# Preface

This book grew out of several courses given at Karl-Franzens-Universität Graz in 2008–2012 and has since been augmented with additional material. It should be of interest both to people new to the field variously known as Additive Number Theory, Additive Combinatorics, Additive Group Theory and Combinatorial Number Theory—as a basic introduction to the area—as well as to the more seasoned researcher, in view of the unified presentation of material previously only available in research articles combined with a fair amount of new material. As there seems no real consensus on whether additive problems belong to Combinatorics, Number Theory, Group Theory, or even Analysis, as they often find themselves lying between all these more established areas of mathematics, we will refer to the broader area of mathematics dealt with by this book simply as Additive Theory.

In recent years, the first few comprehensive texts on what has become a rapidly developing subject have begun to be published in Additive Theory. With so few texts on the subject, I have made little attempt to compete with these already established works. Indeed, the focus of this text is specifically on those areas of Additive Theory that have not been treated in detail by previous books, and even when treating more basic results also found in other texts, I have endeavored to present such results either in greater generality or with new proof variations.

Rather than focus on the great achievements in approximate results—such as Freiman's Theorem, Szemeredi's Theorem, and results achieved via Fourier Analytic/Ergodic Theory breakthroughs—results that require little hypotheses but, at this price, yield only rough results with imprecise constants that leave much room for future improvement, I have instead focused on the more exact and refined results in the area, results which have some satisfactory air of completeness to themselves, even if they may yet one day fit into a wider landscape. This reflects both a personal bias as well as the aforementioned wish to complement, and not compete with, the current material available.

On top of this, I have not shied away from presenting important results even when they have notoriously complicated proofs. There is little prerequisite for this book apart from a solid background in undergraduate mathematics, particularly in the theory of finitely generated abelian groups and Linear Algebra, which makes the material suitable for a graduate or advanced undergraduate course. Knowledge of the basic algebraic concepts of groups, rings, fields, vector spaces and modules is assumed, though very little beyond the definitions and basics would be needed. The main prerequisite might best be described as a fair amount of mathematical maturity. I suspect that the intricacy of the more difficult proofs treated here may be the like of which a less experienced student has not yet seen. For this reason, I have taken great care to include as many details as reasonable, many more than is common in other texts, when presenting each and every proof. While this may expand the length of each proof in print, I hope that it will also reduce the time needed to absorb the complicated combinatorial arguments that often arise. Many of the chapters are self contained or come in a series of interconnected topics, which gives the lecturer a bit of freedom in the selection of chapters to present. Unlike many other texts, if a theorem is not quoted in a given section, it is not used, which should make it quite easy for a lecturer to peruse the material and decide what to present.

While many mathematical subjects are marked by a slew of tiny lemmas and propositions slowly building up sufficient machinery to begin tackling larger problems, Additive Theory is marked by having its fundamental results more like solitary columns, each of which must be summitted by whatever creative means is available. Once enough such results are proven, it slowly becomes possible to build up the more general theory, a process which is still underway, to solve other problems in, and out of, the field. For this text, I have often chosen to present the more foundational results, rather than their applications, collecting together those results which can be used to prove other results in the field and further. Putting the theorems into practice has been left mostly, though not entirely, to the exercises. All this results in a slightly different structure, namely, many chapters are devoted to proving one or two key results rather than presenting related results on a common theme. The thematic grouping occurs at a much higher level, and the book can be divided into three major parts: one dealing with sumsets of sets, one with subsequence sums of sequences, and one introducing some of the more advanced methods from the field. For the latter, a fair amount of linear algebra and the occasional other random subject is helpful. However, while I have not endeavored to prove every basic result used in the proofs, I have attempted to make it clear exactly which result is being used at any given point. Thus, a student less familiar with these more advanced topics can simply take the prerequisites here as given, and this may actually facilitate their absorbtion when seen in a later course on the subject. Regardless, this is a rare occurrence, and the few non-self contained pre-requisites for this final part can be quickly reviewed on the side if needed.

