

Chapter 1

Introduction

1.1 Motivation of the Exit Time Problem from Climate Dynamics

Our primary interest in this book lies in the study of dynamical properties of reaction-diffusion equations perturbed by Lévy noise of intensity ε in the small noise limit $\varepsilon \rightarrow 0$. The material of the book is based on the Ph.D. thesis [Hög11] by M. Högele. Typically, a reaction diffusion equation we consider is supposed to possess two domains of attraction connected by a separating manifold. Without perturbations by noise, the system's solution trajectories would relax to the stable equilibrium of the domain of attraction in which they are started. If noise is turned on, spontaneous transitions from one domain of attraction to the other one become possible, through large deviations of the noisy system in the Gaussian case, and eventually through jumps in the case of more general Lévy noise. In any case, additive noise transforms the stable states in the domains of attraction into *metastable* ones with characteristic transition times depending on the noise amplitude. One of the main problems we shall address is concerned with describing the asymptotic order of time as a function of noise amplitude ε it takes the system to switch from one domain of attraction to the other one—or from one *metastable regime* to the other one—in the small noise limit $\varepsilon \rightarrow 0$. In the Gaussian case, the transition dynamics has been intensively studied and well understood mainly on the basis of the Freidlin–Wentzell theory of noisy perturbations of dynamical systems. As will become clear below, in the case of non-Gaussian Lévy noise, this involves detailed and subtle estimates on the time the system will spend in neighborhoods of the separating manifold. Primarily for this reason, we chose to restrict our attention on one particular class of reaction-diffusion equations, the Chafee–Infante equation described in more detail below. As one of its main features, the Chafee–Infante equation exhibits two domains of attraction connected by a smooth separating manifold the globally complex structure of which is well understood. This will enable us to assess questions about residence times in its small

neighborhoods to a degree that suffices to derive the global features of the dynamics of transitions. The need to have a more detailed understanding of the meandering of trajectories of the noisy system near parts of a complex separating manifold is the only reason for us to confine our study to this particular class of reaction-diffusion equations with two domains of attraction. We are confident that our general line of reasoning applies to a much more general class of reaction-diffusion equations for instance with finitely many domains of attraction. The main obstacle to overcome in a generalization consists in formulating conditions on the noise which guarantee that the system does not get caught for too long in neighborhoods of manifolds separating domains of attraction the structure of which should be sufficiently well described for this purpose. We refrain from formulating such conditions here, and leave generalizations to other systems of reaction-diffusion equations for further research. Our initial motivation to look for problems of this kind originates in a climate dynamics context. Roughly, the two domains of attraction have to be interpreted as two stable climate states in a conceptual energy balance type climate model. In a noisy environment, they describe metastable states of the global averaged temperature, typically cold and warm states. The guiding question asked concerns typical times for transitions between them triggered by noise.

Let us introduce the main object of our study, the Chafee–Infante equation perturbed by Lévy noise, as one of the simplest idealized semilinear stochastic reaction-diffusion equations. Of course, the asymptotic study of its dynamics in the small noise limit possesses interest independently of any particular background in which it may arise. Some of the intuition behind its main terms will be motivated by briefly looking at this simple climate dynamics background.

Noisy energy balance models aim at describing qualitative features of the global temperature, seasonally and longitudinally averaged, as a function $X^\varepsilon(t, \zeta)$ of time and the zonal position ζ identified with a point on the unit interval, perturbed by spatial–temporal noise of (small) intensity $\varepsilon > 0$. The underlying temporal evolution of temperature on the interval $[0, 1]$ limited by the poles involves random processes taking their real values in sets of functions on compact domains. This leads directly to equations in infinite-dimensional spaces, and infinite-dimensional models of noise, formally to an SPDE. In the light of our guiding example, its three components may be interpreted in the following way.

1. A *reaction* term f of the evolution equation may be seen as expressing a deterministic forcing of temperature. It derives heuristically from simple assumptions on the balance between absorbed and emitted solar radiation energy as a function of time (see [Imk01]). Absorbed energy is qualified as a function of the temperature dependent albedo function, and emitted energy by the Stefan–Boltzmann law for black body radiators as being proportional to the forth power of temperature. The resulting energy balance as a function of temperature has two stable and one unstable zero representing equilibria of a dynamical system. Hence the resulting reaction term can be described as the negative gradient of a potential function $f = -U'$ with two local minima representing a cold and a warm basic climate state.

