Chapter 1
Introduction and Statement of the Main Results

The braid groups B_n of the plane were introduced by E. Artin in 1925 and further studied in 1947 [1, 2]. They were later generalised by Fox to braid groups of arbitrary topological spaces via the following definition [3]. Let M be a compact, connected surface, and let $n \in \mathbb{N}$. We denote the set of all ordered n-tuples of distinct points of M, known as the nth *configuration space of* M, by:

$$F_n(M) = \left\{(p_1, \ldots, p_n) \mid p_i \in M \quad \text{and} \quad p_i \neq p_j \quad \text{if} \quad i \neq j\right\}.$$

Configuration spaces play an important rôle in several branches of mathematics and have been extensively studied, see [4–7] for example.

The symmetric group S_n on n letters acts freely on $F_n(M)$ by permuting coordinates. The corresponding quotient space $F_n(M)/S_n$ will be denoted by $D_n(M)$. The nth *pure braid group* $P_n(M)$ (respectively the nth *braid group* $B_n(M)$) is defined to be the fundamental group of $F_n(M)$ (respectively of $D_n(M)$). We refer the reader to [8] for a recent survey on surface braid groups and the computation of the lower algebraic K-theory of their group rings.

Together with the real projective plane $\mathbb{R}P^2$, the braid groups of the 2-sphere $\mathbb{S}^2$ are of particular interest, notably because they have non-trivial centre [9, 10], and torsion elements [11, 12]. Indeed, Fadell and Van Buskirk showed that among the braid groups of compact, connected surfaces, $B_n(\mathbb{S}^2)$ and $B_n(\mathbb{R}P^2)$ are the only ones to have torsion [12, 13]. Let us recall briefly some of the properties of $B_n(\mathbb{S}^2)$ [9, 12, 13].

If $\mathbb{D}^2 \subseteq \mathbb{S}^2$ is a topological disc, there is a homomorphism $\iota\colon B_n \to B_n(\mathbb{S}^2)$ induced by the inclusion. If $\beta \in B_n$ then we shall denote its image $\iota(\beta)$ simply by β. Then $B_n(\mathbb{S}^2)$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$ which are subject to the following relations:

$$\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i \text{ if } |i-j| \geqslant 2 \text{ and } 1 \leqslant i, j \leqslant n-1\\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for all } 1 \leqslant i \leqslant n-2, \text{ and}\\ \sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \cdots \sigma_1 &= 1. \end{aligned} \tag{1.1}$$

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Consequently, $B_n(\mathbb{S}^2)$ is a quotient of B_n. The first three sphere braid groups are finite: $B_1(\mathbb{S}^2)$ is trivial, $B_2(\mathbb{S}^2)$ is cyclic of order 2, and $B_3(\mathbb{S}^2)$ is a ZS-metacyclic group (a group whose Sylow subgroups, commutator subgroup and commutator quotient group are all cyclic) of order 12, isomorphic to the semi-direct product $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ of cyclic groups, the action being the non-trivial one. For $n \geqslant 4$, $B_n(\mathbb{S}^2)$ is infinite. The following projection:

$$\begin{cases} \xi: \ B_n(\mathbb{S}^2) \to \mathbb{Z}_{2(n-1)} \\ \qquad\qquad \sigma_i \mapsto \overline{1} \quad \text{for all } 1 \leqslant i \leqslant n-1 \end{cases}$$

is the Abelianisation homomorphism whose kernel is the commutator subgroup $\Gamma_2\left(B_n(\mathbb{S}^2)\right)$ of $B_n(\mathbb{S}^2)$. Note that if $w \in B_n(\mathbb{S}^2)$ then $\xi(w)$ is the exponent sum (relative to the σ_i) of w modulo $2(n-1)$. Further, we have a natural short exact sequence:

$$1 \to P_n(\mathbb{S}^2) \to B_n(\mathbb{S}^2) \xrightarrow{\pi} S_n \to 1, \tag{1.2}$$

π being the homomorphism that sends σ_i to the transposition $(i, i+1)$.

Gillette and Van Buskirk showed that if $n \geqslant 3$ and $k \in \mathbb{N}$ then $B_n(\mathbb{S}^2)$ has an element of order k if and only if k divides one of $2n$, $2(n-1)$ or $2(n-2)$ [9]. The torsion elements of $B_n(\mathbb{S}^2)$ and $B_n(\mathbb{R}P^2)$ were later characterised by Murasugi:

Theorem 1 (Murasugi [11]). *Let $n \geqslant 3$. Then up to conjugacy, the torsion elements of $B_n(\mathbb{S}^2)$ are precisely the powers of the following three elements:*

(a) $\alpha_0 = \sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}$ *(of order $2n$).*
(b) $\alpha_1 = \sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2$ *(of order $2(n-1)$).*
(c) $\alpha_2 = \sigma_1 \cdots \sigma_{n-3}\sigma_{n-2}^2$ *(of order $2(n-2)$).*

So the maximal finite cyclic subgroups of $B_n(\mathbb{S}^2)$ are isomorphic to $\mathbb{Z}_{2n}$, $\mathbb{Z}_{2(n-1)}$ or $\mathbb{Z}_{2(n-2)}$. In [14], we showed that $B_n(\mathbb{S}^2)$ is generated by α_0 and α_1. Let $\Delta_n^2 = (\sigma_1 \cdots \sigma_{n-1})^n$ denote the so-called 'full twist' braid of $B_n(\mathbb{S}^2)$. If $n \geqslant 3$, Δ_n^2 is the unique element of $B_n(\mathbb{S}^2)$ of order 2, and it generates the centre of $B_n(\mathbb{S}^2)$ [9]. It is also the square of the 'half twist' element defined by:

$$\Delta_n = (\sigma_1 \cdots \sigma_{n-1})(\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1\sigma_2)\sigma_1. \tag{1.3}$$

