On the Formulation of Inverse Problem in Electrical Prospecting

V.P. Gubatenko

Abstract The following inverse problem can be formulated for the isotropic geological medium with applications in electrical prospecting: *The electromagnetic field is measured on the surface of the ground. Find the distribution of electrical conductivity σ and magnetic permeability μ of the geological medium.* We consider a simplified mathematical formulation of this problem in the frequency domain, assuming that the parameters of the geological medium σ and μ possess the frequency dispersion.

1 The First Inverse Problem

Assume, x, y, and z are the Cartesian coordinates in Euclidean space. Our goal is to find the coefficients σ and μ of Maxwell equations

$$\operatorname{rot}\mathbf{H} = \sigma\,\mathbf{E}, \tag{1}$$

$$\operatorname{rot}\mathbf{E} = i\omega\mu\,\mathbf{H} \tag{2}$$

in region $V = \left\{M(x,y,z) \in R^3 \,|\, z > 0\right\}$ (in the ground). Here, $E = E(M,i\omega) = E(x,y,z,i\omega) = (E_x,E_y,E_z)$ and $H = H(M,i\omega) = H(x,y,z,i\omega) = (H_x,H_y,H_z)$ are the complex amplitudes of electric and magnetic fields in the ground, respectively, i is the imaginary unit, and ω is the angular frequency.

Let the unknown parameters $\sigma = \sigma(x,y,z,i\omega)$ and $\mu = \mu(x,y,z,i\omega)$ of the medium satisfy the conditions

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$$\sigma\left(M,i\omega\right)\neq 0, \qquad \mu\left(M,i\omega\right)\neq 0, \tag{3}$$

$$\mathrm{Re}\,\sigma\left(M,i\omega\right)\geq 0, \qquad \mathrm{Im}\,\mu\left(M,i\omega\right)\leq 0, \tag{4}$$

$$\sigma\left(M,i\omega\right)\in C^{k-1}\left(V\right),\ \mu\left(M,i\omega\right)\in C^{k-1}\left(V\right), \quad k\geq 3. \tag{5}$$

Here, the restrictions (3) and (4) indicate the feasibility of the physical parameters of the medium, and the Eq. (5) is the condition of smoothness.

Since the parameters of the medium make available measurements near the ground, we assume that their distributions on the surface $z=+0$ are known:

$$\sigma=\sigma^{0}\left(x,y,+0,i\omega\right), \quad \mu=\mu^{0}\left(x,y,+0,i\omega\right). \tag{6}$$

Suppose also that vector fields $\mathbf{E}$ and $\mathbf{H}$ are known on the surface $z=+0$:

$$\mathbf{E}=\mathbf{E^{0}}\left(x,y,+0,i\omega\right), \quad \mathbf{H}=\mathbf{H^{0}}\left(x,y,+0,i\omega\right), \tag{7}$$

where $\mathbf{E}=\mathbf{E^{0}}\left(x,y,+0,i\omega\right)=\left(E_x^0,E_y^0,E_z^0\right)$, $\mathbf{H}=\mathbf{H^{0}}\left(x,y,+0,i\omega\right)=\left(H_x^0,H_y^0,H_z^0\right)$. Then Eqs. (1) and (2) on the surface $z=+0$ can be written as

$$\begin{aligned}
&\frac{\partial H_z^0}{\partial y}-\left.\frac{\partial H_y}{\partial z}\right|_{z=+0}=\sigma^0 E_x^0,\\
&\left.\frac{\partial H_x}{\partial z}\right|_{z=+0}-\frac{\partial H_z^0}{\partial x}=\sigma^0 E_y^0,\\
&\frac{\partial H_y^0}{\partial x}-\frac{\partial H_x^0}{\partial y}=\sigma^0 E_z^0,
\end{aligned} \tag{8}$$

$$\begin{aligned}
&\frac{\partial E_z^0}{\partial y}-\left.\frac{\partial E_y}{\partial z}\right|_{z=+0}=i\omega\mu^0 H_x^0,\\
&\left.\frac{\partial E_x}{\partial z}\right|_{z=+0}-\frac{\partial E_z^0}{\partial x}=i\omega\mu^0 H_y^0,\\
&\frac{\partial E_y^0}{\partial x}-\frac{\partial E_x^0}{\partial y}=i\omega\mu^0 H_z^0.
\end{aligned} \tag{9}$$

Here, $\left.\frac{\partial E_x}{\partial z}\right|_{z=+0}$, $\left.\frac{\partial E_y}{\partial z}\right|_{z=+0}$, $\left.\frac{\partial H_x}{\partial z}\right|_{z=+0}$ and $\left.\frac{\partial H_y}{\partial z}\right|_{z=+0}$ are the partial derivatives of the electromagnetic field components along the coordinate z on the surface $z=+0$.