2. A spatial *diffusion term* $\frac{\partial^2}{\partial \zeta^2} X^\varepsilon$ may be seen in our model motivation as representing heat diffusion between equator and poles which is caused by different rates of insolation due to different angles of incidence of sunlight.
3. The energy balance based reaction term and the heat diffusion term lead—in an idealized version—to a deterministic Chafee–Infante equation. According to Hasselmann’s approach (see Arnold [Arn01] and Hasselmann [Has76]) this equation may be seen to be perturbed by an *additive stochastic process* L of small intensity $\varepsilon > 0$ taking values in an appropriate function space on the interval $[0, 1]$. It represents unresolved solar and atmospheric forcing. Following the suggestion in Ditlevsen [Dit99] and Gairing et al. [GHIP11] we may take L to be of Lévy type with jump measure tails of polynomial order. The most prominent example is the case of α -stable noise.

With this motivating example in mind, let us now turn to the investigation of the dynamics of the Chafee–Infante equation from a general perspective, in particular its exit and transition dynamics between the domains of attraction of the metastable states. We will denote the solution of the deterministic Chafee–Infante equation by $u = X^0$. It formally satisfies

$$\begin{aligned} \frac{\partial}{\partial t} u(t, \zeta) &= \frac{\partial^2}{\partial \zeta^2} u(t, \zeta) + f(u(t, \zeta)), \quad \zeta \in [0, 1], \quad t > 0, \\ u(t, 0) &= u(t, 1) = 0, \quad t > 0, \\ u(0, \zeta) &= x(\zeta), \quad \zeta \in [0, 1], \end{aligned} \tag{1.1}$$

where $U(y) = (\lambda/4)y^4 - (\lambda/2)y^2$ for $\lambda > 0$ fixed, and $f = -U'$.

The solution takes values in an infinite-dimensional function space, as for example $L^2(0, 1)$, $H_0^1(0, 1)$ or $\mathcal{C}_0(0, 1)$, where also the initial state x is taken (see [Tem92] or [SY02]). Since its pure reaction term f has two zeros given by the minima of U , apart from singular values of λ , the Chafee–Infante equation possesses in a generic setting two hyperbolic stable states $\phi^+, \phi^- \in \mathcal{C}^\infty(0, 1)$. Nevertheless, there may be several unstable saddles, depending on the value of the parameter λ .

If the additive Lévy noise term of intensity $\varepsilon > 0$ is added as a perturbation to the deterministic equation, we obtain the stochastic Chafee–Infante equation of the form

$$\begin{aligned} \frac{\partial}{\partial t} X^\varepsilon(t, \zeta) &= \frac{\partial^2}{\partial \zeta^2} X^\varepsilon(t, \zeta) + f(X^\varepsilon(t, \zeta)) + \varepsilon \dot{L}(t, \zeta), \quad \zeta \in [0, 1], \quad t > 0, \\ X^\varepsilon(t, 0) &= X^\varepsilon(t, 1) = 0, \quad t > 0, \\ X^\varepsilon(0, \zeta) &= x(\zeta), \quad \zeta \in [0, 1], \end{aligned} \tag{1.2}$$

where $\lambda > 0$ and $f = -U'$. The noise term \dot{L} formally represents the generalized derivative of a pure jump Lévy process in the Sobolev space $H = H_0^1(0, 1)$ with Dirichlet boundary conditions, regularly varying Lévy measure of index $\alpha \in (0, 2)$ and initial value $x \in H$.

For the one-dimensional counterpart of (1.2) without diffusion term Imkeller and Pavlyukevich investigate the asymptotic behavior of exit and transition times in the small noise limit in [IP06a, IP08] and [IP06b]. In contrast to the Wiener case, for which exponential growth with respect to the noise intensity is observed in [FV98], these models feature exit rates with polynomial growth in the limit of small noise. Accordingly, the critical time scale in which the global metastable behavior of the jump diffusion can be reduced to a finite state Markov chain jumping between the metastable states (see also [BEGK04]) is equally polynomial in the noise intensity.

