

Chapter 1

Preliminaries

In this chapter, we first recall some basic covering lemmas and notions of doubling cubes, using these we further establish the Lebesgue differentiation theorem and the Calderón–Zygmund decomposition.

1.1 Covering Lemmas

This section is devoted to some basic covering lemmas. We first recall the following Besicovitch covering theorem which is very important and useful in our context.

Theorem 1.1.1. *Let E be a bounded set in \mathbb{R}^D . If, for every $x \in E$, there exists a closed cube $Q(x)$ centered at x , then it is possible to choose, from among the given cubes $\{Q(x)\}_{x \in E}$, a subsequence $\{Q_k\}_k$ (possibly finite) such that*

- (i) $E \subset \bigcup_k Q_k$;
- (ii) *no point of \mathbb{R}^D is in more than N_D (a number that only depends on D) cubes of the sequences $\{Q_k\}$, namely, for every $z \in \mathbb{R}^D$,*

$$\sum_k \chi_{Q_k}(z) \leq N_D;$$

- (iii) *the sequence $\{Q_k\}_k$ can be distributed in B_D (a natural number that only depends on D) families of disjoint cubes.*

Proof. For any set $\Omega \subset \mathbb{R}^D$, denote by d_Ω the diameter of Ω . Now let

$$a_0 := \sup\{d_{Q(x)} : x \in E\}.$$

If $a_0 = \infty$, then we can take a single cube $Q(x)$ to cover E and the conclusions of Theorem 1.1.1 hold true. Assume that $a_0 < \infty$. We choose $Q_1 \in \{Q(x)\}_{x \in E}$ with center $x_1 \in E$ such that $d_{Q_1} > a_0/2$. Let

$$a_1 := \sup\{d_{Q(x)} : x \in (E \setminus Q_1)\}.$$

We now choose Q_2 with center $x_2 \in (E \setminus Q_1)$ such that $d_{Q_2} > a_1/2$. Going on in this way, if there exists some $m \in \mathbb{N} := \{1, 2, \dots\}$ such that

$$E \setminus \left(\bigcup_{k=1}^m Q_k \right) = \emptyset, \quad (1.1.1)$$

then the selection process is finished. Otherwise, we go on our selection and obtain a sequence of points, $\{x_k\}_k$, and cubes, $\{Q_k\}_k$, such that, for all i, j with $i \neq j$,

$$\frac{1}{3}Q_i \cap \frac{1}{3}Q_j = \emptyset. \quad (1.1.2)$$

To see this, we first observe that, for all $k \in \mathbb{N}$, it holds true that $a_k \leq a_{k-1} < \infty$ and

$$a_{k-1}/2 < d_{Q_k} \leq a_{k-1}.$$

From this observation, we further deduce that, for all $0 \leq j < i$, $d_{Q_i}/2 < d_{Q_j}$, which is equivalent to the fact that $\ell(Q_i)/2 < \ell(Q_j)$. Combining this with the fact that $x_i \notin Q_j$, we obtain (1.1.2).

From (1.1.2) and the fact that E is a bounded set, it follows that the sequence $\{\ell(Q_k)\}_k$ is either finite or $\ell(Q_k) \rightarrow 0$ as $k \rightarrow \infty$ (For otherwise, (1.1.1) does not hold true for all $m \in \mathbb{N}$ and there exists $\epsilon \in (0, \infty)$ such that, for any $N \in \mathbb{N}$, there exists $k \in \mathbb{N}$ satisfying that $k > N$ and $\ell(Q_k) \geq \epsilon$. We then choose a subsequence $\{Q_{k_N}\}_{N=1}^\infty$ of $\{Q_k\}_{k=1}^\infty$ such that, for any k_N , $\ell(Q_{k_N}) \geq \epsilon$. This, together with (1.1.2) and the fact that E is bounded, further implies (1.1.1) for some $m \in \mathbb{N}$, which is impossible). If the selection process stops, the conclusion (i) is trivial. If the sequence $\{\ell(Q_k)\}_k$ is infinite and $\ell(Q_k) \rightarrow 0$, then $d_{Q_k} \rightarrow 0$ and hence $a_k \rightarrow 0$. Thus, there exists $x \in E \setminus (\bigcup_{k=1}^\infty Q_k)$ and hence there exists k_0 such that $a_{k_0} < d_{Q(x)}$, which is contradictory to our selection. Thus, $E \subset \bigcup_{k=1}^\infty Q_k$ and (i) holds true in this case.

To see (ii), fix $z \in \mathbb{R}^D$ and draw D hyperplanes through z and consider the 2^D closed “hyperquadrants” through z determined by them. Fix k with Q_k including z . Let

$$\mathcal{J} := \{j \in \mathbb{N} : Q_j \ni z \text{ and } x_j \text{ lies in the same “hyperquadrants” as } x_k\}.$$

By the fact that $x_i \notin Q_j$ and $\ell(Q_i)/2 < \ell(Q_j)$ for all $i, j \in \mathbb{N}$ with $i > j$, we see that

$$\ell(Q_k) < \ell(Q_j) < 2\ell(Q_k)$$

when $j \in \mathcal{J}$ and $j > k$, and

$$\ell(Q_j) < \ell(Q_k) < 2\ell(Q_j)$$

when $j \in \mathcal{J}$ and $j < k$. This further implies that $\frac{1}{3}Q_j \subset \frac{8}{3}Q_k$ for all $j \in \mathcal{J}$, which, together with (1.1.2), implies that there exists a positive constant N depending on D such that the cardinality of \mathcal{J} is at most $N + 1$. Thus, the cardinality of cubes containing z is at most $N_D := 2^D(N + 1)$, which completes the proof of (ii).

