Chapter 2
Theory of Thin Plates for Laminates

Thin isotropic, orthotropic or laminate plates with constant or widthwise variable material properties are considered in this monograph. Thin-walled beam-columns or girders composed of the above-mentioned plates are also analysed. In order to take into account all buckling modes (global, local and their interaction), the plate two-dimensional theory has been adopted to model the structures under analysis.

2.1 Basic Assumptions

Basic assumptions for thin plates were given by Kirchhoff for the linear classical thin plate theory (CPT) and by von Kármán and Marquerre for the nonlinear CPT. They made their assumptions for isotropic materials. Numerous authors have extended those assumptions for orthotropic or even for composite multilayer thin plates [1, 5]. The assumptions are as follows:

- the plate is homogeneous (for example, orthotropic homogenisation is made for a fibre composite—resin matrix and fibre-reinforcement);
- the plate is thin—other dimensions (length and width) are at least 10 times higher than the plate thickness;
- the material of the plate is deformable and it is subjected to Hooke's law;
- the plane stress state is considered for the plate—the stress acting in the plate plane dominates the plate behaviour, stresses acting in the direction normal to the plate plane are assumed to be zero;
- all strains (normal and shear) in the plate plane are low compared to unity and they are linear;
- the strains normal to the plate mid-surface are neglected (the plate thickness does not change after deformation)—this assumption is made according to Kirchhoff-Love;

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- the straight lines normal to the mid-surface of the plate remain straight and normal to the mid-surface after deformation;
- there are no interactions in the normal direction between layers parallel to the middle surface;
- deflections of the plate can be considered in terms of nonlinear geometrical relations.

Additionally, it is assumed that the principal axes of orthotropy do not need to be parallel to the edges of analysed structures (a plate, a beam, column, a beam-column or a girder).

2.2 Geometrical Equations for Thin Plates

A two dimensional model of the plate has been assumed for thin plates and thin-walled beam-columns or girders. For a simpler description, a single plate (Fig. 2.1a) or each *i*-th strip (Fig. 2.1b) of the plate (or a wall of the girder) or each *i*-th wall of the girder (Fig. 2.1c) is referred to as a plate [3, 4].

To describe the middle surface strains for each plate, a complete strain tensor, i.e., with all nonlinear terms, has been assumed [1, 2]:

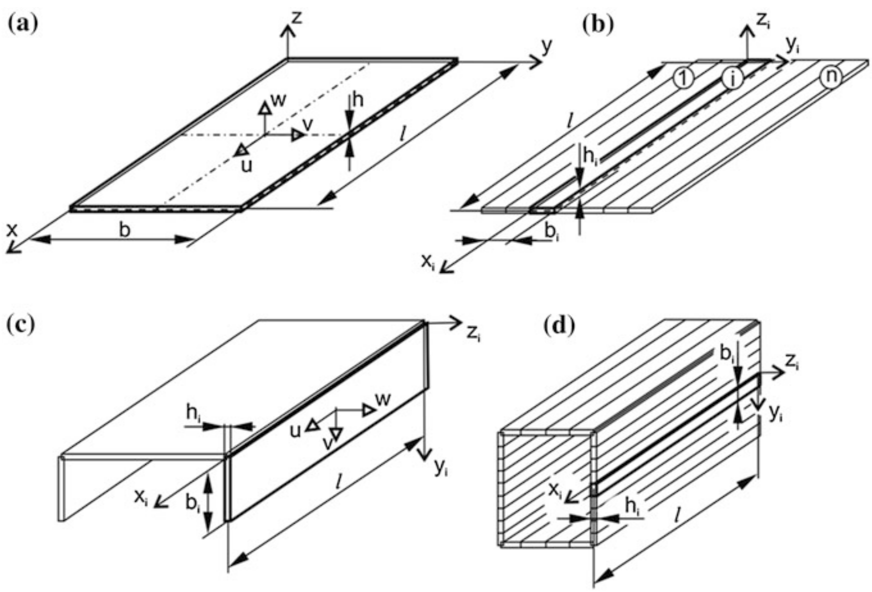

Fig. 2.1 2D plate model for plates and girders with the assumed coordinate system

$$\begin{aligned}
\varepsilon_{ix}^{m} &= \frac{\partial u_i}{\partial x_i} + \frac{1}{2}\left(\frac{\partial w_i}{\partial x_i}\right)^2 + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_i}\right)^2 + \frac{1}{2}\left(\frac{\partial v_i}{\partial x_i}\right)^2, \\
\varepsilon_{iy}^{m} &= \frac{\partial v_i}{\partial y_i} + \frac{1}{2}\left(\frac{\partial w_i}{\partial y_i}\right)^2 + \frac{1}{2}\left(\frac{\partial u_i}{\partial y_i}\right)^2 + \frac{1}{2}\left(\frac{\partial v_i}{\partial y_i}\right)^2, \\
2\varepsilon_{ixy}^{m} &= \gamma_{ixy}^{m} = \frac{\partial u_i}{\partial y_i} + \frac{\partial v_i}{\partial x_i} + \frac{\partial w_i}{\partial x_i}\frac{\partial w_i}{\partial y_i} + \frac{\partial u_i}{\partial x_i}\frac{\partial u_i}{\partial y_i} + \frac{\partial v_i}{\partial x_i}\frac{\partial v_i}{\partial y_i},
\end{aligned} \tag{2.1}$$

or in a shorter form:

$$\begin{aligned}
\varepsilon_{ix}^{m} &= u_{i,x} + \frac{1}{2}(w_{i,x}^2 + u_{i,x}^2 + v_{i,x}^2), \\
\varepsilon_{iy}^{m} &= v_{i,y} + \frac{1}{2}(w_{i,y}^2 + u_{i,y}^2 + v_{i,y}^2), \\
\gamma_{ixy}^{m} &= u_{i,y} + v_{i,x} + w_{i,x}w_{i,y} + u_{i,x}u_{i,y} + v_{i,x}v_{i,y},
\end{aligned} \tag{2.2}$$

where: u_i, v_i, w_i are displacements parallel to the respective axes x_i, y_i, z_i of the local Cartesian system of co-ordinates, whose plane x_iy_i coincides with the middle surface of the *i*-th plate before its buckling (Fig. 2.1).

In the majority of publications devoted to the structure stability, the terms$(u_{i,x}^2 + v_{i,x}^2)$, $(u_{i,y}^2 + v_{i,y}^2)$ and $(u_{i,x}u_{i,y} + v_{i,x}v_{i,y})$, i.e., the strain tensor components in (2.2), in general are neglected for ε_{ix}, ε_{iy}, γ_{ixy}, respectively.