In this book we shall be primarily concerned with the question: *To which extent do these results still hold true in the infinite dimensional Chafee–Infante reaction-diffusion framework, with corresponding infinite-dimensional noise?*

We shall show in Theorem 5.11 that the expected exit time from (reduced) domains of attraction of the metastable states ϕ^+, ϕ^- increases polynomially of order $\varepsilon^{-\alpha}$ in the limit of small noise intensity ε , and characterize the exit scenarios. We shall also show in Theorem 7.10 that for this time scale of ε the jump diffusion system reduces to a finite state Markov chain with values in the set of stable states $\{\phi^+, \phi^-\}$. Our analysis can be considered as a starting point for studying metastable behavior of dynamical systems induced by reaction-diffusion equations perturbed by Lévy jump noise on a more general basis. We also note that our model gives rise to order preserving random dynamical systems (see [Chu01]). This property potentially has in store further information on qualitative asymptotic behavior of the system, for instance on the structure of its pullback attractors.

1.2 Heuristics for the First Exit Times: Noise Decomposition into Small and Large Jumps

The study of exit times from domains of attraction will be the main ingredient of our investigation of the dynamical properties of the Chafee–Infante equation perturbed by Lévy noise. In this section we explain the heuristics of the method to determine the expected first exit time for a domain of attraction of the stable states ϕ^\pm in the asymptotics of small noise intensity. In doing this, we extend the arguments given in [IP08] for dimension 1 which proceed along the following lines.

Step 1. A detailed study of the stable solutions as well as the separating manifold of the deterministic Chafee–Infante equation leads to the construction of reduced versions $D^\pm(\varepsilon^\gamma) \subset D^\pm$ of the domains of attraction D^\pm of the stable solutions ϕ^\pm such that the solution $u(t; x)$ of the Chafee–Infante equation starting in $x \in D^\pm(\varepsilon^\gamma)$ finds itself within a small neighborhood $B_{\varepsilon^{2\gamma}}(\phi^\pm)$ at times t exceeding $T_{\text{rec}} + \kappa\gamma |\ln \varepsilon|$. Here T_{rec} is a global relaxation time and $\kappa > 0$ a global constant, formally

$$u(t; x) \in B_{\varepsilon^{2\gamma}}(\phi^\pm) \quad \text{for all } t \geq T_{\text{rec}} + \kappa\gamma |\ln \varepsilon| \quad \text{and } x \in D^\pm(\varepsilon^\gamma). \quad (1.3)$$

Step 2. For a threshold $c > 0$ we recursively define the sequence of jump times of the driving Lévy process L with values in H exceeding c by

$$T_{i+1} := \inf\{t > T_i \mid \|\Delta_t L\| > c\}, \quad T_0 = 0,$$

where for $t \geq 0$ and a process Y we write $\Delta_t Y = Y(t) - Y(t-)$. If $(S(t))_{t \geq 0}$ is the Markovian semigroup associated with the diffusion operator on $(0, 1)$, and we use the mild solution formulation following [PZ07], the jumps of X^ε are just the jumps of L , i.e.

$$\Delta_{T_i} X^\varepsilon = \Delta_{T_i} \int_0^\cdot S(\cdot - s) dL(s) = \Delta_{T_i} L, \quad i \in \mathbb{N}. \quad (1.4)$$

We let the threshold c depend on ε , and choose $c = c(\varepsilon) = \frac{1}{\varepsilon^\rho}$ for $\rho \in (0, 1)$ to split $L(t) = \xi^\varepsilon(t) + \eta^\varepsilon(t)$ into a small jump part ξ^ε , with

$$\varepsilon \|\Delta_t \xi^\varepsilon\| \leq \varepsilon \frac{1}{\varepsilon^\rho} \rightarrow 0, \quad \varepsilon \rightarrow 0+ \quad (1.5)$$

and a large jump part η^ε , with $\eta^\varepsilon(t) = \sum_{i: T_i \leq t} \Delta_{T_i} L$, $t \geq 0$. Between two large jump times T_i and T_{i+1} , the strong Markov property allows us to consider X^ε as being driven by the small jump component $\varepsilon \xi^\varepsilon$ alone. Denote this process by Y^ε . In finite dimensions Y^ε is directly seen to deviate after a deterministic uniform relaxation time s_{r^*} to a large ball $B_{r^*}(0)$ only negligibly from the deterministic solution u uniformly on time intervals of the order of its inter-jump waiting times $t_{i+1} = T_{i+1} - T_i$. Formally

