horizon" physics down the throat from what happens further away. This "decoupling" is an essential feature of AdS/CFT so we will always work, for all values of λ, in the limit

[20] In natural units $\hbar = c = 1$, energy is measured in dimensions of inverse length $[E] = L^{-1}$.

$$E^{\infty}\ell_s \ll \min(1, 1/\sqrt{\lambda}). \tag{2.175}$$

Another way to phrase this is that we consider the theory defined by excitations whose (asymptotic) energy remains finite as we send $\ell_s \to \infty$.

So far we have phrased this limit in terms of the energy measured at infinity and shown that asymptotically stringy excitations become infinitely massive and can be ignored in this limit. What about the near-horizon throat region?

In a gravitational theory energy can only be defined locally. The redshift relates the energy between two observers at r_1 and r_2 as $\int_{r_1}^{r_2}\sqrt{g_{tt}}$. Approximating this integral by its value down the throat, the energy E_0 of a local observer at say $r = 1$ in the throat is related to the asymptotically measured energy as

$$E^{\infty} \sim (Z_1 Z_5)^{1/4} E_0 = \sqrt{\lambda} E_0. \tag{2.176}$$

What does this imply about the energy of excitations down the throat in our limit (2.175)? Consider the two cases:

1. $\lambda > 1$: Then by (2.175) we have $\sqrt{\lambda}E_0\ell_s \ll 1/\sqrt{\lambda}$, which can be written as:

$$E_0\ell_s \ll 1/\lambda < 1. \tag{2.177}$$

   There are no stringy excitations down the throat.
2. $\lambda \ll 1$; Then by (2.175) we have $\sqrt{\lambda}E_0\ell_s \ll 1$, or:

$$E_0\ell_s \ll 1/\sqrt{\lambda}, \tag{2.178}$$

   but we also have $1/\sqrt{\lambda} \gg 1$ and thus we can have stringy modes down the throat. This happens because the energy of these modes is so red-shifted that, at infinity, we still have $E^{\infty} = \sqrt{\lambda}E_0 \ll 1$ even if we consider excitations with e.g. $E_0\ell_s \sim n \gg 1$ so long as $n\sqrt{\lambda} \ll 1$.

We conclude that there *can* be stringy excitations down the throat only when $\lambda \ll 1$. These are decoupled from the asymptotic region due to the redshift.

Thus the closed string picture we arrive at is one where the spacetime has a throat region and an asymptotically flat region and, at low energies, these regions are decoupled from each other. As in the open picture we are interested in the throat region near the brane itself let us examine what that region looks like in more detail (Fig. 2.26).

Throat Geometry

Let us consider the geometry of the throat. First we put $Z_p = 1$ effectively setting $Q_p = 0$. We can later add the momentum as excitations on the throat geometry. Deep in the throat we have $r \ll \sqrt{\lambda}\,\ell_s$ and hence

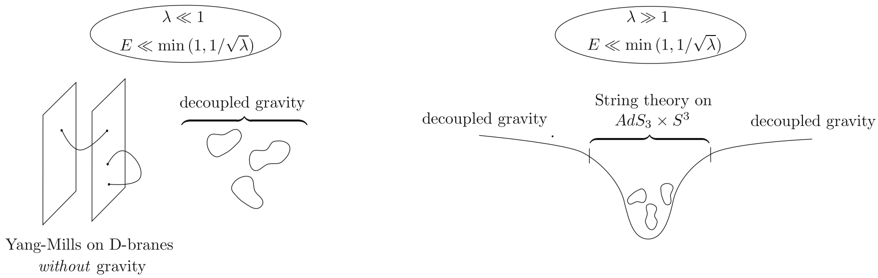


Fig. 2.26 We consider low-energy excitations $E^{\infty}\ell_s < \min(1, 1/\sqrt{\lambda})$. *Left* in the regime $\lambda \gg 1$, we have a field theory describing open string theory, *right* for $\lambda \ll 1$, we can have a "stringy" *black hole*, with (open) string excitations and gravitons down the throat, which decouple from the asymptotic geometry

$$Z_{1,5} \sim \frac{\lambda(\ell_s)^2}{r^2}. \tag{2.179}$$

The geometry becomes

$$ds^2 = \frac{r^2}{\lambda \ell_s^2}(-dt^2 + dx_5^2 + \ldots) + (\lambda \ell_s^2)\frac{dr^2}{r^2} + \lambda \ell_s^2 d\Omega_3^2 + ds^2(T^4). \tag{2.180}$$

This is the geometry of $AdS_3 \times S^3$ (times a constant volume T^4). The radius of anti-de Sitter space and the three-sphere are equal and set by λ in string units:

$$R_{AdS} = R_S = \sqrt{\lambda}\ell_s. \tag{2.181}$$

Note that the geometry $AdS_3 \times S^3 \times T^4$ is a solution to the equations of motion itself, essentially because the equations for the warp factors

$$\Delta Z_i = 0, \tag{2.182}$$

are insensitive to the presence or absence of the integration constant, h, in the harmonic functions $Z = h + Q/r^2$ and it is this feature which distinguishes the $AdS_3 \times S^3$ solution from the flat-space one.

2.5.2 Putting it All Together: AdS/CFT

Let us now put together the various pieces we have assembled. Recall that we claim that there is an open-closed duality meaning that we are free to use open or closed strings to describe a given system. The system we are interested in is the D1-D5-P system. We study this system in the particular low-energy limit (2.169). Note that this limit is phrased in terms of dimensionless parameters $E\ell_s$ so it is a consistent "decoupling" limit; we can formally take a limit sending $E^{\infty}\ell_s \to 0$ and this defines a completely independent subsector of the original string theory.

When taking this limit what we find is that:

- In the *open description* the open strings on the brane decouple from the physics off the brane and furthermore only the massless open strings survive. Thus open string theory reduces to supersymmetric Yang-Mills on the D-brane. At $\lambda \ll 1$ this theory is weakly coupled and can be studied. When $\lambda \gg 1$ this becomes a strongly coupled gauge theory and it is hard to compute anything.
- In the *closed description* the closed strings near the horizon (down the throat) decouple from those asymptotically far away so there is a self-contained closed string theory living on $AdS_3 \times S^3 \times T^4$. When $\lambda \gg 1$ only light excitations survive the low-energy limit so we are left with supergravity on the aforementioned spacetime but when $\lambda \ll 1$ stringy modes *can* be excited so the theory really is a full string theory.

The statement of AdS/CFT, which we see is just a low-energy limit of open-closed duality, is that the two descriptions listed above are in fact equivelent! Put another way supersymmetric Yang-Mills on a D-brane is equivelant to a string theory on an AdS spacetime. When the gauge theory is weakly coupled ($\lambda \ll 1$) the AdS is very stringy and thus its hard to study it (many massive string modes are excited). On the other hand, when the gauge theory is strongly coupled ($\lambda \gg 1$) the closed string theory on AdS reduces to supergravity leading to the remarkable observation that we can understand strongly coupled gauge theories by studying semi-classical supergravity! This is the primary reason why AdS/CFT has been so fruitful in the last years.

It should be emphasized that all the statements made above were made in the limit of sending $N \to \infty$ and $g_s \to 0$ while keeping the combination $\lambda = g_s N$ as a free parameter. Thus the gauge theories above always have very large gauge groups $SU(N)$ with $N \to \infty$. The duality between gauge theory and closed string theory is believed to hold even for finite N and there are numerous computations checking $1/N$ corrections to the above. This regime is much harder to study however as making N and λ finite means that g_s can no longer be zero and we need to consider higher loop diagrams in string theory or supergravity and this is quite hard.

2.5.3 AdS/CFT Dictionary

In Table 2.7 we collect the various equivalences implied by AdS/CFT. When the field theory is weakly coupled, the AdS space has a very small radius L and string theory corrections are important (strongly coupled string theory on AdS). When the field theory is strongly coupled, the AdS space is large and well described by classical supergravity. In terms of couplings this means that for small λ, we have good control

Table 2.7 Equivalence between open string and closed string theory for various values of λ

Yang-Mills on a D-brane	Closed string theory on AdS
Decoupled sector:	Closed strings
gauge theory on a brane	down the throat
No strings/no gravity	*Full, closed string theory*
λ: gauge ('t Hooft) coupling constant	$\lambda = L/\ell_s$: size of AdS in string units
N: rank of gauge group	$N = L/\ell_p$: size of AdS in Planck units
λ small: weakly coupled	AdS small $\rightarrow$ stringy
Control	*No control*
λ large: strongly coupled	AdS large $\rightarrow$ Supergravity
No control	*Control*

of the gauge theory, whereas for large λ, we have good control of the gravitational anti-de Sitter physics.

Note that unlike string theory in flat space where the only parameter is $g_s \sim f(\ell_s/\ell_p)$ in AdS there is an additional dimensionful scale, L, the AdS radius, allowing us to define two independent parameters: N and λ. Following the discussion above we see that in gauge theory, N is the rank of the gauge group while in gravity, N is the size of the AdS space in Planck units (while λ is the size of AdS in string units). The inverse AdS radius measured in Plank units, $1/N$, provides the natural perturbative parameter for quantum gravity in the bulk; i.e. this parameter enters in loop corrections for both gravity and string theory. Thus the limit of an infinite number of colors, $N \rightarrow \infty$ is nothing other than the classical limit in the AdS theory! While this may seem like a somewhat strange statement it in fact parallels a well known statement in gauge theory that at large N the dynamics of the gauge theory become much simpler (see [27] for an explanation).

Exercise 2.5.13 *Show that the AdS length (size of the D1-D5 black hole throat) in string units is set by λ, and in Planck length by N:*

$$\lambda = L/\ell_s, \qquad N = (L/\ell_P)^n \tag{2.183}$$

for some number n. Find n.

Because supergravity is only valid at large N, we only understand large N gauge groups from supergravity. On the other hand, we could invert this to maybe learn quantum gravity from small N gauge groups. For instance, for $N = 2, 3$ the size of AdS space is a few Planck units and gravity is strongly coupled.

Note that the AdS/CFT correspondence is a conjecture. We haven't proven anything, we have just given motivation! It is very hard to prove: a proof would require a detailed knowledge of strongly coupled field theories. However, it is very well established as many very non-trivial computations (not necessarily protected by symmetry) have been found to match on both sides and thus most string theorists hold it to be true. In some sense it is nothing more than the low energy limit of the much more powerful open/closed string duality hinted at by Fig. 2.18. The closed string exchange between D-branes, which can be interpreted as a tree level open string diagram, has all the massive modes implicit. For AdS/CFT, we only consider the massless, non-oscillatory modes.

Formal AdS/CFT Duality

The correspondence can be formalized by equating the path integrals of the two theories:

$$\boxed{Z_{\mathrm{CFT}}(\lambda, N) = Z_{\mathrm{IIB}}^{\mathrm{string}}(\lambda, N)|_{\text{on asympt. AdS space}}}. \tag{2.184}$$

This equality summarizes the AdS/CFT conjecture.

We often restrict to λ very large, and then we get an equivalence between large 't Hooft coupling CFT and IIB supergravity on an asymptotically AdS space:

$$Z_{\mathrm{CFT}}(\lambda \to \infty, N) = Z_{\mathrm{IIB}}^{\mathrm{sugra}}(N)|_{\mathrm{AdS}}, \tag{2.185}$$

where sugra stands for supergravity. Schematically, we can write the supergravity path integral as

$$Z_{\mathrm{IIB}}^{\mathrm{sugra}}(N) = \int \mathcal{D}g \, \exp\left(-\int \sqrt{-g}(\text{gravitons} + \ldots)\right), \tag{2.186}$$

there are other fields besides the metric g, but let's just forget about them for the sake of the argument. When N is large, we are doing classical supergravity: at fixed $\lambda = g_s N$, loops are suppressed because g_s is small. Then we can perform a saddle point approximation around the minima of the action (the classical solutions to the equations of motion), and the large N approximation is

$$Z_{\mathrm{IIB}}^{\mathrm{sugra}}(N \to \infty) = \sum_i e^{-S_i}, \tag{2.187}$$

The sum runs over solutions to the equations of motion (saddle points) and it is actually possible to calculate its main contributions. In the limit $\lambda \to \infty$ (large 't Hooft coupling) and $N \to \infty$ (planar limit), states in the CFT are hence related to classical solutions in AdS.

The left-hand side of (2.185) is always well-defined because CFTs are formally well-defined objects. Thus, as a consequence of the AdS/CFT correspondence, the right-hand side is also well defined: quantum gravity on AdS spaces is hence better defined than a generic QFT!

2.5.4 Entropy Counting

A black hole solution has an entropy and a temperature. Thus the natural candidate dual in the CFT is an ensemble of states corresponding to a thermal density matrix with the same quantum numbers as the black hole (in particular the mass). Such a density matrix has the following form

$$\rho_{BH} = \sum_{\psi} e^{-\beta H} |\psi\rangle\langle\psi|. \tag{2.188}$$

At high temperature there is no difference between the microcanonical and the canonical ensemble. Therefore we can work with the temperature, the thermodynamic dual of the mass, rather than with the mass itself.

Remember the set-up of the D1-D5-P system wrapped on $T^4 \times S^1$ of Fig. 2.20. The CFT that describes this system lives on the two-dimensional spacetime formed by the common circle on which the branes are wrapped and the time direction: $S^1 \times \mathbb{R}_t$. (This is the CFT dual to the AdS_3 near-horizon geometry of the D1-D5 black hole.)

Cardy gave us a formula for the entropy in a CFT at high temperature, irrespective of the coupling:

$$S \sim \sqrt{\frac{cL_0}{6}}, \tag{2.189}$$

where L_0 is the momentum along one direction, and c is the central charge. Although we will not justify this formula (it is a standard result in the study of 2d CFTs) let us note that it gives the number of states at a given level, L_0, in a CFT with central charge c. Because we are assuming the black hole to correspond to a thermal ensemble which is essentially a sum over all states we can use this formula and simply substitute in the black hole quantum numbers that give c and L_0 via AdS/CFT.

Note that Cardy's formula has the same form as the entropy computed using our a simple free oscillator counting. There c was the "entropy density". For a boson in a free theory, $c = 1$, for a free fermion one has $c = 1/2$. But the CFT we are considering here is strongly coupled since we want a large classical black hole so $\lambda \gg 1$ (as is N). Thus we cannot simply model the system using free fields but the great virtue of Cardy's formula is that it holds for any CFT, even a strongly coupled one. Moreover, it does not rely on any assumption of supersymmetry so this is a qualitatively different way of computing the degeneracy (recall that we were able to use a "free" open string picture in our previous counting because we argued, via supersymmetry, that we could work in the small $\lambda = g_s N$ regime and then simply tune λ to large values without changing the number of supersymmetric states).

For gravity on an AdS space, the central charge of the dual CFT is the AdS length in Planck units (we will motivate this partially below):

$$c = \frac{L}{\ell_P}, \tag{2.190}$$

Note, as expected (for the entropy to be invariant), this quantity is independent of the coupling but depends only on the D1-D5 charges:

$$c = N_1 N_5. \tag{2.191}$$

On the other hand this would not be the case if the central charge was the AdS length in string units, because then c would be equal to $\sqrt{g_s}\sqrt{N_1 N_5}$ and hence coupling-dependent. The fact that c is independent of the string coupling g_s is very important, because it assures that the entropy (through the Cardy formula) is independent of the coupling as well.

If we put the momentum excitations on the D1-D5 AdS_3 throat to match the full D1-D5-P black hole solution then in the dual CFT this corresponds adding light-like momentum along the string that the dual CFT lives on. Although we will not review this in detail it simply follows because the quantum numbers in the CFT can be matched to those in AdS and under this identification momentum waves in the bulk simply correspond to momenta along the CFT worldsheet. Thus, like the spacetime momentum[21] the momentum in the CFT is chiral and thus corresponds to a state with non-vanishing $L_0 \sim N_p$. Thus Cardy's formula gives the entropy

$$S \sim \sqrt{\frac{N_1 N_5 N_p}{6}}. \tag{2.193}$$

This does not rely on weak coupling but rather is valid for any value of g_s.

We have now argued, via AdS/CFT correspondence, that a thermal ensemble in a strongly coupled CFT is dual to a black hole geometry, and that we can use the Cardy formula to compute the entropy. Let us briefly motivate the identification of the central charge which we recall is

$$c = N_1 N_5. \tag{2.194}$$

From the original brane theory this is not hard to believe as the dominant degrees of freedom are the 1-5 strings and there are $N_1 N_5$ of them (recall that the central charge of a CFT is some measure of the degrees of freedom). This can be seen another way: in a 2d CFT, the partition function at high temperature goes as

$$Z_{\text{CFT}} \sim e^{cT}, \tag{2.195}$$

and hence the entropy goes as

$$S \sim \log Z_{\text{CFT}} \sim cT. \tag{2.196}$$

---

[21] Remember that the metric of the D1-D5-P system looks like

$$ds^2 = -dt^2 + dx_5^2 + Z_p dx_-^2, \tag{2.192}$$

with $dx_- = dt - dx_5$. This fixes a particular chirality of the plane wave.

Table 2.8 For supersymmetric black holes, we can match the Bekenstein-Hawking entropy from a weak coupling computation

$S_{\text{BH}}^{\text{micro}}$	=	$S_{\text{BH}}^{\text{macro}}$
$\downarrow$		$\downarrow$
$\log(N)$		$A_H/4G_N$
Weak coupling		**Strong coupling**

This also shows why we can interpret c as the entropy density.

Questions from the audience:

- *The black hole is extremal. How can there be a (CFT) temperature?* In CFT, there is a left and a right temperature, related to the total amount of left- and right moving excitations. Using the null circle x_- (or x_+ if we would have that coordinate in the metric), gives a length of this thermal circle that gives a temperature T_L (T_R for x_+). The total temperature of a thermal ensemble of states is related to those temperatures as
$$\frac{1}{T} = \frac{1}{T_R} + \frac{1}{T_L}. \tag{2.197}$$
T_L and T_R are in fact chemical potentials for the quantum numbers L_0 and $\bar{L}_0$ in the CFT; these measure the number of left and right moving light-light momentum waves. In the extremal D1-D5 setup, we only have left-moving excitations and hence $T_L \neq 0$, but still $T_R = 0$. Therefore the BH temperature T is zero, even though there is a "left-moving temperature" T_L.
- *We have treated AdS/CFT. Here we had AdS_3 of the near-horizon plus the dual CFT. What happens if you insert a black hole inside an asymptotically AdS space?* Consider AdS with a black hole inside it. This corresponds to a CFT at a non-zero temperature T (so both T_L and T_R are non-zero), see Table 2.5.