The changes in the bending and twisting curvatures of the middle surface are assumed according to [6, 7] as follows:

$$\begin{aligned}
\kappa_{ix} &= -\frac{\partial^2 w_i}{\partial x_i^2} = -w_{i,xx}, \\
\kappa_{iy} &= -\frac{\partial^2 w_i}{\partial y_i^2} = -w_{i,yy}, \\
\kappa_{ixy} &= -\frac{\partial^2 w_i}{\partial x_i \partial y_i} = -w_{i,xy}.
\end{aligned} \tag{2.3}$$

The geometrical relationships given by Eqs. (2.2) and (2.3) allow one to consider both out-of-plane and in-plane bending of the plate.

For the laminated plate (Fig. 2.2), where there is a *p* number of plies, the strains of the *k*-th ply can be related to the strains and the curvatures of the middle surface of the laminate at $z = 0$ in the form [1]:

$$\{\bar{\varepsilon}\} = \begin{Bmatrix} \varepsilon_{ix} \\ \varepsilon_{iy} \\ \gamma_{ixy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{ix}^{m} \\ \varepsilon_{iy}^{m} \\ \gamma_{ixy}^{m} \end{Bmatrix} + z\begin{Bmatrix} \kappa_{ix} \\ \kappa_{iy} \\ 2\kappa_{ixy} \end{Bmatrix}, \tag{2.4}$$

where $z_{k-1} \le z \le z_k$ (Fig. 2.2) and for $k = p = 1$, $z0 = -h/2$, $z_1 = h/2$.

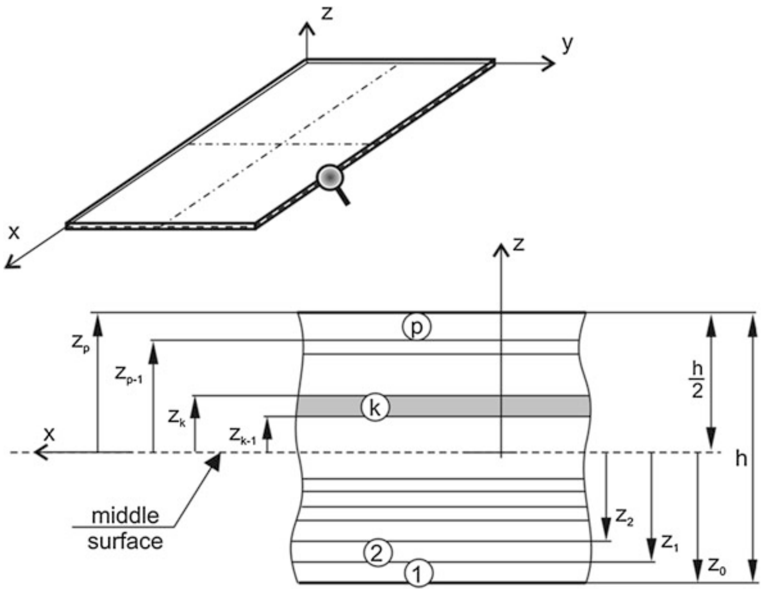

Fig. 2.2 Assumed coordinate system for the layered plate

2.3 Constitutive Equations for Laminates

Let us consider one rectangular ply of the laminate with principal axes of orthotropy *1* and *2* parallel to ply edges (Fig. 2.3).

Similarly as in the previous paragraph, let us consider an *i*-th plate or strip of the structure under analysis. The stress-strain relationship for such a plate is the same as for an orthotropic plate and can be written in the following form [1]:

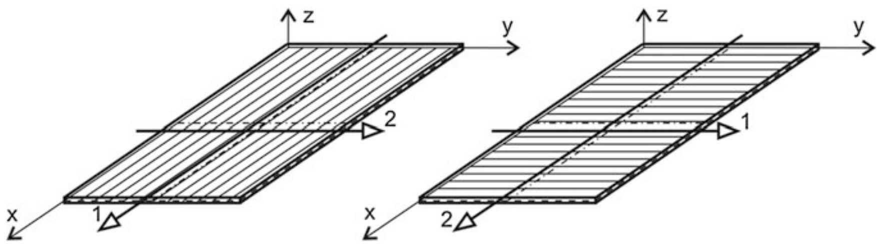

Fig. 2.3 Principal axes of orthotropy for lamina

$$\begin{aligned}\sigma_{i1} &= \frac{E_{i1}}{1 - \nu_{i12}\nu_{i21}}(\varepsilon_{i1} + \nu_{i21}\varepsilon_{i2}),\\ \sigma_{i2} &= \frac{E_{i2}}{1 - \nu_{i12}\nu_{i21}}(\nu_{i12}\varepsilon_{i1} + \varepsilon_{i2}),\\ \tau_{i12} &= G_{i12}\gamma_{i12} = 2G_{i12}\varepsilon_{i12},\end{aligned} \tag{2.5}$$

where E_{i1}, E_{i2} is the Young's modulus in longitudinal *1* and transverse *2* direction, correspondingly; ν_{i12} is the Poisson's ratio for which strains are in longitudinal direction *1* and stress in transverse direction *2*, G_{i12} is the shear modulus (Kirchhoff's modulus) in plane *12*.

In further equations in this section, the subscript i denoting an i-th plate or strip is omitted because all the equations presented correspond to one plate or strip only.

Equations (2.5) written in a matrix form are as follows:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}, \tag{2.6}$$

or in a more convenient form:

$$\{\sigma\} = [Q]\{\varepsilon\}, \tag{2.7}$$

where:

$$\begin{aligned}Q_{11} &= \frac{E_1}{1 - \nu_{12}\nu_{21}},\\ Q_{12} &= Q_{21} = \nu_{21}\frac{E_1}{1 - \nu_{12}\nu_{21}} = \nu_{12}\frac{E_2}{1 - \nu_{12}\nu_{21}},\\ Q_{22} &= \frac{E_2}{1 - \nu_{12}\nu_{21}},\\ Q_{66} &= G_{12}.\end{aligned} \tag{2.8}$$

The Young's modulus and Poisson's ratios occurring in (2.5) and (2.8) according to the Betty-Maxwell theorem or according to the symmetry condition of the stress tensor ($Q_{12} = Q_{21}$) should fulfil the following relation:

$$E_1\nu_{21} = E_2\nu_{12}. \tag{2.9}$$

Fibres in individual plies of laminates are arranged at different angles to the plate edges. It means that the principal axes of orthotropy are rotated at an angle θ in relation to the coordinate system adopted for the entire plate (Fig. 2.4).