$$\sup_{x \in D^\pm(\varepsilon^\gamma) \cap B_{r^*}(0)} \sup_{T_i \leq t \leq T_{i+1}} \|Y^\varepsilon(t) - u(t)\| \rightarrow 0 \quad \text{for } \varepsilon \rightarrow 0+ \quad (1.6)$$

in probability. This means that as long as there are no large jumps the solution of the Chafee–Infante equation follows the deterministic solutions on their way to relaxation in the neighborhoods of stable equilibria. Therefore they cannot contribute essentially to exits from their domains of attraction. Exits from these domains will thus be triggered by large jumps. Since in infinite dimensions we solve our equation in a mild sense we establish instead of (1.6) that the small deviation result for Y^ε is implied by

$$\varepsilon \xi^*(t) \rightarrow 0, \quad \varepsilon \rightarrow 0+, \text{ for } t \geq 0.$$

Here $\xi^*(t) = \int_0^t S(t-s) d\xi^\varepsilon(s)$ is the stochastic convolution with respect to ξ^ε (see Sect. 3.3).

Step 3. The inter-jump waiting times of η^ε are all independent and possess exponential laws of parameter β_ε , where

$$\beta_\varepsilon := \nu \left(\frac{1}{\varepsilon^\rho} B_1^c(0) \right) \approx \varepsilon^{\alpha\rho},$$

and ν is the jump measure of L for which we assume that it varies regularly of index α . In accordance with exponential laws, they are therefore expected to be of order $\frac{1}{\varepsilon^\alpha}$. For small ε this quantity is much bigger than the ε -dependent component of the relaxation time $T_{rec} + \kappa\gamma|\ln \varepsilon|$ of the deterministic solution u to $B_{\varepsilon^{2\gamma}}(\phi^\pm)$. We can therefore expect that Y^ε has had enough time to relax to a neighborhood of a stable solution before the next big jump occurs, without leaving the reduced domain in the meantime. This jump therefore originates from a position close to an equilibrium. The effects sketched in (1.4), (1.3) and (1.6) therefore combine, and imply that for small ε exit events start in $B_{\varepsilon^{2\gamma}}(\phi^\pm)$ and are most probably triggered by the large jump part $\varepsilon\eta^\varepsilon$. Hence the first exit time $\tau(\varepsilon)$ from D^\pm is expected to be roughly

$$\tau(\varepsilon) \approx \inf\{T_i = \sum_{j=1}^i t_j \mid \phi^\pm + \varepsilon\Delta_{t_i}L \notin D^\pm\}.$$

Step 4. Using the regular variation of the Lévy measure ν of L we obtain for the probability of large jumps big enough to trigger exits

$$\begin{aligned} \mathbb{P}(\phi^\pm + \varepsilon\Delta_{t_i}L \notin D^\pm) &= \mathbb{P}\left(\Delta_{t_i}L \in \frac{1}{\varepsilon}((D^\pm)^c - \phi^\pm)\right) \\ &= \frac{\nu\left(\frac{1}{\varepsilon}((D^\pm)^c - \phi^\pm) \cap \frac{1}{\varepsilon^\rho}B_1^c(0)\right)}{\nu\left(\frac{1}{\varepsilon^\rho}B_1^c(0)\right)} \approx \varepsilon^{\alpha(1-\rho)}. \end{aligned}$$

Therefore exits times from reduced domains of attraction of the stable equilibria in the limit of small noise are given by

$$\begin{aligned} \mathbb{E}[\tau(\varepsilon)] &\approx \sum_{i=1}^{\infty} \mathbb{E}[T_i] \mathbb{P}(\inf\{j : \phi^\pm + \varepsilon\Delta_{t_j}L \notin D^\pm\} = i) \\ &\approx \mathbb{E}[t_1] \mathbb{P}(\phi^\pm + \varepsilon\Delta_{t_1}L \notin D^\pm) \sum_{i=1}^{\infty} i (1 - \mathbb{P}(\phi^\pm + \varepsilon\Delta_{t_1}L \notin D^\pm))^{i-1} \\ &\approx \frac{1}{\varepsilon^{\alpha\rho}} \varepsilon^{\alpha(1-\rho)} \left(\frac{1}{\varepsilon^{\alpha(1-\rho)}}\right)^2 = \frac{1}{\varepsilon^\alpha}. \end{aligned}$$

1.3 A Glance at Related Literature

To the best of our knowledge the method of this work sketched in Sect. 1.2 has not been used in the context of SPDEs so far. We shall therefore only give an overview over parts of the literature to which our attention had been drawn in the course of these studies. We do not claim completeness.