It is well known that:

$$\Delta_n \sigma_i \Delta_n^{-1} = \sigma_{n-i} \quad \text{for all } i = 1, \ldots, n-1. \tag{1.4}$$

The uniqueness of the element of order 2 in $B_n(\mathbb{S}^2)$ implies that the three elements α_0, α_1 and α_2 are respectively nth, $(n-1)$th and $(n-2)$th roots of Δ_n^2, and this yields the useful relation:

$$\Delta_n^2 = \alpha_i^{n-i} \quad \text{for all } i \in \{0, 1, 2\}. \tag{1.5}$$

In what follows, if $m \geqslant 2$, Dic_{4m} will denote the *dicyclic group* of order $4m$. It admits a presentation of the form

$$\left\langle x, y \;\middle|\; x^m = y^2,\ yxy^{-1} = x^{-1} \right\rangle. \tag{1.6}$$

If in addition m is a power of 2 then we will also refer to the dicyclic group of order $4m$ as the *generalised quaternion group* of order $4m$, and denote it by $\mathcal{Q}_{4m}$. We remark that some authors use the terminology 'generalised quaternion group' to be what we refer to as 'dicyclic group', but we follow the terminology of [15, pp. 68 and 82], [16, p. 140], [17, p. 351] and [18, pp. 189 and 252]. As an example, if $m = 2$ then we obtain the usual quaternion group $\mathcal{Q}_8$ of order 8. Further, T^* (resp. O^*, I^*) will denote the *binary tetrahedral group* of order 24 (resp. the *binary octahedral group* of order 48, the *binary icosahedral group* of order 120). We will refer collectively to T^*, O^* and I^* as the *binary polyhedral groups*. More details on these groups may be found in [15, 19–21], as well as in Sect. 2.3 and the Appendix.

In order to understand better the structure of $B_n(\mathbb{S}^2)$, one may study (up to isomorphism) the finite subgroups of $B_n(\mathbb{S}^2)$. From Theorem 1, it is clear that the finite cyclic subgroups of $B_n(\mathbb{S}^2)$ are isomorphic to the subgroups of $\mathbb{Z}_{2(n-i)}$, where $i \in \{0, 1, 2\}$. Motivated by a question of the realisation of $\mathcal{Q}_8$ as a subgroup of $B_n(\mathbb{S}^2)$ of Brown [22] in connection with the Dirac string trick [23, 24], we obtained partial results on the classification of the isomorphism classes of the finite subgroups of $B_n(\mathbb{S}^2)$ in [25, 26]. We remark that the case $n = 4$ was studied by Thompson [27]. The complete classification was given in [28]:

Theorem 2 ([28]). *Let $n \geqslant 3$. Up to isomorphism, the maximal finite subgroups of $B_n(\mathbb{S}^2)$ are:*

(a) $\mathbb{Z}_{2(n-1)}$ *if* $n \geqslant 5$.
(b) Dic_{4n}.
(c) $\mathrm{Dic}_{4(n-2)}$ *if* $n = 5$ *or* $n \geqslant 7$.
(d) T^* *if* $n \equiv 4 \bmod 6$.
(e) O^* *if* $n \equiv 0, 2 \bmod 6$.
(f) I^* *if* $n \equiv 0, 2, 12, 20 \bmod 30$.

Remarks 3 (a) By studying the subgroups of dicyclic and binary polyhedral groups, it is not difficult to show that any finite subgroup of $B_n(\mathbb{S}^2)$ is cyclic, dicyclic or binary polyhedral (see Proposition 85).

(b) As we showed in [25, 28], for $i \in \{0, 2\}$,

$$\Delta_n \alpha_i' \Delta_n^{-1} = \alpha_i'^{-1}, \quad \text{where } \alpha_i' = \alpha_0 \alpha_i \alpha_0^{-1} = \alpha_0^{i/2} \alpha_i \alpha_0^{-i/2}, \tag{1.7}$$

and the dicyclic group of order $4(n-i)$ is realised in terms of the generators of $B_n(\mathbb{S}^2)$ by:

$$\langle \alpha_i', \Delta_n \rangle,$$

which we shall refer to hereafter as the *standard copy* of $\mathrm{Dic}_{4(n-i)}$ in $B_n(\mathbb{S}^2)$.

A key tool in the proof of Theorem 2 is the close relationship due to Magnus of $B_n(\mathbb{S}^2)$ with the mapping class group $\mathcal{MCG}(\mathbb{S}^2, n)$ of the n-punctured sphere, $n \geqslant 3$, given by the short exact sequence [29, 30]:

$$1 \to \langle \Delta_n^2 \rangle \to B_n(\mathbb{S}^2) \xrightarrow{\varphi} \mathcal{MCG}(\mathbb{S}^2, n) \to 1. \tag{1.8}$$

As we shall see, it will also play an important rôle in various parts of this paper, notably in the study of the centralisers and conjugacy classes of the finite order elements in Chap. 2, as well as in some of the constructions in Chap. 3. There is a short exact sequence for the mapping class group analogous to Eq. (1.2); the kernel of the homomorphism $\mathcal{MCG}(\mathbb{S}^2, n) \to S_n$ is the pure mapping class group $\mathcal{PMCG}(\mathbb{S}^2, n)$, which may also be seen as the image of $P_n(\mathbb{S}^2)$ under φ. In particular, since for $n \geqslant 4$, $P_n(\mathbb{S}^2) \cong P_{n-3}(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\}) \times \mathbb{Z}_2$ [10], where the second factor is identified with $\langle \Delta_n^2 \rangle$, it follows from the restriction of Eq. (1.8) to $P_n(\mathbb{S}^2)$ that

$$\mathcal{PMCG}(\mathbb{S}^2, n) \cong P_{n-3}(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\}),$$

in particular $\mathcal{PMCG}(\mathbb{S}^2, n)$ is torsion free for all $n \geqslant 4$.