For the plate presented in Fig. 2.4, stress-strain relationship (2.5) should be transformed from the local *1-2* coordinate system to the global *xy* one. The constitutive equations in the local coordinate system are given by (2.6), and in the global coordinates, they can be written as follows:

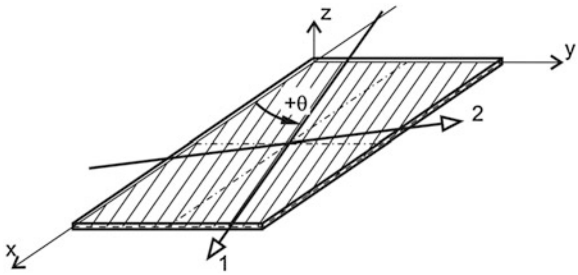

Fig. 2.4 Fibre orientation in the composite ply

$$\left\{\begin{array}{c}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{array}\right\}=\begin{bmatrix}\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16}\\ \overline{Q}_{21} & \overline{Q}_{22} & \overline{Q}_{26}\\ \overline{Q}_{61} & \overline{Q}_{62} & \overline{Q}_{66}\end{bmatrix}\left\{\begin{array}{c}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{array}\right\}, \tag{2.10}$$

or shorter:

$$\{\overline{\sigma}\}=\left[\overline{Q}\right]\{\overline{\varepsilon}\}, \tag{2.11}$$

where the elements of the elasticity matrix $\left[\overline{Q}\right]$are expressed by material properties (E_1, E_2 v_{12} and G_{12}) and the angle of the declination θ between the global and local coordinate systems. The relation between the elasticity matrix in local [Q] and global $\left[\overline{Q}\right]$ coordinate systems can be derived taking into account the relation between stresses in both coordinate systems and strains in both the systems. The stress transformation equations are as follows:

$$\begin{aligned}\sigma_1&=\sigma_x\cos^2\theta+\sigma_y\sin^2\theta+2\tau_x\cos\theta\sin\theta,\\ \sigma_2&=\sigma_x\sin^2\theta+\sigma_y\cos^2\theta-2\tau_x\cos\theta\sin\theta,\\ \tau_{12}&=-\sigma_x\cos\theta\sin\theta+\sigma_y\cos\theta\sin\theta+\tau_x(\cos^2\theta-\sin^2\theta),\end{aligned} \tag{2.12}$$

or in the matrix form:

$$\left\{\begin{array}{c}\sigma_1\\ \sigma_2\\ \tau_{12}\end{array}\right\}=\begin{bmatrix}c^2 & s^2 & 2cs\\ s^2 & c^2 & -2cs\\ -cs & cs & c^2-s^2\end{bmatrix}\left\{\begin{array}{c}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{array}\right\}, \tag{2.13}$$

or shorter:

$$\{\sigma\}=[T]\{\overline{\sigma}\}, \tag{2.14}$$

where: $\{\sigma\}$, $\{\overline{\sigma}\}$ are vectors of stresses in the local and global coordinate systems, correspondingly, $c=\cos\theta$, $s=\sin\theta$ and [T] is the transformation matrix. To find the stress in the global coordinate system having the stress in the coordinate system corresponding to the principle axes of orthotropy, the following relation should be used:

$$\{\overline{\sigma}\}=[T]^{-1}\{\sigma\}, \tag{2.15}$$

For strains, the transformation equations can be written as follows:

$$\begin{aligned}
\varepsilon_1 &= \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + 2\varepsilon_{xy}\cos\theta\sin\theta,\\
\varepsilon_2 &= \varepsilon_x \sin^2\theta + \varepsilon_y \cos^2\theta - 2\varepsilon_{xy}\cos\theta\sin\theta,\\
\varepsilon_{12} &= -\frac{\gamma_{12}}{2} = -\varepsilon_x\cos\theta\sin\theta + \varepsilon_y\cos\theta\sin\theta + \varepsilon_{xy}(\cos^2\theta - \sin^2\theta),
\end{aligned} \tag{2.16}$$

or in the matrix form:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2 - s^2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix}, \tag{2.17}$$

or

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_{12} \end{Bmatrix} = [T]\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} = [T]\{\bar{\varepsilon}\}. \tag{2.18}$$

From the theory of elasticity, it is well known that strain and stress transformations are made in the same way—the transformation matrix [T] is the same (see (2.13) and (2.17)). It should be noted that in (2.16)–(2.18), the shear strains ε_{12}, ε_{xy} in the strain vectors appear instead of the shear angles γ_{12}, γ_{xy}, which are used in constitutive equations (2.6) and (2.10). The relation between strain vectors including the shear strain or the shear angle can be written as follows:

$$\begin{aligned}
\{\varepsilon\} &= \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_{12} \end{Bmatrix} = [R]\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_{12} \end{Bmatrix},\\
\{\bar{\varepsilon}\} &= \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} = [R]\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix}.
\end{aligned} \tag{2.19}$$

Now, using (2.18) and (2.19), the strain transformation can be written as follows:

$$\{\varepsilon\} = [R]\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_{12} \end{Bmatrix} = [R][T]\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} = [R][T][R]^{-1}\{\bar{\varepsilon}\}. \tag{2.20}$$

However, $[R][T][R]^{-1}$ can be shown to be $[T]^{-T}$, then (2.20) has the form:

$$\{\varepsilon\} = [T]^{-T}\{\bar{\varepsilon}\}. \tag{2.21}$$

Substituting the constitutive equations in local coordinate system (2.7) into stress transformation equations (2.15), the following is obtained:

$$\{\overline{\sigma}\} = [T]^{-1}[Q]\{\varepsilon\}. \tag{2.22}$$

After substituting the strain transformation, i.e., (2.21) into (2.22), we obtain the following relation:

$$\{\overline{\sigma}\} = [T]^{-1}[Q][T]^{-T}\{\overline{\varepsilon}\}, \tag{2.23}$$

which is a constitutive equation in the global coordinate system. Comparing (2.23) and (2.11), one obtains the elasticity matrix transformation:

$$[\overline{Q}] = [T]^{-1}[Q][T]^{-T}, \tag{2.24}$$

or in the full form:

$$\begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{21} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{61} & \overline{Q}_{62} & \overline{Q}_{66} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ cs & -cs & c^2-s^2 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} c^2 & s^2 & cs \\ s^2 & c^2 & -cs \\ -2cs & 2cs & c^2-s^2 \end{bmatrix}. \tag{2.25}$$