2.5.5 Non-supersymmetric Black Holes

For supersymmetric black holes, we have seen that the microscopic entropy matches the macroscopic one as in Table 2.8.

We have seen two arguments why the weak-coupling, microscopic calculation gives the correct result for the entropy of the black hole at strong coupling:

- An index which is protected by supersymmetry: it can be calculated at weak coupling and continued to strong coupling.
- AdS/CFT correspondence. The result for the entropy uses the Cardy formula and can be calculated regardless of the coupling, as long as we high temperature states in the CFT (here temperature includes left or right moving temperature).

Both these arguments rely on supersymmetry but in different ways. The first argument requires supersymmetry by construction whereas Cardy's formula holds in any 2d CFT, even one without supersymmetry. Unfortunately only supersymmetric black holes have near-horizon AdS$_3$ factor which allow us to use AdS$_3$/CFT$_2$ and invoke Cardy's formula. What about non-supersymmetric solutions in asymptotically flat spacetime? The index will no longer be protected, and we cannot rely on the AdS/CFT correspondence anymore, because the near-horizon solution of a non-extremal black hole does not have an AdS factor.

On the other hand we can consider non-supersymmetric asymptotically AdS black holes (black holes embedded in an AdS spacetime rather than flat space). We can put a non-extremal black hole (black hole with a non-zero temperature) in $AdS_5 \times S^5$. Without the black hole, the geometry is dual to a conformal field theory, namely $\mathcal{N} = 4$ Super-Yang Mills theory. It is a supersymmetric and conformal (there is no dimensionful scale) field theory that is very similar to QCD.

When we put a black hole in spacetime, this is dual by the AdS/CFT correspondence to heating up the CFT, and hence introducing a scale. This is the setup of Table 2.5.

A high temperature excites the many states of this field theory (gluons, fermions…), and therefore you get an entropy, a number of states that are excited at a given temperature. The temperature breaks both conformal invariance and supersymmetry in the field theory and we get a non-supersymmetric state corresponding to the black hole.

We can repeat the counting of the previous section and find the entropy, both in the field theory (a non-trivial calculation involving fermions, $SU(N)$ gauge groups and so on) and in gravity (an easy calculation using the horizon entropy). One finds:

$$\begin{array}{ll} \mathcal{N} = 4 \text{ SYM} & \left\{ \begin{array}{l} S^{\text{micro}} = a(N)T^3 \\[2ex] S^{\text{macro}} = \frac{3}{4}a(N)T^3 \end{array} \right. \\ \text{Supergravity} & \end{array}$$

with

$$a(N) = \frac{2\pi^2}{3} N^2. \tag{2.198}$$

A pedagogical derivation of this result can be found in [29].

The supergravity entropy only sees three quarters of the entropy of the microscopic counting. We can interpret this as the degrees of freedom that are changing. The field theory computation above is done at weak coupling ($\lambda \ll 1$) where we can easily compute whereas the black hole, which must be large in string units, corresponds to large values of λ. Thus there is some less of states as the spectrum shifts about from weak to strong coupling in Fig. 2.27. Note that this does not happen in the supersymmetric case because of the supersymmetric index is a protected quantity.

The fact that the entropy at a fixed energy changes as we vary the coupling should not be too surprising. As λ is increased various states will receive quantum corrections to their energy and the spectrum will shift about in a complicated way.

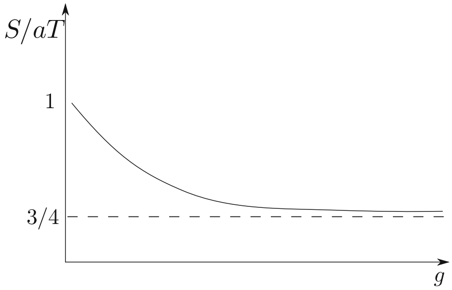


Fig. 2.27 A sketch of the entropy as a function of the coupling for the *black hole* in $AdS_5 \times S^5$ (see Fig. 2 in [30] for a detailed graph)

It turns out that there are other quantities which are also relatively robust so we may hope to compute them using AdS/CFT. That is to say there are quantities which are shared by a large class of theories—a *universality class*—which we may hope contains both $\mathcal{N} = 4$ SYM (the CFT which is dual to string theory on AdS_5) and other more physically relevant theories like QCD (or perhaps all Yang-Mills like theories). Since such quantities don't depend strongly on the detailed structure of the theory we can try to apply AdS/CFT to compute them even if we do not yet know the dual of QCD. Another way of thinking of this is that the gravity dual of $\mathcal{N} = 4$ SYM captures the strong coupling dynamics of a gauge theory and it may be that at strong coupling gauge theories display certain universal behavior.

As an example, take two fundamental properties of fluids in such theories: the entropy density s and the viscosity η. The entropy to viscosity ratio η/s for the quark gluon plasma of QCD can be observed experimentally. In the large N limit the value of η/s can be found exactly in $\mathcal{N} = 4$ SYM, from a weakly coupled gravity computation, and this value is of the same order as the observed value in the RHIC collider, see Table 2.9. Moreover, any calculation in the string theory ballpark always gives the same value of $\eta/s = 1/4\pi$. This is all the more intriguing because existing QCD theories (in which it is difficult to compute strongly coupled quantities) find a number which is off by an order of magnitude.

For this reason, people use AdS/CFT to describe strongly coupled QCD, and also strongly coupled condensed matter theories (so-called AdS/CMT, for instance for superconductors at strong coupling). In fact, this has been the main use of the AdS/CFT correspondence so far and this entire field can be put under the name "holography". There are many articles which can lead you in this direction, see for instance the previous courses on holography at IPhT [31, 32] (see also [33]).

Table 2.9 Entropy to viscosity ratio

η/s	Theory/Experiment
$1/4\pi \cong 0.0796$	$\mathcal{N} = 4$ SYM
$0.12 \pm \ldots$	QCD (Experiment)
$\mathcal{O}(1)$	QCD (Theory)

2.6 The Fuzzball Proposal and Black Hole Hair

In this section, we elucidate the idea that black hole entropy is explained by the existence of a large number of 'black hole microstate' solutions. These are geometries that are solutions to the equations of motion of string theory, have no horizon themselves, but should come in large enough numbers to account for the black hole entropy.

Let us get back to the main problem. We have a microscopic and a microscopic entropy, which agree numerically, but both are valid in different regimes. As an example, think about the air in a room. It is made up out of many molecules. Still, we can extract the entropy without reference to the microscopic state of the molecules through equations of state:

$$\begin{aligned} pV &= nRT, \\ dE &= TdS + pdV. \end{aligned} \tag{2.199}$$

We can determine the entropy S without knowing what air is made of—thermody namically, the entropy is a measure of the energy change in a system on which we have no control or understanding (in contrast to the work term pdV, which we control very well).

So much for thermodynamics, on to statistical mechanics. Boltzmann has taught us that for a given energy E and temperature T, all N different states of the molecules in the room make up the entropy as:

$$S^{\text{micro}} = \log(N). \tag{2.200}$$

This connection between statistical mechanics and thermodynamics is already 150 years old. Does it work for a black hole too? Can we find a number of microstates N for a black hole with a given set of mass and charges, such that $S_{BH} = \log(N)$?

At this point, the programme we followed so far is incomplete. The microscopic calculation ("statistical mechanics") takes place in one regime, but this statistical description is not valid when $g_s N \ll 1$. We have the following question:

- Say you take a state that makes up the entropy in the microscopic calculation. What happens if you follow such states one by one and bring them over to strong coupling?

People believed for a long time that as gravity grows stronger, a horizon forms around the D-branes and the objects end up "being" the black hole [34–38], see Fig. 2.28. Because gravity is always attractive, you expect that as you make Newton's constant larger, increasing the gravitational attraction, "normal" objects only becomes smaller. Only a black hole grows with increasing G_N, as the horizon radius for a (Schwarzschild) black hole scales with Newton's constant as

$$r_H = 2G_N M, \tag{2.201}$$

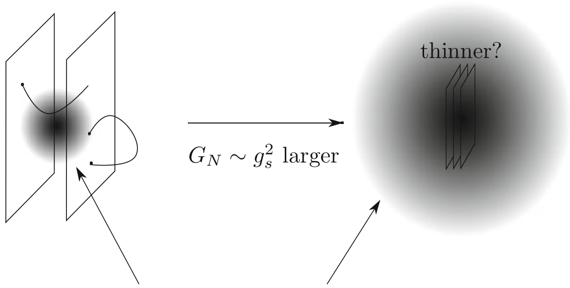


Fig. 2.28 At low G_N ($g_s \ll 1$), the would-be *black hole* horizon is of smaller or equal size as the brane system. For large G_N, the *black hole* horizon is much bigger than the size of the D-brane system at weak coupling

with M the mass of the black hole. Thus the horizon actually *grows* when you make gravity stronger. Take for instance a neutron star. This is a charge neutral object. Imagine a thought experiment in which we scale up Newton's constant G_N. The horizon radius of a black hole that has the same mass as the neutron star will become larger until for a certain large value of G_N, the neutron star collapses into a black hole. This intuition caused people to think for a long time that whatever state you take out of the $\exp(2\pi\sqrt{N_1N_2N_3})$ black hole microstates in the weak coupling description, all of them become a black hole with a singularity in the middle.

We can represent this pictorially. Say we have three microstates made up out of open strings on D1-D5 branes in the decoupled regime, as in Fig. 2.29. As we make gravity stronger, all of these would seem to fall behind the horizon and the information of the state making up the black hole is in the region near the singularity.

We discussed earlier the information paradox: We can throw anything into the black hole, but within GR, this information gets lost and never comes out, as the black hole evaporates into thermal radiation. Since the Hawking radiation process deals with the region around the black hole horizon, the intuitive picture of what happens to a brane microstate does not solve the problem. The horizon region is in the causal past of the singularity and physics in this region has no idea of what happens at the

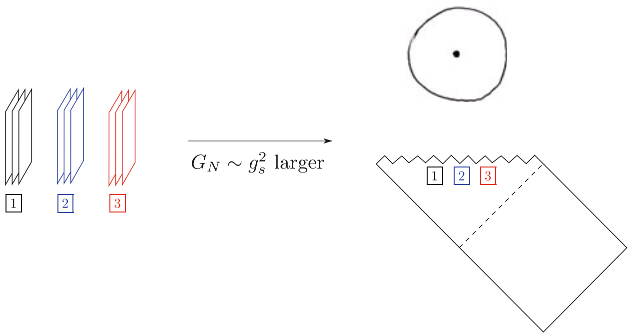


Fig. 2.29 In the naive picture, cranking up G_N puts the information of the microstate 1,2 or 3 into the garbage near the singularity

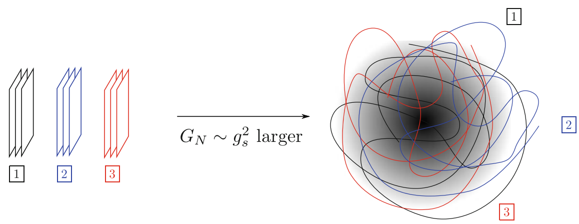


Fig. 2.30 The 'fuzzball proposal': cranking up g_s gives a complicated state of strings and branes of horizon size

singularity. All information still sits near the singularity and the information paradox is still there.[22] In fact, if we want to evade the information problem, arguments by Mathur show that one needs large corrections to the black hole geometry near the horizon [39].

Through the D1-D5-P black hole and the AdS/CFT duality, we should be able to find the CFT process dual to Hawking radiation. In CFT, we can actually address this problem.

In recent years, it has become clear that certain black hole microstates actually grow with G_N just as the black hole does! Look at a microstate. As g_s grows large, they actually become *bigger* and will be of the *same size* as the would-be black hole horizon, see Fig. 2.30. It is an ongoing task to find the actual geometries describing the strong coupling equivalent of the D-brane microstates. For such microstates that are of a size comparable to the black hole's, Hawking evaporation will know about what information made the black hole.

The main problem with this proposal is that you need to explicitly construct 'microstates' of the same size as the black hole horizon. The black hole horizon grows as G_N, but most things get smaller for increasing G_N. We need some very special objects. We will show how to build such growing states that correspond to the CFT we counted at g_s small. These will not have a horizon at large g_s.

The largest success of this proposal has been in the constructing of supersymmetric microstate geometries, see [40–45] for reviews. However, supersymmetric black holes do not radiate, and there is no comparison of the Hawking process. For non-supersymmetric radiating black holes, some large G_N microstates ('microstate geometries') have been constructed [46, 47]. They radiate and the Hawking radiation rate of the black hole agrees nicely with the decay of these states [45, 48–51].

We will review how to count the number of microstate geometries for supersymmetric black holes, using an appropriate quantization technique. So far, the number of microstate geometries found is subleading when compared to the black hole entropy. Ongoing research tries to construct more microstate geometries, see [52–54]. For work on non-supersymmetric multi-center solutions and microstate geometries, see [55–58].

[22] The information paradox leads to a breakdown of unitarity in quantum theory and hence a breakdown of quantum mechanics itself. If we want to save quantum mechanics, we need to make sure there is no information loss.

Note that we have come at the frontier of research: we have some hints about it, but people do not know yet if the proposal is generally true or not. In the next section we will show how to build (certain) fuzzball solutions for the supersymmetric 3-charge black hole.

2.7 Multi-Center Solutions

In this section, we show how to construct five-dimensional multi-center solutions that generalize the string theory black holes we have seen earlier. The microstate geometries for the black hole, or classical fuzzballs, will be in this class.

2.7.1 Preliminaries

In this section, we discuss some necessary basics on differential forms and their application in electromagnetism, and we explain the notation we use in the remainder of the text. We also give some exercises that illustrate an important new term (as opposed to Maxwell theory) in the supergravity action, the Chern-Simons term. This new terms allows for solutions with 'charge dissolved in fluxes', a crucial ingredient for the construction of microstate geometries. The reader familiar with these concepts can skip to the next section on the construction of multi-center solutions.

Differential Forms, Einstein-Maxwell, Sources

We review the following notions, by means of exercises:

- Differential forms, form notation and the definition of the Hodge star operator $\star$.
- 'True' magnetic sources (monopoles) versus 'moving electrons'.
- Sourced electromagnetism in flat space and in curved space (Einstein-Maxwell).

Consider electromagnetism. The anti-symmetric two-form is related to the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ as

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{2.202}$$

In terms of this matrix, the Maxwell equations in vacuum are:

$$\partial_\mu F^{\mu\nu} = 0,$$
$$\partial_{[\mu} F_{\mu\nu]} = 0. \tag{2.203}$$

In form notation they are equivalent to

$$d \star F = 0,$$
$$dF = 0. \tag{2.204}$$

The first expression is the equation of motion that follows from the Lagrangian of electromagnetism:

$$S = \frac{1}{2} \int \star F \wedge F = \frac{1}{4} \int F_{\mu\nu} F^{\mu\nu}. \tag{2.205}$$

The second equation is the Bianchi identity, which just says that locally F is the exterior derivative of a potential $F = dA$, or $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ in form notation.

Exercise 2.7.14 *If you are not familiar with the expressions (2.204) (exterior derivative, Hodge star operator $\star$), read up on it in a book on differential geometry and show that the Eqs. (2.203) and (2.204) are equivalent.*

In particulate, we normalize m-forms as

$$A = \frac{1}{m!} A_{\mu_1 \ldots \mu_m} dx^{\mu_1} \wedge \ldots dx^{\mu_m}. \tag{2.206}$$

The exterior derivative acts on an m-form to produce an $(m+1)$-form as

$$dA_m = \frac{\partial A_{\mu_1 \ldots \mu_m}}{\partial x^\nu} dx^\nu \wedge dx^{\mu_1} \wedge \ldots dx^{\mu_m}, \tag{2.207}$$

and in d dimensions the Hodge star $\star$ takes an m-form to an $n = d - m$ form as follows

$$(\star\lambda)_{\mu_1 \ldots \mu_n} := \frac{1}{m!} \sqrt{g}\, \epsilon_{\mu_1 \ldots \mu_n \nu_1 \ldots \nu_m}\, g^{\nu_1 \rho_1} \ldots g^{\nu_m \rho_m} \lambda_{\rho_1 \ldots \rho_m}. \tag{2.208}$$

Here ϵ is the totally antisymmetric tensor.

Recall that in electromagnetism we can generate a magnetic field by accelerating an electron. However, while a speeding electron generates a magnetic field it does not generate a *magnetic charge*. This is because electric charge only appears in the equation

$$d \star F = q\, \delta(x), \tag{2.209}$$

whereas the magnetic charge sources the Bianchi identity

$$dF = m\, \delta(x). \tag{2.210}$$

The difference between these two is the following. If $m = 0$ then $dF = 0$ everywhere. In flat space this implies there exists a globally defined one-form, $A = A_\mu dx^\mu$, the vector potential, such that $F = dA$. If on the other hand $m \neq 0$ then at the origin F is not closed. Hence there is no globally defined object A such that $F = dA$. However, we can still define an object A everywhere away from the origin (or define it patch-wise). As a side note one might object that solving the electric equation requires something like $A_0 = q/r$, which is singular at the origin. However, we can always smoothen this singular source by allowing a charge distribution (for instance by replacing $q\delta(x)$ with a Gaussian qe^{-qr^2}). The same trick will *not* work for m because the Eq. (2.210) has $dF = ddA$ which is identically zero if A is globally defined.

To write a general field strength that includes both electric and magnetic charge we can do the following. We write

$$F = dA + \Theta, \tag{2.211}$$

with A a global one form encoding the electric charge (and perhaps some part of the magnetic field) via $d\star dA = q\delta(x)$. The two-form Θ on the other hand is *not* globally of the form $d(something)$ but rather satisfies $d\Theta = m\delta(x)$ and hence encodes the part of the field strength coming from the magnetic charge. To see this recall that the definition of the magnetic charge is the integral of the flux through an S^2 around the origin:

$$m = \frac{1}{4\pi}\int_{S^2} F = \frac{1}{4\pi}\int_{S^2}(dA + \Theta) = \frac{1}{4\pi}\int_{S^2}\Theta, \tag{2.212}$$

where the last equality follows because S^2 is a compact manifold without boundary and dA is a total derivative of a globally defined object. Hence the integral $\int dA$ vanishes by Stokes' theorem.