In this book, we go a stage further by classifying (up to isomorphism) the virtually cyclic subgroups of $B_n(\mathbb{S}^2)$. Recall that a group is said to be *virtually cyclic* if it contains a cyclic subgroup of finite index (see also Sect. 2.1). It is clear from the definition that any finite subgroup is virtually cyclic, so in view of Theorem 2, it suffices to concentrate on the *infinite* virtually cyclic subgroups of $B_n(\mathbb{S}^2)$, which are in some sense its 'simplest' infinite subgroups. The classification of the virtually cyclic subgroups of $B_n(\mathbb{S}^2)$ is an interesting problem in its own right. As well as helping us to understand better the structure of these braid groups, the results of this book give rise to some K-theoretical applications. We remark that our work was partially motivated by a question of S. Millán-López and S. Prassidis concerning the calculation of the algebraic K-theory of the braid groups of $\mathbb{S}^2$ and $\mathbb{R}P^2$. It was shown recently that the full and pure braid groups of these two surfaces satisfy the Fibred Isomorphism Conjecture of Farrell and Jones [31–33]. This implies that the algebraic K-theory groups of their group rings (over $\mathbb{Z}$) may be computed by means of the algebraic K-theory groups of their virtually cyclic subgroups via the so-called 'assembly maps'. More information on these topics may be found in [8, 34–36]. The main theorem of this book, Theorem 5, is currently being applied to the calculation of the lower algebraic K-theory of $\mathbb{Z}[B_n(\mathbb{S}^2)]$ [8, 37], which generalises results of the thesis of Millán-López [38, 39] who calculated the lower algebraic K-theory of the group rings of $P_n(\mathbb{S}^2)$ and $P_n(\mathbb{R}P^2)$, making use of our classification of the virtually cyclic subgroups of $P_n(\mathbb{R}P^2)$ in the latter case [40]. This application to K-theory thus provides us with additional reasons to find the virtually cyclic subgroups of $B_n(\mathbb{S}^2)$.

As we observed previously, if $n \leqslant 3$ then $B_n(\mathbb{S}^2)$ is a known finite group, and so we shall suppose in this book that $n \geqslant 4$. Our main result is Theorem 5, which yields the complete classification of the infinite virtually cyclic subgroups of $B_n(\mathbb{S}^2)$, with a small number of exceptions, that we indicate below in Remark 6. Recall that by results of Epstein and Wall [41, 42] (see also Theorem 17 in Sect. 2.1), any infinite virtually cyclic group G is isomorphic to $F \rtimes \mathbb{Z}$ or $G_1 *_F G_2$, where F is finite and $[G_i : F] = 2$ for $i \in \{1, 2\}$ (we shall say that G is of *Type I* or *Type II* respectively). Before stating Theorem 5, we define two families of virtually cyclic groups. If G is a group, let $\operatorname{Aut}(G)$ (resp. $\operatorname{Out}(G)$) denote the group of its automorphisms (resp. outer automorphisms).

Definition 4 Let $n \geqslant 4$.

(1) Let $\mathbb{V}_1(n)$ be the family comprised of the following Type I virtually cyclic groups:
  (a) $\mathbb{Z}_q \times \mathbb{Z}$, where q is a strict divisor of $2(n-i)$, $i \in \{0, 1, 2\}$, and $q \neq n-i$ if $n-i$ is odd.
  (b) $\mathbb{Z}_q \rtimes_\rho \mathbb{Z}$, where $q \geqslant 3$ is a strict divisor of $2(n-i)$, $i \in \{0, 2\}$, $q \neq n-i$ if n is odd, and $\rho(1) \in \operatorname{Aut}(\mathbb{Z}_q)$ is multiplication by -1.
  (c) $\mathrm{Dic}_{4m} \times \mathbb{Z}$, where $m \geqslant 3$ is a strict divisor of $n-i$ and $i \in \{0, 2\}$.
  (d) $\mathrm{Dic}_{4m} \rtimes_\nu \mathbb{Z}$, where $m \geqslant 3$ divides $n-i$, $i \in \{0, 2\}$, $(n-i)/m$ is even, and where $\nu(1) \in \operatorname{Aut}(\mathrm{Dic}_{4m})$ is defined by:

$$\begin{cases} \nu(1)(x) = x \\ \nu(1)(y) = xy \end{cases} \tag{1.9}$$

  for the presentation (1.6) of Dic_{4m}.
  (e) $\mathcal{Q}_8 \rtimes_\theta \mathbb{Z}$, for n even and $\theta \in \operatorname{Hom}(\mathbb{Z}, \operatorname{Aut}(\mathcal{Q}_8))$, for the following actions:
    (i) $\theta(1) = \mathrm{Id}$.
    (ii) $\theta = \alpha$, where $\alpha(1) \in \operatorname{Aut}(\mathcal{Q}_8)$ is given by $\alpha(1)(i) = j$ and $\alpha(1)(j) = k$, where $\mathcal{Q}_8 = \{\pm 1, \pm i, \pm j, \pm k\}$.
    (iii) $\theta = \beta$, where $\beta(1) \in \operatorname{Aut}(\mathcal{Q}_8)$ is given by $\beta(1)(i) = k$ and $\beta(1)(j) = j^{-1}$.
  (f) $T^* \times \mathbb{Z}$ for n even.
  (g) $T^* \rtimes_\omega \mathbb{Z}$ for $n \equiv 0, 2 \bmod 6$, where $\omega(1) \in \operatorname{Aut}(T^*)$ is the automorphism defined as follows. Let T^* be given by the presentation [21, p. 198]:

$$\left\langle P, Q, X \;\middle|\; X^3 = 1,\ P^2 = Q^2,\ PQP^{-1} = Q^{-1},\ XPX^{-1} = Q,\ XQX^{-1} = PQ \right\rangle, \tag{1.10}$$

  and let $\omega(1) \in \operatorname{Aut}(T^*)$ be defined by

$$\begin{cases} P \mapsto QP \\ Q \mapsto Q^{-1} \\ X \mapsto X^{-1}. \end{cases} \tag{1.11}$$

  More details concerning this automorphism will be given in Sect. 2.3.
  (h) $O^* \times \mathbb{Z}$ for $n \equiv 0, 2 \bmod 6$.
  (i) $I^* \times \mathbb{Z}$ for $n \equiv 0, 2, 12, 20 \bmod 30$.