Taking into account (2.25), all elements of the elasticity matrix $[\bar{Q}]$ in the global coordinate system can be calculated and they are as follows:

$$\begin{aligned}
\overline{Q}_{11} &= Q_{11}c^4 + 2(Q_{12} + 2Q_{66})c^2s^2 + Q_{22}s^4, \\
\overline{Q}_{12} &= \overline{Q}_{21} = (Q_{11} + Q_{22} - 4Q_{66})c^2s^2 + Q_{12}(c^4 + s^4), \\
\overline{Q}_{22} &= Q_{11}s^4 + 2(Q_{12} + 2Q_{66})c^2s^2 + Q_{22}c^4, \\
\overline{Q}_{16} &= \overline{Q}_{61} = (Q_{11} - Q_{12} - 2Q_{66})c^3s + (Q_{12} - Q_{22} + 2Q_{66})cs^3, \\
\overline{Q}_{26} &= \overline{Q}_{62} = (Q_{11} - Q_{12} - 2Q_{66})cs^3 + (Q_{12} - Q_{22} + 2Q_{66})c^3s, \\
\overline{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})c^2s^2 + Q_{66}(c^4 + s^4).
\end{aligned} \tag{2.26}$$

Summing the above, the constitutive equations can be rewritten as follows:

- for the k-th ply of laminates:

$$\begin{aligned}
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k &= \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{21} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{61} & \overline{Q}_{62} & \overline{Q}_{66} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_k \\
&= \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{21} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{61} & \overline{Q}_{62} & \overline{Q}_{66} \end{bmatrix}_k \left\{ \begin{Bmatrix} \varepsilon_x^m \\ \varepsilon_y^m \\ \gamma_{xy}^m \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ 2\kappa_{xy} \end{Bmatrix} \right\},
\end{aligned} \tag{2.27}$$

It should be noted that for each ply, the elasticity matrix $[\overline{Q}]_k$ can be different, then the stress can vary through the thickness of the laminate, not necessarily linearly, as it is in the case of strain (if all plies are in the elastic range).

- for the orthotropic plate with the principal axes of orthotropy parallel to the plate (strip) edges:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{E_x}{1-\nu_{xy}\nu_{yx}} & \nu_{yx}\frac{E_x}{1-\nu_{xy}\nu_{yx}} & 0 \\ \nu_{xy}\frac{E_y}{1-\nu_{xy}\nu_{yx}} & \frac{E_y}{1-\nu_{xy}\nu_{yx}} & 0 \\ 0 & 0 & G_{xy} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}, \tag{2.28}$$

- for the isotropic plate (wall of beam-columns):

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}. \tag{2.29}$$

2.4 Generalized Sectional Forces

By generalized sectional forces are meant here sectional forces and moments (Fig. 2.5) dependent on the stress in the section under consideration.

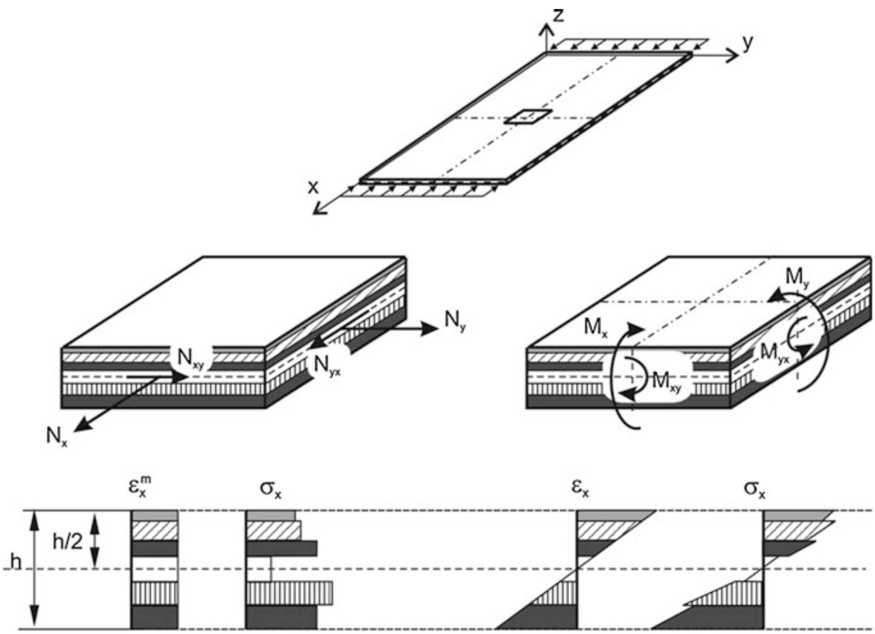

Fig. 2.5 Sectional forces and strains and a stress distribution in laminates

In the case of an isotropic plate or an orthotropic plate with the principal axes of orthotropy parallel to plate edges, the resultant moments and forces can be calculated as follows:

$$\begin{gathered}\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \{N_i\} = \int_{-\frac{h_i}{2}}^{-\frac{h_i}{2}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} dz, \\ \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \{M_i\} = \int_{-\frac{h_i}{2}}^{-\frac{h_i}{2}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} z dz,\end{gathered} \tag{2.30}$$

where $N_{\mathrm{xy}} = N_{\mathrm{yx}}$ and $M_{\mathrm{xy}} = M_{\mathrm{yx}}$.

Substituting the stress-strain relations from the previous sections (Sect. 2.3), the sectional moments and forces:

- for the i-th isotropic strip or wall of the beam-column are expressed by:

$$\begin{gathered}\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \frac{Eh}{1 - v^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^m \\ \varepsilon_y^m \\ \gamma_{xy}^m \end{Bmatrix}, \\ \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = D \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & 1 - v \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix},\end{gathered} \tag{2.31}$$

where: $D = \frac{Eh^3}{12(1-v^2)}$

- for the i-th orthotropic strip or wall, they are:

$$\begin{gathered}\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \frac{h}{1 - v_{xy} v_{yx}} \begin{bmatrix} E_x & v_{yx} E_x & 0 \\ v_{xy} E_y & E_y & 0 \\ 0 & 0 & \left(1 - v_{xy} v_{yx}\right) G_{xy} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^m \\ \varepsilon_y^m \\ \gamma_{xy}^m \end{Bmatrix}, \\ \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_x & v_{yx} D_x & 0 \\ v_{xy} D_y & D_y & 0 \\ 0 & 0 & D_{xy} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix},\end{gathered} \tag{2.32}$$

where: $D_x = \frac{E_x h^3}{12\left(1-v_{xy} v_{yx}\right)}$, $D_y = \frac{E_y h^3}{12\left(1-v_{xy} v_{yx}\right)}$, $D_{xy} = \frac{G_{xy} h^3}{6}$.