In order to prove (iii), we rearrange the sequence $\{Q_k\}_k$ such that the side length of the new sequence, which is still denoted by $\{Q_k\}_k$, is decreasing in k . We fix a cube Q_j of the sequence $\{Q_k\}_k$. By (ii), at most N_D members of the sequence contain a fixed vertex of Q_j . Observe that every cube Q_k with $k < j$ is of a size not smaller than that of Q_j . Thus, if $Q_k \cap Q_j \neq \emptyset$ and $k < j$, then Q_k contains at least one of the 2^D vertices of Q_j . This implies that there exist at most $2^D N_D$ sets of the collection $\{Q_1, \dots, Q_{j-1}\}$ with non empty intersection with Q_j . Consequently, we distribute the sequence $\{Q_k\}_k$ in $2^D N_D + 1$ disjoint sequences in the following way: we let $Q_i \in Q_i$ for $i \in \{1, \dots, 2^D N_D + 1\}$. Since $Q_{2^D N_D + 2}$ is disjoint with Q_{k_0} for some $k_0 \leq 2^D N_D + 1$, we let $Q_{2^D N_D + 2} \in Q_{k_0}$. In the same way, $Q_{2^D N_D + 3}$ is disjoint with all sets in some $Q_{\tilde{k}}$, and we let $Q_{2^D N_D + 3} \in Q_{\tilde{k}}$, and so on. This finishes the proof of (iii), and hence Theorem 1.1.1. \square

Remark 1.1.2. (i) Theorem 1.1.1 is not valid anymore, if x can be in the boundary of $Q(x)$ or arbitrarily close to it. However, if the point x is “far” from the boundary of $Q(x)$ (for example, $x \in \rho^{-1}Q(x)$ for a fixed $\rho \in (1, \infty)$ and any point x and $Q(x)$), then Theorem 1.1.1 also holds true.¹

(ii) We remark that, if E in Theorem 1.1.1 is not bounded, but

$$\sup_{x \in E} \{\ell(Q(x))\} =: M < \infty,$$

then Theorem 1.1.1 still holds true with N_D and B_D replaced by some positive constants \tilde{N}_D and \tilde{B}_D . Indeed, it suffices to partition \mathbb{R}^D in cubes of side length M and then apply Theorem 1.1.1 to the intersection of E with each one of these cubes. We omit the details.

Let $\rho \in (1, \infty)$. For any $f \in L^1_{\text{loc}}(\mu)$ and $x \in \mathbb{R}^D$, let

$$\mathcal{M}^{(\rho)} f(x) := \sup_{\rho^{-1}Q \ni x} \frac{1}{\mu(Q)} \int_Q |f(y)| d\mu(y),$$

where the supremum is taken over all cubes Q satisfying that $\rho^{-1}Q \ni x$. As an application of Theorem 1.1.1, we obtain the boundedness of $\mathcal{M}^{(\rho)}$ from $L^1(\mu)$ to $L^{1,\infty}(\mu)$ and on $L^p(\mu)$ for $p \in (1, \infty]$ as follows.

Corollary 1.1.3. *Let $\rho \in (1, \infty)$ and $p \in (1, \infty]$. Then $\mathcal{M}^{(\rho)}$ is bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$ and on $L^p(\mu)$.*

¹See [23, p. 7].

Proof. Assume that $f \in L^1(\mu)$. For each $t \in (0, \infty)$, let

$$E_t := \{x \in \mathbb{R}^D : \mathcal{M}^{(\rho)} f(x) > t\}.$$

By applying Theorem 1.1.1 to E_t , it is not difficult to see that $\mathcal{M}^{(\rho)}$ is of weak type $(1, 1)$. Observe that $\mathcal{M}^{(\rho)}$ is bounded on $L^\infty(\mu)$. These two facts, together with the Marcinkiewicz interpolation theorem, imply that $\mathcal{M}^{(\rho)}$ is also bounded on $L^p(\mu)$ for any $p \in (1, \infty)$, which completes the proof of Corollary 1.1.3. \square

Also, we need the following Whitney decomposition.²

Proposition 1.1.4. *Let $\Omega \subset \mathbb{R}^D$ be open and $\Omega \neq \mathbb{R}^D$. Then Ω can be decomposed as*

$$\Omega = \bigcup_{i \in I} Q_i,$$

where $\{Q_i\}_{i \in I}$ are cubes with disjoint interiors, $20Q_i \subset \Omega$ for all $i \in I$, and there exist some constants $\beta \in (20, \infty)$ and $N_W \in \mathbb{N}$ such that, for all $k \in I$, $\beta Q_k \setminus \Omega \neq \emptyset$ and there are at most N_W cubes Q_i with $10Q_k \cap 10Q_i \neq \emptyset$ (in particular, the family of cubes $\{10Q_i\}_{i \in I}$ has finite overlapping).

1.2 Doubling Cubes

In this section, we aim to introduce the notion of doubling cubes. A *non-doubling measure* μ on \mathbb{R}^D means that μ is a nonnegative Radon measure which only satisfies the *polynomial growth condition* (0.0.1). Also, let $Q(x, r)$ be the cube centered at x with side length r . Moreover, we always assume that the constant C_0 in (0.0.1) has been chosen big enough such that, for all cubes $Q \subset \mathbb{R}^D$,

$$\mu(Q) \leq C_0[\ell(Q)]^n,$$

where $n \in (0, D]$. Observe that, if (0.0.1) holds true for any ball $B(x, r)$, then, for any cube $Q(x, r)$,

$$\mu(Q(x, r)) \leq \mu\left(B\left(x, \frac{\sqrt{D}}{2}r\right)\right) \leq C_0\left(\frac{\sqrt{D}}{2}\right)^n r^n.$$

Conversely, if we have $\mu(Q(x, r)) \leq C_0 r^n$ for any $x \in \mathbb{R}^D$ and $r \in (0, \infty)$, then, for any ball $B(x, r)$,

$$\mu(B(x, r)) \leq \mu(Q(x, 2r)) \leq C_0 2^n r^n.$$

²See [121, p. 15].

The measure in (0.0.1) is not necessary to satisfy the following *doubling condition* that there exists a positive constant C such that, for all balls B ,

$$\mu(2B) \leq C\mu(B), \quad (1.2.1)$$

where above and in what follows, for all balls $B := B(x, r)$ and positive constant λ , $\lambda B := B(x, \lambda r)$. Though (1.2.1) is not assumed uniformly for all balls, it turns out there exist some cubes satisfying such an inequality.

Definition 1.2.1. Let $\alpha \in (1, \infty)$ and $\beta \in (\alpha^n, \infty)$. A cube Q is called an (α, β) -doubling cube if $\mu(\alpha Q) \leq \beta\mu(Q)$.

Proposition 1.2.2. Let $\alpha \in (1, \infty)$ and $\beta \in (\alpha^n, \infty)$. Then the following two statements hold true:

- (i) For any $x \in \text{supp } \mu$ and $R \in (0, \infty)$, there exists some (α, β) -doubling cube Q centered at x with $\ell(Q) \geq R$;
- (ii) If $\beta > \alpha^D$, then, for μ -almost every $x \in \mathbb{R}^D$, there exists a sequence of (α, β) -doubling cubes, $\{Q_k\}_{k \in \mathbb{N}}$, centered at x with $\ell(Q_k) \rightarrow 0$ as $k \rightarrow \infty$.