(2) Let $\mathbb{V}_2(n)$ be the family comprised of the following Type II virtually cyclic groups:

 (a) $\mathbb{Z}_{4q} *_{\mathbb{Z}_{2q}} \mathbb{Z}_{4q}$, where q divides $(n-i)/2$ for some $i \in \{0, 1, 2\}$.
 (b) $\mathbb{Z}_{4q} *_{\mathbb{Z}_{2q}} \mathrm{Dic}_{4q}$, where $q \geqslant 2$ divides $(n-i)/2$ for some $i \in \{0, 2\}$.
 (c) $\mathrm{Dic}_{4q} *_{\mathbb{Z}_{2q}} \mathrm{Dic}_{4q}$, where $q \geqslant 2$ divides $n-i$ strictly for some $i \in \{0, 2\}$.
 (d) $\mathrm{Dic}_{4q} *_{\mathrm{Dic}_{2q}} \mathrm{Dic}_{4q}$, where $q \geqslant 4$ is even and divides $n-i$ for some $i \in \{0, 2\}$.
 (e) $O^* *_{T^*} O^*$, where $n \equiv 0, 2 \bmod 6$.

Finally, let $\mathbb{V}(n)$ be the family comprised of the elements of $\mathbb{V}_1(n)$ and $\mathbb{V}_2(n)$. Unless indicated to the contrary, in what follows, ρ, ν, α, β and ω will denote the actions defined in parts (1)(b), (d), (e)(ii), (e)(iii) and (g) respectively.

The main result of this book is the following, which classifies (up to a finite number of exceptions) the infinite virtually cyclic subgroups of $B_n(\mathbb{S}^2)$.

Theorem 5 *Suppose that $n \geqslant 4$.*

(1) If G is an infinite virtually cyclic subgroup of $B_n(\mathbb{S}^2)$ then G is isomorphic to an element of $\mathbb{V}(n)$.
(2) Conversely, let G be an element of $\mathbb{V}(n)$. Assume that the following conditions hold:

 (a) if $G \cong Q_8 \rtimes_\alpha \mathbb{Z}$ then $n \notin \{6, 10, 14\}$.
 (b) if $G \cong T^* \times \mathbb{Z}$ then $n \notin \{4, 6, 8, 10, 14\}$.
 (c) if $G \cong O^* \times \mathbb{Z}$ or $G \cong T^* \rtimes_\omega \mathbb{Z}$ then $n \notin \{6, 8, 12, 14, 18, 20, 26\}$.
 (d) if $G \cong I^* \times \mathbb{Z}$ then $n \notin \{12, 20, 30, 32, 42, 50, 62\}$.
 (e) if $G \cong O^* *_{T^*} O^*$ then $n \notin \{6, 8, 12, 14, 18, 20, 24, 26, 30, 32, 38\}$.

 Then there exists a subgroup of $B_n(\mathbb{S}^2)$ isomorphic to G.
(3) Let G be isomorphic to $T^* \times \mathbb{Z}$ (resp. to $O^* \times \mathbb{Z}$) if $n = 4$ (resp. $n = 6$). Then $B_n(\mathbb{S}^2)$ has no subgroup isomorphic to G.

Remark 6 Together with Theorem 2, Theorem 5 yields a complete classification of the virtually cyclic subgroups of $B_n(\mathbb{S}^2)$ with the exception of a small (finite) number of cases for which the problem of their existence is open. These cases are as follows:

(a) Type I subgroups of $B_n(\mathbb{S}^2)$ (see Propositions 62 and 66, as well as Remarks 64 and 67):

 (i) the realisation of $Q_8 \rtimes_\alpha \mathbb{Z}$ as a subgroup of $B_n(\mathbb{S}^2)$, where n belongs to $\{6, 10, 14\}$ and $\alpha(1) \in \mathrm{Aut}\,(Q_8)$ is as in Definition 4(1)(e)(ii).
 (ii) the realisation of $T^* \times \mathbb{Z}$ as a subgroup of $B_n(\mathbb{S}^2)$, where n belongs to $\{6, 8, 10, 14\}$.
 (iii) the realisation of $T^* \rtimes_\omega \mathbb{Z}$ as a subgroup of $B_n(\mathbb{S}^2)$, where the action ω is given by Definition 4(1)(g), and $n \in \{6, 8, 12, 14, 18, 20, 26\}$.
 (iv) the realisation of $O^* \times \mathbb{Z}$ as a subgroup of $B_n(\mathbb{S}^2)$, where n belongs to $\{8, 12, 14, 18, 20, 26\}$.

(v) the realisation of $I^* \times \mathbb{Z}$ as a subgroup of $B_n(\mathbb{S}^2)$, where n belongs to $\{12, 20, 30, 32, 42, 50, 62\}$.

(b) Type II subgroups of $B_n(\mathbb{S}^2)$ (see Remark 72 and Proposition 73):

(i) for $n \in \{6, 8, 12, 14, 18, 20, 24, 26, 30, 32, 38\}$, the realisation of the group $O^* *_{T^*} O^*$ as a subgroup of $B_n(\mathbb{S}^2)$.