Relations (6)–(9) can be taken as the boundary conditions of the inverse problem. However, comparing the expression (7)–(9), we see that the functions appearing in them are dependent. For example, if $E_z^0\neq 0$, then as independent functions can be selected $\mu = \mu^0(x,y,+0,i\omega), E_x^0, E_y^0, E_z^0$, $\left.\frac{\partial E_x}{\partial z}\right|_{z=+0}$, and $\left.\frac{\partial E_y}{\partial z}\right|_{z=+0}$. In this case, the function H_x^0, H_y^0, H_z^0 is defined by conditions (9). Then, $\sigma = \sigma^0(x,y,+0,i\omega)$ is determined from the last equality of Eq. (8), and functions $\left.\frac{\partial H_x}{\partial z}\right|_{z=+0}$, $\left.\frac{\partial H_y}{\partial z}\right|_{z=+0}$ are determined from the first and second equations of the same conditions.

Thus, the boundary conditions for the inverse problem can be written as

$$\mu = \mu^0(x,y,+0,i\omega), \qquad E_x = E_x^0(x,y,+0,i\omega), \qquad E_y = E_y^0(x,y,+0,i\omega),$$
$$E_z = E_z^0(x,y,+0,i\omega), \quad \left.\frac{\partial E_x}{\partial z}\right|_{z=+0} = \varphi(x,y,+0,i\omega), \quad \left.\frac{\partial E_y}{\partial z}\right|_{z=+0} = \psi(x,y,+0,i\omega), \tag{10}$$

where the functions in the right-hand sides of equalities are known. In the case of $E_z^0 = 0$ to the boundary conditions (10) we add

$$\sigma = \sigma^0(x,y,+0,i\omega).$$

Note, that from Eqs. (1), (2) and conditions (4), (5) we obtain

$$\mathbf{E}(M,i\omega) \in C^k(V), \ \mathbf{H}(M,i\omega) \in C^k(V), \tag{11}$$

$$\lim_{z\to+\infty} \mathbf{E}(M,i\omega) = 0, \ \lim_{z\to+\infty} \mathbf{H}(M,i\omega) = 0. \tag{12}$$

We can assume that the vector fields $\mathbf{E}$ and $\mathbf{H}$ are not identically zero in the region V, and it follows from conditions (3). The same is true for $\operatorname{rot}\mathbf{E}$ and $\operatorname{rot}\mathbf{H}$.

As follows from the formulation of the inverse problem, the solution to this problem exists, but not its uniqueness is obvious. Clearly, if we could found any solution $\sigma = \tilde{\sigma}(E,y,z,i\omega)$ and $\mu = \tilde{\mu}(E,y,z,i\omega)$ of this problem, then for these parameters there exists a unique solution of Maxwell equations (1) and (2) with respect to the vector fields $\mathbf{E}$, $\mathbf{H}$. The following question arises: can we reduce the first inverse problem to the problem of finding the vector field $\mathbf{E}$? To answer this question, we formulate the next inverse problem.

2 The Second Inverse Problem

Let the scalar functions σ, μ and vector fields $\mathbf{E}$, $\mathbf{H}$ still satisfy the conditions (3), (5) and (11), (12), and at the same time the vector fields are not identically equal to zero in the region V. Let us formulate the following inverse problem:

Suppose, in region V is given a vector field $\mathbf{E}$. *Find in the region V the field scalar functions* σ, μ *and vector* $\mathbf{H}$, *turning the relationships (1) and (2) to identity.*

A similar problem can be considered for the *given* vector $\mathbf{H}$ and the unknown functions σ, μ, $\mathbf{E}$. However, we will not discuss this problem separately, taking into account the symmetry of the Eqs. (1) and (2) with respect to the formal replacement $\mathbf{E}\leftrightarrow\mathbf{H}$, $i\omega\mu\leftrightarrow\sigma$, called the principle of duality commutes [6].