The Chafee–Infante equation has been extensively studied, starting with the article by [CI74]. Its most interesting feature is a bifurcation in the system parameter representing the steepness of the potential. This considerably changes the dynamics in comparison to the finite dimensional case, see for example [CP89]. Other classical references are the books by [Hen83] and Hale [Hal83]. Existence and regularity of its solutions have been investigated, as well as the fine structure of the attractor. We refer to the books [Tem92, CH98, Rob01, Chu02] and references therein.

SPDE with Gaussian noise go back to the seventies with early works by the authors of [Par75, KR07] and [Wal81, Fre85, Wal86]. Since then the field has expanded enormously in depth and variety, as is impressively documented recently for example in [KRAD⁺08]. More recent treatments can be found among others for instance in the books and articles [DZ92, Cho07, PR07, Kot08, CF11, CFO11, Hai11, HRW12, Hai13] and references therein.

The treatment of the asymptotic dynamical behavior for finite dimensional Gaussian diffusions mainly by techniques related to large deviations was developed in [FV70, FV98]. In [FJL82], the authors use methods based on large deviations in order to analyze the stochastic dynamics for SPDE with Gaussian noise. The *tunneling effects* they discover interpret the phenomenon of metastable behavior of solutions switching between stable equilibria at time scales exponential in the noise intensity. Additionally they show that the transitions asymptotically take place at the saddle points, the number of which varies according to the bifurcation scenarios of the deterministic part. Martinelli et al. [MOS89] show that suitably normalized exit times are asymptotically exponential. Brascosco [Bra91] shows that the process is asymptotically concentrated in balls around the stable states and that the average along trajectories remains close to the stable state before the switching time.

SPDEs with jump noise have been studied since the late eighties, see for example [CM87] and [KPA88]. At the end of the nineties the subject is taken up again in a rich and ongoing series of articles for example by the authors of [AWZ98, Mue98, Bie98, AW00, FR00, Fou00, Fou01, Myt02, Kno04, Sto05, Hau05, Hau06, BW06, PZ06, RZ07, MPR10, FTT10a, FTT10b, DX10, Pré10, Xie10, Wu10, PZ10, PXZ11]. We refer to the monograph [PZ07] for a comprising view on SPDEs with Lévy noise and the bibliography therein.

1.4 Organization of the Book

The material in this book is organized as follows.

In Chap. 2 we study properties of the solution of the deterministic Chafee–Infante equation (1.1). Some of them, which are useful for our purposes and well-known in the literature for a long time are collected in Sect. 2.1. Among them are for instance the uniform absorption of a large ball by the global attractor in H , as well as its precise complex geometric structure. The subsequent Sect. 2.2 is dedicated to the construction of forward invariant subdomains of attraction with respect to the solution flow, appropriately reduced in several steps with respect to a parameter ε .

In fact, the aim is to retain a fortiori forward invariance for these reduced domains of attraction with respect to ε -dependent tubes around trajectories of the deterministic solution.

The remainder of the section combines several concepts in order to prove *Proposition 2.12*, the main result of the chapter. It states that there are constants $T_{rec}, \kappa, \varepsilon_0 > 0$ such that for all $\gamma \in (0, 1)$, $0 < \varepsilon \leq \varepsilon_0$, the deterministic solution $u(t, \zeta; x) = X^0(t, \zeta; x)$ starting from x in a reduced domain $D^\pm(\varepsilon^\gamma)$ is absorbed by the open ball $B_{\varepsilon^{2\gamma}}(\phi^\pm)$ centered in a stable fixed point after time $T_{rec} + \kappa\gamma|\ln \varepsilon|$. Formally

$$u(t; x) \in B_{\varepsilon^\gamma}(\phi^\pm) \quad \forall t \geq T_{rec} + \kappa\gamma|\ln \varepsilon|, \quad x \in D^\pm(\varepsilon^\gamma).$$

This is actually a forward analogue to the absorption result in finite dimension. But since in infinite dimensions the attractor contains generically heteroclinic connections between unstable states of the system, the question of the exit from neighborhoods of unstable states in the separating manifold has to be carefully treated. In particular for the linearization of the system in the vicinity of unstable points the Hartman–Grobman result is not appropriate due to the lack of smoothness of the conjugation maps. Instead we construct the stable and unstable manifolds and exploit their transversality in order to prove exponential repulsion from unstable states sitting on the separating manifold in Sect. 2.2.4.

