Chapter 2
Linear Operators on Banach Spaces

Let $(\mathscr{X}, \|\cdot\|)$ and $(\mathscr{Y}, \|\cdot\|_1)$ be two Banach spaces over the same field $\mathbb{F}$. A mapping $A : D(A) \subset \mathscr{X} \to \mathscr{Y}$ satisfying

$$A(\alpha x + \beta y) = \alpha Ax + \beta Ay$$

for all $x,\ y \in D(A)$ and $\alpha, \beta \in \mathbb{F}$, is called a linear operator or a linear transformation.

In this chapter we study various properties of bounded and unbounded linear operators needed in the sequel.

2.1 Bounded Linear Operators

Definition 2.1. A linear operator $A : \mathscr{X} \mapsto \mathscr{Y}$ satisfying the following property: there exists $K \geq 0$ such that

$$\|Ax\|_1 \leq K\|x\|$$

for all $x \in \mathscr{X}$, is called a bounded (or continuous) linear operator.

The collection of all bounded linear operators from $\mathscr{X}$ into $\mathscr{Y}$ will be denoted by $B(\mathscr{X}, \mathscr{Y})$ with $B(\mathscr{X}, \mathscr{X}) := B(\mathscr{X})$.

2.1.1 Examples of Bounded Linear Operators

Example 2.2. Let $\mathscr{X} = \mathscr{Y} = \mathbb{F}^n$ equipped with its natural norm given by

$$\|x\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$$

for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{F}^n$.

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Consider the canonical base for $\mathbb{F}^n$, that is, $e_1 = (1,0,0,\ldots,0)$, $e_2 = (0,1,0,\ldots,0)$, …, $e_n = (0,0,\ldots,1)$. Thus every $x \in \mathbb{F}^n$ can be written in the orthonormal basis $(e_j)_{j=1,\ldots,n}$ as follows:

$$x = \sum_{j=1}^{n} x_j e_j \text{ for some } x_j \in \mathbb{F},\ j = 1,2,\ldots,n.$$

In particular, there exists $a_{ij} \in \mathbb{K}$ for $i, j = 1,2,\ldots,n$ such that

$$Ae_j = \sum_{i=1}^{n} a_{ij} e_i.$$

We will show that every linear operator from $\mathbb{F}^n$ into $\mathbb{F}^n$ is necessarily bounded. Indeed, let $A : \mathbb{F}^n \mapsto \mathbb{F}^n$ be a linear operator. Then, for all $x, y \in \mathbb{F}^n$ with

$$x = \sum_{j=1}^{n} x_j e_j \text{ and } y = \sum_{j=1}^{n} y_j e_j,$$

we have

$$\begin{aligned}
\|Ax - Ay\| &= \left\| \sum_{j=1}^{n} (x_j - y_j) Ae_j \right\| \\
&\le C \max(|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|) \\
&\le C\|x - y\|,
\end{aligned}$$

where $C = \sum_{j=1}^{n} \|Ae_j\| = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} |a_{ij}|^2 \right)^{1/2} < \infty$. Therefore, A is continuous.

Example 2.3. Fix $p, q \ge 1$ such that $p^{-1} + q^{-1} = 1$. Let $\mathscr{X} = L^p(\mathbb{R}^n)$ and $\mathscr{Y} = L^1(\mathbb{R}^n)$ be equipped with their natural topologies. Consider the so-called *multiplication operator* defined by $Af = Qf$ where $Q \in L^q(\mathbb{R}^n)$. Using Hölder's inequality (Proposition 1.81), it easily follows that $\|Af\|_1 \le \|Q\|_q \|f\|_p$ for all $f \in L^p(\mathbb{R})$ and hence $A \in B(L^p(\mathbb{R}^n), L^1(\mathbb{R}^n))$.

Example 2.4. Fix $a, b \in \mathbb{R}$ with $a < b$. Let $\mathscr{X} = \mathscr{Y} = C[a,b]$, that is, the collection of all continuous functions from $[a,b]$ into $\mathbb{F}$, which we equip with the sup-norm, that is, $\|\cdot\|_\infty$. Consider the so-called *Volterra operator* defined by

$$(Af)(t) = \int_a^t f(s)ds$$

for all $t \in [a,b]$ and $f \in C[a,b]$. It can be easily seen that $\|Af\|_\infty \le (b-a)\|f\|_\infty$ for all $f \in C[a,b]$, which yields $A \in B(C[a,b])$.

Example 2.5. Let $\Omega \subset \mathbb{R}^N$ be a bounded closed subset. Fix $m \in \mathbb{Z}_+$ and let $C^{(m)}(\Omega)$ be the collection of all functions $f : \Omega \mapsto \mathbb{F}$ such that $D^\alpha f$ exists and belongs to $C(\Omega)$ for $|\alpha| \leq m$. The space $C^{(m)}(\Omega)$ equipped with the norm

$$\|f\|_{m,\infty} := \max_{|\alpha| \leq m} \sup_{x \in \Omega} |D^\alpha f(x)|$$

is a Banach space.

Now let $\mathscr{X} = C(\Omega)$ be equipped with its corresponding sup-norm and let $\mathscr{Y} = C^{(m)}(\Omega)$ be equipped with its above-mentioned norm. Consider the differential operator given by

$$Af = \sum_{|\alpha| \leq m} a_\alpha D^\alpha f(x),$$

where the coefficients $a_\alpha \in \mathbb{F}$ are constants.

Obviously, A is a linear operator from $C^{(m)}(\Omega)$ to $C(\Omega)$. Moreover, it is not hard to show that A is continuous.

2.1.2 *Properties of Bounded Operators*

The identity and zero operators of $\mathscr{X}$ will be respectively denoted by I and O and are defined by $I(x) = x$ and $O(x) = 0$ for all $x \in \mathscr{X}$.

Now, if $A : \mathscr{X} \mapsto \mathscr{Y}$ is a bounded linear operator, then its kernel and range are respectively defined by $N(A) = \{x \in \mathscr{X} : Ax = 0\}$ and $R(A) = \{Ax : x \in \mathscr{X}\}$.

Proposition 2.6. *If $A, B \in B(\mathscr{X}, \mathscr{Y})$ and $\gamma \in \mathbb{F}$, then the following properties hold,*

(a) $A + B \in B(\mathscr{X}, \mathscr{Y})$.
(b) $\gamma A \in B(\mathscr{X}, \mathscr{Y})$.
(c) If $A, B \in B(\mathscr{X})$, then $AB, BA \in B(\mathscr{X})$.

Proof. (a) Using the linearity of both A and B it follows that

$$\begin{aligned}(A+B)(\lambda x + \mu y) &= A(\lambda x + \mu y) + B(\lambda x + \mu y) \\ &= \lambda Ax + \mu Ay + \lambda Bx + \mu By \\ &= \lambda (A+B)x + \mu (A+B)y,\end{aligned}$$

and hence $A + B$ is linear.

Similarly, using the triangle inequality and the continuity of both A and B, we obtain

$$\|(A+B)x\| = \|Ax + Bx\|$$

$$
\begin{aligned}
&\leq \|Ax\| + \|Bx\| \\
&\leq K\|x\| + K'\|x\| \\
&= (K+K')\|x\|,
\end{aligned}
$$

which yields $A+B$ is continuous.

(b) The proof is obvious and hence is omitted.

(c) We will prove it for AB as the proof for BA is quite similar. Using the linearity of both A and B and (b) it follows that

$$(AB)(\lambda x + \mu y) = A[B(\lambda x + \mu y)] = A[\lambda Bx + \mu By] = \lambda(AB)x + \mu(AB)y$$

and hence AB is linear. Similarly, using the continuity of both A and B, we obtain that

$$\|(AB)x\| = \|A(Bx)\| \leq K\|Bx\| = (KK')\|x\|,$$

which yields AB is bounded.

In view of Proposition 2.6, $B(\mathscr{X},\mathscr{Y})$ is a vector space. Moreover, if $A \in B(\mathscr{X},\mathscr{Y})$, then we define

$$\|A\| = \sup_{0\neq x\in\mathscr{X}} \frac{\|Ax\|_1}{\|x\|}$$

which turns out to be a norm on $B(\mathscr{X},\mathscr{Y})$ commonly called *operator-norm*. Further, it can be shown that

$$
\begin{aligned}
\|A\| &= \sup_{0\neq x\in\mathscr{X}} \frac{\|Ax\|_1}{\|x\|} \\
&= \sup_{x\in\mathscr{X}} \{\|Ax\|_1 : \|x\| \leq 1\} \\
&= \sup_{x\in\mathscr{X}} \{\|Ax\|_1 : \|x\| = 1\}.
\end{aligned}
$$

By definition of the operator-norm, $\|Ax\|_1 \leq \|A\|\cdot\|x\|$ for all $x \in \mathscr{X}$. Moreover, if $A, B \in B(\mathscr{X})$, then

$$\|AB\| \leq \|A\|\|B\|.$$

Theorem 2.7. *The space $B(\mathscr{X},\mathscr{Y})$ is a Banach space when it is equipped with its operator norm.*

Proof. In view of Proposition 2.6 it follows that $B(\mathscr{X},\mathscr{Y})$ is a normed vector space. Now let $(A_n)_{n\in\mathbb{N}} \subset B(\mathscr{X},\mathscr{Y})$ be a Cauchy sequence, that is, $\|A_n - A_m\| \to 0$ as

$n, m \to \infty$. To complete the proof we have to show that there exists $A \in B(\mathscr{X}, \mathscr{Y})$ such that $\|A_n - A\| \to 0$ as $n \to \infty$. Indeed, for all $x \in \mathscr{X}$, we have,

$$\|(A_n - A_m)x\|_1 \leq \|A_n - A_m\| \cdot \|x\| \to 0$$

as $n, m \to \infty$ and so $(A_n x)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathscr{Y}$. Since $\mathscr{Y}$ is a Banach space, it follows that there exists $\xi \in \mathscr{Y}$ such that $\|A_n x - \xi\|_1 \to 0$ as $n \to \infty$. Thus

$$Ax = \xi = \lim_{n \to \infty} A_n x$$

defines a linear operator $A : \mathscr{X} \mapsto \mathscr{Y}$. It remains to show that A is bounded. Using the fact that $(A_n)_{n \in \mathbb{N}}$ is a Cauchy sequence it follows that $\sup_{n \in \mathbb{N}} \|A_n\| < \infty$. Consequently,

$$\|Ax\|_1 = \lim_{n \to \infty} \|A_n x\|_1 \leq K\|x\|$$

for all $x \in \mathscr{X}$, that is, $A \in B(\mathscr{X}, \mathscr{Y})$.