The 'electric part' of the gauge field, A, solves $d \star dA = q\delta(\boldsymbol{x})$. It can be found by thinking of A as harmonic $\nabla^2 A = \delta(\boldsymbol{x})$. This equation has solutions of the form $A = \frac{q}{r}dt$ (actually there is a larger class of solutions constructed of polynomials of the coordinates but the latter are not normalizable). For the solution of Θ, we refer to Exercise 2.7.15.

Exercise 2.7.15 *Write $\Theta = dB$ where B is only locally defined such that the integral (2.212) gives the magnetic charge m. (Find the form of B first).* Hint: *Explicitly construct*

$$B = f(\theta)d\theta \wedge d\phi, \tag{2.213}$$

using polar coordinates for the flat metric $ds^2_{3,flat} = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$, such that $\int dB = 4\pi m$, with m a constant.

Note that when we couple electromagnetism to gravity (Einstein-Maxwell theory), the equation $d \star F = \delta(\boldsymbol{x})$ involves the metric via the Hodge star. Hence the solution becomes more complicated. It turns out that the metrics of the D-brane type have solutions that look like

$$A = H^{-1} dt, \tag{2.214}$$

where H is some harmonic function that determines the solutions and appears in the metric. Typically, in four dimensions harmonic functions are $H = 1 + q/r$, and asymptotically ($r \to \infty$), we recover the flat space solution $A = -q/r\, dt$.

Important Exercises: Chern-Simons Action

We show how the appearance of new terms in the supergravity Lagrangians (compared to electromagnetism) can allow for 'fuzzball' solutions.

The Lagrangian of electromagnetism coupled to gravity in four dimensions is

$$\mathcal{L}_4 = \tfrac{1}{4}\sqrt{-g} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \tag{2.215}$$

$$= \tfrac{1}{2} \star F \wedge F. \tag{2.216}$$

This is the gauge and Lorentz invariant action for the Maxwell field A_μ. In five dimensions, an extra term is possible:

$$\begin{aligned} \mathcal{L}_5 &= \tfrac{1}{4}\sqrt{-g} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{12}\epsilon^{\mu\nu\rho\sigma\tau} A_\mu F_{\nu\rho} F_{\sigma\tau} \\ &= \tfrac{1}{2} \star F \wedge F + \tfrac{1}{3} A \wedge F \wedge F. \end{aligned} \tag{2.217}$$

This new term seems to be breaking gauge invariance. Consider the gauge transformation:

$$A_\mu \to A_\mu + \partial_\mu \lambda, \tag{2.218}$$

with λ a function. The field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is clearly gauge invariant. The second term in the five-dimensional Lagrangian has a "naked" A_μ and you might expect it is gauge non-invariant. The exercise asks you to prove this intuition wrong.

Exercise 2.7.16 *Show that in five dimensions, the Chern-Simons action*

$$S_{CS} = \int \epsilon^{\mu\nu\rho\sigma\tau} A_\mu F_{\nu\rho} F_{\sigma\tau}. \tag{2.219}$$

is invariant under gauge transformations (2.218). It suffices to show that the integrand is invariant up to a total derivative.

Most extensions of general relativity based on string theory (in particular supergravity) have such a term. So it is important to study its physical consequences.[23]

[23] It is also important for confinement in supersymmetric holographically dual gauge theories through the AdS/CFT correspondence, but that is another matter. See [59, 60].

Choose coordinates x^0, x^1, x^2, x^3, x^4 in five dimensions. Remember that a static electron couples to the gauge field as

$$\int A_0 dt. \tag{2.220}$$

Because of the term (2.219), a non-trivial A_0 is sourced by magnetic terms $F_{12}F_{34}$ through the equations of motion, which schematically have the form $\partial_i F_{0i} = F_{12}F_{34}$ (see Exercise 2.7.17). Even if you don't have electrons, but just magnetic fields of two different kinds, you can have electric fields!

Exercise 2.7.17 *Derive the equations of motion for A_μ following from the action (2.217):*

$$d \star F = F \wedge F. \tag{2.221}$$

Show that you can source electric fields with magnetic fields along different directions, by working this out in components (including the metric components involved in the Hodge star operation).

In the literature, one refers to solutions with this mechanism (magnetic fluxes giving a net electric charge) as solutions with charges dissolved in fluxes.

We will use this kind of solutions with charge dissolved in flux to build microstate geometries. In fact, this mechanism is crucial for the existence of microstate geometries. The absence of such a term in regular electromagnetism is also the reason people had not found black hole microstate geometries before the advent of string theory. This mechanism is widely used in other solutions as well, such as flux compactifications used for the construction of string vacua, see [61] for a review.

2.7.2 Building General Solutions

We discuss how to obtain new solutions with 'charge dissolved in fluxes'. We do this in a stepwise fashion: first we discuss the five-dimensional black hole (without and with rotation), and then we show how to put in magnetic charges.

M2-M2-M2 Black Hole

Let us write down a five dimensional electrically charged black hole by starting in M-theory (11-dimensions) and writing a solution down that involves a compact six-torus. Recall in particular, the supergravity solution for the (supersymmetric) M2-M2-M2 brane system

$$\begin{aligned} ds^2 = & -(Z_1Z_2Z_3)^{-2/3}dt^2 + (Z_1Z_2Z_3)^{1/3}ds^2(\mathbb{R}^4) \\ & + \frac{(Z_2Z_3)^{1/3}}{Z_1^{1/3}}(dx_1^2 + dx_2^2) + \frac{(Z_1Z_3)^{1/3}}{Z_2^{1/3}}(dx_3^2 + dx_4^2) + \frac{(Z_1Z_2)^{1/3}}{Z_3^{1/3}}(dx_5^2 + dx_6^2). \end{aligned} \tag{2.222}$$

This solution describes five space-time dimensions because we actually take the coordinates $x_1, \ldots, x_6$ to be compact ($x_i \sim x_i + 2\pi L_i$ for $i = 1, \ldots, 6$. They describe a six-torus T^6. We write the T^6 as the product of three two-tori T^2.

The M2-branes are all point-like in the transverse $\mathbb{R}^4$ spanned by x^7, x^8, x^9, x^{10} which we can write in radial coordinates

$$ds_4^2 = d\rho^2 + \rho^2 d\Omega_3^2 \tag{2.223}$$

and the five-dimensional black hole is determined by the functions:

$$Z_1 = 1 + \frac{Q_1}{\rho^2}, \qquad Z_2 = 1 + \frac{Q_2}{\rho^2}, \qquad Z_3 = 1 + \frac{Q_3}{\rho^2}. \tag{2.224}$$

The unusual power 2 rather than 1 in the denominator is because we are solving this equation in four rather than three space dimensions. Note that we refer to the radius in $\mathbb{R}^4$ as ρ, to avoid confusion with r for the radius of $\mathbb{R}^3$.

These functions are defined simply by requiring them to solve the equation:

$$\Box_4 Z_I(x) = Q_I \delta(\rho) \tag{2.225}$$

where $\Box_4 \cdot = \sqrt{g_4}^{-1} \partial_i (\sqrt{g_4} g_4^{ij} \partial_j \cdot)$ is defined with respect to the four-dimensional flat metric in the solution above (on $\mathbb{R}^4$). This equation says that we have M2 sources sitting at the origin of $\mathbb{R}^4$ with charges Q_I. The 1 in the equation above is simply a homogeneous solution we are free to add to any given solution to the Eq. (2.225). Since this equation is linear we are free to superimpose solutions (adding delta function sources). Hence the most general solution corresponds to an arbitrary number of M2 sources at various positions $\boldsymbol{\rho}_p \in \mathbb{R}^4$ and p labels the "centers":

$$Z_I = \text{constant} + \sum_p \frac{Q_p}{|\rho - \rho_p|^2} \tag{2.226}$$

See Fig. 2.31.

Recall that in M-theory we have a 3-form gauge potential and for the solution above it has the following form

$$C_{012} = Z_1^{-1}, \qquad C_{034} = Z_2^{-1}, \qquad C_{056} = Z_3^{-1}. \tag{2.227}$$

By "compactifying" on the $x_1, \ldots, x_6$ directions we can think of this as a five-dimensional solution times T^6 and one can show that this six-torus is actually small

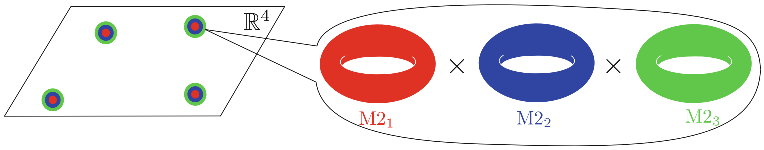


Fig. 2.31 Multiple M2-brane sources in $\mathbb{R}^4$. Each source can correspond to three types of M2-branes wrapped on a T^2 inside T^6, and smeared in the other torus directions

(the length of each cycle is order 1 in string units) so at low energies this space-time looks five-dimensional. In this case the different components of the three-form C_3 reduce to three independent gauge fields A^I_μ in five dimensions:

$$A^{(1)}_\mu = C_{\mu 12}, \qquad A^{(2)}_\mu = C_{\mu 34}, \qquad A^{(3)}_\mu = C_{\mu 56} \tag{2.228}$$

And likewise there are three field-strengths, $F^{(I)} = dA^{(I)}$ with $I = 1, 2, 3$. In form notation, the four-form $F_4 = dC_3$ of M-theory is then given by

$$F_4 = F^{(I)} \wedge \omega_I = d(Z_I^{-1} dt) \wedge \omega_I = (\partial_\rho Z_I^{-1}) d\rho \wedge dt \wedge \omega_I , \tag{2.229}$$

where we defined the volume forms on eqch two-torus:

$$\omega_1 = dx_1 \wedge dx_2 \, , \qquad \omega_2 = dx_3 \wedge dx_4 \, , \qquad \omega_3 = dx_5 \wedge dx_6 . \tag{2.230}$$

In five dimensions the solution given by the functions Z_I of (2.224) is a spherically symmetric, electrically charged black hole in $\mathbb{R}^{1,4}$. We can generalize this solution in three ways, and we will do so in the remainder of this section, by:

- Adding angular momentum
- Adding magnetic charge
- Adding a more complicated base space (instead of $\mathbb{R}^4$)
- (Adding a more general internal space that preserves supersymmetry in five dimensions: a Calabi-Yau manifold instead of a T^6. We will not do this explicitly in these lectures.)

Multi-center solutions with these ingredients can describe black hole microstate geometries.

Adding Angular Momentum

The first generalization is to add angular momentum to this solution. We do this by replacing dt in the metric with $dt + k$ where $k = k_i(x)dx^i$ $(i = 7, 8, 9, 10)$ is a one-form in the four-dimensional base space:

$$ds_5^2 = -(Z_1 Z_2 Z_3)^{-2/3}(dt + k)^2 + (Z_1 Z_2 Z_3)^{1/3} ds^2(\mathbb{R}^4) \tag{2.231}$$

We will only consider the five non-compact directions from now on. Since the gauge field and metric are coupled via the equations of motion, adding angular momentum to the metric modifies the gauge field as well:

$$F^{(I)} = d(Z_I^{-1}(dt + k)) = d(Z_I^{-1}) \wedge (dt + k) + Z_I^{-1} dk. \tag{2.232}$$

Note this field strength has magnetic $F_{ij}^{(I)}$ components (from $\partial_i k_j$), because we have a moving charge. Remember from Sect. 2.7.1 that this does not represent a genuine magnetic monopole charge. This setup allows to describe a rotating supersymmetric black hole [62].

By adding a $k = k_i(x)dx^i$ term to the metric we get non-vanishing g_{ti} cross-terms in the metric. Such terms imply that the space-time itself carries angular momentum. This is *not* to be confused with being time-dependent. None of the fields above, including the metric, contains any explicit dependence on the time coordinate. A rather good analogy is to consider a *featureless* spinning ring in for instance $\mathbb{R}^3$, see Fig. 2.32. Since the ring is featureless nothing changes in time: the ring is always just sitting there spinning and from one instance to the next everything looks identical. Nonetheless, this solution carries angular momentum. In GR, such solutions with mixed g_{ti} components but no time-dependence are referred to as *stationary*. Solutions with no time dependence and $g_{ti} = 0$ are *static*.[24]

In $\mathbb{R}^4$ there are two independent angular momenta, because we can think of $\mathbb{R}^4$ as $\mathbb{R}^2 \times \mathbb{R}^2$: we have one independent angular momentum in each plane. For a single centered black hole, supersymmetry, is only preserved if we force these two angular momenta to be equal. This condition can be generalized as

$$(1 + \star_4)dk = 0 \tag{2.233}$$

which implies k is self-dual. Here $\star_4$ is the Hodge dual defined on the flat $\mathbb{R}^4$ given by $x_7, \ldots, x_{10}$. Note that acting on this with d we find $d \star dk = 0$, meaning k is

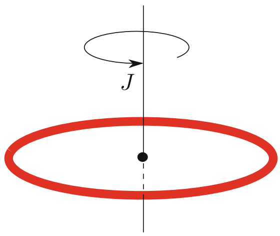


Fig. 2.32 A uniformly spinning ring with angular momentum J around its symmetry axis

[24] Stated without reference to a set of coordinates, 'static' means that the metric admits a global, nowhere zero, time-like hypersurface orthogonal Killing vector field. A generalization are the 'stationary' space-times, which admit a global, nowhere zero time-like Killing vector field.

a harmonic one-form. We will see later that by turning on additional fields, we can relax the condition of equal angular momenta for supersymmetric solutions.

Exercise 2.7.18 *Show that Eq. (2.233) is solved by (2.235). The constant J is proportional to the angular momentum of space-time. See for instance Sect. 2.2 in [20] for more information on asymptotic charges.*

Recall that without k we had the entropy $S_{BH} = \sqrt{Q_1 Q_2 Q_3}$ (up to numerical factors). When we turn on k we get an asymptotic angular momentum J. It can be read off from the asymptotic expansion of k in terms of the angles ϕ_1 and ϕ_2 in the two orthogonal $\mathbb{R}^2$-planes.

If we write the metric on $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ as

$$ds^2 = d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi_1^2 + \cos^2\theta d\phi_2^2), \tag{2.234}$$

the asymptotically leading terms of the momentum one-form k are

$$k = \frac{J}{\rho^2}\sin^2\theta d\phi_1 + \frac{J}{\rho^2}\cos^2\theta d\phi_2, \tag{2.235}$$

with J a constant.

One can compute the horizon area to be (up to a numerical prefactor)

$$S = \sqrt{Q_1 Q_2 Q_3 - J^2}, \tag{2.236}$$

We see that angular momentum reduces the entropy. From a macroscopic point of view this is not hard to understand as the horizon is spinning very fast and this causes it to Lorentz contract and shrink. If we try to spin it up too fast, to the point that $J^2 = Q_1 Q_2 Q_3$, the horizon shrinks to zero size and we cannot go further (at least not with this ansatz). Although we will not say much about it, it is possible to reproduce this entropy using techniques quite similar to those of Sect. 2.4 (and indeed this was done shortly after the $J = 0$ entropy was first reproduced in [62]). The supersymmetric black hole with rotation is often called BMPV black hole after the authors of [62]. The interested reader can read more on microstate counting for these rotating black holes in [20].

Magnetic Charges

Above we added angular momentum to the metric. Even though this sourced magnetic components of the field strength, this was only so in much the same way as a moving electron generates a magnetic field. While a speeding electron generates a magnetic field it does not generate a *magnetic charge* as discussed in the preliminaries of Sect. 2.7.1.

If we want magnetic charges we need to add a closed but not exact term to each of the electromagnetic fields $F^{(I)}$ which we denote by $\Theta^{(I)}$. The field strengths becomes

$$F^{(I)} = d\left(Z_I^{-1}(dt+k)\right) + \Theta^{(I)}. \tag{2.237}$$

Of course this would not be consistent without modifying the form of the metric as well. It turns out this modification is rather straightforward. Recall that in the original metric, the Z_I were potentials sourced by delta-function sources at the locations of the M2's:

$$\Box_4 Z_I(x) = \sum_p Q_I \delta(x_p) \tag{2.238}$$

This source naturally corresponds to an electric field which must satisfy

$$d \star F^{(I)} = \sum_p Q_I \delta(x_p), \qquad dF^{(I)} = 0. \tag{2.239}$$

Recall that in string theory we have peculiar terms in the action such as

$$S = \frac{1}{2}\int F \wedge \star F + \frac{1}{3}\int A \wedge F \wedge F, \tag{2.240}$$

which implies that magnetic flux in this theory can source electric charge via the equation of motion

$$d \star F = F \wedge F. \tag{2.241}$$

This equation translates, in this setting, into a constraint on the functions Z_I which now are no longer simply sourced by a delta-function but look like

$$\Box_4 Z_I(x) = Q_I \delta(x) + \left| \star_4[\Theta^{(J)} \wedge \Theta^{(K)}] \right|, \tag{2.242}$$

with I, J, K all different.

It is important to realize that what is happening here is that if we have two pairs of magnetic charges in this theory they can induce *electric* charge. Thus even if our solution has no explicit electric source (no delta function on the right-hand side of (2.242)) there can be non-trivial electric charge carried by the fields $F^{(I)}$ themselves. Note that this phenomenon, and even the equation above, should look very familiar from non-abelian gauge theories where the gauge field sources itself and carries electric charge (think of glueballs in QCD). The difference is that here we are dealing with an *abelian* theory, and the non-linear interactions arise because of the strange second term in the action (2.240).

While it is obvious that $\Theta^{(I)}$ must be closed away from sources this is not the only constraint they must satisfy. It is harder to show but it turns out that supersymmetry also imposes that the Θ's appearing above are self-dual so that

$$\Theta^{(I)} = \star_4 \Theta^{(I)}. \tag{2.243}$$

Angular Momentum from Crossed Fields

Recall from electromagnetism that when the electromagnetic field has both an electric and magnetic component it carries angular momentum in the form of a Poynting vector

$$\boldsymbol{J} = \boldsymbol{E} \times \boldsymbol{B}. \tag{2.244}$$

While the original solution given above had angular momentum coming from the metric (2.231) encoded in the mixed metric components $g_{ti} \sim k_i$, the addition of a magnetic field changes the angular momentum. This comes from the supergravity equation

$$(1 + \star_4) dk = Z_1 \Theta^{(1)} + Z_2 \Theta^{(2)} + Z_3 \Theta^{(3)}, \tag{2.245}$$

which modifies (2.233) in a way that is essentially analogous to (2.244) with Z_I encoding the electric field and $\Theta^{(I)}$ the magnetic.