Since the above open cases occur for even values of n, the complete classification of the infinite virtually cyclic subgroups of $B_n(\mathbb{S}^2)$ for all $n \geqslant 5$ odd is an immediate consequence of Theorem 5.

Theorem 7 *Let $n \geqslant 5$ be odd. Then up to isomorphism, the following groups are the infinite virtually cyclic subgroups of $B_n(\mathbb{S}^2)$.*

(I) (a) $\mathbb{Z}_m \rtimes_\theta \mathbb{Z}$, where $\theta(1) \in \{Id, -Id\}$, m is a strict divisor of $2(n-i)$, for $i \in \{0, 2\}$, and $m \neq n - i$.

(b) $\mathbb{Z}_m \times \mathbb{Z}$, where m is a strict divisor of $2(n-1)$.

(c) $Dic_{4m} \times \mathbb{Z}$, where $m \geqslant 3$ is a strict divisor of $n - i$ for $i \in \{0, 2\}$.

(II) (a) $\mathbb{Z}_{4q} *_{\mathbb{Z}_{2q}} \mathbb{Z}_{4q}$, where q divides $(n-1)/2$.

(b) $Dic_{4q} *_{\mathbb{Z}_{2q}} Dic_{4q}$, where $q \geqslant 2$ is a strict divisor of $n - i$, and $i \in \{0, 2\}$.

Most of this book is devoted to proving Theorem 5, and is broadly divided into two Chaps. 2 and 3, together with a short Appendix. The aim of Chap. 2 is to prove Theorem 5(1). In conjunction with Theorem 2, Theorem 17 gives rise to a family $\mathcal{VC}$ of virtually cyclic groups, defined in Sect. 2.1, with the property that any infinite virtually cyclic subgroup of $B_n(\mathbb{S}^2)$ belongs to $\mathcal{VC}$. In that section, we shall discuss a number of properties pertaining to virtually cyclic groups. Proposition 26 describes the correspondence in general between the virtually cyclic subgroups of a group G possessing a unique element x of order 2 and its quotient $G/\langle x \rangle$. By the short exact sequence (1.8), this proposition applies immediately to $B_n(\mathbb{S}^2)$ and $\mathcal{MCG}(\mathbb{S}^2, n)$, and will be used at various points, notably to obtain the classification of the virtually cyclic subgroups of $\mathcal{MCG}(\mathbb{S}^2, n)$ from that of $B_n(\mathbb{S}^2)$. Two other results of Sect. 2.1 that will prove to be useful in Sect. 3.8 are Proposition 20 which shows that almost all elements of $\mathbb{V}_2(n)$ of the form $G *_H G$ may be written as a semi-direct product $\mathbb{Z} \rtimes G$, and Proposition 27 which will be used to determine the number of isomorphism classes of the elements of $\mathbb{V}_2(n)$.

The principal difficulty to proving Theorem 5 is to decide which of the elements of $\mathcal{VC}$ are indeed realised as subgroups of $B_n(\mathbb{S}^2)$. This is achieved in two stages, *reduction* and *realisation*. In the first stage, we reduce the subfamily of $\mathcal{VC}$ of Type I groups in several ways. To this end, in Sect. 2.2, we obtain a number of results of independent interest concerning structural aspects of $B_n(\mathbb{S}^2)$. The first of these is the calculation of the centraliser and normaliser of its maximal finite cyclic and dicyclic subgroups. Note that if $i \in \{0, 1\}$, the centraliser of α_i, considered as an element of B_n, is equal to $\langle \alpha_i \rangle$ [43, 44]. A similar equality holds in $B_n(\mathbb{S}^2)$ and is obtained using Eq. (1.8) and the corresponding result for $\mathcal{MCG}(\mathbb{S}^2, n)$ due to Hodgkin [45]:

Proposition 8 *Let $i \in \{0, 1, 2\}$, and let $n \geqslant 3$.*

(a) The centraliser of $\langle \alpha_i \rangle$ in $B_n(\mathbb{S}^2)$ is equal to $\langle \alpha_i \rangle$, unless $i = 2$ and $n = 3$, in which case it is equal to $B_3(\mathbb{S}^2)$.
(b) The normaliser of $\langle \alpha_i \rangle$ in $B_n(\mathbb{S}^2)$ is equal to:

$$\begin{cases} \langle \alpha_0, \Delta_n \rangle \cong Dic_{4n} & \text{if } i = 0 \\ \left\langle \alpha_2, \alpha_0^{-1} \Delta_n \alpha_0 \right\rangle \cong Dic_{4(n-2)} & \text{if } i = 2 \\ \langle \alpha_1 \rangle \cong \mathbb{Z}_{2(n-1)} & \text{if } i = 1, \end{cases}$$

unless $i = 2$ and $n = 3$, in which case it is equal to $B_3(\mathbb{S}^2)$.
(c) If $i \in \{0, 2\}$, the normaliser of the standard copy of $Dic_{4(n-i)}$ in $B_n(\mathbb{S}^2)$ is itself, except when $i = 2$ and $n = 4$, in which case the normaliser is equal to $\alpha_0^{-1}\sigma_1^{-1}\langle \alpha_0, \Delta_4 \rangle \sigma_1 \alpha_0$, and is isomorphic to $\mathcal{Q}_{16}$.

If F is a maximal dicyclic or finite cyclic subgroup of $B_n(\mathbb{S}^2)$, parts (a) and (b) imply immediately that $B_n(\mathbb{S}^2)$ has no Type I subgroup of the form $F \rtimes \mathbb{Z}$.