Lemma 1. *For the existence of the second inverse problem is necessary and sufficient that the unknown scalar function* μ *is a solution of the differential equation*

$$\mathbf{E}\times \operatorname{rot}\left(\frac{1}{\mu}\operatorname{rot}\mathbf{E}\right)=0 \tag{13}$$

except for solutions μ *of the equation*

$$\operatorname{rot}\left(\frac{1}{\mu}\operatorname{rot}\mathbf{E}\right)=0\,. \tag{14}$$

Remark to Lemma 1. If the function μ is a solution of Eq. (14), the unknown vector $\mathbf{H}$ is identically zero.

Lemma 2. *If for a given vector E there exists a solution of the second inverse problem, then at any point* $M\in V$ *or*

$$(\mathbf{E}(M),\operatorname{rot}\mathbf{E}(M))=0,\quad (\operatorname{rot}\mathbf{E}(M),\operatorname{rot}\operatorname{rot}\mathbf{E}(M))=0,$$

or

$$(\mathbf{E}(M),\operatorname{rot}\mathbf{E}(M))\neq 0,\quad (\operatorname{rot}\mathbf{E}(M),\operatorname{rot}\operatorname{rot}\mathbf{E}(M))\neq 0.$$

Remark to Lemma 2. If $(\mathbf{E},\operatorname{rot}\mathbf{E})\neq 0$, $(\operatorname{rot}\mathbf{E},\operatorname{rot}\operatorname{rot}\mathbf{E})\neq 0$ at some point $M\in V$ then since the scalar functions are continuous, there exists a neighborhood of this point at which these inequalities are true. Therefore, when setting the vector $\mathbf{E}$ in the second inverse problem, we consider two cases: or $(\mathbf{E},\operatorname{rot}\mathbf{E})\equiv 0$, $(\operatorname{rot}\mathbf{E},\operatorname{rot}\operatorname{rot}\mathbf{E})\equiv 0$ in the area, or $(\mathbf{E},\operatorname{rot}\mathbf{E})$, $(\operatorname{rot}\mathbf{E},\operatorname{rot}\operatorname{rot}\mathbf{E})$ are not identically zero in any subregion V. The first case corresponds to the orthogonal vectors $\mathbf{E}$ and $\mathbf{H}$, but the second case is not orthogonal. The first case includes, for example, three-component flat and axisymmetric electromagnetic fields, and the second- five-component transverse electric and transverse magnetic fields [8].

Theorem 1. *If the vector field* $\mathbf{E}$ *is a solution of the nonlinear equations*

$$(\mathbf{E},\operatorname{rot}\mathbf{E})=0,\quad (\operatorname{rot}\mathbf{E},\operatorname{rot}\operatorname{rot}\mathbf{E})=0, \tag{15}$$

then for a given vector $\mathbf{E}$*, the solution of the second inverse problem exists and is not unique.*

The proof of this theorem is based on the theory of the linear partial differential equations of the first order and common solutions to these equations by means of characteristic systems [5]. After finding the solutions of Eq. (13) and after removing solutions of Eq. (14) from solutions of Eq. (13), we can determine functions σ and $\mathbf{H}$ as follows:

$$\sigma = \frac{1}{i\omega\mathbf{E}^2}\left(\mathbf{E}, \mathrm{rot}\left(\frac{1}{\mu}\mathrm{rot}\,\mathbf{E}\right)\right), \quad \mathbf{H} = \frac{1}{i\omega\mu}\mathrm{rot}\,\mathbf{E}. \tag{16}$$

We note here that the first equality of Eq. (16) is determined from

$$\mathrm{rot}\left(\frac{1}{\mu}\mathrm{rot}\,\mathbf{E}\right) = i\omega\sigma\mathbf{E}. \tag{17}$$