Exercise 2.7.19 *For a flavour of why a constraint like (2.243) might follow from supersymmetry consider the action for electromagnetism in four space-time dimensions*

$$S = \int F \wedge \star F \tag{2.246}$$

and decompose $F = F^+ + F^-$ *into self-dual and anti-self-dual parts* $F^\pm = \frac{1}{2}(1 \pm \star) F$. *Rewriting the action in terms of* $F^\pm$ *show that it takes the form*

$$S = \int \left(F^+ \wedge F^+ - F^- \wedge F^- \right). \tag{2.247}$$

If we put $F = F^+$ *(or put otherwise* $F^- = 0$*) then the action is a positive definite perfect square. This is related, morally, to supersymmetry because the latter has a Hamiltonian* $H = \{Q^\dagger, Q\}$ *which is also a sum of squares implying that the energy is always greater an zero. In both cases solving the quadratic equations can be reduced to solving linear ones:*

$$F^+ = 0, \qquad vs. \qquad Q|\phi\rangle = 0, \tag{2.248}$$

and the solutions are minimal action and *minimal energy configurations.*

Overview Before Continuing

We have derived the following system of equations that describes a solution with 3 electric charges, 3 magnetic charges and angular momentum:

$$\Theta^{(I)} = \star_4 \Theta^{(I)},$$
$$\Box_4 Z_I(x) = \frac{1}{2} C_{IJK} \left| \star_4 [\Theta^{(J)} \wedge \Theta^{(K)}] \right|,$$
$$(1+\star) dk = Z_I \Theta^{(I)}, \tag{2.249}$$

where $C_{IJK} = 1$ when all I, J, K are different and zero otherwise and the sum over repeated indices is implied. On the right-hand side of the last two equations, we silently assume the possibility of delta-function sources as well.

We wrote the equations in a suggestive order. To solve these equations, we first have to find a set of self-dual two-forms $\Theta^{(I)}$ on $\mathbb{R}^4$. Then we can solve the functions Z_I in terms of those two-forms. Finally, we need to construct the momentum k from Z_I and $\Theta^{(I)}$. Amazingly, this is a solution of supergravity, the low-energy limit of string theory, and a solution to these equations is a supersymmetric supergravity solution.

Before we solve this system in the specified order, we extend the four-dimensional space $\mathbb{R}^4$ to a non-trivial base space.

Non-trivial Base Space

So far we have taken the four-dimensional metric ds_4^2 to be flat. However, it turns out that supersymmetry does not require this space to be trivial but to be a more general metric of hyperkähler type [63].

An interesting and pretty general class of four-dimensional metrics that are hyperkähler are the Gibbons-Hawking and Taub-NUT metrics which take the form of a circle fibre (coordinate ψ) over flat three-dimensional $\mathbb{R}^3$:

$$ds_4^2 = V^{-1}(d\psi + A)^2 + V(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)), \tag{2.250}$$

where V depends only on the three-dimensional coordinates r, θ, ϕ and the one-form A satisfies

$$\nabla \times A = \nabla V. \tag{2.251}$$

The fibre coordinate is periodically identified as $\psi \sim \psi + 4\pi$.

The harmonic V on this space has the general form

$$V = \epsilon_0 + \sum_i \frac{q_i^0}{r_i} \tag{2.252}$$

where now $r_i = |\boldsymbol{r} - \boldsymbol{r}_i|$ and $\boldsymbol{r}_i \in \mathbb{R}^3$. When working on $\mathbb{R}^3$ space instead of $\mathbb{R}^4$ we will use the Hodge dual $\star_3$ and radial coordinate r instead of $\star_4$ and ρ.

Near a pole of V, the Gibbons-Hawking metric looks like $\mathbb{R}^4$, as Exercise (2.7.20) asks you to show. Asymptotically, at large r, the four-dimensional space is $\mathbb{R}^3 \times S^1$.

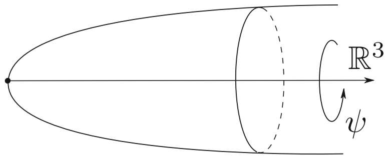


Fig. 2.33 Taub-NUT space with the harmonic function $V = 1 + n/r$ looks like a cigar. Near $r \to 0$, the ψ *circle* shrinks to zero size smoothly and space is locally $\mathbb{R}^4/\mathbb{Z}_n$. Asymptotically, the ψ *circle* is of constant radius and space-time asymptotes to $\mathbb{R}^3 \times S^1$

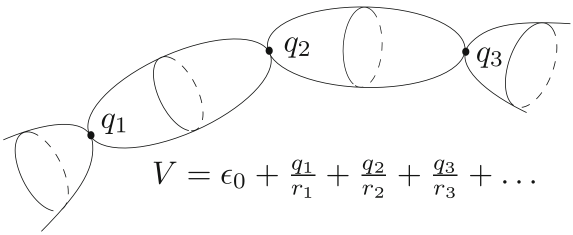


Fig. 2.34 Multi-center taub-NUT space is a "bubbled geometry". At each center, the size of the ψ *circle* goes to zero and the geometry looks like smooth $\mathbb{R}^4/\mathbb{Z}_n$. Asymptotically, the geometry is $\mathbb{R}^3 \times S^1$

We can read the radius of S^1 from the asymptotic expansion of the metric as the constant $1/\sqrt{\epsilon_0}$. By varying ϵ_0, we can thus interpolate between a compactification to three dimensional flat space, and $\mathbb{R}^4$ asymptotics by taking ϵ_0 to be zero. See Figs. 2.33 and 2.34 for depictions of single and multi-centered Taub-NUT spaces.

Exercise 2.7.20 *Show that if we choose $V = 1/r$ (with r the radial distance in the $\mathbb{R}^3$) we recover the trivial metric on $\mathbb{R}^4$ globally.* Hint: *Change coordinates to $\rho = 2\sqrt{r}$ and show that the metric for small ρ becomes*

$$ds_4^2 = d\rho^2 + \rho^2 d\Omega_3^2, \tag{2.253}$$

with $d\Omega_3^2$ the metric on an S^3 of unit radius. In doing so you show that the ψ circle shrinks to zero size smoothly *at the location of any pole in V since, whatever the form of V, near a pole it looks like $V = 1/r$. Hence* **the space-time is smooth at the location of the poles.** *(In fact, near a pole the function V looks like n/r for some charge n. This leads to an orbifold singularity $S^3/\mathbb{Z}_n$. Since string theory is well-defined on orbifold backgrounds, we still consider this as a regular space-time.)*

Exercise 2.7.21 invites you to explore the full eleven- and ten-dimensional solution with a Taub-NUT center and no M2-branes. They give respectively the eleven-dimensional Kaluza-Klein monopole and the 6-brane of IIA supergravity.

Exercise 2.7.21 *If we set $Z_I = 1$ and $V = 1 + \frac{n}{r}$ and we take the product of the space-time (2.6) with $\mathbb{R}^{1,6}$ then we get an 11-dimensional metric that is a solution* *of M-theory. As shown in the previous exercise this metric is* smooth *since the poles in V actually do not give any singularities in space-time. Now check that we can reduce on ψ and get a 10-dimensional solution corresponding to a D6-brane in IIA supergravity (*Hint: *see Sect. 2.4 of Amanda Peet's lecture notes [20] or Polchinski [6] Chap. 8 to see how to do the dimensional reduction). As a consequence, D6-branes in M-theory lift to smooth geometries in M-theory since the D6-brane poles correspond to poles in the V function which are smooth in 11-dimensional space-time.*

2.7.3 Solutions to the Equations of Motion and Supersymmetry

We specify how to find the complete solution to the equations of motion and the supersymmetry equations. These five-dimensional solutions were first described in [64, 65]. First we repeat the ansatz for a torus compactification of M-theory to a five-dimensional supersymmetric solution:

$$ds^2 = -(Z_1 Z_2 Z_3)^{-2/3} dt^2 + (Z_1 Z_2 Z_3)^{1/3} ds_4^2 + \sum_{I=1}^{3} \frac{(Z_1 Z_2 Z_3)^{1/3}}{Z_I^{2/3}} ds_I^2, \quad (2.254)$$

where ds_I, $I = 1, 2, 3$ are the metrics on three T^2's (for example $ds_1^2 = dx_1^2 + dx_2^2$). The four-form field strength decomposes into three two-form field strengths as:

$$F_4 = F^{(I)} \wedge \omega_I, \qquad F^{(I)} = d\left(Z_I^{-1}(dt + k)\right) + \Theta^{(I)}. \quad (2.255)$$

For the four-dimensional base space, we take the general class of Gibbons-Hawking or multi-centered Taub-NUT metrics[25]

$$ds_4^2 = V^{-1}(d\psi + A)^2 + V \underbrace{(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2))}_{\mathbb{R}^3}. \quad (2.256)$$

For the rest of this chapter we work directly in five dimensions and will no longer consider the compact part of the geometry (though that is easy to add in).

The solutions above involve unknowns $k = k_i dx^i$, Z_I and $\Theta^{(I)} = \frac{1}{2}\Theta_{ij}^{(I)} dx^i \wedge dx^j$. They only depend on the coordinates of the three-dimensional flat base space (the Taub-NUT angle ψ is an isometry of the solution). We take the base space to be fixed but of course this means we should specify a V and then fix A via $\nabla \times A = \nabla V$. When the base space is Taub-NUT (asymptotically $\mathbb{R}^3 \times S^1$), the five-dimensional solutions can be compactified to the four-dimensional solutions found in [66, 67].

[25] In fact, the requirement of supersymmetry only requires the base space to be hyperkähler. The additional constraint of a Taub-NUT of Gibbons-Hawking metric makes it possible to solve for the metric explicitly. For more information, see [41] and reference therein.

Supersymmetry and the equations of motion can be simply repackaged into the following conditions

$$\Theta^I = \star_4 \Theta^{(I)}, \tag{2.257}$$

$$\nabla^2 Z_I = \tfrac{1}{2} C_{IJK} \left| \star_4 [\Theta^{(J)} \wedge \Theta^{(K)}] \right|, \tag{2.258}$$

$$(1 + \star_4) dk = Z_I \Theta^{(I)}, \tag{2.259}$$

where C_{IJK} is a completely symmetric tensor. For a more general supersymmetry-preserving compactification of M-theory on a six-dimensional Calabi-Yau manifold, C_{IJK} is given by the triple intersection products of a basis of two-cycles on the Calabi-Yau. We restrict to T^6 compactifications, for which $C_{IJK} = |\epsilon_{IJK}|$. Note that in the second equation we write no longer $\Box_4 Z_I$ but $\nabla^2 Z_I$, since the solution does not depend on the Gibbons-Hawking coordinate ψ. We will also omit the explicit possible delta function sources from now on.

As we noted before, now that we have specified the base space, we can solve this system in three steps: first we need to give the self-dual closed two-forms $\Theta^{(I)}$, then we solve functions Z_I, and then we can solve k. Note that in every step, the procedure is linear in the "new" unknown; hence this is a very tractable problem. We follow the three steps now.

1. Self-Dual Two-Forms

First we construct the $\Theta^{(I)}$. On Taub-NUT space, like $\mathbb{R}^4$, it is not hard to solve $\Theta = \star_4 \Theta$. First define the vielbeins

$$e^0 = V^{-1/2}(d\psi + A), \quad e^i = V^{1/2} dy_i, \tag{2.260}$$

such that the four-dimensional Taub-NUT metric (2.6) is written as a sum of squares:

$$ds^2 = (e^0)^2 + (e^1)^2 + (e^2)^2 + (e^3)^2. \tag{2.261}$$

Then one can check that the two-form

$$\Omega = (e^0 \wedge e^1 + e^2 \wedge e^3) \tag{2.262}$$

is self-dual ($\Omega = \star_4 \Omega$). There are actually three such self-dual Ω's we can construct by permuting the indices on the first term (the sign of the permuted second term is fixed by self-duality).

Exercise 2.7.22 *Check the above statement. First prove that*

$$\star_4 \, (e^A \wedge e^B) = \frac{1}{2} \epsilon^{ABCD} (e^C \wedge e^D) \tag{2.263}$$

for A, B, C, D from 0 to 3. Then prove that the three Ω^a defined as

$$\Omega^1 = e^0 \wedge e^1 + e^2 \wedge e^3, \quad \Omega^2 = e^0 \wedge e^2 + e^3 \wedge e^1, \quad \Omega^3 = e^0 \wedge e^3 + e^1 \wedge e^2, \tag{2.264}$$

are self-dual two-forms under $\star_4$.

The two-forms $\Theta^{(I)}$ must not only be self-dual but also locally closed (and hence co-closed because they are harmonic). Thus we start with Ω^a, $a = 1, 2, 3$ and construct a closed self-dual two-form Θ as

$$\Theta = \partial_a \left(\frac{K}{V} \right) \Omega^a \tag{2.265}$$

Exercise 2.7.23 asks you to prove that Θ is closed only if K is harmonic on the flat three-dimensional space.

Exercise 2.7.23 *Show that Θ defined in (2.265) is closed if K is harmonic on $\mathbb{R}^3$ ($\nabla^2 K = 0$).*

Recall that a harmonic function K on $\mathbb{R}^3$ satisfies $\nabla^2 K = 0$ which has the general solution

$$K = h + \sum_q \frac{p_q}{|\boldsymbol{r} - \boldsymbol{r}_q|}. \tag{2.266}$$

where $\boldsymbol{r}_p$ are arbitrary vectors in $\mathbb{R}^3$ at which H can be singular and the charges p_p and asymptotic value h are constants. In fact $\nabla^2 H = 0$ only holds away from $\boldsymbol{r}_p$ and this equation should be understood as $\nabla^2 K = p_p\, \delta(r - r_p)$. We see that our solution can have an arbitrary number of centers ('sources') on $\mathbb{R}^3$.

Hence the magnetic fluxes of the solution are the self-dual and closed two-forms

$$\Theta^{(I)} = \partial_a \left(\frac{K^I}{V} \right) \Omega^a, \tag{2.267}$$

with K^I three harmonic functions. We will write the harmonic function K^I in terms of charges and asymptotic constants as:

$$K^I = h^I + \sum_{q=1}^{N} \frac{p_q^I}{|\boldsymbol{r} - \boldsymbol{r}_q|}. \tag{2.268}$$

2. Warp Factors

The system of Eq. (2.257) is essentially *linear* if solved in the right order (there are no quadratic interactions or fields sourcing *themselves* quadratically). So once we have Θ we can plug it into (2.258) and solve for the 'warp factors' Z_I. The solution must be sourced by the right-hand side of (2.258) but can also include a homogeneous contribution that solves the equation $\nabla^2 Z_I = 0$. Combining these we get

$$Z_I = \frac{C_{IJK} K^J K^K}{V} + L_I, \tag{2.269}$$

where L_I are three more independent harmonic functions (on $\mathbb{R}^3$) satisfying $\nabla^2 L_I = 0$:

$$L_I = h_I + \sum_{p=1}^{N} \frac{q_{I,p}}{|\boldsymbol{r} - \boldsymbol{r}_p|} \tag{2.270}$$

Exercise 2.7.24 *Check that Z_I given in Eq. (2.269) satisfies (2.258).*

3. Rotation One-Form

The final Eq. (2.259) simply reproduces the (anti-)self-duality condition we mentioned above ($dk = - \star dk$) in the absence of explicit magnetic source ($\Theta = 0$). When such sources are turned on we solve this equation by decomposing k through the following ansatz:

$$k = \mu(d\psi + A) + \omega, \tag{2.271}$$

with $\omega = \omega_i dx^i$ a form on $\mathbb{R}^3$ and μ a function of the three-dimensional coordinates.

Exercise 2.7.25 *Show that plugging the ansatz (2.271) into (2.259) yields an equation for ω and μ:*

$$\nabla \times \boldsymbol{\omega} = (V \nabla \mu - \mu \nabla V) - V\, Z_I \nabla \left(\frac{K^I}{V} \right) \tag{2.272}$$

where as always we sum over $I = 1, \ldots, 3$.

To solve the Eq. (2.272) for ω we take a further divergence and use $\nabla \cdot (\nabla \times \boldsymbol{\omega}) = 0$ to obtain

$$V \nabla^2 \mu = \nabla \cdot \left(V\, Z_I \nabla \left(\frac{K^I}{V} \right) \right). \tag{2.273}$$

Exercise 2.7.26 *Show that this can be solved as*

$$\mu = \frac{1}{6} C_{IJK} \frac{K^I K^J K^K}{V^2} + \frac{1}{2} \frac{K^I L_I}{V} + M, \tag{2.274}$$

with M a harmonic function. The corresponding solution for ω satisfies

$$\nabla \times \boldsymbol{\omega} = V \nabla M - M \nabla V + \tfrac{1}{2}(K^I \nabla L_I - L_I \nabla K^I) \tag{2.275}$$

There is a nice and clean way of writing the solution for ω in terms of the harmonic functions. Write the harmonic functions as a vector

$$H \equiv (V, L_1, L_2, L_3; M, K_1, K_2, K_3). \tag{2.276}$$

Then the right-hand side of (2.275) defines a symplectic product of such matrices:

$$\nabla \times \omega = \langle H, \nabla H \rangle. \tag{2.277}$$

While it is possible to get an explicit form for ω in simple examples, one generally has to resort to patches to specify the solution for ω given the harmonics V, K^I, L_I and M.

Exercise 2.7.27 *Show that on a flat base in absence of magnetic charges ($\Theta^{(I)} = 0$), you reproduce the earlier expression for k of Eq. (2.235). Use Exercise 2.7.20 for the coordinate transformation to flat space*

$$ds_4^2 = d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta\, d\phi_1^2 + \cos^2\theta\, d\phi_2^2), \tag{2.278}$$

and take a single center with $M = m/r$. Determine the relation between J and m.