Proof. We first prove (i). To this end, assume that (i) does not hold true. Then there exist some positive constant C and $x_0 \in \text{supp } \mu$ such that, for any cube Q centered at x_0 with $\ell(Q) \geq C$, we have $\mu(\alpha Q) > \beta\mu(Q)$. Now we take Q_0 be such a cube with $\mu(Q_0) > 0$. Then, by our assumption and the growth condition, we see that, for any $k \in \mathbb{N}$,

$$\beta^k \mu(Q_0) < \mu(\alpha^k Q_0) \leq C_0 \alpha^{kn} [\ell(Q_0)]^n,$$

which in turn implies that

$$\mu(Q_0) < C_0 \left(\frac{\alpha^n}{\beta} \right)^k [\ell(Q_0)]^n.$$

Letting $k \rightarrow \infty$, we have $\mu(Q_0) = 0$, which contracts to $\mu(Q_0) > 0$. This implies that there exists some (α, β) -doubling cube Q centered at x_0 with $\ell(Q) \geq C_0$. Thus, (i) holds true.

To prove (ii), for any fixed $\alpha \in (1, \infty)$ and $\beta \in (\alpha^D, \infty)$, let

$$\Omega := \{x \in \mathbb{R}^D : \text{there does not exist any sequence of } (\alpha, \beta) \text{ -- doubling cubes centered at } x \text{ whose side lengths tend to zero}\}.$$

We show that $\mu(\Omega) = 0$. For any $m \in \mathbb{N}$, let

$$\Omega_m := \{x \in \mathbb{R}^D : \text{all cubes centered at } x \text{ with side lengths less than } 1/m \text{ are not } (\alpha, \beta) \text{ -- doubling cubes}\}.$$

Observe that $\Omega = \cup_{m=1}^{\infty} \Omega_m$. It suffices to prove that $\mu(\Omega_m) = 0$ for any $m \in \mathbb{N}$. To this end, we fix a cube Q with $\ell(Q) \leq \frac{1}{2m}$ and denote by Q_x^N the cube centered at x whose side length is $\alpha^{-N}\ell(Q)$ for any $x \in \Omega_m \cap Q$ and $N \in \mathbb{N}$. By Theorem 1.1.1, there exists a sequence of cubes, $\{Q_k^N\}_{k \in I_N}$, such that

$$\Omega_m \cap Q \subset \bigcup_{k \in I_N} Q_k^N \quad \text{and} \quad \sum_{k \in I_N} \chi_{Q_k^N} \lesssim 1.$$

Since the center of Q_k^N is in Ω_m and $\ell(Q_k^N) \leq \frac{1}{2m}$, Q_k^N is not a (α, β) -doubling cube for each k . Therefore, from this and the fact that $\alpha^N Q_k^N \subset 3Q$, it follows that

$$\mu(Q_k^N) < \beta^{-1} \mu(\alpha Q_k^N) < \cdots < \beta^{-N} \mu(\alpha^N Q_k^N) \leq \beta^{-N} \mu(3Q). \quad (1.2.2)$$

On the other hand, by the facts $\sum_{k \in I_N} \chi_{Q_k^N} \lesssim 1$ and $Q_k^N \subset 3Q$, we conclude that

$$\sum_{k \in I_N} |Q_k^N| \lesssim |3Q|, \quad (1.2.3)$$

where $|\cdot|$ denotes the D -dimensional Lebesgue measure. The inequality (1.2.3) is equivalent to that

$$\#(I_N) \alpha^{-ND} [\ell(Q)]^D \lesssim 3^D [\ell(Q)]^D,$$

where above and in what follows, for any set E , $\#(E)$ denotes its *cardinality*. Then we have $\#(I_N) \lesssim \alpha^{ND}$, which, together with (1.2.2), in turn implies that

$$\mu\left(\Omega_m \cap Q\right) \leq \sum_{k \in I_N} \mu(Q_k^N) \lesssim \alpha^{ND} \beta^{-N} \mu(3Q).$$

Letting $N \rightarrow \infty$, we see that $\mu(\Omega_m \cap Q) = 0$.

Notice that, for each $m \in \mathbb{N}$, $\mathbb{R}^D = \cup_i Q_{m,i}$, where $\{Q_{m,i}\}_i$ are cubes with

$$\ell(Q_{m,i}) = \frac{1}{2m}$$

for all i . We then find that

$$\mu(\Omega_m) \leq \sum_i \mu\left(\Omega_m \cap Q_{m,i}\right) = 0.$$

This further implies that $\mu(\Omega) = 0$ and finishes the proof of (ii) and hence Proposition 1.2.2. \square

Let $\rho \in (1, \infty)$. In the following, we always take $\beta_\rho := \rho^{D+1}$. For any cube Q , let \tilde{Q}^ρ be the *smallest* (ρ, β_ρ) -doubling cube which has the form $\rho^k Q$ with $k \in \mathbb{N} \cup \{0\} =: \mathbb{Z}_+$. If $\rho = 2$, we denote the cube \tilde{Q}^ρ simply by \tilde{Q} . Moreover, by a doubling cube Q , we always mean a $(2, 2^{D+1})$ -doubling cube.

Example 1.2.3. Let

$$\mu := \chi_Q dx dy + \chi_I dx,$$

where $Q := [-1, 1] \times [-1, 1]$ and $I := Q \cap \mathbb{R} = \{(x, 0) : -1 \leq x \leq 1\}$. If B is the disc centered at $(x, y) \in Q$, $y \in (0, \infty)$, of radius y , then $\mu(B) \sim y^2$ while $\mu(2B) \sim y$ with the implicit equivalent positive constants independent of x and y , and hence μ is a non-doubling measure.

Example 1.2.4. Let $E \subset \mathbb{C}$ be compact. Define the *capacity*

$$\alpha_+(E) := \sup\{\mu(E) : \mu \text{ is a positive Radon measure supported on } E \text{ such that } \mathcal{C}\mu \text{ is a continuous function on } \mathbb{C} \text{ and } \|\mathcal{C}\mu\|_{L^\infty(\mathbb{C})} \leq 1\},$$

where $\mathcal{C}\mu$ is the *Cauchy transform* defined by setting, for all $x \notin \text{supp } \mu$,

$$\mathcal{C}\mu(x) := \int_{\mathbb{C}} \frac{1}{z - x} d\mu(z).$$

Now let μ_0 be a Radon measure supported on E such that $\mathcal{C}\mu_0$ is a continuous function on \mathbb{C} , $\|\mathcal{C}\mu_0\|_{L^\infty(\mathbb{C})} \leq 1$ and $\mu_0(E) \geq \alpha_+(E)/2$. Then we conclude that, for all $x \in \mathbb{C}$ and $r \in (0, \infty)$, $\mu_0(B(x, r)) \leq r$.³

1.3 The Lebesgue Differentiation Theorem

In this section, we establish the Lebesgue differentiation theorem. To begin with, we recall the fact that continuous functions are dense in $L^p(\mu)$ for any $p \in [1, \infty)$.⁴

Lemma 1.3.1. *Let $p \in [1, \infty)$ and $f \in L^p(\mu)$. Then, for any $\epsilon \in (0, \infty)$, there exists a continuous function g with compact support on \mathbb{R}^D such that $\|f - g\|_{L^p(\mu)} < \epsilon$.*

The main result of this section is as follows.