In Chap. 3 we collect some basic and more advanced material about stochastic equations in infinite dimensions, with a particular view towards solutions X^ε of the stochastic Chafee–Infante equation. We introduce Lévy processes with values in Hilbert spaces, and discuss their decomposition into appropriate compound Poisson large jump components and small jump components. We give a brief introduction to the theory of stochastic integration for Lévy processes, and of global existence and uniqueness of solutions X^ε with respect to the concept of mild solutions. This is discussed along with stochastic convolutions with Lévy noise. The chapter ends with a discussion of the strong Markov property and its consequences in the particular case of the stochastic Chafee–Infante equation, and the presentation of basic material on slowly and regularly varying functions. These concepts are needed in the context of the jump measures of the driving Lévy processes arising in our stochastic equations.

Chapter 4 is devoted to the derivation of the crucial small deviation result of the solution of the Chafee–Infante equation perturbed only by the small jump part of the driving Lévy process from the solution of the deterministic equation. It is here that the technique of decomposition of the Lévy process into a small and large jump component starts taking effect. Assume for simplicity that the Lévy process L is a pure jump process with symmetric Lévy measure ν , which is regularly varying of index $\alpha \in (0, 2)$. Then $L = \xi^\varepsilon + \eta^\varepsilon$ can be decomposed into the martingale ξ^ε with jumps bounded from above $\|\Delta.L\| \leq \frac{1}{\varepsilon^\rho}$, $\rho \in (0, 1)$, and the compound Poisson process η^ε with finite intensity $\beta_\varepsilon = \nu\left(\frac{1}{\varepsilon^\rho} B_1^c(0)\right)$ and the jump measure

$\nu(\cdot \cap \frac{1}{\varepsilon^\rho} B_1^c(0)) / \beta_\varepsilon$. By regular variation of ν , the rate β_ε turns out to be of the order $\varepsilon^{\alpha\rho}$ for small ε .

For $n \in \mathbb{N}$ let T_n be the n -th jump of η^ε . Then due to the structure of the mild solution X^ε the increments $X^\varepsilon(T_1; x) - X^\varepsilon(T_1-; x)$ and $\varepsilon(L(T_1) - L(T_1-))$ coincide. By the strong Markov property it follows for $t \leq T_1$ that $X^\varepsilon(t; x) = Y^\varepsilon(t; x)$, if $Y^\varepsilon(\cdot; x)$ is the solution of (1.2), where L is replaced by the small jump martingale ξ^ε . Since $\varepsilon\xi^\varepsilon$ is of pure jump type for $t \leq T_1$ the jump increments $\|X^\varepsilon(t) - X^\varepsilon(t-)\| = \|Y^\varepsilon(t) - Y^\varepsilon(t-)\|$ equal $\varepsilon\|\xi^\varepsilon(t) - \xi^\varepsilon(t-)\|$ and hence are bounded by $\varepsilon^{1-\rho} \searrow 0$ as $\varepsilon \rightarrow 0+$. It is therefore reasonable to expect the convergence $Y^\varepsilon(t; x) \rightarrow u(t; x)$ in an appropriate sense as $\varepsilon \rightarrow 0+$. In fact in *Proposition 4.7* this turns out to true for fixed time horizon T and initial values x in a bounded subset of $D^\pm(\varepsilon^\gamma)$. In order to ensure the mentioned boundedness condition on the initial values we prove in Sect. 4.1 with the help of perturbation arguments that in the presence of bounded noise $\|\varepsilon\xi^*\| \leq 1$ the small noise solution Y^ε enters a ball $B_{r^*}(0)$ before a deterministic time $s_{r^*} > 0$.