Theorem 2.8. *If $A : \mathscr{X} \to \mathscr{Y}$ is a linear operator, then the following statements are equivalent:*

(a) A is continuous.
(b) A is continuous at 0.
(c) There exists a constant $C > 0$ such that $\|Au\|_1 \leq C . \|u\|$ for each $u \in \mathscr{X}$.

Proof. It is not hard to see that (a) yields (b). Now if (b) holds, then there exists $\eta > 0$ such that $\|Ax\|_1 \leq 1$ whenever $\|x\| \leq \eta$. Thus for each $0 \neq x \in \mathscr{X}$, we have

$$\left\| \frac{\eta x}{\|x\|} \right\| = \eta.$$

Now

$$1 \geq \left\| A\left(\frac{\eta x}{\|x\|}\right) \right\|_1 = \frac{\eta \|Ax\|_1}{\|x\|},$$

and hence $\|Ax\|_1 \leq \eta^{-1}\|x\|$, that is (c) holds.

Now if (c) holds, then

$$\|Ax - Ax_0\|_1 = \|A(x - x_0)\|_1 \leq C\|x - x_0\|.$$

Consequently, for each $\varepsilon > 0$ one can find an $\eta = \varepsilon C^{-1}$ such that $\|Ax - Ax_0\|_1 < \varepsilon$ whenever $\|x - x_0\| \leq \eta$. Therefore, A is continuous at x_0, which yields A is continuous on $\mathscr{X}$ as x_0 is arbitrary.

In the rest of this chapter, the spaces $(\mathscr{H}_1, \langle\cdot,\cdot\rangle_1, \|\cdot\|_1)$, $(\mathscr{H}_2, \langle\cdot,\cdot\rangle_2, \|\cdot\|_2)$, and $(\mathscr{H}_3, \langle\cdot,\cdot\rangle_3, \|\cdot\|_3)$ stand for Hilbert spaces over the same field $\mathbb{F}$.

2.1.2.1 Adjoint for Bounded Operators

Let $A \in B(\mathscr{H}_1, \mathscr{H}_2)$. For each $y \in \mathscr{H}_2$, the functional $x \mapsto \xi_y(x) := \langle Ax, y\rangle_2$ is linear and bounded. Therefore, from the Riesz representation theorem (Theorem 1.108), there exists a unique $y^* \in \mathscr{H}_1$ such that

$$\langle Ax, y\rangle_2 = \xi_y(x) = \langle x, y^*\rangle_1$$

for all $x \in \mathscr{H}_1, y \in \mathscr{H}_2$.

The transformation $A^* : \mathscr{H}_2 \mapsto \mathscr{H}_1$ defined by $y \mapsto y^* = A^*y$ is called the adjoint of the linear operator A. In view of the above,

$$\langle Ax, y\rangle_2 = \langle x, A^*y\rangle_1$$

for all $x \in \mathscr{H}_1, y \in \mathscr{H}_2$.

Proposition 2.9. *If $A : \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear operator, then $A^* \in B(\mathscr{H}_2, \mathscr{H}_1)$. Furthermore, $\|A\| = \|A^*\|$.*

Proof. First of all, let us show that A^* is a bounded linear operator. Indeed, given $v, w \in \mathscr{H}_2$ and $\alpha, \beta \in \mathbb{F}$, we have

$$\begin{aligned}
\langle u, A^*(\alpha v + \beta w)\rangle_2 &= \langle Au, \alpha v + \beta w\rangle_2 \\
&= \overline{\alpha}\langle Au, v\rangle_2 + \overline{\beta}\langle Au, w\rangle_2 \\
&= \overline{\alpha}\langle u, A^*v\rangle_1 + \overline{\beta}\langle u, A^*w\rangle_1 \\
&= \langle u, \alpha A^*v\rangle_1 + \langle u, \beta A^*w\rangle_1 \\
&= \langle u, \alpha A^*v + \beta A^*w\rangle_1
\end{aligned}$$

for all $u \in \mathscr{H}_1$. Hence $A^*(\alpha v + \beta w) = \alpha A^*v + \beta A^*w$, that is, A^* is a linear operator from $\mathscr{H}_2$ into $\mathscr{H}_1$.

Now

$$\begin{aligned}
\|A^*u\|_1^2 &= \langle A^*u, A^*u\rangle_1 \\
&= \langle AA^*u, u\rangle_2 \\
&\le \|AA^*u\|_2 \cdot \|u\|_2 \\
&\le \|A\| \cdot \|A^*u\|_1 \cdot \|u\|_2
\end{aligned}$$

and hence $\|A^*u\|_1 \le \|A\| \cdot \|u\|_2$, that is, $\|A^*\| \le \|A\|$. Similarly, $\|(A^*)^*\| \le \|A^*\|$. Now using the fact $(A^*)^* = A$ it follows that $\|A\| \le \|A^*\|$. Combining, we obtain the desired result.

Proposition 2.10 ([18]). *If $A : \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear operator, then $A^*A \in B(\mathscr{H}_1)$ and $AA^* \in B(\mathscr{H}_2)$. Moreover, $\|AA^*\| = \|A^*A\| = \|A^*\|^2 = \|A\|^2$.*

Proposition 2.11 ([18]). *If A, B are bounded linear operators on $\mathscr{H}$ and if $\lambda \in \mathbb{C}$, then,*

(a) $I^* = I$.
(b) $O^* = O$.
(c) $(A+B)^* = A^* + B^*$.
(d) $(\lambda A)^* = \overline{\lambda}\, A^*$.
(e) $(AB)^* = A^*\, B^*$.

Definition 2.12. A bounded linear operator $A : \mathscr{H} \mapsto \mathscr{H}$ is called self-adjoint or symmetric if $A = A^*$.

Proposition 2.13 ([70]). *If $A \in B(\mathscr{H}_1, \mathscr{H}_2)$, then both AA^* and A^*A are self-adjoint.*

Proposition 2.14 ([18]). *Let $A : \mathscr{H} \mapsto \mathscr{H}$ be a bounded self-adjoint operator. Then the following properties hold:*

(a) $\langle Ax, x\rangle \in \mathbb{R}$ *for all* $x \in \mathscr{H}$.
(b) $\|A\| = \sup\limits_{x\neq 0} \dfrac{|\langle Ax, x\rangle|}{\|x\|^2}$.
(c) If $B \in B(\mathscr{H})$ is also self-adjoint and if $AB = BA$, then AB is also self-adjoint.

Example 2.15. Let $\mathscr{H}_1 = \mathscr{H}_2 = L^2[a,b]$, which we equip with the L^2-topology. Consider the integral operator defined by

$$(Af)(t) = \int_a^b K(t,s) f(s) ds$$

for all $f \in L^2[a,b]$, where $K \in L^2([a,b] \times [a,b])$.

It can be easily seen that A is a bounded linear operator on $L^2[a,b]$. Moreover, the adjoint of A is given by

$$(A^* f)(t) = \int_a^b \overline{K(s,t)} f(s) ds$$

for all $f \in L^2[a,b]$. In particular, A is self-adjoint if and only if

$$K(t,s) = \overline{K(s,t)}$$

for all $t, s \in [a,b]$.

2.1.2.2 The Inverse Operator

Definition 2.16. An operator $A \in B(\mathscr{X})$ is called invertible if there exists $B \in B(\mathscr{X})$ such that

$$AB = BA = I.$$

In that event, the operator B is called the inverse operator of A and denoted by $B = A^{-1}$.

Theorem 2.17 ([18]). *If $A \in B(\mathscr{X})$ and if $\|A\| < 1$, then the linear operator $I - A$ is invertible and its inverse is given by*

$$(I-A)^{-1} = \sum_{k=0}^{k} A^k.$$

Proof. First of all, observe that $(I-A)(I+A^2+\cdots+A^n) = I - A^{n+1}$ and that $\|A^{n+1}\| \leq \|A\|^{n+1} \mapsto 0$ as $n \mapsto \infty$ as $\|A\| < 1$. Consequently,

$$\lim_{n\to\infty}(I-A)(I+A^2+\cdots+A^n) = I \ \text{ in } \ B(\mathscr{X}).$$

Again from $\|A\| < 1$ and the fact that $B(\mathscr{X})$ is a Banach algebra, then the quantity

$$S := \lim_{n\to\infty}(I+A^2+\cdots+A^n)$$

does exist and $(I-A)S = I$. In fact, we have

$$(I-A)S = I + (I-A)\left[S - (I+A^2+\cdots+A^n)\right].$$

On other hand,

$$\begin{aligned}\|(I-A)\left[S-(I+A^2+\cdots+A^n)\right]\| &\leq \|I-S\| \\ &\quad \cdot \ \|S-(I+A^2+\cdots+A^n)\| \\ &\quad \mapsto 0 \ \text{ as } \ n \mapsto \infty,\end{aligned}$$

hence $(I-A)S = I$.