2.7.4 Physical Solution and Fuzzballs

Above we have shown that the solution can be specified in terms of eight harmonic functions V, K^I, L_I and M. We started with a black hole with harmonic functions $Z_I = L_I$, encoding three electric charges, and angular momentum encoded by the harmonic function M. In terms of eleven-dimensional M-theory, we have the brane interpretation:

M2's: L_1, L_2, L_3	Angular Momentum: M

Now we have also 3 magnetic fields, given by the harmonic functions K^I, and a magnetic geometric charge (of the Gibbons-Hawking space), encoded by V. The black hole charge can be dissolved in the magnetic fields. In M-theory language, these correspond to

M5's: K^1, K^2, K^3	Kaluza-Klein monopole: V

For concreteness, we fix a notation for the charges and constants of the harmonic functions. We organize the harmonic functions in a symplectic vector H:

$$H = (H^0, H^I, H_I, H_0) \equiv (V, K^I, L_I, M). \tag{2.279}$$

The symplectic vector of harmonic functions is written in terms of a symplectic array of constants h and charges Γ at each center:

$$H = h + \sum_{q=1}^{N} \frac{\Gamma_q}{|\boldsymbol{r} - \boldsymbol{r}_q|}, \tag{2.280}$$

with

$$h = (h^0, h^I; h_I, h_0), \qquad \Gamma \equiv (p^0, p^I, q_I, q_0). \tag{2.281}$$

For later use, we define the symplectic product of any two symplectic vectors A, B as:

$$\langle A, B \rangle = A^0 B_0 - A_0 B^0 + \tfrac{1}{2}(A^I B_I - A_I B^I). \tag{2.282}$$

In the remainder of this section, we give the physical requirements one has to impose on the solutions, and we show how we can construct microstate geometries.

Physical Requirements

At this point, getting the solution from harmonic functions is like blindly using a computer. We still have many questions: Are these solution physical? What are their properties? Are there singularities? We will answer these questions now.

We start with the vector $\boldsymbol{\omega}$ that describes the angular momentum of the metric in $\mathbb{R}^3$. To have it well-defined in space-time, the divergence of (2.277) should be zero:

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\omega}) = 0. \tag{2.283}$$

This gives a condition on the harmonic functions. First we write them as the symplectic product of the vector of harmonic functions H:

$$H = h + \sum_{i} \frac{\Gamma_i}{|\boldsymbol{r} - \boldsymbol{r}_i|}. \tag{2.284}$$

Then (2.283) gives the condition:

$$\langle H, \nabla^2 H \rangle = V \nabla^2 M - M \nabla^2 V + \tfrac{1}{2}(K^I \nabla^2 L_I - L_I \nabla^2 K^I) = 0. \tag{2.285}$$

The leading terms are those at the positions of the centers. Writing the charges for a harmonic functions at each center as $\Gamma_i = (p_i^0, p_i^I, q_{I,i}, q_{0,i})$, we have

$$\sum_j \langle H, \Gamma_j \rangle \delta(\boldsymbol{r}_j) = 0. \tag{2.286}$$

Demanding that each delta function contribution is zero gives one condition for each center $\boldsymbol{r}_i$:

$$0 = \langle \Gamma_i, h \rangle + \sum_j \frac{\langle \Gamma_i, \Gamma_j \rangle}{r_{ij}}, \tag{2.287}$$

with the relative distances

$$r_{ij} = |\boldsymbol{r}_i - \mathbf{r}_j|. \tag{2.288}$$

The physical interpretation of these equations is to assure there are no Dirac-Misner strings in the geometry (such that there is no source on the right-hand side of (2.283)).

Once the charges are fixed, the Eq. (2.286) then give constraints on the center positions $\boldsymbol{r}_i$: these equations tell you where the points are. We call these 'bubble equations' (giving r_p's in terms of Q's), because the resulting geometries have 'bubbles' (non-trivial two-cycles). Other names for these equations are 'integrability equations' (term coined by the original discoverer, Denef [66, 67]) and 'Denef equations', in the context of the related four-dimensional solutions.

Two-Center Solution Space

What is the space of solutions of the bubble equations? For simplicity, we restrict to two centers first. Then there is only one equation:

$$\frac{\langle \Gamma_1, \Gamma_2 \rangle}{r_{12}} + \langle \Gamma_1, h \rangle = 0. \tag{2.289}$$

We should have $\langle \Gamma_1, h \rangle \langle \Gamma_1, \Gamma_2 \rangle < 0$ to find a solution. This equation then fixes the distance r_{12}. The space of solutions is given by 2 points fixed by a rigid rod. The system has two degrees of freedom: two points in space-time have three degrees of freedom in $\mathbb{R}^3$ (three for each point, minus three for the center of mass), and the bubble equation fixes one. The solution space is the S^2 of possible positions of the second point at a distance r_{12} of the first one.

The vector of constants, h, determines the asymptotics of the harmonic functions through $H_{r\to\infty} = h$ and it determines what the space looks like asymptotically (for instance it contains a constant h^0 for the harmonic function $V = h^0 + p^0/r$ in the metric). For fixed charges Γ_1, Γ_2, the constants h also describe an interesting moduli space. Fix the charges such that $\langle \Gamma_1, \Gamma_2 \rangle > 0$. The value of h then determines if we can find a solution to the bubble Eq. (2.289). Take for instance a geometry with constants h such that $\langle \Gamma_1, h \rangle < 0$ and the bubble Eq. (2.289) have a solution. By tuning the asymptotic parameters h, we could go from $\langle \Gamma_1, h \rangle < 0$ to $\langle \Gamma_1, h \rangle = 0$ and even $\langle \Gamma_1, h \rangle > 0$: the solution disappears. It is no longer a valid physical solution. If we look at the solution space in function of the asymptotic parameters, the boundary $\langle \Gamma_1, h \rangle = 0$ determines a "wall of marginal stability". When crossing a wall of marginal stability ("wall-crossing"), these states just disappear. When $\langle \Gamma_1, h \rangle < 0$, the solution is part of the solution space, and we have an entropy associated to them (the 'number' of such states). When we cross the wall of marginal stability in the moduli space of allowed constant parameters h, the solution is gone and the entropy that counts all allowed solutions jumps.

Three-Center Solution Space

We turn to the more interesting solution space for three centers. The vector of harmonic functions is.

$$H = \frac{\Gamma_1}{|\boldsymbol{r}-\boldsymbol{r}_1|} + \frac{\Gamma_2}{|\boldsymbol{r}-\boldsymbol{r}_2|} + \frac{\Gamma_3}{|\boldsymbol{r}-\boldsymbol{r}_3|} + h. \tag{2.290}$$

From (2.286), we get three equations, one at each center (from the $\delta(\boldsymbol{r}_i)$-contributions)

$$\begin{aligned}
&\frac{\langle\Gamma_1,\Gamma_2\rangle}{r_{12}} + \frac{\langle\Gamma_1,\Gamma_3\rangle}{r_{13}} + \langle\Gamma_1,h\rangle = 0,\\
&\frac{\langle\Gamma_2,\Gamma_1\rangle}{r_{12}} + \frac{\langle\Gamma_2,\Gamma_3\rangle}{r_{23}} + \langle\Gamma_2,h\rangle = 0,\\
&\frac{\langle\Gamma_1,\Gamma_3\rangle}{r_{13}} + \frac{\langle\Gamma_2,\Gamma_3\rangle}{r_{23}} + \langle\Gamma_3,h\rangle = 0.
\end{aligned} \tag{2.291}$$

These equations can be thought of as describing a balance of forces. The symplectic products pairs electric with magnetic charges (M, L_I are electric, K^I, V magnetic). We get a huge angular momentum forcing the points away from each other. But because of supersymmetry, all forces cancel and any solution is perfectly stable.

Define

$$A_{ij} \equiv \langle\Gamma_i,\Gamma_j\rangle. \tag{2.292}$$

Note that the symplectic product is antisymmetric and hence so is the matrix A. By a cyclic permutation of charges at the different centers, we can always take

$$A_{12} > 0, \qquad A_{23} > 0, \qquad A_{31} > 0. \tag{2.293}$$

Then the bubble equations are

$$\begin{aligned}
\frac{A_{12}}{r_{12}} - \frac{A_{31}}{r_{13}} + h_1 = 0,\\
-\frac{A_{12}}{r_{12}} + \frac{A_{23}}{r_{23}} + h_2 = 0,\\
\frac{A_{12}}{r_{12}} - \frac{A_{23}}{r_{23}} + h_3 = 0,
\end{aligned} \tag{2.294}$$

where the constants h_i are defined as $h_i = \langle\Gamma_i, h\rangle$. Only two of these equations are independent (for instance the sum of the first two gives the third one), and they leave only one of the distances r_{ij} unfixed. In total, three centers in $\mathbb{R}^3$ have 6 degrees of freedom (or "dof's"), three for each center minus three for the center of mass (only relative positions are important). The bubble equations fix two more. We thus have 4 degrees of freedom left. We can take these to be

- The radius r_{13} (1 dof)
- The orientation of r_{13} (2 dof's)
- The $U(1)$ angle around r_{13} (1 dof)

When we would consider n points instead of 3, the bubble equations allow for a $2(n-1)$-dimensional space of solutions (Fig. 2.35).

Scaling Solutions

One solution looks very interesting. If the triangle inequalities are satisfied:

$$|A_{12}| + |A_{23}| \geq |A_{31}|, \tag{2.295}$$

(and cyclic), there is a limit where the radii go to zero:

$$\begin{aligned} r_{12} &= |A_{12}|\epsilon + \mathcal{O}(\epsilon^2), \\ r_{13} &= |A_{13}|\epsilon + \mathcal{O}(\epsilon^2), \\ r_{23} &= |A_{23}|\epsilon + \mathcal{O}(\epsilon^2). \end{aligned} \tag{2.296}$$

As $\epsilon \to 0$, the bubble equations are satisfied up to first order, because the constants h_i can be suitable 'eaten up' by order $\mathcal{O}(\epsilon)$ terms in $\frac{A_{ij}}{r_{ij}} = \frac{1}{\epsilon} + \mathcal{O}(\epsilon)$. The r_{ij}'s are the lengths of the sides of a triangle and always satisfy triangle inequalities. The limit $\epsilon \to 0$ can only be done when also the $|A_{ij}|$ satisfy the triangle inequalities. We then have a limit where all radii go to zero. The points sit on a fixed triangle which gets smaller and smaller. If the triangle inequalities are not satisfied, we cannot have such a scaling limit.

Scaling Solutions

What is so special about these solutions? We have stated before the idea to replace the black hole geometry with some other object. In this section, we have made this more concrete. We can find an object with the same (electric/M2) charges as the black hole, but which also has magnetic dipole charges. The black hole is replaced

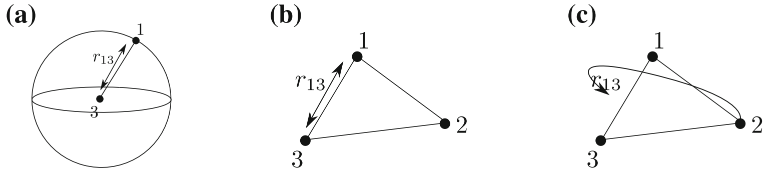


Fig. 2.35 A three-center configuration has 4 free parameters by the bubble equations. **a** S^2 of orientations of r_{13}. **b** Scale of r_{13}. **c** $U(1)$ angle around r_{13}

by a solution with many centers and magnetic charges, by finding the solution from the harmonic functions $H = (V, K^I, L_I, M)$.[26]

The solutions can 'go scaling', such that the several centers can come closer and closer, by sending some control parameter $\epsilon \to 0$, as in Eq. (2.296). When $\epsilon = 0$ and the centers are on top of each other, we recover the black hole (Fig. 2.36).

Remember that we were considering extremal black holes. These have an infinitely deep throat.[27] A scaling solution with scaling size ϵ, has a throat of length $L \propto -\ln \epsilon$. As $\epsilon \to 0$, you get a throat with a cap that gets longer and longer. These solutions form an infinite family, see Fig. 2.37 for an illustration.

A paradox

The scaling solutions form an infinite family: we can make ϵ smaller and smaller, we always find good solutions. But from AdS/CFT, we know that there is a finite entropy

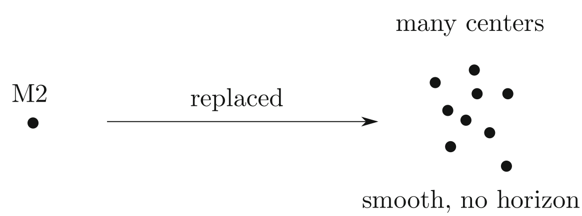
$$S = \sqrt{Q_1 Q_2 Q_3}, \tag{2.297}$$

which tells us there is a finite number of states. This is a puzzle [68]:

- $N_{\text{micro}} = e^{S_{BH}}$ is large but finite.
- $N_{\text{class. grav.}}$ (number of smooth solutions) is infinite.[28]

How to reconcile these pictures? That's for the next section!

Fig. 2.36 Replacing the *black hole* with a multi-center configuration



[26] In fact, there are certain conditions the harmonic functions H have to obey such that the multi-center geometry is also smooth and horizonless at each center. We will not dwell on that, see [41] for more information.

[27] By 'infinite throat', people mean that the spatial metric distance $\int ds$ to the horizon from any point outside the horizon blows up.

[28] Note: only a subset of this multi-center solutions are actual fuzzballs. We need some more information to discuss them, we will leave it at this for the moment.

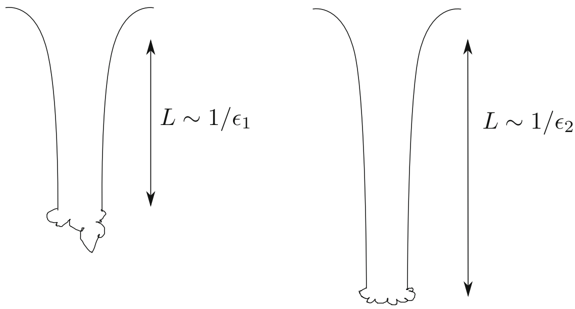


Fig. 2.37 For every value of ϵ we find a scaling solution with a deep throat. As $\epsilon \to 0$, we recover the infinitely deep *black hole* throat

2.8 Quantizing Geometries

So far we have studied a large class of supersymmetric multi-centered solutions and have suggested that they are related to the microstates of large supersymmetric black holes. But to make this connection between classical geometries and quantum states we have to "quantize". Since these are gravitational solutions quantizing them seems rather daunting and certainly we do not know how to do this in full generality. Rather here we will introduce a powerful covariant formalism for quantizing systems without resorting to a Hamiltonian formulation (which would be tedious in this case). In particular we will show how the solution space of a system is formally equivalent to the phase space and how we can thus construct states directly on this space. This construction usually goes under the name of "geometric quantization" but we will eschew many of the mathematical technicalities that usually are associated with this. Rather we will focus on explaining why this makes sense.

Note that we will make heavy use of supersymmetry as we do not have access to the full solution space of the theory but rather only some supersymmetric truncation of the latter. Quantizing a sub-space of a system is not necessarily a consistent thing to do but in this case we can rely on supersymmetry-based arguments (and explicit matching with expectations) to see that the Hilbert spaces we generate are a good approximation to the actual Hilbert space of the system.

2.8.1 Constraint Equations and Solution Space

To keep this chapter well-contained, we choose to recall the necessary background material discussed in previous sections.

We start in eleven-dimensions from the metric and gauge field

$$
\begin{aligned}
ds^2 &= (Z_1 Z_2 Z_3)^{-2/3} dt^2 + (Z_1 Z_2 Z_3)^{1/3} ds_4^2 + ds^2(T^6),\\
F_4 &= [d(Z_1^{-1}(dt + k)) + \Theta^I] \wedge dx_1 \wedge dx_2 + \cdots
\end{aligned}
\tag{2.298}
$$

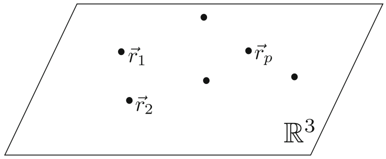


Fig. 2.38 The multi-center solutions are sourced on multiple positions in the $\mathbb{R}^3$ base of Taub-NUT space

with the four-dimensional multi-center Taub-NUT metric

$$ds_4^2 = V^{-1}(d\psi + A) + V ds^2(\mathbb{R}^3). \tag{2.299}$$

The functions Z_I, one-form k and two-forms Θ^I that determine the solution are found from the harmonic functions

$$H \equiv (V, K^I, L_I, M), \tag{2.300}$$

as explained in the previous section (Fig. 2.38).

The harmonic functions satisfy a sourced harmonic equation:

$$\nabla^2 H = \sum_i \Gamma_p \delta(\boldsymbol{r} - \boldsymbol{r}_p). \tag{2.301}$$

The solution is

$$H = \sum_{p=1}^{N} \frac{\Gamma_p}{|\boldsymbol{r} - \boldsymbol{r}_p|} + h_0, \tag{2.302}$$

where $\boldsymbol{r}_p$ are the position vectors of the different centers in $\mathbb{R}^3$ and h_0 is a vector of constants for the different harmonic functions. The charges at each center give poles in the harmonic functions, corresponding to multiple sources, and each may or may not have a horizon (depending on the charge Γ_p at the center).

Given a set of asymptotic charges $\Gamma = \sum_{p=1}^{N} \Gamma_p$ the space of all possible solutions with N centers is given by all the possible ways of arranging these centers in $\mathbb{R}^3$.

At first glance, we would think this space is $\mathbb{R}^{3N-3}$, the space of locations of N centers on $\mathbb{R}^3$.[29]

However, the positions of the centers are constrained in terms of the charges, by the bubble or Denef equations introduced in the last section:

[29] Only the relative positions are of importance, hence the degrees of freedom of one of the centers do not count and we get $3N - 3$ coordinates that specify a physical solution with N centers.

$$\forall p: \qquad \sum_{\substack{q=1 \\ q \neq p}}^{N} \frac{\langle \Gamma_p, \Gamma_q \rangle}{|\boldsymbol{r}_p - \boldsymbol{r}_q|} + \langle \Gamma_p, h \rangle = 0. \tag{2.303}$$

We write the harmonic functions and charges as symplectic vectors:

$$H = (\underbrace{V,\ K^I}_{\text{elec.}},\ \underbrace{L_I,\ M}_{\text{magn.}}), \qquad \Gamma = (p^0, p^I, q_I, q_0). \tag{2.304}$$

with $I = 1, 2, 3$ giving us either possible charges at each center.

Given two symplectic vectors of harmonic functions H and H' recall that there exists a symplectic inner product that couples electric and magnetic components

$$\langle H, H' \rangle = VM' - MV' + K^I L_I' - L_I K'^I. \tag{2.305}$$

Note that this pairing is antisymmetric. You should think of it as giving momentum from crossed electric and magnetic fields, similar to the Poynting vector in electromagnetism:

$$\boldsymbol{J} = \boldsymbol{E} \times \boldsymbol{B}. \tag{2.306}$$

The constraints (2.303) have a clear physical meaning. The first way to understand them is through supersymmetry. Each individual center breaks $\mathcal{N} = 2$ supersymmetry of the supergravity theory to a *particular* $\mathcal{N} = 1$ subgroup. Generically all the centers break $\mathcal{N} = 2$ to a different residual $\mathcal{N} = 1$ (encoded in a $U(1)$ valued phase) but when the distances between the centers satisfy the Eq. (2.303) the $\mathcal{N} = 1$ supersymmetry preserved by all the centers are compatible and thus the combined system preserves an overall $\mathcal{N} = 1$ supersymmetry.