Eventually, proceeding from deterministic to random time intervals T_1 in Sects. 4.2 and 4.3 we prove in the crucial *Proposition 4.5* that there are right choices of ρ, γ providing a constant $\vartheta > \alpha(1 - \rho)$ such that the *small deviations event*

$$E_x := \left\{ \sup_{s \in [0, s_{r^*}]} \|\varepsilon\xi^*\| \leq \varepsilon^{2\gamma}, \quad \sup_{s \in [s_{r^*}, T_1]} \|Y^\varepsilon(s; x) - u(s - s_{r^*}; Y^\varepsilon(s_{r^*}; x))\| \leq (1/2)\varepsilon^{2\gamma} \right\},$$

has small probability uniformly in the initial position x . More precisely there exists $C_\theta > 0$ and ε_0 such that for $0 < \varepsilon \leq \varepsilon_0$

$$\mathbb{P}(\cup_{x \in D^\pm(\varepsilon^\gamma)} E_x^c) \leq C_\theta \varepsilon^\vartheta.$$

Chapter 5 starts with estimates of probabilities for exit events of X^ε by those of events of the form $\{T_1 > s_{r^*} + T_{rec} + \kappa|\ln \varepsilon|\}$, E_x and $\{\phi^\pm + \varepsilon W_1 \in D^\pm(\varepsilon^\gamma)\}$, where $W_1 = X^\varepsilon(T_1) - X^\varepsilon(T_1-)$ is the size of the first big jump. Under some mild non-degeneracy conditions on the Lévy characteristics of our noise process, we are able to prove the main *Theorem 5.11* about exponential convergence of first exit times of the reduced domains of attraction $D^\pm(\varepsilon^\gamma)$. This is done in a sequence of theorems along the lines of arguments explained in Sect. 1.2, and via a calculation of Laplace transforms of exit times in the small noise limit. We eventually construct a family of random variables $\bar{\tau}(\varepsilon)_{\varepsilon > 0}$ with $\mathcal{L}(\bar{\tau}(\varepsilon)) = EXP(1)$ for all $\varepsilon > 0$ such that for all $\theta < 1$

$$\lim_{\varepsilon \rightarrow 0+} \mathbb{E} \left[\exp(\theta \lambda^\pm(\varepsilon) \tau^\pm(\varepsilon)) - \exp(\theta \bar{\tau}(\varepsilon)) \right] = 0.$$

In Chap. 6 exit times are used to investigate the asymptotic behavior of transition times between different domains of attraction of the Chafee–Infante equation. We first apply the results obtained before to estimate entering times into different

reduced domains of attraction (*Theorem 6.3*). This again leads to the description of the asymptotic behavior of the transition times between small balls around different stable equilibrium states in the small noise limit (*Theorem 6.7*).

Chapter 7 starts with a detailed discussion of an additional hypothesis on the jump characteristics of the driving Lévy process, which provides an upper bound for the time to leave neighborhoods of the separating manifold between the domains of attraction.

In Sect. 7.2 we derive two localization results for the solution of the stochastic Chafee–Infante equation on subcritical and critical time scales. Section 7.3 is devoted to the main result of this work, the description of the metastable behavior of the stochastic Chafee–Infante equation (*Theorem 7.10*). It states the convergence of the solution of the stochastic Chafee–Infante equation to a continuous time Markov chain switching between the stable states ϕ^\pm on a critical time scale which corresponds to the typical exit time scale of Chap. 5. The Markov chain’s switching rates are directly related to the mass of the centered domains of attraction $D^\pm - \phi^\pm$ with respect to the limiting measure of the regularly varying Lévy jump measure ν .

The appendix provides a more detailed treatment of some aspects of the climate physics background leading to the study of the dynamics of one-dimensional stochastic differential equations perturbed by Lévy noise. It is derived from energy balance models in [IP08], and—in an idealized version—the dynamics of the Chafee–Infante equation. We briefly review basic ideas of low dimensional models, and explain the heuristics of coupled atmosphere-ocean models investigated by Hasselmann [Has76] which in a scaling limit are believed to provide nonlinear S(P)DE describing qualitative features of climate dynamics. We finally discuss the simple class of noisy energy-balance models which, if Milankovich cycles as a source of periodic forcing are fed into the system, lead to a qualitatively correct explanation of the dynamics of global glacial periods.

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