The second reduction, given in Proposition 35 in Sect. 2.3 will make use of the fact that if $\theta : \mathbb{Z} \to \operatorname{Aut}(F)$ is an action of $\mathbb{Z}$ on the finite group F, the isomorphism class of the semi-direct product $F \rtimes_\theta \mathbb{Z}$ depends only on the class of $\theta(1)$ in $\operatorname{Out}(F)$. Since we are interested in the realisation of isomorphism classes of virtually finite subgroups in $B_n(\mathbb{S}^2)$, it will thus be sufficient to study the Type I groups of the form $F \rtimes_\theta \mathbb{Z}$, where $\theta(1)$ runs over a transversal of $\operatorname{Out}(F)$ in $\operatorname{Aut}(F)$. To this end, in Sect. 2.3, we recall the structure of $\operatorname{Out}(F)$ for the binary polyhedral groups. One could also carry out this analysis for the other finite subgroups of $B_n(\mathbb{S}^2)$ given by Theorem 2, but the resulting conditions on θ are weaker than those obtained from a generalisation of a second result of L. Hodgkin concerning the powers of α_i that are conjugate in $B_n(\mathbb{S}^2)$. More precisely, in Sect. 2.4, we prove the following proposition.

Proposition 9 *Let $n \geqslant 3$ and $i \in \{0, 1, 2\}$, and suppose that there exist $r, m \in \mathbb{Z}$ such that α_i^m and α_i^r are conjugate in $B_n(\mathbb{S}^2)$.*

(a) If $i = 1$ then $\alpha_1^m = \alpha_1^r$.
(b) If $i \in \{0, 2\}$ then $\alpha_i^m = \alpha_i^{\pm r}$.

In particular, conjugate powers of the α_i are either equal or inverse. So if F is a finite cyclic subgroup of $B_n(\mathbb{S}^2)$ then by Theorem 1 the only possible actions of $\mathbb{Z}$ on F are the trivial action and multiplication by -1. This also has consequences for the possible actions of $\mathbb{Z}$ on dicyclic subgroups of $B_n(\mathbb{S}^2)$. As in Proposition 8, the proof of Proposition 9 will make use of a similar result for the mapping class group and the relation (1.8).

The final reduction, described in Sect. 2.5.2, again affects the possible Type I subgroups that may occur, and is a manifestation of the periodicity (with least period 2 or 4) of the subgroups of $B_n(\mathbb{S}^2)$ that was observed in [28] for the finite subgroups. The following proposition will be applied to rule out Type I subgroups of $B_n(\mathbb{S}^2)$ isomorphic to $F \rtimes_\theta \mathbb{Z}$ with non-trivial action θ, where F is either O^* or I^* (one could also apply the result to the other possible finite groups F, but this is not necessary in our context in light of the consequences of Proposition 9 mentioned above). The following proposition may be found in [46, 47], and may be compared with the analogous result for $\mathbb{R}P^2$ [48, Proposition 6]. We shall give an alternative proof in Sect. 2.5.1.

Proposition 10 ([46, 47])

(a) The space $F_2(\mathbb{S}^2)$ (resp. $D_2(\mathbb{S}^2)$) has the homotopy type of $\mathbb{S}^2$ (resp. of $\mathbb{R}P^2$). Hence the universal covering space of $D_2(\mathbb{S}^2)$ is $F_2(\mathbb{S}^2)$.

(b) If $n \geqslant 3$, the universal covering space of $F_n(\mathbb{S}^2)$ or $D_n(\mathbb{S}^2)$ has the homotopy type of the 3–sphere $\mathbb{S}^3$.

Putting together these reductions will allow us to prove Theorem 5(1), first for the groups of Type I in Sect. 2.6.1, and then for those of Type II in Sect. 2.6.2. The structure of the finite subgroups of $B_n(\mathbb{S}^2)$ imposes strong constraints on the possible Type II subgroups, and the proof in this case is more straightforward than that for Type I subgroups.

The second part of the manuscript, Chap. 3, is devoted to the analysis of the realisation of the elements of $\mathbb{V}(n)$ as subgroups of $B_n(\mathbb{S}^2)$ and to proving parts (2) and (3) of Theorem 5. With the exception of the values of n excluded by the statement of part (2), we prove the existence of the elements of $\mathbb{V}(n)$ as subgroups of $B_n(\mathbb{S}^2)$, first those of Type I in Sects. 3.1–3.4 and then those of Type II in Sect. 3.6. The results of these sections are gathered together in Proposition 68 (resp. Proposition 73) which proves Theorem 5(2) for the subgroups of Type I (resp. Type II). The construction of the elements of $\mathbb{V}(n)$ involving finite cyclic and dicyclic groups are largely algebraic, and will rely heavily on Lemma 51, as well as on Lemma 29 which describes the action by conjugation of the α_i on the generators of $B_n(\mathbb{S}^2)$. In contrast, the realisation of the elements of $\mathbb{V}(n)$ involving the binary polyhedral groups is geometric in nature, and occurs on the level of mapping class groups via the relation (1.8). The constraints involved in the constructions indicate why the realisation of such elements is an open problem for the values of n given in Remark 6. For $n \in \{4, 6\}$, in Proposition 62(d) we are also able to rule out the existence of the virtually cyclic groups given in Theorem 5(3).

In Sect. 3.8, we discuss the isomorphism problem for the amalgamated products that occur as elements of $\mathbb{V}_2(n)$. It turns out that with one exception, abstractly there is only one way (up to isomorphism) to embed the amalgamating subgroup in each of the two factors. With the help of Proposition 27, we are able to prove the following result.