Thus $(I-A)$ is invertible and $(I-A)^{-1} = S$, where

$$S = \sum_{k=0}^{\infty} A^k \ \ (A^0 \ \text{ being } \ I).$$

Observe that if $A, B \in B(\mathscr{X})$ are invertible, so is AB with $(AB)^{-1} = B^{-1}A^{-1}$.

Proposition 2.18 ([102]). *If $A \in B(\mathscr{X})$ is invertible and if $B \in B(\mathscr{X})$ is given such that*

$$\|A - B\| < \frac{1}{\|A^{-1}\|},$$

then B is invertible and its inverse is given by

$$B^{-1} = \sum_{k=0}^{\infty} [A^{-1}(A-B)]^k A^{-1}$$

and

$$\|A^{-1} - B^{-1}\| \le \frac{\|A^{-1}\|^2 \cdot \|A-B\|}{1 - \|A^{-1}\| \cdot \|A-B\|}.$$

Proof. It suffices to write $B = A[I - A^{-1}(A-B)]$. From $\|A^{-1}(A-B)\| < 1$ and Theorem 2.17 it follows that $I - A^{-1}(A-B)$ is invertible. Using the fact that A is invertible it follows that $A[I - A^{-1}(A-B)]$ is invertible. Further, using Theorem 2.17 it follows that

$$B^{-1} = [I - A^{-1}(A-B)]^{-1} A^{-1} = \sum_{k=0}^{\infty} [A^{-1}(A-B)]^k A^{-1}.$$

Moreover,

$$\|A^{-1} - B^{-1}\| \le \frac{\|A^{-1}\|^2 \cdot \|A-B\|}{1 - \|A^{-1}\| \cdot \|A-B\|}.$$

2.1.2.3 Spectrum for Bounded Linear Operators

Definition 2.19. If $A : \mathscr{X} \to \mathscr{X}$ is a bounded linear operator, then its spectrum $\sigma(A)$ is defined by

$$\sigma(A) = \{\lambda \in \mathbb{C} : \ \lambda I - A \ \text{ is not invertible}\}.$$

Similarly, the resolvent $\rho(A)$ of A is defined by $\rho(A) = \mathbb{C} \backslash \sigma(A)$, that is, the collection of all $\lambda \in \mathbb{C}$ such that the operator $A - \lambda I$ is one-to-one ($N(A - \lambda I) = 0$), onto ($R(A - \lambda I) = \mathscr{X}$), and bounded.

Example 2.20 ([18]). Let $q : [\alpha, \beta] \to \mathbb{C}$ be a continuous function. Define the bounded linear operator M_q on $\mathscr{X} = L^2([\alpha, \beta])$ by

$$(M_q \phi)(s) = q(s)\phi(s), \ \ \forall s \in [\alpha, \beta].$$

It can be shown that $\lambda I - M_q$ is invertible on $L^2([\alpha,\beta])$ if and only if

$$\lambda - q(s) \neq 0, \ \ \forall s \in [\alpha,\beta]. \tag{2.1}$$

The inverse $(\lambda I - M_q)^{-1}$ of $\lambda I - M_q$ is defined by

$$\left[(\lambda I - M_q)^{-1}\right] \psi(s) = \left[\frac{1}{\lambda - q(s)}\right] \psi(s)$$

with the following estimate:

$$\left\|(\lambda I - M_q)^{-1}\right\| \leq \max_{s\in[\alpha,\beta]} \left[\frac{1}{|\lambda - q(s)|}\right].$$

The spectrum $\sigma(M_q)$ of M_q is given by

$$\sigma(M_q) = \{q(s) : s \in [\alpha,\beta]\}.$$

2.1.2.4 Compact Operators

Definition 2.21 ([102]). An operator $A \in B(\mathscr{H}_1,\mathscr{H}_2)$ is said to be compact if for each sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathscr{H}_1$ with $\|x_n\|_1 = 1$ for each $n \in \mathbb{N}$, the sequence $(Ax_n)_{n\in\mathbb{N}}$ has a subsequence which converges in $\mathscr{H}_2$.

The collection of all compact operators from $\mathscr{H}_1$ to $\mathscr{H}_2$ is denoted by $\mathscr{K}(\mathscr{H}_1,\mathscr{H}_2)$.

Theorem 2.22 ([102]). *If $K,L \in B(\mathscr{H}_1,\mathscr{H}_2)$ are compact linear operators, then*

(a) *αL is compact.*
(b) *$K+L$ is compact.*
(c) *If $A \in B(\mathscr{H}_3,\mathscr{H}_1)$ and $B \in B(\mathscr{H}_2,\mathscr{H}_3)$, then KA and BK are compact.*

Proof. (a) This is straightforward.

(b) Let $(u_n) \in \mathscr{H}_1$ with $\|x_n\|_1 = 1$. Since K is compact, $(Ku_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(Ku_{n_k})_{k\in\mathbb{N}}$. Similarly, $(Lu_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(Lu_{n_k})_{k\in\mathbb{N}}$. Therefore, $((K+L)u_{n_k})_{k\in\mathbb{N}}$ converges.

(c) Let $(v_n)_{n\in\mathbb{N}} \subset \mathscr{H}_3$ with $\|v_n\|_3 = 1$ for each $n \in \mathbb{N}$. Thus $(Av_n)_{n\in\mathbb{N}}$ is bounded. Now since K is compact, it is clear $(KAv_n)_{n\in\mathbb{N}}$ has a convergent subsequence and hence KA is a compact operator.

Let $(w_n)_{n\in\mathbb{N}} \subset \mathscr{H}_1$ with $\|w_n\|_1 = 1$ for each $n \in \mathbb{N}$. Now since K is compact, then $(Kw_n)_{n\in\mathbb{N}}$ has a convergent subsequence, say, $(Kw_{n_k})_{k\in\mathbb{N}}$. Now by the continuity of B it follows that $(BKw_{n_k})_{k\in\mathbb{N}}$ converges and hence BK is a compact operator.

Example 2.23. In $\mathscr{H} = L^2[a,b]$, define the integral operator A by

$$(Af)(t) := \int_a^b V(t,\tau)f(\tau)d\tau \quad \text{for each} \quad f \in L^2[a,b].$$

Assuming that $V \in L^2([a,b] \times [a,b])$, it can be shown that A is compact.

Definition 2.24. A bounded linear operator is of finite rank if its image is a finite-dimensional Banach space. The collection of all finite rank operators from $\mathscr{H}_1$ into $\mathscr{H}_2$ is denoted $\mathscr{F}(\mathscr{H}_1, \mathscr{H}_2)$.

Proposition 2.25 ([102]). *Let $A : \mathscr{H}_1 \mapsto \mathscr{H}_2$ be an operator of finite rank n. Then there exist vectors $x_1, x_2, \ldots, x_n$ in $\mathscr{H}_1$ and vectors $y_1, y_2, \ldots, y_n$ in $\mathscr{H}_2$ such that*

$$Ax = \sum_{k=1}^{n} \langle x, x_k \rangle y_k.$$

The vectors $y_1, y_2, \ldots, y_n$ can be chosen to be an orthonormal base for $R(A)$.

Proposition 2.26. $\mathscr{F}(\mathscr{H}_1, \mathscr{H}_2) \subset \mathscr{K}(\mathscr{H}_1, \mathscr{H}_2)$.

Proposition 2.27 ([102]). *An operator $A \in B(\mathscr{H}_1, \mathscr{H}_2)$ is compact if and only if its adjoint A^* is compact.*

Proof. If A is compact, then AA^* is compact. Thus for any $(x_n)_{n\in\mathbb{N}} \subset \mathscr{H}_2$ with $\|x_n\|_2 = 1$ there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $(AA^*x_{n_k})_{k\in\mathbb{N}}$ converges. Now using the fact

$$\|A^*x_{n_k} - A^*x_{n_l}\| \leq 2\|AA^*x_{n_k} - AA^*x_{n_l}\|$$

it follows that $(A^*x_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence, which converges as $\mathscr{H}_1$ is complete. Therefore, A^* is compact. Similarly, if A^* is compact, then using the above proved result, it follows that $A = (A^*)^*$ is compact.

2.2 Unbounded Linear Operators

Let $A : D(A) \subset \mathscr{X} \mapsto \mathscr{Y}$ be a linear operator. If A is not continuous, then A is called an unbounded operator. The collection of all unbounded linear operators from $\mathscr{X}$ into $\mathscr{Y}$ will be denoted $\mathscr{U}(\mathscr{X}, \mathscr{Y})$ with $\mathscr{U}(\mathscr{X}, \mathscr{X}) = \mathscr{U}(\mathscr{X})$. Because of the domain involved, elements of $\mathscr{U}(\mathscr{X})$ need to be manipulated with a great care. For instance, if $A, B \in \mathscr{U}(\mathscr{X}, \mathscr{Y})$, then their (algebraic) sum $A + B$ may or may be trivial depending on $D(A) \cap D(B)$. Note that the (algebraic) sum operator $A + B$ is defined by

$$D(A+B) = D(A) \cap D(B) \quad \text{and} \quad (A+B)x = Ax + Bx$$

for all $x \in D(A) \cap D(B)$.

Similarly, the product operator AB is defined by,

$$D(AB) = \{x \in D(B) : Bx \in D(A)\} \text{ and } ABx = A(Bx)$$

for all $x \in D(AB)$.

As for bounded linear operators, if $A : D(A) \subset \mathscr{X} \mapsto \mathscr{Y}$ is an unbounded linear operator, then we define its kernel $N(A)$ and range $R(A)$ as follows:

$$N(A) := \{x \in D(A) : Ax = 0\} \text{ and } R(A) = \{Ax : x \in D(A)\}.$$

It is clear that $N(A) \subset \mathscr{X}$ while $R(A) \subset \mathscr{Y}$. The operator A is said to be *one-to-one* if $N(A) = 0$. The operator A is said to be *onto* if $R(A) = \mathscr{Y}$. Of course, a linear operator A is *bijective* or *invertible* if it is one-to-one and onto. If A is invertible, then its inverse will be denoted by A^{-1}. If A is invertible, then $D(A^{-1}) = R(A)$ and $R(A^{-1}) = D(A)$.