There is a second interpretation of the constraints (2.303). Consider for concreteness a solution with two centers. The Poynting vector gives an angular momentum "binding". For electromagnetism in flat space, we get for a magnetic charge m and an electric charge q that

$$J = \frac{qm}{2}, \tag{2.307}$$

no matter what the distance is between the two centers. With gravity, the angular momentum depends on the distance between the centers:

$$J = \frac{qm}{r}, \tag{2.308}$$

and there is a non-zero force. The constraint equations can be interpreted as the condition for all those forces to balance.

Exercise 2.8.28 *Show that the sum over p (from 1 tot N) of (2.303) is zero.*

From Exercise 2.8.28, we see that there are in fact only $N-1$ independent constraints. Therefore, the solution space is a $(2N-2)$ dimensional submanifold of $\mathbb{R}^{3N-3}$:

$$M_{2N-2} \subset \mathbb{R}^{3N-3}. \tag{2.309}$$

For instance, for two centers we get

$$M_2 \subset \mathbb{R}^3. \tag{2.310}$$

The constraint fixes the distance $r_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$ so M_2 corresponds to the possible rotations of the position $\boldsymbol{r}_2$ around $\boldsymbol{r}_1$ with fixed inter-center separation r_{12}. This is of course nothing but a two-sphere

$$M_2 = S^2. \tag{2.311}$$

The constraint equations should be understood as follows. When we fix the asymptotic charges, there is still a continuous family of positions we can vary. Hence the solution space itself is a function of the charges $M_{2N-2}(\Gamma_p)$.

Our goal here will be to calculate the "number of states" in a fixed solution space. The reason to undertake such a computation is the following. For a given charge vector Γ, if we consider all possible decompositions in to multiple centers $\Gamma = \sum_p \Gamma_p$ and compute the states from each such solution space, we may hope that this can reproduce the entropy of a single center black hole with total charge Γ. If so then we have a found a good *supergravity* realization of the black hole microstates. But to convert the solutions above into "microstates" we have to quantize the solution space. Therefore we first give some basic quantum mechanics to see how to get a quantum space out of a classical solution space.

2.8.2 Basic Quantum Mechanics

We recall classical mechanics in the Hamiltonian symplectic formalism, its quantization and the concepts of phase space and its relation to the space of solutions.

Hamiltonian Formulation

Let us recall the basic simple formulation of quantum mechanics (which is not covariant) and then try to modify it to make it more covariant. If we start with a Lagrangian of a system with positions q:

$$L(q^i, \dot{q}^i). \tag{2.312}$$

with $i = 1, \cdots, n$ then the generalized momenta are

$$p_i = \frac{\partial L}{\partial \dot{q}^i}. \tag{2.313}$$

From this Lagrangian we can derive an associated Hamiltonian which is a function of the positions and generalized momenta only (for ease of notation we will mostly suppress indices on position and momentum vectors)

$$H(q, p) = p\dot{q} - L. \tag{2.314}$$

In terms of which the equations of motion are

$$\begin{aligned} \dot{p} &= -\frac{\partial H}{\partial q}, \\ \dot{q} &= \frac{\partial H}{\partial p}, \end{aligned} \tag{2.315}$$

Of course we could have foregone a Lagrangian and simply postulated a Hamiltonian system directly but the connection with a Lagrangian formulation will be important in what follows. The Hamiltonian formulation is based on the phase space which is the space of positions q and momenta p on a fixed time slice. It is this dependence of a choice of time slice (and direction) that makes the formulation non-covariant.

An essential ingredient in the Hamiltonian formulation of classical mechanics is the Poisson bracket, defined on any functions on the phase space, via

$$\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial q}. \tag{2.316}$$

In the simple systems first encountered in physics we often have $\{q, p\} = 1$ but this need not always be the case and this is one of the reasons a more general formulation is necessary. More generally we expect some bivector ω such that

$$\{q^i, p^j\} = \omega^{ij}. \tag{2.317}$$

While locally we can find coordinates such that ω is diagonal this need not hold globally. It is very important, however, that ω^{ij} be invertible as this allows us to find a symplectic two-form:

$$\omega \equiv \omega_{ij} dq^i \wedge dp^j. \tag{2.318}$$

which defines a symplectic structure on the phase space. Thus in general the Hamiltonian formulation requires the set of data (p, q, H, ω^{ij}).

We have tacitly assumed above that there is some natural choice of p's and q's on the entire phase space but if the latter is some non-trivial manifold then we need to cover it with patches. How then does one define, on each patch, which local coordinates should be thought of as positions and which momenta?

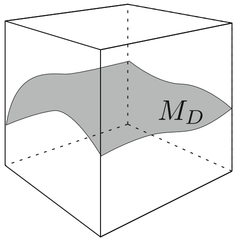


Fig. 2.39 A n-dimensional subspace M_n

A more covariant way to do this is to consider n-dimensional subspace M_n of the $2n$-dimensional phase space, as in Fig. 2.39, on which the pullback of the symplectic form vanishes:

$$\omega|_{M_n} = 0, \tag{2.319}$$

Such subspaces are referred to as Lagrangian submanifolds and they are interesting because if we consider any local coordinates, x^i, on them then by virtue of (2.319) we have

$$\{x^i, x^j\} = 0, \tag{2.320}$$

This is non-trivial because the x may be some non-trivial combination of p and q. The fact that they nonetheless have vanishing Poisson brackets mean they can be thought of as a new set of canonical positions. Thus Lagrangians in phase space are a covariant generalization of the splitting of phase space coordinates into canonical position and momenta.

So far we have used classical notions such as Poisson brackets but this discussion generalizes to quantum mechanics. To quantize a classical system we replace the Poisson bracket by a commutator (or anti-commutator for fermions)

$$[q, p] = i\hbar. \tag{2.321}$$

Thus the p's and q's can no longer correspond to $2n$ numbers but rather half of them are now operators. Normally, we take the q's to be commuting numbers, and p are their derivatives

$$p = \frac{\hbar}{i}\frac{\partial}{\partial q}. \tag{2.322}$$

Thus we see a Lagrangian subspace is nothing other than a space of mutually commuting variables

$$[x^i, x^j] = 0, \tag{2.323}$$

Once more such manifolds define (in a covariant way) natural slices of phase space that we can think of as position spaces.

This notion is quite important because in quantum mechanics states must be functions of only one set of canonical variables – the position or the momenta but not both. Thus Lagrangian submanifolds allow us to define the Hilbert space of states in a nice covariant way as the space of (wave) functions on a Lagrangian submanifolds

$$\mathcal{H} = \{\psi(x) \in \mathcal{L}^2(M_n, \mathbb{C})\}. \tag{2.324}$$

The advantage here over the usual formulation is that we have covariantized our approach as the Eq. (2.319) is a coordinate-invariant statement. Moreover this approach generalizes to more complex systems where the phase space (the space of (q^i, p^i)) is not merely $\mathbb{R}^{2d}$ but some more complex manifold. Of course we are implicitly assuming there is some nice foliation of the phase space into time slice $M_n(t)$ where t is some parameterization of time.

A consequence of this more formal description of the quantum phase space is that it yields another way to compute the number of states. This is simply the symplectic volume of the phase space: (up to some subtleties that we can neglect)

$$\#\,\text{states} = \int_{\text{phase space}} \omega^n. \tag{2.325}$$

where we note that ω^n is a $2n$-form that we can integrate over the entire space. Classically this does not count states because it is not integer quantized. In quantum mechanics, however, we think of ω as partitioning the phase space into Plank-sized cells. As a consequence its volume must be normalized such that the volume is integrally quantized (Fig. 2.40).

Mathematically, this can be justified because the wave functions are actually sections of a bundle defined on M_n and associated with ω (which is essentially its curvature). Thus the integral above computes (again, up to some subtleties) the index of an operator D associated with this bundle:

$$\text{ind}\, D = \int (\ldots). \tag{2.326}$$

Recall that an index counts the number of (chiral) zero modes of a particular operator and this is an integral quantity. In our setup, things are simple enough that the $(\ldots)$ are just ω^n .

The current treatment raises an important question:

- *Classically, we expect an infinite number of states (everything is continuous). Hence we should be able to go anywhere in phase space and have an infinite number of allowed states. But $\int \omega^n$ should be finite? Is there a clash?*

We will answer this question explicitly in an example below. Yes, classically the number of states is infinite, but the *volume* of phase space is finite. Only in quantum mechanics, the volume is the number of states.

Exercise 2.8.29 *Consider a particle in a box of length L.*

1. *Compute the number of quantum states: calculate the integral*

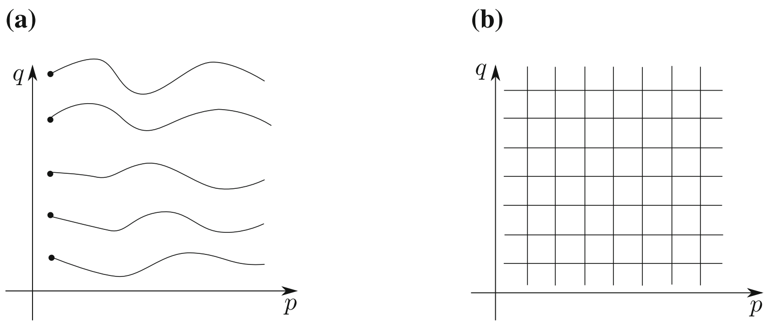


Fig. 2.40 Classical versus quantum phase space. The volume of classical phase space can be a real number, in quantum mechanics it is an integer. **a** Classically, we can continuously integrate histories. **b** In quantum mechanics, phase space is a discrete grid of points

$$\int_0^L \int_0^{p_{\max}} \omega, \tag{2.327}$$

with

$$[x, p] = \omega^{-1} \tag{2.328}$$

and $p_{\max}$ *should be allowed quantum values (see a textbook on quantum mechanics). Convince yourself this integral counts the number of states.*

2. *Repeat the calculation for a two-dimensional box.*

Let us consider a simple example to get a better feel for this formalism. Take the Hamiltonian of a free particle

$$H = \tfrac{1}{2}p^2. \tag{2.329}$$

Given q and p, we can always define the complex coordinates on phase space:

$$z = q + ip\,, \qquad \bar{z} = q - ip. \tag{2.330}$$

Then we have the commutation relation

$$[z, \bar{z}] = 1. \tag{2.331}$$

In terms of z, $\bar{z}$ it is no longer obvious which coordinate is a "position" and which a "momentum" and we must make an arbitrary choice. We can, for instance, take wave functions to depend only on z:

$$\psi(z). \tag{2.332}$$

Now the number of states is counted by an index

$$\mathrm{ind}\,(\bar{\partial}) = \#\,\mathrm{states}, \tag{2.333}$$

with $\bar{\partial}$ the Dolbeault operator because clearly $\bar{\partial}\psi(z) = 0$ so wave functions are simply functions annihilated by $\bar{\partial}$. Note that this method needs a complex structure on phase space, which can not always be defined. For a simple manifold like $\mathbb{R}^{2n}$ it can be done. If there is a complex structure, then it turns out that the above gives a good way to quantize.

Consider now a slight extension of the free particle model. Couple it to an electromagnetic field. The Lagrangian is

$$L = \frac{1}{2}(\dot{q} + Aq)^2. \tag{2.334}$$

The canonical momentum is

$$p = \dot{q} + Aq. \tag{2.335}$$

This is very different from previous examples! Even if there is no velocity, $\dot{q} = 0$, there is still a non-vanishing momentum. When there are space components of the gauge field

$$A_i \neq 0, \tag{2.336}$$

the position themselves no longer commute:

$$\omega^{ij} = [q^i, p^j] = A_j[q^i, q^j] \neq 0. \tag{2.337}$$

The non-commutativity of phase space becomes a non-commutativity of the physical space due to the magnetic field A_i.

From Phase Space to Solution Space

So far we have reformulated quantum mechanics in a slightly more covariant and general language but let us see what this is useful for. Here we will try to prove the following claims:

1. The number of states is the symplectic volume of phase space.
2. Phase space is isomorphic to solution space (up to some caveats).

and hence:

- **The number of states is the symplectic volume of solution space.**

The first claim we have already argued in the previous section. The last one follows trivially from the other two. Thus we are left with demonstrating the validity of our second claim above.

Given any initial point in phase space $\{q_0, p_0\}$ there is a prescription to generate an entire "history": namely we integrate using the equations of motion with initial conditions $\{q_0, p_0\}$. The p's act morally as velocities, and they allow us to integrate

$q(t_0)$ for any t_0 to a further time step (see Fig. 2.41). Thus any point in phase space corresponds to a full solution to the equations of motion (a "history" of the particle or system).

Conversely, given a solution $q(t)$ to the equations of motion and a choice of time slice at for instance t_0, we can unique extract a point in the phase space by simply reading off $\{q(t_0), p(t_0)\}$ evaluated on the solution $q(t)$ at time t_0. Thus, once a time-slice is fixed, each solution uniquely maps to a point in the phase space (Fig. 2.41). Combining these observations we have now proved our second claim above.

What's more there is a natural way to compute the symplectic form directly in the Lagrangian formulation. This allows us to use the solution space to compute both the number of states and their explicit form without ever needing to use a Hamiltonian formulation (going to the phase space and formulating everything in terms of conjugate variables).

An important subtlety, however, is that the arguments made above apply to the full solution space and phase space—it is these full spaces that are isomorphic. It is not clear, if we restrict to a subspace of the solution space, whether this maps to a proper phase space. This is important in this situation because the supersymmetric solution space is exactly such a truncation.

2.8.3 Intermezzo: From QM to QFT and GR

We want to go from quantum mechanics (QM) to Quantum Field Theory (QFT). In QM, the points at time t are unconstrained, and the wave function $\psi(x)$ is a function of the unconstrained positions. In QFT, the points on each time slice are now fields ϕ that are constrained by the equations of motion, and the wave functional $\Psi(\phi)$ is a function of those constrained fields. Note that we use the formulation of time slices and evolution of the fields from one to the other defining wave functions on each slice. This is equivalent to the path integral formulation

$$\langle\psi'|e^{iHt}|\psi\rangle = \int \mathcal{D}e^{-S}. \tag{2.338}$$

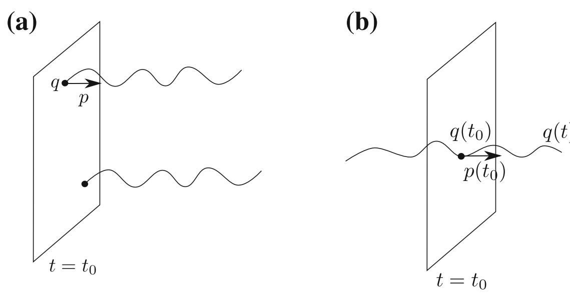


Fig. 2.41 *Left* given an initial configuration at $t = t_0$, we can integrate the equations of motion to obtain the full solution $q(t)$, $p(t)$. *Right* given a solution $q(t)$, we have a phase space at every t

In field theory, the coordinates and momenta are replaced by fields:

$$q \to \phi(x)$$
$$p \to \Pi(x) = \frac{\partial L}{\partial \dot{\phi}}. \tag{2.339}$$

As before for quantum mechanics, in field theory we consider the fields on a spatial slice such as the one in Fig. 2.42.

In GR, things are a little more tricky than in field theory because the background is not fixed. We will not address these subtleties here but will simply assume we find a nice foliation of all the space-times we consider. We define spatial slices Σ such as the one in Fig. 2.43 and we use a metric adapted to the slices

$$ds^2 = (N^2 + \beta_k \beta^k)dt^2 + 2\beta_k dx^k dt + h_{ij} dx^i dx^j, \tag{2.340}$$

in terms of the data

$$(h_{ij}, \beta_k, N). \tag{2.341}$$

where now h_{ij} is a metric on the spatial slice.

One finds that β_k and N are non-dynamical variables as their momenta are zero:

$$\Pi^\beta = 0, \qquad \Pi^N = 0. \tag{2.342}$$

These equations can be interpreted as constraints on the other fields. The only dynamical variables are then the three-dimensional metric h_{ij} and its momenta Π_h:

$$\Pi_h^{ij} \equiv \frac{\delta L}{\delta \partial_t h_{ij}}. \tag{2.343}$$

What terms contribute to the momentum Π_h? These are terms in the Lagrangian of the form:

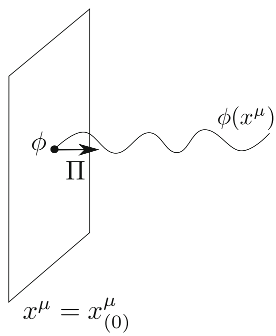


Fig. 2.42 Fields on a spatial slice of constant t

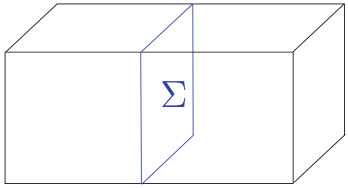


Fig. 2.43 GR on a spatial slices Σ

$$L = \cdots + \partial_t h_{ij} \Omega^{t,ij} + \cdots \tag{2.344}$$

Assume first that $\beta_k = 0$. Then the metric has no mixed spatial-temporal components:

$$g_{\mu\nu} = g_{ij} + g_{tt}, \tag{2.345}$$

and $\partial_t h_{ij}$ can only talk to something else ($\Omega^{t,ij}$) with another time derivative and hence

$$\Pi^{ij} \sim \dot{h}^{ij}. \tag{2.346}$$

For time-independent solutions we would thus have $\Pi^{ij} = 0$. Thus if we consider families of static solutions (time-independent and no mixed terms in the metric) they cannot map to a full phase space as they contain no momentum-like variables. Instead such solutions map to a Lagrangian submanifold of the full phase space (they form a "configuration space" rather than a phase space).

If, on the other hand, $\beta_k \neq 0$ then $\partial_t h_{ij}$ can couple to terms like $\partial^i g^{tj}$ etc., with *spatial* derivatives. Therefore,

$$\Pi^{ij} \sim \text{time independent terms}, \tag{2.347}$$

which means $\Pi^{ij} \neq 0$ even for time-independent solutions. Remember that the multi-center metrics we were looking are of this sort since they are stationary (time-independent with mixed terms $g_{ti} \sim k_i$ terms coming from a $(dt + k_i dx^i)^2$).