For the seasoned researcher, there is quite a bit more in this text than you might expect. Indeed, many chapters contain entire new proofs, some more major than others, for classical results. At other times, older results have been given slight generalizations, and many new results not found in other texts have been collected here. The following are some highlights:

- An entire extended theory of Freiman homomorphisms, developed not just for $n$-fold sumsets $nA$, but also for sumsets having distinct summands $\sum_{i=1}^{n} A_i$. Weaker, more flexible forms of restricted homomorphisms are also treated. The universal ambient group is introduced in normalized form, an upper bound on its torsion subgroup is given, several short exact sequences are derived, and its value is explicitly calculated for small torsion-free sumsets (below the $3k-4$ bound) and for general sumsets below the Cauchy-Davenport bound.
- The entire theory of finite sumsets is extended to include certain infinite summands. This novel viewpoint gives a natural way to view containment by, and containment of, arithmetic progressions as two cases of the very same phenomenon.
- An entire chapter devoted to the basics of the Isoperimetric method (often neglected in other texts), including applications to direct additive questions involving Sidon sets showing that a Sidon set (from an arbitrary abelian group) must have large sumset with any other set.
- A simplified proof of Kemperman's Structure Theorem—which is often avoided solely owing to its complexity—based off the recent ideas used to extend it.
- The DeVos-Goddyn-Mohar Theorem presented in full with a new proof variation.
- A unified presentation of the Partition Theorem (currently spread across several research articles) including a strengthened form that implies and generalizes the subsequence sum case of the DeVos-Goddyn-Mohar Theorem. A general weighted version valid for $R$-modules (and more general homomorphism weights) is also presented.
- The recent partial generalization of Pollard's Theorem to General Abelian groups, as well as related progress by Hamidoune and Serra, is given with improved bounds.
- The $3k-4$ Theorem for distinct summands in $\mathbb{Z}$ is given in a slightly more general form and also includes the recent complementary result concerning containment of long arithmetic

progressions in the sumset. A more general version of the technique of modular reduction is included.
- The $3k-4$ Theorem in $C_p$, with $p$ prime, is given in several new variations in which the logarithmic (resp. linear) restriction is shifted from the cardinalities of the sets onto the 'additive constant' $|A+B|-|A|-|B|+1$. In particular, no assumption on the relative size of $|A|$ versus $|B|$ is required.
- Recent precise bounds for a sumset being isomorphic to a torsion-free sumset are given as a simple consequence of the bound on the torsion subgroup of the universal ambient group.
- Snevily's Conjecture, only recently proven, is included.
- The Savchev-Chen Structure result for long zero-sum free sequences is also included.
- Weighted subsequence sums are treated in the general context, recently introduced by Yuan and Zheng, of using homomorphisms as weights.
- The Combinatorial Nullstellensatz is included along with the recently proven punctured version.

There is, of course, much more in the text, but for this, the reader should now press ahead into the main bulk of the text. Worth noting is that definitions for symbolic notation can be found via the index, though most recurring notation is introduced in Chapters 1, 2 and 10, with some notable exceptions in Section 4.3 (stabilizers and periodicity), Section 7.2 (relative complements), and Section 13.1 (additional setpartition notation). As regards suggestions for using the text in a course, there are several possibilities. Regardless of what is covered, it will be helpful to review the notational conventions contained in the just mentioned chapters/sections. Part I can be used simply as an introductory course in basic structural sumset results. Alternatively, the material from Part I can be treated much more rapidly by presenting only the finite sumset cases of results (thus skipping Chapters 3 and 4) and avoiding certain purposely placed redundancies. For instance, the Multiplicity and Pigeonhole Bounds as well as Vosper's Theorem (the bulk of Chapters 5 and 8) can be derived as respective consequences of Kneser's Theorem and the Kemperman Structure Theorem. They have been given separate proofs both to highlight the importance of these special cases and the methods used there as well as to gradually accustom the student to the complexities of the more general results. For students already familiar with basic structural sumset results (either by prerequisite or during the second half of a course), the chapters from Parts II and III can combined in most any way. However, certain chapters fit well in succession. For instance, Chapters 10 and 13–16 work well together as do Chapters 7–8 and 19–21. In fact, if an instructor is somewhat careful about which topics from Parts II and III are chosen, then only a minimal amount of Part I need be covered beforehand (Kneser's Theorem being rather indispensable, and Chapter 7 and Section 8.1 needed if the build up to the $3k-4$ Theorem for $C_p$ is to be followed through Chapters 19–21), giving another possibility for an introductory course using the material from this text. Alternatively, a course focussed on Freiman Homomorphisms can be constructed from Chapters 1–3, 6–7, 9 and 20, with extra material regarding homomorphisms for infinite sumsets also in Chapter 4.

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Graz, Austria David J. Grynkiewicz

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