2.2.1 Examples of Unbounded Operators

Example 2.28 ([136]). Let $\mathscr{X} = \mathscr{Y} = L^2(\mathbb{R})$ be equipped with its natural topology and consider the one-dimensional Laplace operator defined by

$$D(A) = H^2(\mathbb{R}) \text{ and } Au = -u''$$

for all $u \in H^2(\mathbb{R})$.

Consider the sequence of functions defined by $\psi_n(t) = e^{-n|t|}, \quad n = 1,2,\ldots$. Clearly, for each $n = 1,2,\ldots$, $\psi_n \in D(A) = H^2(\mathbb{R})$. Furthermore,

$$\|\psi_n\|_2^2 = \int_{-\infty}^{+\infty} e^{-2n|t|}dt = \frac{1}{n}$$

and

$$\|A\psi_n\|_2^2 = \int_{-\infty}^{+\infty} n^4 e^{-2n|t|}dt = n^3.$$

Therefore, $\dfrac{\|A\psi_n\|_2}{\|\psi_n\|_2} = n \to \infty$ as n goes to ∞, that is, A is an unbounded linear operator on $L^2(\mathbb{R})$.

Example 2.29 ([136]). Let $\mathscr{X} = \mathscr{Y} = L^2(0,1)$ be equipped with its natural topology and consider the derivative operator defined by

$$D(A) = C^1(0,1) \text{ and } Au = u'$$

for all $u \in C^1(0,1)$, where $C^1(0,1)$ is the collection of continuously differentiable functions over $(0,1)$.

Consider the sequence of functions defined by $\phi_n(t) = t^n$, $n = 1,2,\ldots$. Clearly, for each $n = 1,2,\ldots$, $\phi_n \in C^1(0,1)$. Furthermore,

$$\|\phi_n\|_2^2 = \int_0^1 t^{2n} dt = \frac{1}{2n+1},$$

and

$$\|A\phi_n\|_2^2 = \int_0^1 n^2 t^{2n-2}\, dt = \frac{n^2}{2n-1}.$$

Here again,

$$\frac{\|A\phi_n\|_2}{\|\phi_n\|_2} = n\sqrt{\frac{2n+1}{2n-1}} \to \infty$$

as n goes to ∞, that is, A is an unbounded linear operator on $L^2(\mathbb{R})$.

2.2.2 Closed Linear Operators

Equip $\mathscr{X} \times \mathscr{Y}$ with its natural norm, that is, $\|(x,y)\| = (\|x\|^2 + \|y\|^2)^{1/2}$ for all $(x,y) \in \mathscr{X} \times \mathscr{Y}$.

Definition 2.30. If $A : D(A) \subset \mathscr{X} \to \mathscr{Y}$ is a linear operator, then its graph is defined by

$$\mathscr{G}(A) = \{(x, Ax) \in \mathscr{X} \times \mathscr{Y} :\ x \in D(A)\}.$$

Definition 2.31. If A, B belong to $\mathscr{U}(\mathscr{X}, \mathscr{Y})$, then A is said to be an extension of B if $D(B) \subset D(A)$ and $Au = Bu$ for all $u \in D(B)$. We then denote this by $B \subset A$. Moreover, $B \subset A$ if and only if $\mathscr{G}(B) \subset \mathscr{G}(A)$.

Definition 2.32. A linear operator $A : D(A) \subset \mathscr{X} \mapsto \mathscr{X}$ is called closed if its graph $\mathscr{G}(A) \subset \mathscr{X} \times \mathscr{X}$ is closed. Namely, if $u_n \in D(A)$ such that $u_n \to u$ and $Au_n \to v$ in $\mathscr{X}$ as $n \to \infty$, then $u \in D(A)$ and $Ax = v$. The collection of all closed operators on $\mathscr{X}$ will be denoted $cl(\mathscr{X})$.

It is clear that an operator $A : D(A) \subset \mathscr{X} \mapsto \mathscr{Y}$ is closed if and only if $\mathscr{G}(A) \subset \mathscr{X} \times \mathscr{Y}$ is closed.

2.2.2.1 Examples of Closed Linear Operators

Example 2.33 ([70]). Every $A \in B(\mathscr{X})$ is closed. Indeed, let $(u_n)_{n\in\mathbb{N}} \in D(A)$ such that $u_n \to u$ with $Au_n \to v$ in $\mathscr{X}$ as $n \to \infty$. Now since A is bounded, therefore $D(A) = \mathscr{X}$. Again, from the continuity of A it is clear that $u \in \mathscr{X}$ and $Au = v$.

Example 2.34. If $A \in cl(\mathscr{X})$ and $B \in B(\mathscr{X})$, then $A+B \in cl(\mathscr{X})$. Indeed, suppose $(u_n)_{n\in\mathbb{N}} \in D(A+B) = D(A)$ such that $u_n \to u$ and $(A+B)u_n \to v$ in $\mathscr{X}$ as $n \to \infty$. Now since B is bounded it follows that $Au_n \to v - Bx$ in $\mathscr{X}$ as $n \to \infty$. Since A is closed, then $u \in D(A)$ and $Au = v - Bu$.

Example 2.35. This is an illustration of Example 2.34. Let $\mathscr{X} = L^2(\mathbb{R}^n)$ and define A and B by

$$D(A) = H^2(\mathbb{R}^n),\ \ Au = -\Delta u,\ \ \forall u \in H^2(\mathbb{R}^n),$$

and

$$D(B) = \{u \in L^2(\mathbb{R}^n) : \gamma(x)u \in L^2(\mathbb{R}^n)\},\ \ Bu = \gamma(x)u,\ \ \forall u \in D(B),$$

where Δ is the n-dimensional Laplace operator defined by

$$\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$$

and that $\gamma \in L^\infty(\mathbb{R}^n)$. It is then clear that $B \in B(L^2(\mathbb{R}^n))$ and hence $-\Delta + \gamma \in cl(L^2(\mathbb{R}^n))$ with $D(-\Delta + \gamma) = H^2(\mathbb{R}^n)$.

Example 2.36. Let $\mathscr{X} = \mathscr{Y} = C[a,b]$ where $a, b \in \mathbb{R}$ with $a < b$. Define $C_c^1(a,b]$ to be the set of all $f \in C^1[a,b]$ with $\mathrm{Supp}(f) \subset (a,b]$ where

$$\mathrm{Supp}(f) = \overline{\{x \in [a,b] : f(x) \neq 0\}}.$$

Setting $Au = u'$ with $D(A) = C_c^1(a,b]$ it can be easily shown that $A \notin cl(C[a,b])$.

2.2.3 *Spectral Theory for Linear Operators*

2.2.3.1 Basic Definitions

Definition 2.37. If $A \in cl(\mathscr{X})$, then $\rho(A)$ the resolvent set of A is defined by

$$\rho(A) = \{\lambda \in \mathbb{C} :\ \lambda I - A : D(A) \mapsto \mathscr{X} \text{ is bijective, and } (\lambda I - A)^{-1} \in B(\mathscr{X})\},$$

and $\sigma(A)$ the spectrum of A is the complement of the resolvent set $\rho(A)$ in $\mathbb{C}$.

Note that if $\lambda \in \rho(A)$, then the operator-valued function $R(\lambda,A) := (\lambda I - A)^{-1} : \rho(A) \mapsto B(\mathscr{X})$ is called the resolvent of the operator A. It should be mentioned that $\rho(A) \neq \emptyset$ if $A \in cl(\mathscr{X})$.

Example 2.38. Let $\mathscr{X} = \mathscr{Y} = C[a,b]$ and let $Au = u'$ for all $u \in D(A) = \{u \in C^1[a,b] : u(b) = 0\}$. We want to compute $\rho(A)$. For that, we have to find conditions on $\lambda \in \mathbb{C}$ such that

$$(\lambda I - A)u = f \tag{2.2}$$

where $\lambda \in \mathbb{C}$, $u \in D(A)$, and $f \in C[a,b]$, can be solved.

This is equivalent to solving the differential equation $u' = \lambda u - f$ where $u \in C^1[a,b]$ and $u(b) = 0$, which is also equivalent to

$$u(t) = \int_t^b e^{(t-s)\lambda} f(s)ds$$

for all $t \in [a,b]$.

Thus for each $\lambda \in \mathbb{C}$, Eq. (2.2) can be solved and its solution is given above. This means that $\rho(A) = \mathbb{C}$, which yields $\sigma(A) = \emptyset$. Let us also mention that the operator $(\lambda I - A)^{-1}$ is then defined by

$$(\lambda I - A)^{-1} v(t) = R(\lambda,A)v(t) = \int_t^b e^{(t-s)\lambda} v(s)ds$$

for each $v \in C[a,b]$.