Therefore, the commutation relations go as

$$[h_{ij}, \Pi^{kl}] \sim [h_{ij}, h^{kl}], \tag{2.348}$$

analogous to the previous example of a particle in a magnetic field with

$$[q_i, p_j] \sim [q_i, q_j]. \tag{2.349}$$

The spatial metrics no longer commute on the phase space. This will be very important for getting the number of states.

Crnkovic-Witten-Zuckerman Formalism

Since we are working with solution spaces we want a covariant formalism rather than the non-covariant GR Hamiltonian formalism we discussed above. Let us see how to arrive at this. Consider a class of solutions with a spatial foliation with each time slice being a Cauchy surface

$$\Sigma = \text{Cauchy surface.} \tag{2.350}$$

Define

$$\omega := \int_\Sigma d\Sigma_\ell J^\ell, \tag{2.351}$$

Here J^ℓ is the "symplectic current" associated with the action (see below). We have introduced the $(D-1)$-form

$$d\Sigma_\ell = \Sigma_{\mu_1\ldots\mu_{D-1}\ell} dx^1 \wedge \ldots \wedge dx^{D-1}. \tag{2.352}$$

which is just the volume form on the Cauchy surface. Then ω is a two-form on the space of fields. The symplectic current is

$$J_\ell = \delta\left[\frac{\delta L}{\delta \partial_\ell \phi_k}\right] \wedge \delta\phi^k, \tag{2.353}$$

where ϕ^k runs over the fields. If $\ell = 0$, we get $J_0 = d\Pi \wedge d\phi$, reminiscent of the symplectic form in mechanics $dp \wedge dq$. But unlike the standard formulation this is covariant as we have not fixed a coordinatized notion of time. Rather by using spacelike foliation we get a covariant notion of time as the direction normal to the slices (but with no reference to a coordinate system).

Exercise 2.8.30 *Play around with ω:*

1. *Show that ω is closed under a field variation*

$$\delta_\phi \omega = 0. \tag{2.354}$$

2. *Show that the symplectic current is conserved*

$$\partial_\ell J^\ell = 0. \tag{2.355}$$

You need to impose the equations of motion for one of these.

From the exercise we see that ω does not vary from slice to slice (because it is conserved).

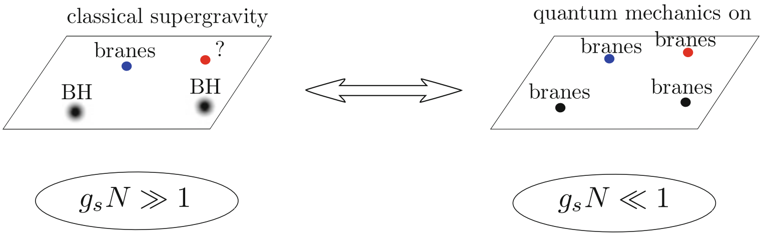


Fig. 2.44 At large $g_s N$, we have the supergravity multi-center solution. Each center can be either a black hole (with a horizon), or some horizonless singularity, or a smooth center etc. For small $g_s N$, we just have non-back-reacting branes at several positions in flat space-time

2.8.4 Back to Solution Space

Now we have the pieces in place to quantize our space of solutions. We begin by evaluating the symplectic form for the Lagrangian of M-theory. The fields are the metric and the four-form and are evaluated at the positions on solution space:

$$\phi_\ell = \{g_{\mu\nu}[\boldsymbol{r}_p], F_{\mu\nu\rho\sigma}[\boldsymbol{r}_p]\}. \tag{2.356}$$

The symplectic form looks like

$$J^\ell = \delta\left[\frac{\delta L}{\delta\partial_\ell g[\boldsymbol{r}_p]}\right] \wedge \delta g[\boldsymbol{r}_p] + \text{ four-form term.} \tag{2.357}$$

The two-form ω will be something like

$$(\ldots) \wedge d\boldsymbol{r}_p, \tag{2.358}$$

where each $\{\boldsymbol{r}_p\}$ parametrizes a metric; these are the "coordinates" of our solution.

How to do this? Remember that the constraint equations come from the integrability condition of the defining equation for $\boldsymbol{\omega}$ (which is part of the metric $g_{\mu\nu}$):

$$\boldsymbol{\nabla} \times \boldsymbol{\omega} = V\boldsymbol{\nabla} M + \cdots. \tag{2.359}$$

We need to find $\boldsymbol{\omega}(\boldsymbol{r}_p)$, construct $g(\boldsymbol{\omega})$ and then we can find J^ℓ. This is very difficult because inverting Eq. (2.359) cannot be easily done.

We will follow the lazy string theorist approach and use supersymmetry to our advantage. The back-reacted supergravity system is valid for $g_s N \gg 1$. As we discussed in previous sections, when $g_s N \ll 1$, we just have a quantum mechanical theory on branes at the positions of the centers on eleven-dimensional flat space-time $\mathbb{R}^3 \times T^6 \times \mathbb{R}_t$, see Fig. 2.44.

It can be shown that on each $g_s N$ side the solution space and the symplectic form are protected because of supersymmetry (the proof uses the fact that both are determined by the certain terms in the Lagrangian whose form is fixed by supersymmetry and thus cannot change even as we vary $g_s N$). Moreover one can check by explicit computation that the solution spaces at strong and weak coupling are exactly the same. For instance, for 2 centers, we still find S^2 as the solution space. Thus we are free to compute the symplectic form directly in the brane quantum mechanics which is a much easier computation.

The result we get from the $g_s N \ll 1$ quantum-mechanics-on-branes calculation is

$$\omega = \frac{1}{4} \sum_{p,q} \langle \Gamma_p, \Gamma_q \rangle \frac{r^i_{pq}}{|r_{pq}|^2} \epsilon_{ijk} \delta r^j_{pq} \wedge \delta r^k_{pq}, \tag{2.360}$$

and we defined

$$\boldsymbol{r}_{pq} = \boldsymbol{r}_p - \boldsymbol{r}_q. \tag{2.361}$$

The real coordinates in this calculation are the $\boldsymbol{r}_{pq}$, vectors between the centers. While we do not show the detailed derivation of this formula here (the interested reader can find it in [69]) its origin is very easy to understand. Recall from the discussion in the previous section that an electrically charge particle in the background of a magnetic field has a coupling $(\dot{q} + eAq)$, with e the electric charge, and this leads to a canonical momentum of the form

$$p = eA(q)\, q. \tag{2.362}$$

The symplectic form (2.360) is exactly of this form: each center feels, via $\langle \Gamma_p, \Gamma_q \rangle$ an electric-magnetic coupling to the gauge field generated by any other center which is "magnetically" charged with respect to it. So (2.360) is really just of the form $\omega = A(q)\delta q \wedge \delta q$ where we have plugged in the appropriate value for $A(q)$.

Morally, the $\delta r^j_{pq} \wedge \delta r^k_{pq}$ are like the $dx^i \wedge dx^j$ contributions in quantum mechanics. As before, this means that coordinates do not commute:

$$[r^i_{pq}, r^j_{pq}] = \omega^{ij} \neq 0. \tag{2.363}$$

Note that the r^i_{pq} only talk with the $r^j_{p'q'}$ when $p = p', q = q'$: the several components of a the vector between the p^{th} and q^{th} centers are non-commutative, but they commute with all the other components of all the other inter-center vectors. There is only pairwise non-commutativity.

The angular momentum is:

$$J = \frac{1}{2} \sum_{p,q} \langle \Gamma_p, \Gamma_q \rangle \frac{\boldsymbol{r}_{pq}}{|r_{pq}|}. \tag{2.364}$$

It is a sum of contributions from each pair of points. Each individual contribution is a vector along the line connecting two points (unit vectors $\frac{\boldsymbol{r}_{pq}}{|r_{pq}|}$) with size the angular momentum from the crossed electric and magnetic fields $\langle \Gamma_p, \Gamma_q \rangle$.

Two-Center Solutions

Let us make things more clear using an explicit example with two centers. Write $J = \langle \Gamma_1, \Gamma_2 \rangle$, then the volume form on phase space is

$$\omega = J \sin\theta \, d\theta \wedge d\phi, \tag{2.365}$$

the standard symplectic form on a two-sphere. (Remember that the solution space for two center is the S^2 of orientations of the fixed rod $\boldsymbol{r}_{12}$.) The normalization of the two-form is the angular momentum between the two centers.

The number of states is then

$$\int_{S^2} \omega = 2|J| + 1. \tag{2.366}$$

We get $2|J| + 1$ rather than $2|J|$ because of subtleties with fermions. This is exactly the number of states for an angular momentum multiplet (Fig. 2.45).

Exercise 2.8.31 *"Meaningless algebra" for the two-center solution space:*

- *Check that*

$$d\omega = 0 \tag{2.367}$$

- *Check that ω_{S^2} defined as (2.360) evaluates to (2.365).*

Three-Center Solutions

Solution space is $2N-2$ dimensional. For $N = 3$, we get a four-dimensional solution space M_4. The bubble equations fix two distances in terms of the third, say $r_{23}(r_{12})$ and $r_{13}(r_{12})$. The four remaining parameters are

- The distance r_{12}.
- The $U(1)$ of orientations around segment r_{12}.
- The orientation of r_{12} in space (an S^2 as for the two-center solution space).

Therefore the solution space is:

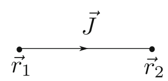


Fig. 2.45 Two-center solution

$$M_4 = I \times U(1) \times S^2, \tag{2.368}$$

where I is the line segment of r_{12}. The second product is a non-trivial fibration.

Note that the size of the angular momentum is a function of the distance r_{12} as well:

$$J(r_{12}). \tag{2.369}$$

By the bubble equations the interval I of allowed r_{12} values is constrained

$$I = [r_{12}^{\min}, r_{12}^{\max}]. \tag{2.370}$$

Hence also the angular momentum is bounded between $J_{\min}$ and $J_{\max}$, see Fig. 2.46.

We can see the system as a whole range of angular momentum multiplets, see Fig. 2.47. Let us note an important caveat here when discussing entropy. We are referring here only to the configuration entropy coming from the different ways of arranging the centers. Each individual center, if it has a horizon, may have additional entropy associated with that horizon. In our discussion of entropy above we neglect this because we are mostly interested in looking for black hole microstates. That is to say we want to find a realization of the black hole entropy via horizonless smooth solutions. If the centers are themselves black holes with horizons, we are not counting the horizon entropy of a single black hole with the total charge of all the centers.

More Centers?

Let us fix the total charge, Γ, and consider an N-center decomposition

$$\Gamma = \sum_{N=1}^{\infty} \left(\sum_{q=1}^{N} \Gamma_q \right). \tag{2.371}$$

For large charge Γ the number of centers N can be quite large and we can also arrange the centers all to be horizonless. What are all possible states corresponding to these charges? We fix Γ first, then we fix the sectors we want to divide over, and we divide the charges. All these states are in one Hilbert space, of total charge Γ. Are all these possible states reproducing the black hole entropy of a single black

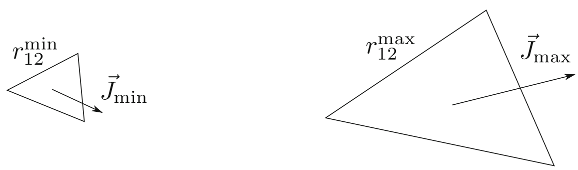


Fig. 2.46 The angular momentum is a function of the size of r_{12}

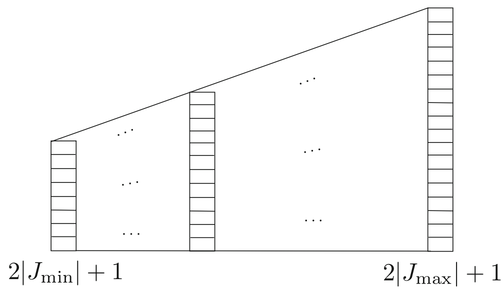


Fig. 2.47 The angular momentum is a function of the size of r_{12}. The states are divided into one angular momentum multiplet for each allowed value of J

hole with charge Γ? Should we use smooth centers? How many can we put? Can we reproduce the entropy?

The result in the literature so far is:

- For fully interacting centers ($\langle \Gamma_p, \Gamma_q \rangle \neq 0$), this counting has only been done in full generality for 2 and 3 centers. It has been extended to $N + 1$ centers, where the first N have all charges equal $\Gamma_1 = \ldots = \Gamma_N$ and the other center has non-vanishing $\langle \Gamma_p, \Gamma_{N+1} \rangle$ with all the others.

Note that classically, there can be a problem due to configurations with runaway behaviour. One of the centers can go off to infinity in the bubble equations, and this screws up the asymptotics, see Fig. 2.48.

After quantization, there is a density on $M_4 = \mathbb{R} \times U(1) \times S^2$. This gives a finite volume. There is no more runaway, because the wave function for the positions of the centers has no support at infinity, 'the tail is vanishing'. This renders $\langle \boldsymbol{r}_p \rangle$ finite. See Fig. 2.49.

2.8.5 Scaling Solutions

Let us go to solutions where the centers can come arbitrarily close. We stay in the three-center example. Remember that the bubble equations look like

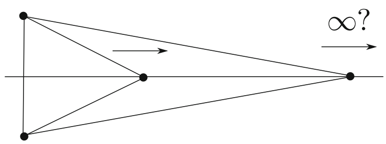
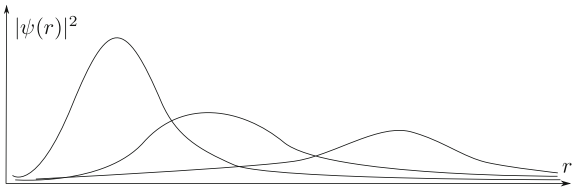


Fig. 2.48 Classically, one of the centers can run of to infinity

Fig. 2.49 In quantum mechanics, the wave function has no support at infinity



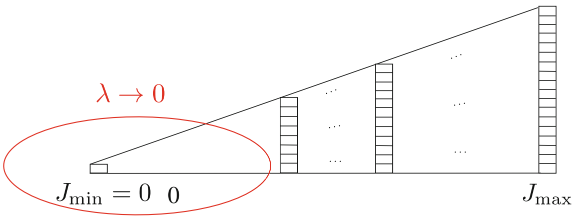


Fig. 2.50 The angular momentum multiplet triangle is completed for scaling solutions, since the solution space contains the limit $\lambda \to 0$, such that $J_{\min} = 0$

$$\begin{aligned}\frac{\langle \Gamma_1, \Gamma_2\rangle}{r_{12}} + \frac{\langle \Gamma_1, \Gamma_3\rangle}{r_{13}} &= c_1,\\ \frac{\langle \Gamma_2, \Gamma_1\rangle}{r_{12}} + \frac{\langle \Gamma_2, \Gamma_3\rangle}{r_{13}} &= c_2,\\ \frac{\langle \Gamma_3, \Gamma_1\rangle}{r_{13}} + \frac{\langle \Gamma_3, \Gamma_2\rangle}{r_{23}} &= c_3,\end{aligned} \tag{2.372}$$

with $c_p = -\langle \Gamma_p, h\rangle$. We look for solutions with

$$r_{pq} = \lambda \langle \Gamma_p, \Gamma_q\rangle + \mathcal{O}(\lambda^2), \tag{2.373}$$

such that we can send $\lambda \to 0$. Then we find that $\langle \Gamma_p, \Gamma_q\rangle = \alpha\, r_{pq}$ for some constant α. Hence we can only take this limit when the $\Gamma_{pq} \equiv \langle \Gamma_p, \Gamma_q\rangle$ satisfy the triangle inequalities.

As a consequence, the angular momentum is zero when $\lambda = 0$:

$$\boldsymbol{J} = \sum \Gamma_{pq} \frac{\boldsymbol{r}_{pq}}{r_{pq}} = \alpha \sum \boldsymbol{r}_{pq} = 0, \tag{2.374}$$

where the last equality follows because the $\boldsymbol{r}_{pq}$ form a closed triangle. Therefore, near $\lambda \to 0$, we have $J \to 0$. This means that we 'complete' the triangle of states in the angular momentum multiplets of Fig. 2.47 to that of Fig. 2.50. We can parametrize the region near $J_{\min} = 0$ by the scaling parameter λ.

When the inter-center distance $r_{pq} \sim \lambda \to 0$, the geometry develops a very deep throat of size proportional to $1/\lambda$, see Fig. 2.51. As the centers come closer and closer, the throat becomes deeper and deeper.

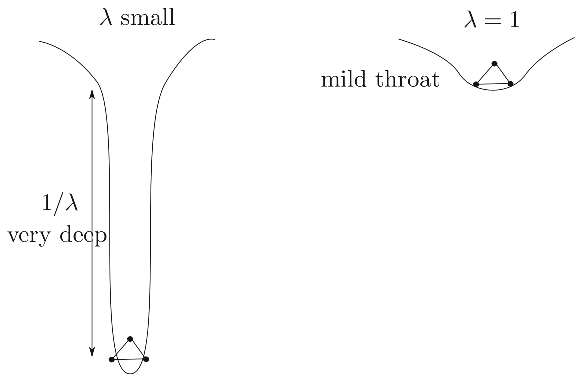


Fig. 2.51 By scaling down the distances between the centers as $\lambda \to 0$, the geometry develops a very deep throat whose size is inversely proportional to λ. When λ is of order 1 on the other hand, we only have a very mild throat

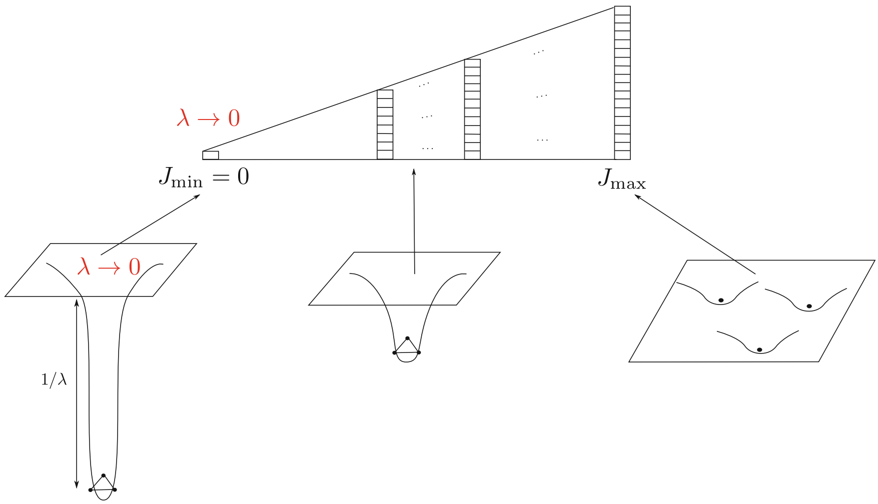


Fig. 2.52 The correspondence of scaling solutions of a certain size to angular momentum multiplets in the quantized solutions space

Putting these things together, gives a situation of the states in solution space as in Fig. 2.52. This reveals a paradox. As $\lambda \to 0$, we get deeper and deeper microstates and we can continue like this forever. On the other hand, the number of states associated to the region of small λ of Fig. 2.52, gives a finite number of states. Stated in a different way, in quantum mechanics, it is meaningless to put states in a cell smaller than $\hbar$-size. Remember that on solution space, we had non-commuting coordinates r^i_{pq} and r^j_{pq}. This translates to the impossibility of localizing r^i_{pq} and r^j_{pq} with a resolution

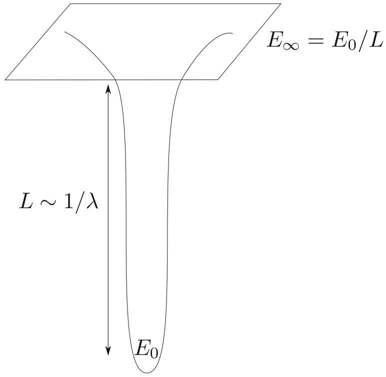


Fig. 2.53 The energy E_0 of an excitation down the throat is redshifted to $E_\infty \sim E_0/L$, with L the throat length

smaller than $\hbar$. Therefore there is some cut-off, and all deeper and deeper microstates must correspond to one quantum state.