Theorem 2. *Let the vector* $\mathbf{E}$ *is given in the region* V *and* $(\mathbf{E}, \mathrm{rot}\,\mathbf{E}) \neq 0$, $(\mathrm{rot}\,\mathbf{E}, \mathrm{rot}\,\mathrm{rot}\,\mathbf{E}) \neq 0$ *in this region. For the existence solution of the inverse problem, it is necessary and sufficient that the vector E is the solution of the nonlinear equation*

$$\mathrm{rot}\,\mathbf{F}^E = 0, \tag{18}$$

where

$$\mathbf{F}^E = \frac{1}{(\mathrm{rot}\,\mathbf{E}, \mathrm{rot}\,\mathrm{rot}\,\mathbf{E})}\mathrm{div}\left[\frac{(\mathrm{rot}\,\mathbf{E}, \mathrm{rot}\,\mathrm{rot}\,\mathbf{E})}{(\mathbf{E}, \mathrm{rot}\,\mathbf{E})}\mathbf{E}\right]\mathrm{rot}\,\mathbf{E} + \frac{1}{(\mathbf{E}, \mathrm{rot}\,\mathbf{E})}(\mathbf{E} \times \mathrm{rot}\,\mathrm{rot}\,\mathbf{E}).$$

The general solution μ of the second inverse problem has the form

$$\mu = \mu_0(i\omega)\exp\left(\int_{M_0}^{M} F_x^E\,dx + F_y^E\,dy + F_z^E\,dz\right) \tag{19}$$

where $\mathbf{F}^E = \left(F_x^E, F_y^E, F_z^E\right)$; $M(x,y,z) \in V$, $M_0(x_0,y_0,z_0) \in V$; the function $\mu_0(i\omega)$ is arbitrary and does not depend on the coordinates. Electrical conductivity and magnetic field are determined by formulas (16). Let us consider the following inverse problem.

3 The Third Inverse Problem

As follows from Theorems 1 and 2, the vector field $\mathbf{E}$ is accompanied by a family of functions $\{\mathbf{E}, \mathbf{H}, \mu, \sigma\}$, which becomes an identity equation (1) and (2) then and only then, when the vector field $\mathbf{E}$ satisfies to the Eqs. (15) or (18). Of course, not every vector field $\mathbf{E}$ under these conditions uniquely determines the scalar functions σ, μ, which obey the conditions of physics Eq. (4). For a single determination of the parameters of the medium in accordance with Theorems 1 and 2, we require a priori information on the distribution of the permeability function μ in the region V.

Suppose, for example, here and below, $\mu = \mu_0 = 4\pi \cdot 10^{-7}$ H/m, which corresponds to the sedimentary rocks studied in the structural electrical prospecting. In this case, as for the orthogonal fields **E**, **H**, and also for the non-orthogonal fields, vector **E** must be a solution of the equation

$$\mathbf{E} \times \operatorname{rot}\operatorname{rot}\mathbf{E} = 0\,. \tag{20}$$

Then the first inverse problem is reduced to the following inverse problem:

The Inverse Problem 3.1. *Find the solution of Eq. (20) satisfying to the boundary conditions*

$$E_x = E_x^0\,(x,y,+0,i\omega)\,, \qquad E_y = E_y^0\,(x,y,+0,i\omega)\,, \tag{21}$$

$$E_z = E_z^0\,(x,y,+0,i\omega) \neq 0,\ \left.\frac{\partial E_x}{\partial z}\right|_{z=+0} = \varphi\,(x,y,+0,i\omega)\,,\ \left.\frac{\partial E_y}{\partial z}\right|_{z=+0} = \psi\,(x,y,+0,i\omega)$$

and in the case when $E_z = 0$

$$\sigma = \sigma^0\,(x,y,+0,i\omega)\,, \qquad E_x = E_x^0\,(x,y,+0,i\omega)\,, \qquad E_y = E_y^0\,(x,y,+0,i\omega)\,, \tag{22}$$

$$\left.\frac{\partial E_x}{\partial z}\right|_{z=+0} = \varphi\,(x,y,+0,i\omega),\ \left.\frac{\partial E_y}{\partial z}\right|_{z=+0} = \psi\,(x,y,+0,i\omega)\,.$$

If the first inverse problem with boundary conditions (10) has a unique solution, then the electric field intensity **E** for the inverse problem 3.1 is also unique. Having determined the field **E** of the inverse problem 3.1, we easily find from (16) functions σ and H.