³See [137, p. 530] and [37, p. 40].

⁴See [111, p. 69].

Theorem 1.3.2. *Let $f \in L^1_{\text{loc}}(\mu)$. Then, for μ -almost every $x \in \text{supp } \mu$ and any sequence of cubes, $\{Q_k(x)\}_k$, centered at x with $\ell(Q_k(x)) \rightarrow 0$, $k \rightarrow \infty$,*

$$\lim_{k \rightarrow \infty} \frac{1}{\mu(Q_k(x))} \int_{Q_k(x)} f(y) d\mu(y) = f(x). \quad (1.3.1)$$

Proof. By a standard localization, it suffices to consider the case when $f \in L^1(\mu)$. We claim that (1.3.1) holds true for any continuous function g . To this end, for any $x \in \mathbb{R}^D$ and each k , let

$$I_k(x) := \left| \frac{1}{\mu(Q_k(x))} \int_{Q_k(x)} g(y) d\mu(y) - g(x) \right|.$$

Since g is continuous, for any $\epsilon \in (0, \infty)$, there exists $K \in \mathbb{N}$, depending on x and ϵ , such that, for any $k > K$ and $y \in Q_k(x)$, $|g(y) - g(x)| < \epsilon$. From this fact, it follows that

$$I_k(x) \leq \frac{1}{\mu(Q_k(x))} \int_{Q_k(x)} |g(y) - g(x)| d\mu(y) \leq \epsilon.$$

Since ϵ is arbitrary, we further conclude that $I_k(x) \rightarrow 0$, $k \rightarrow \infty$. Thus, the claim holds true.

We now show that, for any $f \in L^1(\mu)$ and μ -almost every x ,

$$\limsup_{k \rightarrow \infty} |m_{Q_k(x)}(f) - f(x)| = 0.$$

By Lemma 1.3.1, there exists a sequence of continuous functions, $\{f_n\}_n$, on \mathbb{R}^D such that $\|f - f_n\|_{L^1(\mu)} \rightarrow 0$, $n \rightarrow \infty$. It then follows from the claim that, for each $n \in \mathbb{N}$,

$$\begin{aligned} & \limsup_{k \rightarrow \infty} |m_{Q_k(x)}(f) - f(x)| \\ & \leq \limsup_{k \rightarrow \infty} [|m_{Q_k(x)}(f) - m_{Q_k(x)}(f_n)| + |m_{Q_k(x)}(f_n) - f_n(x)|] \\ & \quad + |f_n(x) - f(x)| \\ & \leq \mathcal{M}^{(2)}(f - f_n)(x) + |f_n(x) - f(x)|. \end{aligned}$$

For any $\epsilon \in (0, \infty)$, let

$$E_\epsilon := \left\{ x \in \mathbb{R}^D : \limsup_{k \rightarrow \infty} |m_{Q_k(x)}(f) - f(x)| > \epsilon \right\}.$$

Then, by Corollary 1.1.3 and Lemma 1.3.1, we see that

$$\begin{aligned} \mu(E_\epsilon) &\leq \mu\left(\left\{x \in \mathbb{R}^D : \mathcal{M}^{(2)}(f - f_n)(x) > \frac{\epsilon}{2}\right\}\right) \\ &\quad + \mu\left(\left\{x \in \mathbb{R}^D : |f_n(x) - f(x)| > \frac{\epsilon}{2}\right\}\right) \\ &\lesssim \frac{1}{\epsilon} \|f_n - f\|_{L^1(\mu)}, \end{aligned}$$

which tends to 0, as $n \rightarrow \infty$. Therefore, we obtain $\mu(E_\epsilon) = 0$. This finishes the proof of Theorem 1.3.2. \square

As a consequence of Theorem 1.3.2, we further obtain the following conclusion.

Corollary 1.3.3. *Let $p \in [1, \infty)$ and $f \in L^p_{\text{loc}}(\mu)$. Then, for μ -almost every $x \in \text{supp } \mu$ and $Q_k(x)$ as in Theorem 1.3.2,*

$$\lim_{k \rightarrow \infty} \frac{1}{\mu(Q_k(x))} \int_{Q_k(x)} |f(y) - f(x)|^p d\mu(y) = 0.$$

Proof. Let $\mathbb{Q} := \{r_i\}_{i \in \mathbb{N}}$ be the set of all rational numbers and, for each i ,

$$\begin{aligned} Z_i := \left\{x \in \text{supp } \mu : \limsup_{k \rightarrow \infty} \frac{1}{\mu(Q_k(x))} \int_{Q_k(x)} |f(y) - r_i|^p d\mu(y) \right. \\ \left. \neq |f(x) - r_i|^p \right\}. \end{aligned}$$

Since $|f(y) - r_i|^p \in L^1_{\text{loc}}(\mu)$, it follows, from Theorem 1.3.2, that $\mu(Z_i) = 0$ for any $i \in \mathbb{N}$. Define

$$Z_0 := \{x \in \text{supp } \mu : |f(x)| = \infty\}.$$

Then $\mu(\cup_{i=0}^\infty Z_i) = 0$ and, to show Corollary 1.3.3, it suffices to prove that, whenever $x \notin \cup_{i=0}^\infty Z_i$,

$$\limsup_{k \rightarrow \infty} \frac{1}{\mu(Q_k(x))} \int_{Q_k(x)} |f(y) - f(x)|^p d\mu(y) = 0. \quad (1.3.2)$$

Now, for any $\epsilon \in (0, \infty)$ and each x , we choose $r_i \in \mathbb{Q}$ such that $|f(x) - r_i|^p < \epsilon$. By the fact that $x \notin \cup_{i=0}^\infty Z_i$, we see that

$$\limsup_{k \rightarrow \infty} \frac{1}{\mu(Q_k(x))} \int_{Q_k(x)} |f(y) - f(x)|^p d\mu(y) = |f(x) - r_i|^p \leq \epsilon.$$

Since ϵ is arbitrary, it follows that (1.3.2) holds true, which completes the proof of Corollary 1.3.3. \square

1.4 The Calderón–Zygmund Decomposition

This section is devoted to the Calderón–Zygmund decomposition.