Proposition 2.39 ([18]). *Let $A, B \in cl(\mathscr{X})$.*

(a) If $\lambda, \mu \in \rho(A)$, then $R(\lambda,A) - R(\mu,A) = (\mu - \lambda)R(\lambda,A)\,R(\mu,A)$. Furthermore, $R(\lambda,A)$ and $R(\mu,A)$ commute.
(b) If $D(A) \subset D(B)$, then for all $\lambda \in \rho(A) \cap \rho(B)$ we have

$$R(\lambda,A) - R(\lambda,B) = R(\lambda,A)(A-B)R(\lambda,B).$$

(c) If $D(A) = D(B)$, then for all $\lambda \in \rho(A) \cap \rho(B)$ we have

$$R(\lambda,A) - R(\lambda,B) = R(\lambda,A)(A-B)R(\lambda,B) = R(\lambda,B)(A-B)R(\lambda,A).$$

Proof. (a) Clearly

$$\begin{aligned} R(\lambda,A) - R(\mu,A) &= R(\lambda,A)[(\mu I - A) - (\lambda I - A)]R(\mu,A) \\ &= (\mu - \lambda)R(\lambda,A)\,R(\mu,A). \end{aligned}$$

Similarly

$$\begin{aligned} R(\lambda,A)\,R(\mu,A) &= \frac{1}{\mu-\lambda}[R(\lambda,A)-R(\mu,A)] \\ &= \frac{1}{\lambda-\mu}[R(\mu,A)-R(\lambda,A)] \\ &= R(\mu,A)\,R(\lambda,A). \end{aligned}$$

(b) We have

$$\begin{aligned} R(\lambda,A)-R(\lambda,B) &= R(\lambda,A)[(\lambda I-B)-(\lambda I-A)]R(\lambda,B) \\ &= R(\lambda,A)(A-B)R(\lambda,B). \end{aligned}$$

(c) We have

$$\begin{aligned} R(\lambda,A)-R(\lambda,B) &= R(\lambda,A)[(\lambda I-B)-(\lambda I-A)]R(\lambda,B) \\ &= R(\lambda,A)(A-B)R(\lambda,B) \\ &= R(\lambda,B)(A-B)R(\lambda,A). \end{aligned}$$

2.2.3.2 Self-Adjoint Linear Operators

Definition 2.40 ([57]). If $A : D(A) \subset \mathscr{H}_1 \to \mathscr{H}_2$ is a densely defined linear operator, then its adjoint denoted by A^* is defined in a unique fashion by

$$D(A^*) = \{v \in \mathscr{H}_2 : u \mapsto \langle Au, v\rangle_2 \text{ is } \mathscr{H}_1\text{—continuous over } D(A)\},$$

and

$$\langle Au, v\rangle_2 = \langle u, A^*v\rangle_1, \;\; \forall u \in D(A), \;\; v \in D(A^*).$$

Proposition 2.41 ([102]). *If $A : D(A) \subset \mathscr{H}_1 \to \mathscr{H}_2$ is a densely defined ($\overline{D(A)} = \mathscr{H}_1$) unbounded linear operator, then A^* is closed.*

Proposition 2.42 ([160]). *If $A, B \in \mathscr{U}(\mathscr{H})$ are densely defined, then*

(a) $A^*B^* \subset (BA)^*$.
(b) If $B \in B(\mathscr{H})$, then $A^*B^* = (BA)^*$.
(c) If $A+B$ is densely defined, we have $(A+B)^* \supset A^* + B^*$.

Proof. (a) Let us show that the operators A^*B^* and BA are adjoint to each other. Indeed, let $u \in D(A^*B^*)$ and $v \in D(BA)$. Clearly, $u \in D(B^*)$ such that $B^*u \in D(A^*)$. Similarly, $v \in D(A)$ such that $Av \in D(B)$. Using the definition of the adjoint it follows

$$\langle A^*B^*u, v\rangle = \langle B^*u, Av\rangle = \langle u, BAv\rangle.$$

(b) Using (a) it is enough to show that $D(BA)^* \subset D(A^*B^*)$. Indeed, let $u \in D(BA)^*$. Now since B^* is bounded it follows that for all $v \in D(BA) = D(A)$, we obtain $\langle (BA)^*u, v\rangle = \langle u, BAv\rangle = \langle B^*u, v\rangle$, and hence $B^*u \in D(A^*)$, that is, $u \in D(A^*B^*)$.

(c) Let $u \in D(A^* + B^*) = D(A^*) \cap D(B^*)$. Clearly, for all $v \in D(A+B) = D(A) \cap D(B)$, we have

$$\langle (A^* + B^*)u, v\rangle = \langle A^*u, v\rangle + \langle B^*u, v\rangle = \langle u, Av\rangle + \langle u, Bv\rangle = \langle u, (A+B)v\rangle$$

and hence $u \in D((A+B)^*)$ and $(A+B)^*u = A^*u + B^*u$.

Definition 2.43. An operator $A : D(A) \subset \mathscr{H} \mapsto \mathscr{H}$ is called self-adjoint if $A = A^*$.

Theorem 2.44 ([160]). *Let $A : D(A) \subset \mathscr{H} \mapsto \mathscr{H}$ be a closed linear operator. Then AA^* and A^*A are self-adjoint linear operators on $\mathscr{H}$.*

Definition 2.45. A linear operator A is said to have a compact resolvent if $\rho(A) \neq \emptyset$ and $R(\lambda, A) = (\lambda I - A)^{-1}$ is a compact operator for all $\lambda \in \rho(A)$. In particular, A is said to have a compact inverse whether A^{-1} is compact.

Theorem 2.46 ([102]). *Suppose $\mathscr{H}$ is an infinite dimensional Hilbert space and $A : D(A) \subset H \mapsto \mathscr{H}$ is self-adjoint and has a compact inverse. Then,*

(a) There exists an orthonormal basis $(f_n)_{n\in\mathbb{N}}$ of $\mathscr{H}$ consisting of eigenvectors of A. If the sequence $(\mu_n)_{n\in\mathbb{N}}$ are the corresponding eigenvalues, then μ_n is a real number and $|\mu_n| \to \infty$ as $n \to \infty$. The numbers of repetitions of μ_n in the sequence $(\mu_n)_{n\in\mathbb{N}}$ is finite and equals the dimension of $N(\mu_n I - A)$.

(b) $D(A) = \left\{ x \in \mathscr{H} : \sum_{n=0}^{\infty} |\mu_n|^2 \cdot |\langle x, f_n\rangle|^2 < \infty \right\}.$

(c) $Ax = \sum_{n=0}^{\infty} \mu_n \langle x, f_n\rangle f_n$ *for all $x \in D(A)$.*

Example 2.47. Let $\lambda = (\lambda_n)_{n\in\mathbb{N}}$ be a real-valued sequence. Define the diagonal operator D_λ on l^2 by $D(D_\lambda) = \{x = (x_k)_{k\in\mathbb{N}} \in l^2 : (\lambda_k x_k)_{k\in\mathbb{N}} \in l^2\}$, $D_\lambda x = (\lambda_k x_k)_{k\in\mathbb{N}}$ for all $x \in D(D_\lambda)$. It can be easily shown that D_λ is a self-adjoint linear operator. Moreover, for all $\mu \in \rho(D_\lambda)$, the linear operator $(\mu I - D_\lambda)^{-1}$ is compact if and only if $\lim_{n\to\infty} |\lambda_n| = \infty$.

Example 2.48. In $L^2(a,b)$ consider the so-called Neumann Laplace operator defined by

$$D(\Delta_N) = \{u \in H^2(a,b) : u'(a) = u'(b) = 0\} \text{ and } \Delta_N u = u''$$

for all $u \in D(\Delta_N)$.

It can be shown that the operator Δ_N is self-adjoint and has compact resolvent.

2.2.4 Sectorial Linear Operators

2.2.4.1 Basic Definitions

Definition 2.49 ([137]). A linear operator $A : D(A) \subset \mathscr{X} \to \mathscr{X}$ (not necessarily densely defined) is said to be sectorial if the following hold: there exist constants $\zeta \in \mathbb{R}$, $\theta \in (\frac{\pi}{2}, \pi)$, and $M > 0$ such that

(a) $\rho(A) \supset S_{\theta,\zeta} := \{\lambda \in \mathbb{C} : \lambda \neq \zeta, \ |\arg(\lambda - \zeta)| < \theta\}$, and

(b) $\|R(\lambda,A)\| \leq \dfrac{M}{|\lambda - \zeta|}$ for each $\lambda \in S_{\theta,\zeta}$.

Most of the unbounded linear operators encountered in the literature are sectorial. Observe that if A is a sectorial operator, then A is necessarily closed as $\rho(A) \neq \emptyset$. Thus the space $(D(A), \|\cdot\|_A)$, where $\|x\|_A = \|x\| + \|Ax\|$ for all $x \in D(A)$, is a Banach space. The norm $\|\cdot\|_A$ is called the graph norm of A.

Proposition 2.50 ([137]). *Let A be a linear operator on $\mathscr{X}$ such that $\rho(A)$ contains the half-plane $\{\lambda \in \mathbb{C} : \Re e\lambda \geq \zeta\}$, and $\|\lambda R(\lambda,A)\| \leq M$ for $\Re e\lambda \geq \zeta$ with $\zeta \in \mathbb{R}$ and $M > 0$. Then A is sectorial.*

2.2.4.2 Examples of Sectorial Operators

Example 2.51. Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset with C^2 boundary $\partial\Omega$. Let $\mathscr{X} = L^2(\Omega)$ and define the second-order differential operator

$$Au = \Delta u \ \text{ for all } \ u \in D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega).$$

It can be shown that the linear operator A is sectorial.

Example 2.52 ([18]). Let $\Omega \subset \mathbb{R}^N$ be a bounded open subset whose boundary $\partial\Omega$ is of class C^2. Let $n(x)$ denote the outer normal to Ω for each $x \in \partial\Omega$. Consider the differential operator defined by

$$A_0 u(x) = \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u(x),$$

where the coefficients a_{ij} and b_i and c are real, bounded, and continuous on $\overline{\Omega}$. Moreover, we suppose that for each $x \in \overline{\Omega}$, the matrix $(a_{ij}(x))_{i,j=1,\ldots,N}$ is symmetric and strictly positive definite, that is,

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \omega|\xi|^2$$

for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^N$.