Let us show on a simple example of the classical model of the magnetotelluric sounding that the solution as of the first inverse problem such that for the inverse problem 3.1 is not unique in the case of the frequency dispersion of the electrical conductivity.

Suppose that in the half-space $z > 0$ is situated nonmagnetic medium with electrical conductivity $\sigma = \sigma\,(z,i\omega)$, and let us initialize an electromagnetic field $\mathbf{E} = (E_x\,(z,i\omega)\,,0,0)$, $\mathbf{H} = (0,H_y\,(z,i\omega)\,,0)$ with orthogonal vectors **E** and **H**. Then in the half-space $z > 0$, Eq. (20) [or Eq. (17)] has the form

$$\frac{d^2E_x}{dz^2} + i\omega\mu_0\sigma E_x = 0\,. \tag{23}$$

Assume that the surface boundary conditions (22) have the form

$$\sigma^0\,(0,i\omega) = \sigma_0,\ E_x = E^0\,(i\omega)\,,\ \left.\frac{dE_x}{dz}\right|_{z=+0} = \varphi\,(i\omega) = -\sqrt{-i\omega\mu_0\sigma_0}E^0\,(i\omega)\,, \tag{24}$$

where $\sigma_0 = \text{const} > 0$; $\text{Re}\sqrt{-i\omega\mu_0} > 0$; $E^0\,(i\omega)$ is an arbitrary complex function of angular frequency ω. It is easy to see that there are at least two solutions of the first inverse problem and the inverse problem 3.1. These solutions are

$$\sigma = \sigma_0\,, \quad E_x = E^0\,(i\omega)\exp\left(-k_0\,z\right) \tag{25}$$

and

$$\sigma = -\frac{1}{i\omega\mu_0}\frac{k_0^2 + k_1\,(k_1 - k_0)\,(2k_0 - k_1)\,z + \frac{k_1(k_1-k_0)^2}{2}z^2}{1 + (k_1 - k_0)\,z + \frac{(k_1-k_0)^2}{2}z^2}, \tag{26}$$

$$E_x = E^0\,(i\omega)\left[1 + (k_1 - k_0)\,z + \frac{(k_1 - k_0)^2}{2}z^2\right]\exp\left(-k_1 z\right),$$

where $k_0 = \sqrt{-i\omega\mu_0\sigma_0}$; $k_1 = \sqrt{-i\omega\mu_0\sigma_1}$, $\sigma_1 = const$, $\sigma_0 < \sigma_1 < 4\sigma_0$.

Solution (25) corresponds to a quasi-stationary field in the homogeneous half-space $z > 0$ with conductivity $\sigma = \sigma_0$ and the solution (26)—the gradient frequency-dispersive medium. It is easy to show that $\text{Re}\sigma > 0$, $\text{Im}\sigma < 0$ for this medium, and all $\omega > 0$, such that this model is the frequency-dispersed medium and this medium is physically realizable. This result, however, does not contradict to the uniqueness theorem [9], since this theorem is proved under the assumption of the absence of the frequency dispersion of electrical conductivity σ.

This example shows that in order to find the unique solution of the inverse problem it is necessary to know the additional information about the nature of the frequency dispersion of conductivity. If, for example, we know that the unknown scalar function does not depend on the angular frequency ω, we use the method developed by Klibanov and Beilina for hyperbolic coefficient inverse problems [1–4, 7].

Indeed, since the conductivity is determined by the formula (16) and does not depend on the angular frequency, then

$$\frac{\partial}{\partial\omega}\left[\frac{1}{\omega\mathbf{E}^2}\,(\mathbf{E}, \text{rot rot}\,\mathbf{E})\right] = 0, \tag{27}$$

and we can formulate the next inverse problem:

The Inverse Problem 3.2. Find the solution $\mathbf{E}$ of Eqs. (20) and (27) satisfying to (21) under the condition $E_z^0\,(x, y, +0, i\omega) \neq 0$, or (22) in the case of $E_z = 0$.

The solution to this inverse problem exists and is unique, at least for the one-dimensional inverse problem of magnetotelluric sounding. After finding a solution to this problem, it is easy to find the electrical conductivity σ of relationship (27).

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