Theorem 1.4.1. *Let $p \in [1, \infty)$. Then, for any $f \in L^p(\mu)$ and any $\lambda \in (0, \infty)$ (with $\lambda \in (2^{D+1}\|f\|_{L^p(\mu)}/\|\mu\|, \infty)$ if $\|\mu\| < \infty$),*

- (a) *there exists a family $\{Q_i\}_i$ of almost disjoint cubes, that is, $\sum_i \chi_{Q_i} \leq C$, such that*

$$\frac{1}{\mu(2Q_i)} \int_{Q_i} |f(x)|^p d\mu(x) > \frac{\lambda^p}{2^{D+1}}, \quad (1.4.1)$$

$$\frac{1}{\mu(2\eta Q_i)} \int_{\eta Q_i} |f(x)|^p d\mu(x) \leq \frac{\lambda^p}{2^{D+1}} \text{ for all } \eta \in (2, \infty) \quad (1.4.2)$$

and

$$|f(x)| \leq \lambda \text{ for } \mu\text{-almost every } x \in \mathbb{R}^D \setminus \left(\bigcup_i Q_i \right); \quad (1.4.3)$$

- (b) *for each i , let R_i be a $(6, 6^{D+1})$ -doubling cube concentric with Q_i with*

$$\ell(R_i) > 4\ell(Q_i) \quad \text{and} \quad \omega_i := \chi_{Q_i} / \left(\sum_k \chi_{Q_k} \right).$$

Then there exists a family $\{\varphi_i\}_i$ of functions such that, for each i and μ -almost every $x \in \mathbb{R}^D$, $\varphi_i(x) = 0$ if $x \notin R_i$, and φ_i has a constant sign on R_i ,

$$\int_{\mathbb{R}^D} \varphi_i(x) d\mu(x) = \int_{Q_i} f(x) \omega_i(x) d\mu(x) \quad (1.4.4)$$

and

$$\sum_i |\varphi_i(x)| \leq B\lambda \text{ for } \mu\text{-almost every } x \in \mathbb{R}^D, \quad (1.4.5)$$

where B is some positive constant and, when $p = 1$, it holds true that

$$\|\varphi_i\|_{L^\infty(\mu)} \mu(R_i) \leq C \int_{Q_i} |f(x)| d\mu(x) \quad (1.4.6)$$

or, when $p \in (1, \infty)$, it holds true that

$$\left[\int_{R_i} |\varphi_i(x)|^p d\mu(x) \right]^{1/p} [\mu(R_i)]^{1/p'} \leq \frac{C}{\lambda^{p-1}} \int_{Q_i} |f(x)|^p d\mu(x), \quad (1.4.7)$$

here above and in what follows, for $p \in [1, \infty]$, p' stands for the conjugate index of p , namely, $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. Since the proof in the case that $\|\mu\| < \infty$ is similar, we only consider the case that $\|\mu\| = \infty$. Taking into account Proposition 1.2.2 and Theorem 1.3.2, for μ -almost every $x \in \mathbb{R}^D$ such that $|f(x)|^p > \lambda^p$, there exists some cube Q_x satisfying that

$$\frac{1}{\mu(2Q_x)} \int_{Q_x} |f(x)|^p d\mu(x) > \frac{\lambda^p}{2^{D+1}} \quad (1.4.8)$$

and such that, if \hat{Q}_x is centered at x with $\ell(\hat{Q}_x) > 2\ell(Q_x)$, then

$$\frac{1}{\mu(2\hat{Q}_x)} \int_{\hat{Q}_x} |f(x)|^p d\mu(x) \leq \frac{\lambda^p}{2^{D+1}}.$$

Now we apply Theorem 1.1.1 to obtain an almost disjoint subfamily $\{Q_i\}_i$ of cubes satisfying (1.4.1), (1.4.2) and (1.4.3). Indeed, if

$$\Omega := \{x \in \mathbb{R}^D : |f(x)|^p > \lambda^p\}$$

is bounded, then the existence of $\{Q_i\}_i$ comes from Theorem 1.1.1 directly. Otherwise, we choose a cube Q_0 centered at the origin big enough such that

$$2^{D+1} \|f\|_{L^p(\mu)}^p / \mu(Q_0) < \lambda.$$

Then, for any cube Q containing Q_0 , we have

$$2^{D+1} \|f\|_{L^p(\mu)}^p / \mu(Q) < \lambda. \quad (1.4.9)$$

For any $m \in \mathbb{Z}_+$, let $Q_m := (5/4)^m Q_0$. Now we apply Theorem 1.1.1 to

$$(Q_m \setminus Q_{m-1}) \cap \Omega$$

(if $m = 0$ then we apply Theorem 1.1.1 to $Q_0 \cap \Omega$) and Q_x centered at

$$x \in \text{supp } \mu \cap (Q_m \setminus Q_{m-1}) \cap \Omega$$

to obtain a sequence $\{Q_{m_i}\}_{i \in \Lambda_m}$.

Now (a) is reduced to showing that the sequence $\{Q_{m_i}\}_{i \in \Lambda_m, m \in \mathbb{Z}_+}$ also has the finite overlapping property. To this end, we first claim that there exists some constant N_0 such that $Q_x \subset Q_{m+N_0}$ for all $m \in \mathbb{Z}_+$ and $x \in Q_m \setminus Q_{m-1}$. Indeed, for any $m \in \mathbb{Z}_+$ and $x \in Q_m$, we see that $Q_0 \subset Q(x, 2\ell(Q_m))$. Then, if $\ell(Q_x) > \ell(Q_m)$, we would have $Q_0 \subset 2Q_x$, which implies that $2Q_x$ satisfies (1.4.9). This contradicts (1.4.8). Thus, we conclude that $\ell(Q_x) \leq \ell(Q_m)$, from which the claim follows. Furthermore, it is not difficult to see that there exist \tilde{N}_0 and M which is big enough and depends on \tilde{N}_0 such that, for all $m \geq M$ and $x \in Q_m \setminus Q_{m-1}$,

$$Q_x \subset Q_{m+N_0} \setminus Q_{m-\tilde{N}_0}.$$

This further implies that, for all $m \geq M$ and $x \in Q_m \setminus Q_{m-1}$,

$$\sum_{m \in \mathbb{Z}_+, m \geq M, i \in \Lambda_m} \chi_{Q_{m_i}}(x) \leq (N_0 + \tilde{N}_0 + 1)N_D,$$

where N_D is as in Theorem 1.1.1. On the other hand, by Theorem 1.1.1, we know that, for all $m \leq M-1$ and $x \in Q_m \setminus Q_{m-1}$,

$$\sum_{m \in \mathbb{Z}_+, m \leq M-1, i \in \Lambda_m} \chi_{Q_{m_i}}(x) \leq MN_D.$$

Thus, by these two facts, we conclude that the sequence $\{Q_{m_i}\}_{m \in \mathbb{Z}_+, i \in \Lambda_m}$ has the finite overlapping property.