Hence even though we can make the throats as deep as we want classically, all these deep throats do not exists after quantization. This is related to the earlier puzzle, that due to redshift, the energy E_∞ would have a continuous spectrum for deeper and deeper throats, see Fig. 2.53: a string stretching between two centers remains massless at spatial infinity.

On the other hand, the CFT should have a discrete spectrum, otherwise the counting of microstates would not give a finite number. So the question is whether there is a cut-off in the throat, and what it is.

While the exact answer to this question depends on the state we consider and is somewhat complicated, a simple order of magnitude estimate can be gleaned as follows. We consider the geometry of the throat up to the scale where λ takes its expectation value in the lowest angular momentum state (the state at $J = J_{min}$; see Fig. 2.53 above). That is, we compute $\langle\lambda\rangle$ in the state $|j = 0\rangle$ and then plug this into the harmonics to yield a solution. This gives a cutoff on the throat and we can determine the mass gap by putting a scalar field on this background and computing the gap in its spectrum (this is analogous to a standard computation to determine the mass gap in global AdS and essentially measures the "size of the box" provided by the gravitational potential).

This computation yields a mass gap that, when measured in AdS units $1/L_{AdS}$, scales as $1/c$. Here c is a dimensionless number given by comparing the AdS length to the plank length $c = L_{AdS}/\ell_P$. Thus the mass gap is

$$\frac{1}{c\,L_{AdS}}. \tag{2.375}$$

whereas the mass gap in global AdS is just

$$\frac{1}{L_{AdS}}. \tag{2.376}$$

The suggestive terminology c alludes to the fact that this number is the central charge of the dual CFT. For example in the case where the AdS_3 is the near horizon of the D1-D5 black hole c is proportional to $Q_1 Q_5$ and is the central charge of the dual D1-D5 CFT.

This is a very significant result. Recall that in our derivation of the black hole entropy in earlier sections a very important role was played by the so called "long string picture" where the entropically dominant sector of the CFT came from a string with a winding numer that is proportional to $Q_1 Q_5$ as well. Consequently the momentum of this string was quantized in units $\frac{1}{Q_1 Q_5 R}$ with R the dimensionful length of the CFT circle $R = 2\pi L_{AdS}$.

This computation thus suggests that the quantum corrections to the deep throat microstates not only discretize the spectrum, hence resolving the issue of a continuous spectrum, but also do this by giving them a mass gap corresponding to the most entropic sector of the CFT. This suggests these states at least occupy the "typical" sector of the CFT and hence are potentially the kind of states that account for the black hole entropy.

References

1. N. Deruelle, *Black Holes in General Relativity*, Course de Physique Théorique at IPhT, CEA Saclay,ipht.cea.fr/Docspht//articles/t09/346/public/Cours2009.pdfipht.cea.fr/Docspht//articles/t09/346/public/Cours2009.pdf
2. J.D. Bekenstein, Black holes and the second law. Nuovo Cim. Lett. **4**, 737–740 (1972)
3. S.W. Hawking, Particle creation by black holes. Commun. Math. Phys. **43**, 199–220 (1975)
4. M.B. Green, J.H. Schwarz, E. Witten, *Superstring Theoy, Introduction*, Cambridge Monographs On Mathematical Physics, vol 1 (Cambridge University Press, UK, 1987), p. 469
5. M.B. Green, J.H. Schwarz, E. Witten, *Superstring Theory, Loop qamplitudes, anomalies and phenomenology*, Cambridge Monographs On Mathematical Physics, vol 2 (Cambridge University Press, UK, 1987), p. 596
6. J. Polchinski, *String Theory, An Introduction to the Bosonic String*, vol 1 (Cambridge University Press, UK, 1998), p. 402
7. J. Polchinski, *String Theory, Superstring Theory and Beyond*, vol 2 (Cambridge University Press, UK, 1998), p. 531
8. K. Becker, M. Becker, J.H. Schwarz, *String Theory and M-theory: A Modern Introduction* (Cambridge University Press, UK, 2007)
9. E. Kiritsis, *String Theory in a Nutshell* (Princeton University Press, USA 2007)
10. D.Z. Freedman, A. Van Proeyen, *Supergravity* (Cambridge University Press, Cambridge, UK, 2012), p. 607
11. E. Cremmer, B. Julia, J. Scherk, Supergravity theory in eleven-dimensions. Phys. Lett. **B76**, 409–412 (1978)
12. G.W. Gibbons, M.B. Green, M.J. Perry, Instantons and seven-branes in type IIB superstring theory. Phys. Lett. **B370**, 37–44 (1996). http://www.arXiv.org/abs/hep-th/9511080 hep-th/9511080
13. F. Denef, *Les Houches Lectures on Constructing String Vacua*, http://www.arXiv.org/abs/0803.1194 0803.1194
14. E.A. Bergshoeff, J. Hartong, T. Ortin, D. Roest, Seven-branes and supersymmetry. JHEP **0702**, 003 (2007). http://www.arXiv.org/abs/hep-th/0612072 hep-th/0612072

15. E. Bergshoeff, J. Hartong, D. Sorokin, Q7-branes and their coupling to IIB supergravity. JHEP **0712**, 079 (2007). http://www.arXiv.org/abs/0708.2287 0708.2287
16. J. Raeymaekers, D. Van den Bleeken, B. Vercnocke, Chronology protection and the stringy exclusion principle. JHEP **1104**, 037 (2011). http://www.arXiv.org/abs/1011.5693 1011.5693
17. E. Bergshoeff, *p-branes, D-branes and M-branes*, http://www.arXiv.org/abs/hep-th/9611099 hep-th/9611099
18. K.S. Stelle, *BPS Branes in Supergravity*, http://www.arXiv.org/abs/hep-th/9803116 hep-th/9803116
19. R. Argurio, *Brane Physics in M-theory*, http://www.arXiv.org/abs/hep-th/9807171 hep-th/9807171
20. A.W. Peet, *TASI Lectures on Black Holes in String Theory*, http://www.arXiv.org/abs/hep-th/0008241 hep-th/0008241
21. C.W. Misner, K. Thorne, J. Wheeler, *Gravitation*
22. A.A. Tseytlin, Harmonic superpositions of M-branes. Nucl. Phys. **B475**, 149–163 (1996). http://www.arXiv.org/abs/hep-th/9604035 hep-th/9604035
23. A. Strominger, C. Vafa, Microscopic origin of the Bekenstein-Hawking entropy. Phys. Lett. **B379**, 99–104 (1996). http://www.arXiv.org/abs/hep-th/9601029 hep-th/9601029
24. J.M. Maldacena, The Large N limit of superconformal field theories and supergravity. Adv. Theor. Math. Phys. **2**, 231–252 (1998). http://www.arXiv.org/abs/hep-th/9711200 hep-th/9711200
25. E. Witten, Anti-de Sitter space and holography. Adv. Theor. Math. Phys. **2**, 253–291, (1998). http://www.arXiv.org/abs/hep-th/9802150 hep-th/9802150
26. S. Gubser, I.R. Klebanov, A.M. Polyakov, Gauge theory correlators from noncritical string theory. Phys. Lett. **B428**, 105–114 (1998). http://www.arXiv.org/abs/hep-th/9802109 hep-th/9802109
27. G. 't Hooft, *Large N*, http://www.arXiv.org/abs/hep-th/0204069 hep-th/0204069
28. I. Bena, M. Berkooz, J. de Boer, S. El-Showk, D. Van den Bleeken, Scaling BPS solutions and pure-Higgs states. JHEP **1211**, 171 (2012). http://www.arXiv.org/abs/1205.5023 1205.5023
29. I.R. Klebanov, *TASI lectures: Introduction to the AdS/CFT Correspondence*, http://www.arXiv.org/abs/hep-th/0009139 hep-th/0009139
30. J.-P. Blaizot, E. Iancu, U. Kraemmer and A. Rebhan, Hard thermal loops and the entropy of supersymmetric Yang-Mills theories, JHEP **0706**, 035 (2007). http://www.arXiv.org/abs/hep-ph/0611393 hep-ph/0611393
31. A. Bernamonti and R. Peschanski, Time-dependent AdS/CFT correspondence and the Quark-Gluon plasma. Nucl. Phys. Proc. Suppl. **216**, 94–120 (2011). http://www.arXiv.org/abs/1102.0725 1102.0725
32. M. Petrini, *The AdS/CFT correspondence*, Course de Physique Théorique at IPhT, CEA Saclay, http://ipht.cea.fr/Docspht/search/article.php?id=t10/312http://ipht.cea.fr/Docspht/search/article.php?id=t10/312
33. J.R. Peschanski Robi, *The dynamics of Quark-Gluon Plasma and AdS/CFT correspondence*, Course de Physique Théorique at IPhT, CEA Saclay, http://ipht.cea.fr/Docspht/search/article.php?id=t10/310http://ipht.cea.fr/Docspht/search/article.php?id=t10/310
34. L. Susskind, *Some Speculations About Black Hole Entropy in String Theory*, http://www.arXiv.org/abs/hep-th/9309145 hep-th/9309145
35. E. Halyo, A. Rajaraman, L. Susskind, Braneless black holes. Phys. Lett. **B392**, 319–322 (1997). http://www.arXiv.org/abs/hep-th/9605112 hep-th/9605112
36. E. Halyo, B. Kol, A. Rajaraman, L. Susskind, Counting Schwarzschild and charged black holes. Phys. Lett. **B401**, 15–20 (1997). http://www.arXiv.org/abs/hep-th/9609075 hep-th/9609075
37. G.T. Horowitz, J. Polchinski, A Correspondence principle for black holes and strings. Phys. Rev. **D55**, 6189–6197 (1997). http://www.arXiv.org/abs/hep-th/9612146 hep-th/9612146
38. T. Damour, G. Veneziano, Selfgravitating fundamental strings and black holes. Nucl. Phys. **B568**, 93–119 (2000). http://www.arXiv.org/abs/hep-th/9907030 hep-th/9907030
39. S.D. Mathur, The Information paradox: A Pedagogical introduction. Class. Quant. Grav. **26**, 224001 (2009). http://www.arXiv.org/abs/0909.1038 0909.1038

40. S.D. Mathur, The fuzzball proposal for black holes: An elementary review. Fortsch. Phys. **53**, 793–827 (2005). http://www.arXiv.org/abs/hep-th/0502050 hep-th/0502050
41. I. Bena, N.P. Warner, Black holes, black rings and their microstates. Lect. Notes Phys. **755**, 1–92 (2008). http://www.arXiv.org/abs/hep-th/0701216 hep-th/0701216
42. S.D. Mathur, *Fuzzballs and the Information Paradox: A Summary and Conjectures*, http://www.arXiv.org/abs/0810.4525 0810.4525
43. V. Balasubramanian, J. de Boer, S. El-Showk and I. Messamah, Black holes as effective geometries. Class. Quant. Grav. **25**, 214004 (2008). http://www.arXiv.org/abs/0811.0263 0811.0263
44. K. Skenderis, M. Taylor, The fuzzball proposal for black holes. Phys. Rept. **467**, 117–171 (2008). http://www.arXiv.org/abs/0804.0552 0804.0552
45. B.D. Chowdhury, A. Virmani, *Modave Lectures on Fuzzballs and Emission from the D1-D5 System*, http://www.arXiv.org/abs/1001.1444 1001.1444
46. V. Jejjala, O. Madden, S.F. Ross, G. Titchener, Non-supersymmetric smooth geometries and D1-D5-P bound states. Phys. Rev. **D71**, 124030 (2005). http://www.arXiv.org/abs/hep-th/0504181 hep-th/0504181
47. S. Giusto, S.F. Ross, A. Saxena, Non-supersymmetric microstates of the D1-D5-KK system. JHEP **12**, 065 (2007). http://www.arXiv.org/abs/0708.3845 0708.3845
48. V. Cardoso, O.J. Dias, J.L. Hovdebo, R.C. Myers, Instability of non-supersymmetric smooth geometries. Phys. Rev. **D73**, 064031 (2006). http://www.arXiv.org/abs/hep-th/0512277 hep-th/0512277
49. B.D. Chowdhury, S.D. Mathur, Radiation from the non-extremal fuzzball. Class. Quant. Grav. **25**, 135005 (2008). http://www.arXiv.org/abs/0711.4817 0711.4817
50. B.D. Chowdhury, S.D. Mathur, Non-extremal fuzzballs and ergoregion emission. Class. Quant. Grav. **26**, 035006 (2009). http://www.arXiv.org/abs/0810.2951 0810.2951
51. B.D. Chowdhury, S.D. Mathur, Pair creation in non-extremal fuzzball geometries. Class. Quant. Grav. **25**, 225021 (2008). http://www.arXiv.org/abs/0806.2309 0806.2309
52. I. Bena, J. de Boer, M. Shigemori, N.P. Warner, Double, Double Supertube Bubble. JHEP **1110**, 116 (2011). http://www.arXiv.org/abs/1107.2650 1107.2650
53. I. Bena, S. Giusto, M. Shigemori, N.P. Warner, Supersymmetric solutions in six dimensions: a linear structure. JHEP **1203**, 084 (2012). http://www.arXiv.org/abs/1110.2781 1110.2781
54. S. Giusto, R. Russo, D. Turton, New D1-D5-P geometries from string amplitudes. JHEP **1111**, 062 (2011). http://www.arXiv.org/abs/1108.6331 1108.6331
55. I. Bena, S. Giusto, C. Ruef, N.P. Warner, Supergravity solutions from floating branes. JHEP **1003**, 047 (2010). http://www.arXiv.org/abs/0910.1860 0910.1860
56. G. Dall'Agata, S. Giusto, C. Ruef, U-duality and non-BPS solutions. JHEP **1102**, 074 (2011). http://www.arXiv.org/abs/1012.4803 1012.4803
57. G. Bossard, C. Ruef, Interacting non-BPS black holes. Gen. Rel. Grav. **44**, 21–66 (2012). http://www.arXiv.org/abs/1106.5806 1106.5806
58. G. Bossard, S. Katmadas, Duality covariant non-BPS first order systems. JHEP **1209**, 100 (2012). http://www.arXiv.org/abs/1205.5461 1205.5461
59. I.R. Klebanov, A.A. Tseytlin, Gravity duals of supersymmetric SU(N) x SU(N+M) gauge theories. Nucl. Phys. **B578**, 123–138 (2000). http://www.arXiv.org/abs/hep-th/0002159 hep-th/0002159
60. I.R. Klebanov, M.J. Strassler, Supergravity and a confining gauge theory: Duality cascades and chi SB resolution of naked singularities. JHEP **0008**, 052 (2000). http://www.arXiv.org/abs/hep-th/0007191 hep-th/0007191
61. M. Grana, Flux compactifications in string theory: a comprehensive review. Phys. Rept. **423**, 91–158 (2006). http://www.arXiv.org/abs/hep-th/0509003 hep-th/0509003
62. J. Breckenridge, R.C. Myers, A. Peet, C. Vafa, D-branes and spinning black holes. Phys. Lett. **B391**, 93–98 (1997). http://www.arXiv.org/abs/hep-th/9602065 hep-th/9602065
63. J.B. Gutowski, H.S. Reall, General supersymmetric AdS(5) black holes. JHEP **0404**, 048 (2004). http://www.arXiv.org/abs/hep-th/0401129 hep-th/0401129
64. J.P. Gauntlett, J.B. Gutowski, Concentric black rings. Phys. Rev. **D71**, 025013 (2005). http://www.arXiv.org/abs/hep-th/0408010 hep-th/0408010

65. I. Bena, P. Kraus, N.P. Warner, Black rings in Taub-NUT. Phys. Rev. **D72**, 084019 (2005). http://www.arXiv.org/abs/hep-th/0504142 hep-th/0504142
66. F. Denef, Supergravity flows and D-brane stability. JHEP **0008**, 050 (2000). http://www.arXiv.org/abs/hep-th/0005049 hep-th/0005049
67. B. Bates, F. Denef, Exact solutions for supersymmetric stationary black hole composites. JHEP **1111**, 127 (2011). http://www.arXiv.org/abs/hep-th/0304094 hep-th/0304094
68. I. Bena, C.-W. Wang, N.P. Warner, Plumbing the Abyss: Black ring microstates. JHEP **0807**, 019 (2008). http://www.arXiv.org/abs/0706.3786 0706.3786
69. J. de Boer, S. El-Showk, I. Messamah, D. Van den Bleeken, Quantizing N = 2 multicenter solutions. JHEP **0905**, 002 (2009). http://www.arXiv.org/abs/0807.4556 0807.4556

http://www.springer.com/978-3-319-00214-9

Black Objects in Supergravity
Proceedings of the INFN-Laboratori Nazionali di Frascati School 2011
Bellucci, S. (Ed.)
2013, VII, 358 p., Hardcover
ISBN: 978-3-319-00214-9