Theorem 2.53 ([137]). *Let $p > 1$.*

(a) Let $A_p : W^{2,p}(\mathbb{R}^N) \mapsto L^p(\mathbb{R}^N)$ be the linear operator defined by $A_p u = A_0 u$. Then the operator A_p is sectorial in $L^p(\mathbb{R}^N)$ and the domain $D(A_p)$ is dense in $L^p(\mathbb{R}^N)$.

(b) Let A_0 be defined as above and let A_p be the linear operator defined by

$$D(A_p) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \ \text{ and } \ A_p u = A_0 u$$

for all $u \in D(A_p)$. Then the linear operator A_p is sectorial in $L^p(\Omega)$. Moreover, $D(A_p)$ is dense in $L^p(\Omega)$.

(c) Let A_0 be defined as above and let A_p be the linear operator defined by

$$D(A_p) = \{u \in W^{2,p}(\Omega) : Bu_{|_{\partial\Omega}} = 0\}, \ \ A_p u = A_0 u, \ \ u \in D(A_p)$$

where

$$Bu(x) = b_0 u(x) + \sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i}$$

with the coefficients b_i $(i = 1, \ldots, N)$ are in $C^1(\overline{\Omega})$ and the condition

$$\sum_{i=1}^{N} b_i(x) n_i(x) \neq 0 \ \ x \in \partial\Omega$$

holds. Then A_p is sectorial in $L^p(\Omega)$ and $D(A_p)$ is dense in $L^p(\Omega)$.

2.2.5 Semigroups of Linear Operators

Definition 2.54. A family of bounded linear operators $(T(t))_{t\in\mathbb{R}^+} : \mathscr{X} \to \mathscr{X}$ is said to be a semigroup if the following hold,

(a) $T(0) = I$; and
(b) $T(t+s) = T(t)T(s)$ for all $s,t \geq 0$.
    If in addition,
(c) $\lim_{t\searrow 0} \|T(t) - I\| = 0$, then the semigroup $T(t)$ is said to be uniformly continuous.

Definition 2.55. A semigroup of bounded linear operators $(T(t))_{t\in\mathbb{R}^+} : \mathscr{X} \mapsto \mathscr{X}$ is said to be a c_0-semigroup (or strongly continuous semigroup of bounded linear operators) if

$$\lim_{t\searrow 0} \|T(t)x - x\| = 0$$

for each $x \in \mathscr{X}$.

Observe that if $(T(t))_{t\in\mathbb{R}^+} : \mathscr{X} \to \mathscr{X}$ is a semigroup of bounded linear operators, one can associate with it an operator $(D(A), A)$ called its infinitesimal generator defined by

$$D(A) := \left\{ u \in \mathscr{X} : \lim_{t \searrow 0} \frac{T(t)u - u}{t} \text{ exists} \right\}, \tag{2.3}$$

and

$$Au := \lim_{t \searrow 0} \frac{T(t)u - u}{t} \tag{2.4}$$

for every $u \in D(A)$.

Remark 2.56. Note that an operator A is the infinitesimal generator of a uniformly continuous semigroup of bounded linear operators $(T(t))_{t\in\mathbb{R}^+}$ if and only if A is bounded. In that event, it can be shown that

$$T(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}.$$

Example 2.57 ([18]). Suppose that $\mathscr{X} = (BUC(\mathbb{R}), \|\cdot\|_\infty)$ is the Banach space of bounded uniformly continuous functions on the real number line equipped with the sup norm. Define

$$(S(t)\varphi)(\sigma) = \varphi(t + \sigma)$$

for all $\varphi \in BUC(\mathbb{R})$. Then, $(S(t))_{t\in\mathbb{R}}$ is a c_0-semigroup with $\|S(t)\| \leq 1$ for each $t \in [0, \infty)$. Moreover, its infinitesimal generator A is defined by

$$D(A) = H^1(\mathbb{R}) \text{ and } A\varphi = \varphi'$$

for all $\varphi \in H^1(\mathbb{R})$.

Example 2.58 ([18]). Let $1 \leq p < \infty$ and let $\mathscr{X} = L^p(\mathbb{R})$ equipped with its natural norm $\|\cdot\|_p$. Define $(S(0))u(x) = u(x)$ for all $x \in \mathbb{R}$, and

$$(S(t))u(x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-|x-y|^2}{4t}} u(y)dy, \ \ t > 0, \ \ x \in \mathbb{R}.$$

Then $S(t)$ is a c_0-semigroup satisfying $\|S(t)u\|_p \leq \|u\|_p$ and whose infinitesimal generator A_p is defined by

$$D(A_p) = W^{2,p}(\mathbb{R}) \text{ and } A_p u = u''$$

for all $u \in D(A_p)$.

Example 2.59 ([18]). This is a generalization of Example 2.58. Let $1 \leq p < \infty$ and let $\mathscr{X} = L^p(\mathbb{R}^N)$ (or $BC(\mathbb{R}^N, \mathbb{C})$ equipped with the sup norm) equipped with its natural norm $\|\cdot\|_p$. Define $(S(0))u(x) = u(x)$ for all $x \in \mathbb{R}^N$, and

$$(S(t))u(x) = \frac{1}{(4\pi t)^{N/2}} \int_{-\infty}^{\infty} e^{\frac{-\|x-y\|^2}{4t}} u(y)dy, \ \ t > 0, \ \ x \in \mathbb{R}.$$

Then $S(t)$ is a c_0-semigroup satisfying $\|S(t)u\|_p \leq \|u\|_p$ whose infinitesimal generator A_p is defined by

$$D(A_p) = W^{2,p}(\mathbb{R}^N) \ \text{ and } \ A_p u = \Delta u$$

for all $u \in D(A_p)$.

2.2.5.1 Basic Properties of Semigroups

Theorem 2.60 ([148]). *Let $(T(t))_{t \in \mathbb{R}^+} : \mathscr{X} \to \mathscr{X}$ be a semigroup of bounded linear operators, then*

(a) there are constants C, ζ such that $\|T(t)\| \leq C\, e^{\zeta t}, \ t \in \mathbb{R}^+$;
(b) the infinitesimal generator A of the semigroup $T(t)$ is a densely defined closed operator;
(c) the map $t \mapsto T(t)x$ which goes from $\mathbb{R}^+$ into $\mathscr{X}$ is continuous for every $x \in \mathscr{X}$;
(d) the differential equation given by

$$\frac{d}{dt} T(t)x = AT(t)x = T(t)Ax,$$

holds for every $x \in D(A)$;
(e) for every $x \in \mathscr{X}$, then $T(t)x = \lim_{s \searrow 0} (\exp(tA_s))x$, with

$$A_s x := \frac{T(s)x - x}{s},$$

where the above convergence is uniform on every compact subset of $\mathbb{R}^+$; and
(f) if $\lambda \in \mathbb{C}$ with $\Re e \lambda > \zeta$, then the integral

$$R(\lambda, A)x := (\lambda I - A)^{-1} x = \int_0^{\infty} e^{-\zeta t}\, T(t)x\, dt,$$

defines a bounded linear operator $R(\lambda, A)$ on $\mathscr{X}$ whose range is $D(A)$ and

$$(\lambda I - A)\, R(\lambda, A) = R(\lambda, A)(\lambda I - A) = I.$$

If $\zeta = 0$ in (a) above, then the corresponding semigroup is uniformly bounded. Moreover, if $C = 1$, then $(T(t))_{t\in\mathbb{R}^+}$ is said to be a c_0-semigroup of contractions.

Theorem 2.61 (Hille–Yosida [148]). *Let $A : D(A) \to \mathscr{X}$ be an unbounded linear operator in a Banach space $\mathscr{X}$. Then A is the infinitesimal generator of a c_0-semigroup of contractions $(T(t))_{t\in\mathbb{R}^+}$ if and only if:*

(a) A is a densely defined closed operator; and
(b) the resolvent $\rho(A)$ of A contains $\mathbb{R}^+$ and

$$\left\|(\lambda I - A)^{-1}\right\| \leq \frac{1}{\lambda}, \quad \forall \lambda > 0. \tag{2.5}$$

Definition 2.62. Let $\mathscr{X}$ be a Banach space. The family of bounded operators $(T(t))_{t\in\mathbb{R}} : \mathscr{X} \to \mathscr{X}$ is said to be a c_0-*group* if the following statements hold:

(a) $T(0) = I$.
(b) $T(t+s) = T(t)T(s)$ for every $s,t \in \mathbb{R}$.
(c) $\lim\limits_{t\to 0} \|T(t)x - x\| = 0$ for $x \in \mathscr{X}$.

Theorem 2.63 ([148]). *Let $A : D(A) \to \mathscr{X}$ be a linear operator on $\mathscr{X}$. Then A is the infinitesimal generator of a c_0-group of bounded linear operators $(T(t))_{t\in\mathbb{R}}$ satisfying $\|T(t)\| \leq C\, e^{\zeta|t|}$ if and only if:*

(a) A is a densely defined closed operator; and
(b) every $\lambda \in \mathbb{R}$ such that $|\lambda| \geq \zeta$ is in $\rho(A)$ and that for such a λ, the following holds:

$$\|(\lambda I - A)^{-n}\| \leq \frac{C}{(|\lambda| - \zeta)^n}. \tag{2.6}$$

Proposition 2.64 ([137]). *Let A be a sectorial operator and let $T(t)$ be the analytic semigroup associated with it. Then the following hold:*

(a) $T(t)u \in D(A^k)$ for all $t > 0$, $u \in \mathscr{X}$, $n \in \mathbb{N}$. If $D(A^n)$, then

$$A^n T(t)u = T(t)A^n u, \quad t \geq 0;$$

(b) there exist constants $M_0, M_1, \ldots$ such that

$$\|T(t)\| \leq M_0 e^{\zeta t}, \quad t > 0, \quad \textit{and}$$

$$\|t^n (A - \zeta I)^n T(t)\| \leq M_n e^{\zeta t}, \quad t > 0; \quad \textit{and}$$

(c) the mapping $t \to T(t)$ belongs to $C^\infty((0,\infty), B(\mathscr{X}))$ and

$$\frac{d^n}{dt^n} T(t) = A^n T(t), \quad t > 0, \ \forall n \in \mathbb{N}.$$

Proposition 2.65 ([137]). *Let $(T(t))_{t>0}$ be a family of bounded linear operators on $\mathscr{X}$ such that $t \mapsto T(t)$ is differentiable, and*

(a) $T(t+s) = T(t)T(s)$ *for all* $t, s > 0$;
(b) there exist $\zeta \in \mathbb{R}$, $M_0, M_1 > 0$ *such that*

$$\|T(t)\| \le M_0 e^{\zeta t}, \quad \|tT'(t)\| \le M_1 e^{\zeta t}, \quad \forall t > 0;$$

(c) either (a) there exists $t > 0$ *such that* $T(t)$ *is one-to-one, or (b) for every* $x \in \mathscr{X}$, $s - \lim\limits_{t \to 0} T(t)x = x$.