To prove (b), assume first that the family of cubes, $\{Q_i\}_i$, is finite. We may further suppose that this family of cubes is ordered in such a way that the sizes of the cubes $\{R_i\}_i$ are non decreasing (namely $\ell(R_{i+1}) \geq \ell(R_i)$ for all i). The functions φ_i that we now construct are of the form $\varphi_i = \alpha_i \chi_{A_i}$ with $\alpha_i \in \mathbb{R}$ and $A_i \subset R_i$ such that $\mu(A_i) \geq \mu(R_i)/2$. We let $A_1 := R_1$ and $\varphi_1 := \alpha_1 \chi_{R_1}$, where the constant α_1 is chosen such that

$$\int_{Q_1} f(z)w_1(z) d\mu(z) = \int_{\mathbb{R}^D} \varphi_1(z) d\mu(z).$$

Suppose that $\varphi_1, \dots, \varphi_{k-1}$ have been constructed, satisfying (1.4.4) and

$$\sum_{i=1}^{k-1} |\varphi_i| \leq B\lambda,$$

where B is some constant which is fixed below.

Let $\{R_{s_1}, \dots, R_{s_m}\}$ be the subfamily of $\{R_1, \dots, R_{k-1}\}$ such that $R_{s_j} \cap R_k \neq \emptyset$ and $\{\varphi_{s_j}\}_{j=1}^m$ the corresponding functions. We claim that there exists some positive constant C_1 such that

$$\mu \left(\left\{ x \in \mathbb{R}^D : \sum_j |\varphi_{s_j}(x)| > 2C_1\lambda \right\} \right) \leq \frac{\mu(R_k)}{2}.$$

Indeed, if all $\{R_1, \dots, R_{k-1}\}$ are disjoint with R_k , then the claim holds true automatically. Otherwise, since $\ell(R_{s_j}) \leq \ell(R_k)$ (because of the non decreasing sizes of $\{R_i\}_i$), it follows that $R_{s_j} \subset 3R_k$. Taking into account that, for $i \in \{1, \dots, k-1\}$,

$$\int_{\mathbb{R}^D} |\varphi_i(x)| d\mu(x) \leq \int_{Q_i} |f(x)| d\mu(x),$$

using that R_k is $(6, 6^{D+1})$ -doubling, together with the finite overlapping property of $\{Q_i\}_i$ and (1.4.2), we conclude that there exists a positive constant C_1 such that

$$\begin{aligned} \sum_j \int_{\mathbb{R}^D} |\varphi_{s_j}(x)| d\mu(x) &\leq \sum_j \int_{Q_{s_j}} |f(x)| d\mu(x) \\ &\lesssim \int_{3R_k} |f(x)| d\mu(x) \\ &\lesssim \left[\int_{3R_k} |f(x)|^p d\mu(x) \right]^{1/p} [\mu(3R_k)]^{1/p'} \\ &\lesssim \lambda [\mu(6R_k)]^{1/p} [\mu(3R_k)]^{1/p'} \\ &\leq C_1 \lambda \mu(R_k). \end{aligned}$$

This implies the claim.

Let

$$A_k := R_k \cap \left\{ x \in \mathbb{R}^D : \sum_j |\varphi_{s_j}(x)| \leq 2C_1\lambda \right\}$$

and $\varphi_k := \alpha_k \chi_{A_k}$, where the constant α_k satisfies that

$$\int_{\mathbb{R}^D} \varphi_k(z) d\mu(z) = \int_{Q_k} f(z) w_k(z) d\mu(z).$$

Notice that $\mu(A_k) \geq \mu(R_k)/2$. By this fact, together with (1.4.2), we then see that there exists a positive constant C_2 such that

$$|\alpha_k| \leq \frac{1}{\mu(A_k)} \int_{Q_k} |f(x)| d\mu(x) \leq \frac{2}{\mu(R_k)} \int_{\frac{1}{2}R_k} |f(x)| d\mu(x) \leq C_2 \lambda$$

(this calculation also applies to $k = 1$). Thus, we find that, for all $x \in \mathbb{R}^D$,

$$|\varphi_k(x)| + \sum_{j=1}^{k-1} |\varphi_j(x)| \leq (2C_1 + C_2)\lambda.$$

Therefore, (1.4.5) holds true for all k , if we take $B := 2C_1 + C_2$. Also, if $p = 1$, then, by the choices of A_i and φ_i , we have

$$\|\varphi_i\|_{L^\infty(\mu)} \mu(R_i) \lesssim |\alpha_i| \mu(A_i) \sim \left| \int_{\mathbb{R}^D} f(x) w_i(x) d\mu(x) \right| \lesssim \int_{Q_i} |f(x)| d\mu(x).$$

This implies (1.4.6). If $p \in (1, \infty)$, then we conclude that

$$\begin{aligned} \left[\int_{R_i} |\varphi_i(x)|^p d\mu(x) \right]^{1/p} [\mu(R_i)]^{1/p'} &= |\alpha_i| [\mu(A_i)]^{1/p} [\mu(R_i)]^{1/p'} \\ &\lesssim |\alpha_i| \mu(A_i) \\ &\sim \left| \int_{Q_i} f(x) w_i(x) d\mu(x) \right| \\ &\lesssim \left[\int_{Q_i} |f(x)|^p d\mu(x) \right]^{1/p} [\mu(Q_i)]^{1/p'}. \end{aligned}$$

On the other hand, from (1.4.1), it follows that

$$\left[\int_{Q_i} |f(x)|^p d\mu(x) \right]^{1/p} [\mu(2Q_i)]^{1/p'} \lesssim \frac{1}{\lambda^{p-1}} \int_{Q_i} |f(x)|^p d\mu(x). \quad (1.4.10)$$

By these two facts, we obtain (1.4.7).