For laminates which are composed of many plies with different orientation and/ or different material properties, stress tensors for each layer can be different. Due to the above-mentioned differences, the resultant moments and forces acting on the laminate should be calculated as a sum of integrals of the stress for all laminate plies, taken in the following manner:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \sum_{k=1}^{p} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k dz$$
$$= \sum_{k=1}^{p} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{21} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{61} & \overline{Q}_{62} & \overline{Q}_{66} \end{bmatrix}_{ik} \left[\int_{z_{k-1}}^{z_k} \begin{Bmatrix} \varepsilon_x^m \\ \varepsilon_y^m \\ \gamma_{xy}^m \end{Bmatrix} dz + \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ 2\kappa_{xy} \end{Bmatrix} zdz \right], \tag{2.33}$$

and

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \sum_{k=1}^{p} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k zdz$$
$$= \sum_{k=1}^{p} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{21} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{61} & \overline{Q}_{62} & \overline{Q}_{66} \end{bmatrix}_{ik} \left[\int_{z_{k-1}}^{z_k} \begin{Bmatrix} \varepsilon_x^m \\ \varepsilon_y^m \\ \gamma_{xy}^m \end{Bmatrix} zdz + \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ 2\kappa_{xy} \end{Bmatrix} z^2dz \right]. \tag{2.34}$$

In the above equations, the stiffness matrices $[\overline{Q}]_k$ are outside the integral over each layer because their elements are constant across the thickness of every particular layer.

As we know, all strains ε and all curvatures κ are not functions of z, but they refer to the middle surface, so they can be drawn outside the summation signs. Thus, (2.33) and (2.34) can be written as:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{21} & A_{22} & A_{26} & B_{21} & B_{22} & B_{26} \\ A_{61} & A_{62} & A_{66} & B_{61} & B_{62} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{21} & B_{22} & B_{26} & D_{21} & D_{22} & D_{26} \\ B_{61} & B_{62} & B_{66} & D_{61} & D_{62} & D_{66} \end{bmatrix}_i \begin{Bmatrix} \varepsilon_x^m \\ \varepsilon_y^m \\ \gamma_{xy}^m \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}, \tag{2.35}$$

or in a more convenient form:

$$\begin{Bmatrix} \{N\} \\ \{M\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{Bmatrix} \{\varepsilon^m\} \\ \{\kappa\} \end{Bmatrix}, \tag{2.36}$$

where:

$$\begin{aligned} A_{pq} &= \sum_{k=1}^{n} \left(\overline{Q}_{pq}\right)_k (z_k - z_{k-1}), \\ B_{pq} &= \frac{1}{2} \sum_{k=1}^{n} \left(\overline{Q}_{pq}\right)_k \left(z_k^2 - z_{k-1}^2\right), \\ D_{pq} &= \frac{1}{3} \sum_{k=1}^{n} \left(\overline{Q}_{pq}\right)_k \left(z_k^3 - z_{k-1}^3\right), \end{aligned} \tag{2.37}$$

and

$$A_{pq} = A_{qp},\ B_{pq} = B_{qp},\ D_{pq} = D_{qp}. \tag{2.38}$$

In (2.36), the sub-matrix [A] is an extensional stiffness matrix, [D] is a bending stiffness matrix, and [B] is a bending-extension coupling the stiffness matrix. If all elements of the sub-matrix [B] are not equal to zero, then, for example, the deformation of the laminate subjected to tension load is not only extension but also bending and/or twisting (Fig. 2.6a). Another example for non-zero sub-matrix [B] elements is such that during bending the laminate is bent and also suffers from extension of the middle surface (Fig. 2.6b).

The laminates can represent a few special cases of layers alignment and these are:

- symmetrical structure—the laminate consists of an even number of layers arranged symmetrically about the middle surface. Thus, the following stiffness matrix elements are equal to zero:

$$B_{pq} = A_{16} = A_{61} = A_{26} = A_{62} = D_{16} = D_{61} = D_{26} = D_{62} = 0, \tag{2.39}$$

- regular symmetric cross-ply laminate—called a quasi-orthotropic material—the laminate consists of an odd number of layers arranged symmetrically with respect to the middle surface and the lamina pairs are oriented in such a way that the principal material direction corresponds to plate edges (plies are oriented at 0 or 90 degrees to the longitudinal direction of the considered plate). For such a laminate, the following elements of the stiffness matrix are equal to zero:

$$B_{pq} = A_{16} = A_{61} = A_{26} = A_{62} = D_{16} = D_{61} = D_{26} = D_{62} = 0, \tag{2.40}$$

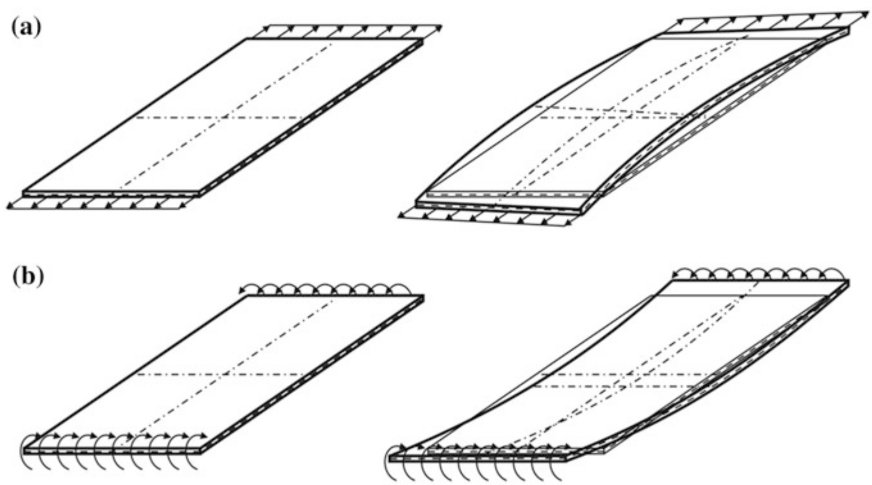

Fig. 2.6 Possible deflection of non-quasi-isotropic laminates

- regular symmetric angle-ply laminate—the laminate consists of an odd number of plies with equal thickness and principal material properties of each layers are arranged with opposite signs of the angle of orientation (for example, [+ θ/−θ/ + θ]). In this case, the following elements of the stiffness matrix diminish:

$$B_{pq} = 0, \tag{2.41}$$

- antisymmetric cross-ply laminate—the laminate consists of an even number of layers of the same thickness, laid on each other with the principal axes of orthotropy alternating at 0 and 90 degrees to the laminate axes. The known information about elements of the stiffness matrix is as follows:

$$\begin{aligned} &A_{16} = A_{61} = A_{26} = A_{62} = 0, \\ &D_{16} = D_{61} = D_{26} = D_{62} = 0, \\ &B_{12} = B_{21} = B_{16} = B_{61} = B_{26} = B_{62} = B_{66} = 0, \\ &B_{11} = -B_{22} \neq 0, \end{aligned} \tag{2.42}$$