Then $t \mapsto T(t)$ *is analytic in* $(0, \infty)$ *with values in* $B(\mathscr{X})$, *and there exists a unique sectorial operator* $A : D(A) \subset \mathscr{X} \to \mathscr{X}$ *such that* $(T(t))_{t \ge 0}$ *is the semigroup associated with* A.

2.2.6 *Intermediate Spaces*

2.2.6.1 Fractional Powers of Sectorial Operators

Let A be a sectorial linear operator on $\mathscr{X}$ whose associated analytic semigroup $T(t)$ satisfies the following: for all $t > 0$,

$$\|T(t)\| \le M_0 e^{-\omega t}, \ \ \|tAT(t)\| \le M_1 e^{-\omega t},$$

where $M_0, M_1, \omega > 0$.

For each $\alpha > 0$ one defines the fractional powers of $-A$ implicitly by

$$(-A)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} T(t) dt, \tag{2.7}$$

where Γ is defined by

$$\Gamma(x) := \int_0^{+\infty} e^{-xt} t^{x-1} dt$$

for each $x > 0$.

Lemma 2.66 ([18]). *For all* $\alpha, \beta > 0$, *then the following properties hold:*

(a) $(-A)^{-\alpha}(-A)^{-\beta} = A^{-(\alpha+\beta)}$.
(b) $\lim\limits_{\alpha \to 0} (-A)^{-\alpha} = I$ *in the strong operator topology.*

Proof. (a) We have,

$$
\begin{aligned}
(-A)^{-\alpha}(-A)^{-\beta} &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{+\infty}\int_0^{+\infty} t^{\alpha-1}s^{\beta-1}T(t)T(s)dtds\\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{+\infty} t^{\alpha-1}\int_t^{+\infty}(r-t)^{\beta-1}T(r)drdt\\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{+\infty}\int_0^{r} t^{\alpha-1}(r-t)^{\beta-1}dtT(r)dr\\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{1}\tau^{\alpha-1}(1-\tau)^{\beta-1}d\tau\int_0^{\infty} r^{\alpha+\beta-1}T(r)dr\\
&= \frac{1}{\Gamma(\alpha+\beta)}\int_0^{+\infty} r^{\alpha+\beta-1}T(r)dr\\
&= (-A)^{-\alpha-\beta}.
\end{aligned}
$$

(b) Since $(-A)^{-\alpha}$ is one-to-one, if $v\in D(A)$, there exists $u\in\mathscr{X}$ such that $v=(-A)^{-\alpha}u$. Thus $(-A)^{-\alpha}v-v=(-A)^{-1-\alpha}u-(-A)^{-1}u\to 0$ as $\alpha\to 0$ by the fact that $(-A)^{-\alpha}$ is continuous with respect to uniform operator norm.

Remark 2.67 ([18]).

(a) Let $\alpha\in(0,1)$. Using the fact that

$$(\lambda I-A)^{-1}=\int_0^{\infty}e^{-\lambda t}T(t)dt,$$

then Eq. (2.7) can be rewritten as

$$(-A)^{-\alpha}=\frac{\sin(\pi\alpha)}{\pi}\int_0^{+\infty}t^{-\alpha}(tI-A)^{-1}dt. \tag{2.8}$$

(b) The operator $(-A)^{-\alpha}$ is one-to-one, and hence has an inverse, which obviously is $(-A)^{\alpha}$. The operator $(-A)^{\alpha}$ is closed with domain $D((-A)^{\alpha})=R((-A)^{-\alpha})$. The operators $(-A)^{\alpha}$ are called fractional powers of $-A$.
(c) If $\alpha>\beta$, then $D((-A)^{\alpha})\subset D((-A)^{\beta})$.
(d) $D((-A)^{\alpha})$ is endowed with the norm $\|u\|_{\alpha}=\|(-A)^{\alpha}u\|$ for each $u\in D((-A)^{\alpha})$.
(e) $(-A)^{\alpha}$ commutes with $T(t)$ on $D(-A)^{\alpha})$ with

$$\|T(t)\|_{B(D(-A)^{\alpha}))}\leq M_0e^{-\omega t},\ \ t>0.$$

2.2.6.2 The Spaces $D_A(\alpha)$, $D_A(\alpha,\infty)$, and $D_A(\alpha+n,p)$

Let A be a sectorial and let $(T(t))_{t\geq 0}$ be the analytic semigroup associated with it. Clearly, $(T(t))_{t\geq 0}$ maps $(0,\infty)$ into $B(\mathscr{X})$ and there exist $M_0, M_1 > 0$ [137] with

$$\|T(t)\| \leq M_0 e^{\omega t}, \quad t > 0, \tag{2.9}$$

$$\|t(A-\omega)T(t)\| \leq M_1 e^{\omega t}, \quad t > 0. \tag{2.10}$$

Definition 2.68. Let $\alpha \in (0,1)$. A Banach space $(\mathscr{X}_\alpha, \|\cdot\|_\alpha)$ is said to be an intermediate space between $D(A)$ and $\mathscr{X}$, or a space of class $\mathscr{J}_\alpha$, if $D(A) \subset \mathscr{X}_\alpha \subset \mathscr{X}$ and there is a constant $c > 0$ such that

$$\|x\|_\alpha \leq C\|x\|^{1-\alpha}\|x\|_A^\alpha, \quad x \in D(A), \tag{2.11}$$

where $\|\cdot\|_A$ is the graph norm of A.

Concrete examples of the intermediate spaces $\mathscr{X}_\alpha$ include $D(A^\alpha)$ for $\alpha \in (0,1)$, the domains of the fractional powers of A, the real interpolation spaces $D_A(\alpha,\infty)$, $\alpha \in (0,1)$, defined as follows:

$$\begin{cases} D_A(\alpha,\infty) := \left\{x \in X : [x]_\alpha = \sup\limits_{0<t\leq 1} \|t^{1-\alpha}(A-\omega)e^{-\omega t}T(t)x\| < \infty\right\} \\ \\ \|x\|_\alpha = \|x\| + [x]_\alpha, \end{cases}$$

and the abstract Hölder spaces $D_A(\alpha) := \overline{D(A)}^{\|\cdot\|_\alpha}$.

One should point out that $D_A(\alpha,\infty)$ is characterized by the behavior of the quantity $t \mapsto \|t^{1-\alpha}AT(t)u\|$ near $t = 0$. Moreover, all the spaces $D_A(\alpha,\infty)$ are subspaces of $\overline{D(A)}$. Namely, the following embedding hold with equivalent norms:

$$D(A) \subset D_A(\beta,\infty) \subset D_A(\alpha,\infty) \subset \overline{D(A)}$$

for all $0 < \alpha < \beta < 1$.

If $\alpha \in (0,1)$, it is not very hard to see that $D_A(\alpha,\infty)$ can be characterized as being the subspace of all $u \in \mathscr{X}$ such that

$$[[u]]_\alpha = \sup_{t\in(0,1]} t^{-\alpha}\|T(t)u - u\| < \infty.$$

Furthermore, the norm defined by $u \mapsto \|u\| + [[u]]_\alpha$ is equivalent to the natural norm of $D_A(\alpha,\infty)$.