Suppose now that the collection $\{Q_i\}_i$ of cubes is not finite. For each fixed N , we consider the family $\{Q_i\}_{1 \leq i \leq N}$ of cubes. Then, by the argument as above, we construct functions, $\varphi_1^N, \dots, \varphi_N^N$, with $\text{supp } \varphi_i^N \subset R_i$ satisfying

$$\begin{aligned} \int_{\mathbb{R}^D} \varphi_i^N(x) d\mu(x) &= \int_{Q_i} f(x) w_i(x) d\mu(x), \\ \sum_{i=1}^N |\varphi_i^N| &\leq B\lambda \end{aligned} \quad (1.4.11)$$

and, when $p = 1$, it holds true that

$$\|\varphi_i^N\|_{L^\infty(\mu)} \mu(R_i) \lesssim \int_{Q_i} |f(x)| d\mu(x)$$

or, when $p \in (1, \infty)$, it holds true that

$$\left[\int_{R_i} |\varphi_i^N(x)|^p d\mu(x) \right]^{1/p} [\mu(R_i)]^{1/p'} \lesssim \frac{1}{\lambda^{p-1}} \int_{Q_i} |f(x)|^p d\mu(x).$$

Notice that the sign of φ_i^N equals the sign of $\int_{Q_i} f(x) w_i(x) d\mu(x)$ and hence it is independent of N .

Assume that $p = 1$. Notice that $\{\varphi_1^N\}_{N \in \mathbb{N}} \subset L^\infty(\mu)$ with uniform bound. By [110, Theorem 3.17], we know that there exists a subsequence $\{\varphi_1^k\}_{k \in I_1}$ which is convergent in the weak-* topology of $L^\infty(\mu)$ to some function $\varphi_1 \in L^\infty(\mu)$. Now we consider a subsequence $\{\varphi_2^k\}_{k \in I_2}$, with $I_2 \subset I_1$, which is also convergent in the weak-* topology of $L^\infty(\mu)$ to some function $\varphi_2 \in L^\infty(\mu)$. In general, for each j , we consider a subsequence $\{\varphi_j^k\}_{k \in I_j}$, with $I_j \subset I_{j-1}$, that converges in the weak-* topology of $L^\infty(\mu)$ to some function $\varphi_j \in L^\infty(\mu)$. Observe that the functions $\{\varphi_j\}_j$ satisfy the required properties. Indeed, it follows that⁵

$$\|\varphi_j\|_{L^\infty(\mu)} \leq \liminf_{k \rightarrow \infty} \|\varphi_j^k\|_{L^\infty(\mu)} \lesssim \frac{1}{\mu(R_j)} \int_{Q_j} |f(x)| d\mu(x),$$

which implies (1.4.6). Similarly, if $p \in (1, \infty)$, then we have (1.4.7).

Fix j . By the argument as above, we may assume that $\{\varphi_j^k\}_k$ are all nonnegative on R_j . The facts that $\{\varphi_j^k\}_k$ converges to φ_j in the weak-* topology of $L^\infty(\mu)$ and $\text{supp } \varphi_j^k \subset R_j$ lead to that, for any $\lambda \in (1, \infty)$,

$$\varphi_j(\chi_{\lambda R_j \setminus R_j} \text{sgn}(\varphi_j)) = 0,$$

where above and in what follows, $\text{sgn}(g)$ denotes the *sign function of the function* g . This implies that $\varphi_j(x) = 0$ for μ -almost every $x \in \mathbb{R}^D \setminus R_j$. Moreover, it is easy to see that φ_j satisfies (1.4.4) and, for μ -almost every $x \in R_j$, $\varphi_j(x) \geq 0$. It remains to show that $\{\varphi_j\}_j$ satisfies (1.4.5). Observe that $\{\varphi_j\}_j \subset L^1_{\text{loc}}(\mu)$. By Theorem 1.3.2, we conclude that, for any $m \in \mathbb{N}$ and μ -almost every $x \in \cup_{j=1}^m R_j$,

$$\begin{aligned} \sum_{j=1}^m |\varphi_j(x)| &= \lim_{r \rightarrow 0} \frac{1}{\mu(Q(x, r))} \int_{Q(x, r)} \sum_{j=1}^m |\varphi_j(y)| d\mu(y) \\ &= \sum_{j=1}^m \lim_{r \rightarrow 0} \frac{1}{\mu(Q(x, r))} \int_{Q(x, r)} \varphi_j(y) \text{sgn}(\varphi_j)(y) \chi_{R_j}(y) d\mu(y) \\ &= \sum_{j=1}^m \lim_{r \rightarrow 0} \lim_{k \rightarrow \infty} \frac{1}{\mu(Q(x, r))} \int_{Q(x, r)} \varphi_j^k(y) \text{sgn}(\varphi_j^k)(y) \chi_{R_j}(y) d\mu(y) \end{aligned}$$

⁵See [157, p. 125].

$$\begin{aligned}
&\leq \sum_{j=1}^m \lim_{k \rightarrow \infty} \lim_{r \rightarrow 0} \frac{1}{\mu(Q(x, r))} \int_{Q(x, r)} |\varphi_j^k(y)| d\mu(y) \\
&\leq B\lambda,
\end{aligned}$$

where, in the third-to-last inequality, we used the fact that

$$\operatorname{sgn}(\varphi_j^k)(x) = \operatorname{sgn}(\varphi_j)(x).$$

This finishes the proof of Theorem 1.4.1. \square

We now establish another version of the Calderón–Zygmund decomposition. To this end, let $\rho \in (1, \infty)$. We introduce the *maximal operator* $\mathcal{M}_{(\rho)}$ by setting, for any $f \in L^1_{\text{loc}}(\mu)$ and $x \in \mathbb{R}^D$,

$$\mathcal{M}_{(\rho)}f(x) := \sup_{Q \ni x} \frac{1}{\mu(\rho Q)} \int_Q |f(y)| d\mu(y). \quad (1.4.12)$$

Theorem 1.4.2. *Let $f \in L^1(\mu)$. For $\lambda \in (0, \infty)$ (with $\lambda \in (2^{D+1}\|f\|_{L^1(\mu)}/\|\mu\|, \infty)$ if $\|\mu\| < \infty$), let*

$$\Omega_\lambda := \{x \in \mathbb{R}^D : \mathcal{M}_{(2)}f(x) > \lambda\}.$$

Then Ω_λ is open and $|f| \leq 2^{D+1}\lambda$ μ -almost everywhere in $\mathbb{R}^D \setminus \Omega_\lambda$. Moreover, if letting the cubes $\{Q_i\}_i$ be the Whitney decomposition of Ω_λ , then

(a) *for each i , there exists a function $\omega_i \in C^\infty(\mathbb{R}^D)$ with $\operatorname{supp} \omega_i \subset \frac{3}{2}Q_i$,*

$$0 \leq \omega_i \leq 1 \text{ and } \|\nabla \omega_i\|_{L^\infty(\mu)} \leq C \ell(Q_i)^{-1}$$

such that $\sum_i \omega_i \equiv 1$ if $x \in \Omega_\lambda$;