Proposition 11 *For each of the amalgamated products given in Definition 4(2), abstractly there is exactly one isomorphism class, with the exception of $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$, for which there are exactly two isomorphism classes.*

Note that Proposition 11 refers to abstract isomorphism classes, and does not depend on the fact that the amalgamated products occurring as elements of $\mathbb{V}_2(n)$ are realised as subgroups of $B_n(\mathbb{S}^2)$. In the exceptional case, that of $\mathcal{Q}_{16} *_{\mathcal{Q}_8} \mathcal{Q}_{16}$, abstractly there are two isomorphism classes defined by Eq. (3.30) and (3.32). In

Corollary 76, we show that abstractly, all but one of the isomorphism classes of the elements of $\mathbb{V}_2(n)$ of the form $G *_H G$ may be written as a semi-direct product of $\mathbb{Z}$ by G. In Propositions 77 and 78, if $n \geqslant 4$ is even we show that one of these isomorphism classes is always realised as a subgroup of $B_n(\mathbb{S}^2)$, while the other isomorphism class is realised as a subgroup of $B_n(\mathbb{S}^2)$ for all $n \notin \{6, 14, 18, 26, 30, 38\}$. It is an open question as to whether this second isomorphism class is realised as a subgroup of $B_n(\mathbb{S}^2)$ for $n \in \{6, 14, 18, 26, 30, 38\}$.

In Sect. 3.9, we deduce the classification of the virtually cyclic subgroups of $\mathcal{MCG}(\mathbb{S}^2, n)$ (with a finite number of exceptions). As we shall see, it will follow from Proposition 26 that the homomorphism φ of the short exact sequence (1.8) induces a correspondence that is one-to-one, with the exception of subgroups of $B_n(\mathbb{S}^2)$ that are isomorphic to $\mathbb{Z}_m \rtimes_\theta \mathbb{Z}$ or $\mathbb{Z}_{2m} \rtimes_\theta \mathbb{Z}$ for m odd, which are sent to the same subgroup $\mathbb{Z}_m \rtimes_{\theta'} \mathbb{Z}$ of $\mathcal{MCG}(\mathbb{S}^2, n)$, the action θ' being given as in Proposition 12(b) below.

Proposition 12 *Let $n \geqslant 4$, and let $\varphi \colon B_n(\mathbb{S}^2) \to \mathcal{MCG}(\mathbb{S}^2, n)$ be the epimorphism given by Eq.* (1.8).

(a) Let H' be an infinite virtually cyclic subgroup of $\mathcal{MCG}(\mathbb{S}^2, n)$ of Type I (resp. Type II). Then $\varphi^{-1}(H')$ is a virtually cyclic subgroup of $B_n(\mathbb{S}^2)$ of Type I (resp. Type II).

(b) Let H be a Type I virtually cyclic subgroup of $B_n(\mathbb{S}^2)$, isomorphic to $F \rtimes_\theta \mathbb{Z}$, where F is a finite subgroup of $B_n(\mathbb{S}^2)$ and $\theta \in \mathrm{Hom}(\mathbb{Z}, Aut\,(F))$. Then $\varphi(H) \cong \varphi(F) \rtimes_{\theta'} \mathbb{Z}$, where $\theta' \in \mathrm{Hom}(\mathbb{Z}, Aut\,(F'))$ satisfies $\theta'(1)(f') = \varphi(\theta(1)(f))$ for all $f' \in F'$ and $f \in F$ for which $\varphi(f) = f'$.

(c) Let H be a Type II virtually cyclic subgroup of $B_n(\mathbb{S}^2)$ isomorphic to $G_1 *_F G_2$, where G_1, G_2 and F are finite subgroups of $B_n(\mathbb{S}^2)$, and F is an index 2 subgroup of both G_1 and G_2. Then $\varphi(H) \cong \varphi(G_1) *_{\varphi(F)} \varphi(G_2)$.

Equation (1.8) and Definition 4 together imply that the following virtually cyclic groups are those that will appear in the classification of the virtually cyclic subgroups of $\mathcal{MCG}(\mathbb{S}^2, n)$. If $m \geqslant 2$, let Dih_{2m} denote the *dihedral group* of order $2m$.

Definition 13 Let $n \geqslant 4$.

(1) Let $\widetilde{\mathbb{V}}_1(n)$ be the family comprised of the following Type I virtually cyclic groups:

(a) $\mathbb{Z}_q \times \mathbb{Z}$, where q is a strict divisor of $n - i$, $i \in \{0, 1, 2\}$.

(b) $\mathbb{Z}_q \rtimes_{\widetilde{\rho}} \mathbb{Z}$, where $q \geqslant 3$ is a strict divisor of $n - i$, $i \in \{0, 2\}$, and $\widetilde{\rho}(1) \in \mathrm{Aut}\left(\mathbb{Z}_q\right)$ is multiplication by -1.

(c) $\mathrm{Dih}_{2m} \times \mathbb{Z}$, where $m \geqslant 3$ is a strict divisor of $n - i$ and $i \in \{0, 2\}$.

(d) $\mathrm{Dih}_{2m} \rtimes_{\widetilde{\nu}} \mathbb{Z}$, where $m \geqslant 3$ divides $n - i$, $i \in \{0, 2\}$, $(n - i)/m$ is even, and where $\widetilde{\nu}(1) \in \mathrm{Aut}\,(\mathrm{Dih}_{2m})$ is defined by:

$$\begin{cases} \widetilde{\nu}(1)(x) = x \\ \widetilde{\nu}(1)(y) = xy \end{cases}$$

for the presentation of Dih_{2m} given by:

$$\left\langle x, y \;\middle|\; x^m = y^2 = 1,\; yxy^{-1} = x^{-1} \right\rangle.$$

(e) $(\mathbb{Z}_2 \oplus \mathbb{Z}_2) \rtimes_{\widetilde{\theta}} \mathbb{Z}$, for n even and $\widetilde{\theta} \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_2 \oplus \mathbb{Z}_2)$, for the following actions:
  (i) $\widetilde{\theta}(1) = \mathrm{Id}$.
  (ii) $\widetilde{\theta} = \widetilde{\alpha}$, where $\widetilde{\alpha}(1) \in \mathrm{Aut}\,(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ is given by $\widetilde{\alpha}(1)((\overline{1}, \overline{0})) = (\overline{0}, \overline{1})$ and $\widetilde{\alpha}(1)((\overline{0}, \overline{1})) = (\overline{1}, \overline{1})$.
  (iii) $\widetilde{\theta} = \widetilde{\beta}$, where $\widetilde{\beta}(1) \in \mathrm{Aut}\,(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ is given by $\widetilde{\beta}(1)((\overline{1}, \overline{0})) = (\overline{1}, \overline{1})$ and $\widetilde{\beta}(1)((\overline{0}, \overline{1})) = (\overline{0}, \overline{1})$.