- antisymmetric angle-ply laminate—the laminate consists of an even number of layers of the same thickness and its plies are arranged in pairs at an angle +θ and −θ, respectively. The elements of the stiffness matrix with the zero value are as follows:

$$\begin{aligned} &A_{16} = A_{61} = A_{26} = A_{62} = 0, \\ &D_{16} = D_{61} = D_{26} = D_{62} = 0, \\ &B_{11} = B_{12} = B_{21} = B_{22} = B_{66} = 0. \end{aligned} \tag{2.43}$$

The examples taken for calculation and presented in this study are obtained only for isotropic structures, orthotropic structures with the principal axes of orthotropy parallel to edges of structures and laminates with a symmetrical arrangement of the layers. For all above-mentioned cases, the expression describing sectional moments and forces can be written as:

$$\begin{aligned} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} &= \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^m \\ \varepsilon_y^m \\ \gamma_{xy}^m \end{Bmatrix}, \\ \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} &= \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{21} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}, \end{aligned} \tag{2.44}$$

where all A_{pq} and D_{pq} should fulfil (2.38) and for:

- symmetric laminates, are described by (2.37),
- orthotropic plates (or strips), they are:

$$
\begin{aligned}
&A_{11} = \frac{E_x h}{1 - \nu_{xy}\nu_{yx}}, && D_{11} = \frac{E_x h^3}{12\left(1 - \nu_{xy}\nu_{yx}\right)},\\
&A_{12} = A_{21} = \frac{\nu_{yx} E_x h}{1 - \nu_{xy}\nu_{yx}} = \frac{\nu_{xy} E_y h}{1 - \nu_{xy}\nu_{yx}}, && D_{12} = D_{21} = \frac{\nu_{yx} E_x h^3}{12\left(1 - \nu_{xy}\nu_{yx}\right)} = \frac{\nu_{xy} E_y h^3}{12\left(1 - \nu_{xy}\nu_{yx}\right)},\\
&A_{22} = \frac{E_y h}{1 - \nu_{xy}\nu_{yx}}, && D_{22} = \frac{E_y h^3}{12\left(1 - \nu_{xy}\nu_{yx}\right)},\\
&A_{66} = G_{xy} h, && D_{66} = \frac{G_{xy} h^3}{6}.
\end{aligned}
\tag{2.45}
$$

- isotropic plates, they are as follows:

$$
\begin{aligned}
&A_{11} = A_{22} = \frac{Eh}{1 - \nu^2}, && D_{11} = D_{22} = \frac{Eh^3}{12(1 - \nu^2)},\\
&A_{12} = A_{21} = \frac{\nu Eh}{1 - \nu^2}, && D_{12} = D_{21} = \frac{\nu Eh^3}{12(1 - \nu^2)},\\
&A_{66} = Gh = \frac{Eh}{2(1 + \nu)}, && D_{66} = \frac{Gh^3}{6} = \frac{Eh^3}{12(1 + \nu)}.
\end{aligned}
\tag{2.46}
$$

2.5 Dynamic Equations of Stability for Thin Orthotropic Plates

Differential equations of motion of the plate have been derived on the basis of the Hamilton's principle. It states that the dynamics of a physical system is determined by a variation problem for the functional based on a single function, the Lagrangian, which contains all physical information concerning the system and the forces acting on it. In the dynamic buckling problem, the motion should be understood as the time dependent deflection.

The Hamilton's principle for conservative systems states that the true evolution (compatible with constrains) of the system between two specific states in a specific time range (t_0, t_1) is a stationary point (a point where the variation is zero) of the action functional Ψ. The action functional Ψ is described by the following equation:

$$
\Psi = \int_{t_0}^{t_1} \Lambda dt = \int_{t_0}^{t_1} (K - \Pi) dt,
\tag{2.47}
$$

where Λ is the Lagrangian function for the system, K is a kinetic energy of the system, and Π is a total potential energy of the system.

Taking the action functional Ψ in form (2.47), the Hamilton's principle can be written as:

$$\delta\Psi = \delta\int_{t_0}^{t_1}\Lambda dt = \delta\int_{t_0}^{t_1}(K-\Pi)dt = 0. \tag{2.48}$$

The total potential energy variation $\delta\Pi$ for the i-th thin plate (or strip) can be written in the form:

$$\delta\Pi = \delta Q - \delta W, \tag{2.49}$$

where δQ is a variation of the internal elastic strain energy:

$$\delta Q = \int_{\Omega}(\sigma_x\delta\varepsilon_x + \sigma_y\delta\varepsilon_y + \tau_{xy}\delta\gamma_{xy})d\Omega, \tag{2.50}$$

where Ω is a volume of the plate and S is its area, thus the volume can be expressed as $\Omega = l{\cdot}b{\cdot}h$ or $\Omega = S{\cdot}h$.

The variation of the internal elastic strain energy for the i-th plate or strip could be expressed by strain and sectional forces and moments in a following way:

$$\begin{aligned}\delta Q =& \delta Q^m + \delta Q^b \\ =& \int_S(N_x\delta\varepsilon_x + N_y\delta\varepsilon_y + N_{xy}\delta\gamma_{xy})dS - \int_S(M_x\delta w_{,xx} + M_y\delta w_{,yy} + 2M_{xy}\delta w_{,xy})dS.\end{aligned} \tag{2.51}$$

The work W of external forces done on the i-th plate can be expressed as:

$$W = \int_0^b h[p^0(y)u + \tau_{xy}^0(y)v]dy + \int_o^{\ell} h[p^0(x)v + \tau_{xy}^0(x)u]dx + \int qwdS, \tag{2.52}$$

if the load perpendicular to the plane of the plate (strip) is neglected, (2.52) can be written as:

$$W = \int_0^b h[p^0(y)u + \tau_{xy}^0(y)v]dy + \int_o^{\ell} h[p^0(x)v + \tau_{xy}^0(x)u]dx, \tag{2.53}$$

where: $p^0(x), p^0(y), \tau_{\mathrm{xy}}^0(x), \tau_{\mathrm{xy}}^0(y)$ are pre-buckling loads applied to the middle surface of the plate (wall or strip) under consideration.