(b) *for each i , let R_i be the smallest $(6, 6^{D+1})$ -doubling cube of the form $6^k Q_i$, $k \in \mathbb{N}$, with $R_i \setminus \Omega_\lambda \neq \emptyset$. Then there exists a sequence $\{\alpha_i\}_i$ of functions such that, for each i and μ -almost every $x \in \mathbb{R}^D$, $\alpha_i(x) = 0$ if $x \notin R_i$,*

$$\int_{\mathbb{R}^D} \alpha_i(x) d\mu(x) = \int_{Q_i} f(x) \omega_i(x) d\mu(x), \quad (1.4.13)$$

$$\|\alpha_i\|_{L^\infty(\mu)} \mu(R_i) \leq C \int_{Q_i} |\alpha_i(x)| d\mu(x) \quad (1.4.14)$$

and

$$\sum_i |\alpha_i(x)| \leq \tilde{B}\lambda \text{ for } \mu\text{-almost every } x \in \mathbb{R}^D, \quad (1.4.15)$$

where C and \tilde{B} are some positive constants;

(c) f can be written as $f := g + b$, where

$$g := f \left[1 - \sum_i \omega_i \right] + \sum_i \alpha_i, \quad b := \sum_i (f \omega_i - \alpha_i)$$

and $\|g\|_{L^\infty(\mu)} \lesssim \lambda$.

Proof. The set Ω_λ is open, because $\mathcal{M}_{(2)}$ is lower semi-continuous. Since, for μ -almost every $x \in \mathbb{R}^D$, there exists a sequence of $(2, 2^{D+1})$ -doubling cubes centered at x with side length tending to zero, it follows that, for μ -almost every $x \in \mathbb{R}^D$ such that $|f(x)| > 2^{D+1}\lambda$, there exists some $(2, 2^{D+1})$ -doubling cube Q centered at x with

$$\int_Q |f| d\mu / \mu(Q) > 2^{D+1}\lambda$$

and hence $\mathcal{M}_{(2)}f(x) > \lambda$. Therefore, for μ -almost every $x \in \mathbb{R}^D \setminus \Omega_\lambda$, we find that $|f(x)| \leq 2^{D+1}\lambda$.

The existence of the function ω_i of (a) is a standard known fact. Moreover, since $R_i \setminus \Omega_\lambda \neq \emptyset$ for each i , we see that

$$\int_{R_i} |f(x)| d\mu(x) \leq \lambda \mu(2R_i).$$

By an argument used in the proofs for (1.4.4), (1.4.5) and (1.4.6), together with this observation, we further obtain (b).

Finally, from (a), we deduce that

$$\text{supp} \left(f \left(1 - \sum_i w_i \right) \right) \subset \mathbb{R}^D \setminus \Omega_\lambda.$$

Observe that $\sum_i w_i \lesssim 1$. Then we have

$$\left\| f \left(1 - \sum_i w_i \right) \right\|_{L^\infty(\mu)} \lesssim \lambda.$$

On the other hand, if (b) holds true, then we see that $\|\sum_i \alpha_i\| \lesssim \lambda$ and hence (c) holds true. This finishes the proof of Theorem 1.4.2. \square

1.5 Notes

- The original theorem of Besicovitch deals with Euclidean balls in \mathbb{R}^D by Besicovitch [5] and with more abstract sets by Morse [98]. Theorem 1.1.1 was given by M. de Guzmán [23, pp. 2–5].
- The maximal functions, $\mathcal{M}^{(\rho)}$ and $\mathcal{M}_{(\rho)}$, were introduced by Tolsa [131]. Tolsa also showed that $\mathcal{M}^{(\rho)}$ and $\mathcal{M}_{(\rho)}$ are both bounded on $L^p(\mu)$ for all $p \in (1, \infty)$ and from $L^1(\mu)$ to $L^{1,\infty}(\mu)$. When $\rho = 1$, Journé [75, p. 10] proved that $\mathcal{M}^{(1)}$ is not bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$. Thus, the assumption that $\rho \in (1, \infty)$ plays a key role here. Sawano [112] also showed that the *non-centered maximal operator* $M_\rho(f)$, with $\rho \in (1, \infty)$, is bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$ by establishing a new covering lemma, where, for all $f \in L^1_{\text{loc}}(\mu)$ and $x \in \mathbb{R}^D$,

$$M_\rho(f)(x) := \sup_{B \ni x} \frac{1}{\mu(\rho B)} \int_B |f(y)| d\mu(y)$$

and the supremum is taken over all the balls B of \mathbb{R}^D such that $B \ni x$.

Let (\mathcal{X}, d, μ) be a metric measure space such that μ only satisfies the *polynomial growth condition* as in (0.0.1) with $B(x, r)$ replaced by

$$B(x, r) := \{y \in \mathcal{X} : d(y, x) < r\}.$$

For all $f \in L^1_{\text{loc}}(\mathcal{X}, \mu)$ and $x \in \mathcal{X}$, the *centered Hardy–Littlewood maximal operator* $\tilde{M}_\rho(f)$, with $\rho \in [2, \infty)$, is defined by setting

$$\tilde{M}_\rho(f)(x) := \sup_{r \in (0, \infty)} \frac{1}{\mu(B(x, \rho r))} \int_{B(x, r)} |f(y)| d\mu(y). \quad (1.5.1)$$

In [103], Nazarov et al. showed that, when $\rho = 3$ in (1.5.1), \tilde{M}_ρ is bounded on $L^p(\mathcal{X}, \mu)$ for all $p \in (1, \infty]$ and from $L^1(\mathcal{X}, \mu)$ to $L^{1,\infty}(\mathcal{X}, \mu)$. Later, using an outer measure, Terasawa in [128] extended the aforementioned result in [103] to any $\rho \in [2, \infty)$. In [112], Sawano further showed that $\rho = 2$ is sharp for the boundedness of \tilde{M}_ρ by giving a counterexample.

- Example 1.2.3 was given by Verdera in [145].
- Example 1.2.4 was given by Tolsa in [137]; see also [37].
- The notion of doubling cubes was introduced by Tolsa in [131].
- Theorem 1.3.2 was established by Tolsa in [131].
- Theorem 1.4.1 was established by Tolsa in [131] (see also [133]), and Theorem 1.4.2 proved by Tolsa in [135]. Another Calderón–Zygmund type decomposition was established by Mateu et al. in [94].

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