(f) $A_4 \times \mathbb{Z}$ for n even.

(g) $A_4 \rtimes_{\widetilde{\omega}} \mathbb{Z}$ for $n \equiv 0, 2 \bmod 6$, where $\widetilde{\omega}(1) \in \mathrm{Aut}\,(A_4)$ is the automorphism defined as follows. Let $A_4 = (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \rtimes \mathbb{Z}_3$ where the action of $\mathbb{Z}_3$ on $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ permutes cyclically the three elements $(\overline{1}, \overline{0})$, $(\overline{0}, \overline{1})$ and $(\overline{1}, \overline{1})$, and let $\widetilde{X}$ be a generator of the $\mathbb{Z}_3$-factor. Then we define $\widetilde{\omega}(1) \in \mathrm{Aut}\,(A_4)$ by:

$$\begin{cases} (\overline{1}, \overline{0}) \mapsto (\overline{1}, \overline{1}) \\ (\overline{0}, \overline{1}) \mapsto (\overline{0}, \overline{1}) \\ \qquad \widetilde{X} \mapsto \widetilde{X}^{-1}. \end{cases}$$

(h) $S_4 \times \mathbb{Z}$ for $n \equiv 0, 2 \bmod 6$.

(i) $A_5 \times \mathbb{Z}$ for $n \equiv 0, 2, 12, 20 \bmod 30$.

(2) Let $\widetilde{\mathbb{V}}_2(n)$ be the family comprised of the following Type II virtually cyclic groups:

(a) $\mathbb{Z}_{2q} *_{\mathbb{Z}_q} \mathbb{Z}_{2q}$, where q divides $(n-i)/2$ for some $i \in \{0, 1, 2\}$.
(b) $\mathbb{Z}_{2q} *_{\mathbb{Z}_q} \mathrm{Dih}_{2q}$, where $q \geqslant 2$ divides $(n-i)/2$ for some $i \in \{0, 2\}$.
(c) $\mathrm{Dih}_{2q} *_{\mathbb{Z}_q} \mathrm{Dih}_{2q}$, where $q \geqslant 2$ divides $n-i$ strictly for some $i \in \{0, 2\}$.
(d) $\mathrm{Dih}_{2q} *_{\mathrm{Dih}_q} \mathrm{Dih}_{2q}$, where $q \geqslant 4$ is even and divides $n-i$ for some $i \in \{0, 2\}$.
(e) $S_4 *_{A_4} S_4$, where $n \equiv 0, 2 \bmod 6$.

Finally, let $\widetilde{\mathbb{V}}(n)$ be the family comprised of the elements of $\widetilde{\mathbb{V}}_1(n)$ and $\widetilde{\mathbb{V}}_2(n)$.

We thus obtain the classification of the virtually cyclic subgroups of $\mathcal{MCG}(\mathbb{S}^2, n)$ (with a finite number of exceptions).

Theorem 14 *Let $n \geqslant 4$. Every infinite virtually cyclic subgroup of $\mathcal{MCG}(\mathbb{S}^2, n)$ is the image under φ of an element of $\mathbb{V}(n)$, and so is an element of $\widetilde{\mathbb{V}}(n)$. Conversely, if G is an element of $\mathbb{V}(n)$ that satisfies the conditions of Theorem 5(2) then $\varphi(G)$ is an infinite virtually cyclic subgroup of $\mathcal{MCG}(\mathbb{S}^2, n)$.*

In Proposition 81, we prove a result similar to that of Proposition 11 for the Type II subgroups of $\mathcal{MCG}(\mathbb{S}^2, n)$ that appear in Definition 13(2), namely that there is a single isomorphism class for such groups, with the exception of the amalgamated product $\mathrm{Dih}_8 *_{\mathrm{Dih}_4} \mathrm{Dih}_8$, for which there are exactly two isomorphism classes. In an analogous manner to that of $B_n(\mathbb{S}^2)$, if n is even then Proposition 83 shows that each of these two classes is realised as a subgroup of $\mathcal{MCG}(\mathbb{S}^2, n)$, with the possible exception of the second isomorphism class when n belongs to $\{6, 14, 18, 26, 30, 38\}$.

As we mentioned previously, the real projective plane $\mathbb{R}P^2$ is the only other surface whose braid groups have torsion. In light of the results of this paper, it is thus natural to consider the problem of the classification of the virtually cyclic subgroups of $B_n(\mathbb{R}P^2)$ up to isomorphism. This is the subject of work in progress [49]. The first step, the classification of the finite subgroups of $B_n(\mathbb{R}P^2)$, was carried out in [50, Theorem 5]. As in this paper, the classification of the infinite virtually cyclic subgroups of $B_n(\mathbb{R}P^2)$ is rather more difficult than in the finite case, but the combination of [50, Corollary 2], which shows that $B_n(\mathbb{R}P^2)$ embeds in $B_{2n}(\mathbb{S}^2)$, with Theorem 5 should be helpful in this respect.

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