For thin plates, it is assumed that the displacements u and v do not depend on the rotations $w_{,x}$ and $w_{,y}$ and, therefore, do not depend on the coordinate z. This approach results in the exclusion of the rotational inertia [7] from the equation for kinetic energy which for the i-th thin plate (strip) can be written as:

$$K = \frac{1}{2}\rho \int\limits_{\Omega} \left((\dot{u})^2 + (\dot{v})^2 + (\dot{w})^2 \right) d\Omega, \tag{2.54}$$

where the dot denotes differentiation with respect to time.

The Hamilton's principle, it is a variation of the action functional $\delta\Psi$ (2.48) for the i-th thin plate (strip or wall) after taking into consideration from (2.49) to (2.54) and assuming a constant density for all layers ($\rho = \rho_{\mathrm{k}} = \mathrm{const}$), can be written as:

$$\delta\Psi = \int\limits_{t_0}^{t_1} \left(-\delta K + \delta Q^m + \delta Q^b - \delta W \right) dt = 0 \tag{2.55}$$

where:

$$\begin{aligned}
\int\limits_{t_0}^{t_1} \delta Q^m dt = & \int\limits_{t_0}^{t_1} \int\limits_{0}^{b} N_x \delta u dy dt|_{x=const} \\
& - \int\limits_{t_0}^{t_1} \int\limits_{S} N_{x,x} \delta u dS dt + \int\limits_{t_0}^{t_1} \int\limits_{0}^{b} N_x u_{,x} \delta u dy dt|_{x=const} \\
& - \int\limits_{t_0}^{t_1} \int\limits_{S} (N_x u_{,x})_{,x} \delta u dS dt \int\limits_{t_0}^{t_1} \int\limits_{0}^{b} N_x v_{,x} \delta v dy dt|_{x=const} - \int\limits_{t_0}^{t_1} \int\limits_{S} (N_x v_{,x})_{,x} \delta v dS dt \\
& + \int\limits_{t_0}^{t_1} \int\limits_{0}^{b} N_x w_{,x} \delta w dy dt|_{x=const} - \int\limits_{t_0}^{t_1} \int\limits_{S} (N_x w_{,x})_{,x} \delta w dS dt + \int\limits_{t_0}^{t_1} \int\limits_{0}^{b} N_y \delta v dy dt|_{x=const} \\
& - \int\limits_{t_0}^{t_1} \int\limits_{S} N_{y,y} \delta v dS dt + \int\limits_{t_0}^{t_1} \int\limits_{0}^{l} N_y u_{,y} \delta u dx dt|_{y=const} - \int\limits_{t_0}^{t_1} \int\limits_{S} (N_y u_{,y})_{,y} \delta u dS dt \\
& + \int\limits_{t_0}^{t_1} \int\limits_{0}^{l} N_y v_{,y} \delta v dx dt|_{y=const} - \int\limits_{t_0}^{t_1} \int\limits_{S} (N_y v_{,y})_{,y} \delta v dS dt + \int\limits_{t_0}^{t_1} \int\limits_{0}^{l} N_y w_{,y} \delta w dx dt|_{y=const} \\
& - \int\limits_{t_0}^{t_1} \int\limits_{S} (N_y w_{,y})_{,y} \delta w dS dt + \int\limits_{t_0}^{t_1} \int\limits_{0}^{l} N_{xy} \delta u dx dt|_{y=const} - \int\limits_{t_0}^{t_1} \int\limits_{S} N_{xy,y} \delta u dS dt \\
& + \int\limits_{t_0}^{t_1} \int\limits_{0}^{b} N_{xy} \delta v dy dt|_{x=const} - \int\limits_{t_0}^{t_1} \int\limits_{S} N_{xy,x} \delta v dS dt + \int\limits_{t_0}^{t_1} \int\limits_{0}^{l} N_{xy} u_{,x} \delta u dx dt|_{y=const} \\
& - \int\limits_{t_0}^{t_1} \int\limits_{S} (N_{xy} u_{,x})_{,y} \delta u dS dt + \int\limits_{t_0}^{t_1} \int\limits_{0}^{b} N_{xy} u_{,y} \delta u dy dt|_{x=const} - \int\limits_{t_0}^{t_1} \int\limits_{S} (N_{xy} u_{,y})_{,x} \delta u dS dt
\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_0}^{t_1}\int_0^l N_{xy}v_{,x}\delta v dxdt|_{y=const}-\int_{t_0}^{t_1}\int_S \left(N_{xy}v_{,x}\right)_{,y}\delta v dSdt\\
&+\int_{t_0}^{t_1}\int_0^b N_{xy}v_{,y}\delta v dydt|_{x=const}-\int_{t_0}^{t_1}\int_S \left(N_{xy}v_{,y}\right)_{,x}\delta v dSdt\\
&+\int_{t_0}^{t_1}\int_0^l N_{xy}w_{,x}\delta w dxdt|_{y=const}-\int_{t_0}^{t_1}\int_S \left(N_{xy}w_{,x}\right)_{,y}\delta w dSdt\\
&+\int_{t_0}^{t_1}\int_0^b N_{xy}w_{,y}\delta w dydt|_{x=const}-\int_{t_0}^{t_1}\int_S \left(N_{xy}w_{,y}\right)_{,x}\delta w dSdt.
\end{aligned}
\tag{2.56}
$$

$$
\begin{aligned}
\int_{t_0}^{t_1}\delta Q^b dt = &-\int_{t_0}^{t_1}\int_0^b M_x\delta w_x dydt|_{x=const}-\int_{t_0}^{t_1}\int_0^b M_{x,x}\delta w dydt|_{x=const}+2\int_{t_0}^{t_1}\int_S M_{x,xx}\delta w dSdt\\
&-\int_{t_0}^{t_1}\int_0^l M_y\delta w_y dxdt|_{y=const}-\int_{t_0}^{t_1}\int_0^l M_{y,y}\delta w dxdt|_{y=const}+2\int_{t_0}^{t_1}\int_S M_{y,yy}\delta w dSdt\\
&-2\int_{t_0}^{t_1}\left[M_{xy}\delta w\right]_{,xy}dt|_{\substack{x=const\\ y=const}}-2\int_{t_0}^{t_1}\int_0^l M_{xy,x}\delta w dxdt|_{y=const}-2\int_{t_0}^{t_1}\int_0^b M_{xy,y}\delta w dydt|_{x=const}\\
&+2\int_{t_0}^{t_1}\int_S M_{xy,xy}\delta w dSdt.
\end{aligned}
\tag{2.57}
$$

$$
\begin{aligned}
\int_{t_0}^{t_1}\delta K dt = &\int_{t_0}^{t_1}\int_S \rho h\,\ddot{u}\,\delta u dSdt+\int_{t_0}^{t_1}\int_S \rho h\,\ddot{v}\,\delta v dSdt+\int_{t_0}^{t_1}\int_S \rho h\,\ddot{w}\,\delta w dSdt\\
&-\int_S \rho h\,\dot{u}\,\delta u dS|_{t=const}-\int_S \rho h\,\dot{v}\,\delta v dS|_{t=const}-\int_S \rho h\,\dot{w}\,\delta w dS|_{t=const}.
\end{aligned}
\tag{2.58}
$$

$$
\begin{aligned}
\int_{t_0}^{t_1}\delta W dt = &\int_{t_0}^{t_1}\int_0^b hp^0(y)\delta u dydt|_{x=const}+\int_{t_0}^{t_1}\int_0^b h\tau_{xy}^0(y)\delta v dydt|_{x=const}\\
&+\int_{t_0}^{t_1}\int_o^\ell hp^0(x)\delta v dxdt|_{y=const}+\int_{t_0}^{t_1}\int_o^\ell h\tau_{xy}^0(x)\delta u dxdt|_{y=const}.
\end{aligned}
\tag{2.59}
$$

The Lagrangian function for the whole system is equal to the sum of the Lagrangian functions of all n plates the system is composed of. To determine the variation of action $\delta\Psi$ for the i-th plate, the following identity:

$$X\,\delta Y = \delta(XY) - Y\delta X \tag{2.60}$$

is used.

In the obtained equation, the terms with the same variation have been grouped and then each of the obtained group of terms (due to the mutual independence of variations) has been equated to zero, giving:

- equilibrium equations:

$$\begin{aligned}
&\int_{t_0}^{t_1}\int_S \{[N_{x,x} + N_{xy,y} + (N_x u_{,x})_{,x} + (N_y u_{,y})_{,y} + (N_{xy} u_{,x})_{,y} + (N_{xy} u_{,y})_{,x}] - h\rho\,\ddot{u}\}\delta u dSdt = 0,\\
&\int_{t_0}^{t_1}\int_S \{[N_{xy,x} + N_{y,y} + (N_x v_{,x})_{,x} + (N_y v_{,y})_{,y} + (N_{xy} v_{,x})_{,y} + (N_{xy} v_{,y})_{,x}] - h\rho\,\ddot{v}\}\delta v dSdt = 0,\\
&\int_{t_0}^{t_1}\int_S \{[M_{x,xx} + M_{y,yy} + 2M_{xy,xy} + (N_x w_{,x})_{,x} + (N_y w_{,y})_{,y} + (N_{xy} w_{,x})_{,y} + (N_{xy} w_{,y})_{,x}]\\
&\quad - h\rho\,\ddot{w}\}\delta w dSdt = 0,
\end{aligned} \tag{2.61}$$

- boundary conditions for lateral edges of the plate ($x = \text{const}$):

$$\begin{aligned}
&\int_{t_0}^{t_1}\int_0^b [N_x + N_x u_{,x} + N_{xy} u_{,y} - hp^0(y)]\delta u dydt|_{x=const} = 0,\\
&\int_{t_0}^{t_1}\int_0^b [N_{xy} + N_x v_{,x} + N_{xy} v_{,y} - h\tau_{xy}^0(y)]\delta v dydt|_{x=const} = 0,\\
&\int_{t_0}^{t_1}\int_0^b M_x \delta w_{,x} dydt|_{x=const} = 0,\\
&\int_{t_0}^{t_1}\int_0^b (M_{x,x} + 2M_{xy,y} + N_x w_{,x} + N_{xy} w_{,y})\delta w dydt|_{x=const} = 0,
\end{aligned} \tag{2.62}$$

- boundary conditions for longitudinal edges of the plate ($y = \text{const}$):

$$\begin{aligned}
&\int_{t_0}^{t_1}\int_0^{\ell} [N_y + N_y v_{,y} + N_{xy} v_{,x} - hp^0(x)]\delta v dx dt\big|_{y=const} = 0,\\
&\int_{t_0}^{t_1}\int_0^{\ell} [N_{xy} + N_y u_{,y} + N_{xy} u_{,x} - h\tau^0_{xy}(x)]\delta u dx dt\big|_{y=const} = 0,\\
&\int_{t_0}^{t_1}\int_0^{\ell} M_y \delta w_{,y} dx dt\big|_{y=const} = 0,\\
&\int_{t_0}^{t_1}\int_0^{\ell} (M_{y,y} + 2M_{xy,x} + N_y w_{,y} + N_{xy} w_{,x})\delta w dx dt\big|_{y=const} = 0,
\end{aligned} \tag{2.63}$$

- boundary conditions for the plate corners (x = const and y = const):

$$\int_{t_0}^{t_1} 2M_{xy}\delta w dt|_{x=const}\big|_{y=const} = 0, \tag{2.64}$$

- initial conditions for t = const:

$$\begin{aligned}
&\int_S h\rho\dot{u}\delta u dS|_{t=const} = 0,\\
&\int_S h\rho\dot{v}\delta v dS|_{t=const} = 0,\\
&\int_S h\rho\dot{w}\delta w dS|_{t=const} = 0,
\end{aligned} \tag{2.65}$$

  which are fulfilled for the entire structure, so if one applies the restrictions at an instant of the initial t_0 and at an instant of the final t_1, then the displacement variations are zero at all points of the structure. Then, system of equations (2.65) vanishes,
- the already used relationship between deformations and internal forces and moments (2.31) or further, up to (2.34) for the orthotropic model of material can be written as follows:

$$\begin{aligned}
&\int_{t_0}^{t_1}\int_S \left(E_x h\varepsilon_x - N_x + \nu_{xy}N_y\right)\delta N_x dSdt = 0,\\
&\int_{t_0}^{t_1}\int_S \left(E_y h\varepsilon_y + \nu_{yx}N_x - N_y\right)\delta N_y dSdt = 0,\\
&\int_{t_0}^{t_1}\int_S \left(2Gh\varepsilon_{xy} - N_{xy}\right)\delta N_{xy} dSdt = 0,
\end{aligned} \tag{2.66}$$

$$\begin{aligned}
&\int_{t_0}^{t_1}\int_S \left(\frac{E_x h^3}{12}\kappa_x - M_x + \nu_{xy}M_y\right)\delta M_x dSdt = 0,\\
&\int_{t_0}^{t_1}\int_S \left(\frac{E_y h^3}{12}\kappa_y + \nu_{yx}M_x - M_y\right)\delta M_y dSdt = 0,\\
&\int_{t_0}^{t_1}\int_S \left(\frac{Gh^3}{6}\kappa_{xy} - M_{xy}\right)\delta M_{xy} dSdt = 0.
\end{aligned} \tag{2.67}$$

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