More generally, if $\alpha \in (0,1)$ and $p \in [1,\infty]$, we define the real interpolation space $D_A(\alpha,p)$ as follows:

$$\begin{cases} D_A(\alpha,p) := \left\{x \in X : t \mapsto v(t) = \|t^{1-\alpha-\frac{1}{p}} AT(t)x\| \in L^p(0,1)\right\} \\ \\ \|x\|_{\alpha,p} = \|x\| + \|v\|_{L^p(0,1)}, \end{cases}$$

and for $n \in \mathbb{N}$,

$$D_A(\alpha+n,p) = \{x \in D(A^n) : A^n x \in D_A(\alpha,p)\}, \ \ \|x\|_{\alpha+n,p} = \|x\| + \|A^n x\|_{\alpha,p}.$$

Proposition 2.69 ([137]). *For* $\alpha \in (0,1)$ *and* $1 \le p \le \infty$ *and for* $(\alpha,p) = (1,\infty)$*, then*

$$D_A(\alpha,p) = (\mathscr{X}, D(A))_{\alpha,p}$$

with equivalent norms. Moreover, for $0 < \alpha < 1$*, then*

$$D_A(\alpha) = (\mathscr{X}, D(A))_{\alpha}.$$

Proposition 2.70 ([137]). *For* $\alpha \in (0,1)$*, then*

$$D_A(\alpha,1) \subset D((-A)^\alpha) \subset (\mathscr{X}, D(A))_{\alpha,p}.$$

Proof. First of all, note that $D((-A)^\alpha)$ belongs to the class J_α. Notice that for each $u \in D(A)$, $(-A)^\alpha u = (-A)^{-(1-\alpha)}(-Au)$ and hence for each $\lambda > 0$,

$$\begin{aligned} \|(-A)^\alpha u\| &= \frac{1}{\Gamma(1-\alpha)} \left\| \left(\int_0^\lambda + \int_\lambda^\infty \right) t^{-\alpha} A e^{tA} u dt \right\| \\ &\le \frac{1}{\Gamma(1-\alpha)} \left(\frac{M_0}{1-\alpha} \|Au\| \lambda^{1-\alpha} + \frac{M_1}{\alpha} \|u\| \lambda^{-\alpha} \right). \end{aligned}$$

Letting $\lambda = \dfrac{\|u\|}{\|Au\|}$ it follows that

$$\|(-A)^\alpha u\| \le c \|Au\|^\alpha \|u\|^{1-\alpha}.$$

It remains to prove that $D(-A)^\alpha$ is continuously embedded in $D_A(\alpha,\infty)$. For that let $u \in D((-A)^\alpha)$ and let $v = (-A)^\alpha u$. So for $0 < \xi \le 1$, we have

$$\begin{aligned} \|\xi^{1-\alpha} A e^{\xi A} u\| &= \|\xi^{1-\alpha} A e^{\xi A} (-A)^\alpha v\| \\ &\le \frac{\xi^{1-\alpha}}{\Gamma(\alpha)} \left\| \int_0^\infty t^{\alpha-1} A e^{(\xi+t)A} v dt \right\| \end{aligned}$$

$$\leq \frac{M_1 \xi^{1-\alpha}}{\Gamma(\alpha)} \int_0^\infty \frac{\xi^{\alpha-1}}{\xi+t} dt \|v\|$$
$$\leq \frac{M_1}{\Gamma(\alpha)} \int_0^\infty \frac{s^{\alpha-1}}{1+s} ds \|(-A)^\alpha u\|$$

and hence $u \in D_A(\alpha, \infty)$.

Lemma 2.71 (Diagana and Nelson [75]). *Suppose A is sectorial. If $n < m$, then,*

$$D_A(\alpha+m, p) \hookrightarrow D_A(\alpha+n, p).$$

Proof. Indeed, $x \in D_A(\alpha+m, p)$ yields $x \in D(A^m) \subset D(A^n)$ and $A^m x \in D_A(\alpha, p)$. Now using the facts that A is invertible and $n - m < 0$ it follows that

$$\begin{aligned} \|t^{1-\alpha-\frac{1}{p}} AT(t)A^n x\| &= \|t^{1-\alpha-\frac{1}{p}} AT(t)A^{n-m}A^m x\| \\ &\leq \|A^{n-m}\| \cdot \|t^{1-\alpha-\frac{1}{p}} AT(t)A^m x\| \in L^p(0,1) \end{aligned}$$

and hence $A^n x \in D_A(\alpha, p)$, that is, $x \in D_A(\alpha+n, p)$.

In this book, the bound of the injection $D_A(\alpha+m, p) \hookrightarrow D_A(\alpha+n, p)$ will be denoted by C'.

Lemma 2.72 (Diagana and Nelson [75]). *Fix $\alpha \in (0,1)$ and $1 \leq p \leq \infty$. Let $A : D(A) \subset \mathscr{X} \mapsto \mathscr{X}$ be a sectorial linear operator. Suppose that the analytic semigroup $(T(t))_{t \geq 0}$ associated with A is exponentially stable, that is, there exists $M, \omega > 0$ such that*

$$\|T(t)\| \leq Me^{-\delta t} \quad \text{for } t \geq 0. \tag{2.12}$$

Then for all $m \in \mathbb{N}$, the function $t \mapsto \|A^m T(t)\|_{B(D_A(\alpha+m-1,p),\mathscr{X})}$ belongs to $L^1(0,\infty)$, that is, there exists a measurable function $H : (0,\infty) \mapsto (0,\infty)$ with

$$\theta_0 := \int_0^\infty H(t)dt < \infty$$

such that

$$\|A^m T(t)\|_{B(D_A(\alpha+m-1,p),\mathscr{X})} \leq H(t), \quad t > 0.$$

Proof. Let $x \in D_A(\alpha+m-1, p)$ for $\alpha \in (0,1)$, $p \in [1,\infty]$, and $n \in \mathbb{N}$. From [137, Corollary 2.2.3] one obtains that $D_A(\alpha, p) \hookrightarrow D_A(\alpha, \infty)$. Now using the definition of $D_A(\alpha, \infty)$ it follows that

$$\begin{aligned} \|A^m T(t)x\| &= \|t^{\alpha-1} t^{1-\alpha} AT(t)A^{m-1}x\| \\ &\leq t^{\alpha-1} \|x\|_{\alpha+m-1,\infty} \\ &\leq Kt^{\alpha-1} \|x\|_{\alpha+m-1,p} \end{aligned}$$

for a.e. $t \in (0,1)$.

Since $\alpha \in (0,1)$ it follows that $t \mapsto A^m T(t) \in L^1((0,1), B(D_A(\alpha+m-1,p), \mathscr{X}))$.

Now using the exponential stability of $T(t)$, it follows that there exists $C_m > 0$ such that for all $x \in D_A(\alpha+m-1,p)$ we have

$$\|A^m T(t)x\| \le C_m t^{-m} e^{-\delta t} \|x\| \le K'_m t^{-m} e^{-\delta t} \|x\|_{\alpha+m-1,p}, \quad t > 0.$$

In particular,

$$\|A^m T(t)x\| \le K'_m t^{-m} e^{-\delta t} \|x\|_{\alpha+m-1,p} \le K'_m e^{-\delta t} \|x\|_{\alpha+m-1,p}, \quad t \ge 1$$

and hence $t \mapsto A^m T(t) \in L^1([1,\infty), B(D_A(\alpha+m-1,p), \mathscr{X}))$.

2.2.6.3 Hyperbolic Semigroups

Definition 2.73 ([15]). Let A be a sectorial operator on $\mathscr{X}$ and let $(T(t))_{t\ge 0}$ be the analytic semigroup associated with it. The semigroup $(T(t))_{t\ge 0}$ is said to be hyperbolic if there exist a projection P and constants $M, \delta > 0$ such that each $T(t)$ commutes with P, $N(P)$ is invariant with respect to $T(t)$, $T(t) : R(Q) \mapsto R(Q)$ is invertible, and

$$\|T(t)Px\| \le Me^{-\delta t}\|x\| \quad \text{for } t \ge 0, \tag{2.13}$$

$$\|T(t)Qx\| \le Me^{\delta t}\|x\| \quad \text{for } \; t \le 0, \tag{2.14}$$

where $Q := I - P$ and $T(t) := (T(-t))^{-1}$ for $t < 0$.

Recall that an analytic semigroup $(T(t))_{t\ge 0}$ is hyperbolic if and only if (see [90])

$$\sigma(A) \cap i\mathbb{R} = \emptyset. \tag{2.15}$$

For the hyperbolic analytic semigroup $T(t)$, we can easily check that estimations similar to Eqs. (2.13) and (2.14) hold also with norms $\|\cdot\|_\alpha$. In fact, as the part of A in $R(Q)$ is bounded, it follows from the inequality Eq. (2.14) that

$$\|AT(t)Qx\| \le c' e^{\delta t}\|x\| \quad \text{for } t \le 0.$$

In view of the above, there exists a constant $c(\alpha) > 0$ such that

$$\|T(t)Qx\|_\alpha \le c(\alpha)e^{\delta t}\|x\| \quad \text{for } t \le 0. \tag{2.16}$$

Similarly,

$$\|T(t)Px\|_\alpha \le \|T(1)\|_{B(\mathscr{X},\mathscr{X}_\alpha)}\|T(t-1)Px\| \quad \text{for } t \ge 1,$$

and then from Eq. (2.13), we obtain

$$\|T(t)Px\|_\alpha \le M' e^{-\delta t}\|x\|, \quad t \ge 1,$$

where M' depends on α.

Clearly,

$$\|T(t)Px\|_\alpha \le M'' t^{-\alpha}\|x\|,$$

and hence there exist constants $M(\alpha) > 0$ and $\gamma > 0$ such that

$$\|T(t)Px\|_\alpha \le M(\alpha)t^{-\alpha}e^{-\gamma t}\|x\| \quad \text{for } t > 0. \tag{2.17}$$

We need the next lemma, which will be very crucial for our computations.

Lemma 2.74 (Diagana [57]). *Let* $0 < \alpha, \beta < 1$. *Then*

$$\|AT(t)Qx\|_\alpha \le ce^{\delta t}\|x\|_\beta \quad \textit{for } t \le 0, \tag{2.18}$$

$$\|AT(t)Px\|_\alpha \le ct^{\beta-\alpha-1}e^{-\gamma t}\|x\|_\beta \quad \textit{for } t > 0. \tag{2.19}$$

Proof. As for Eq. (2.16), the fact that the part of A in $R(Q)$ is bounded yields

$$\|AT(t)Qx\| \le ce^{\delta t}\|x\|_\beta, \quad \|A^2T(t)Qx\| \le ce^{\delta t}\|x\|_\beta \quad \text{for } t \le 0, \tag{2.20}$$

since $\mathscr{X}_\beta \hookrightarrow \mathscr{X}$. Hence, from Eq. (2.11) there is a constant $c(\alpha) > 0$ such that

$$\|AT(t)Qx\|_\alpha \le c(\alpha)e^{\delta t}\|x\|_\beta \quad \text{for } t \le 0. \tag{2.21}$$

Furthermore,

$$\|AT(t)Px\|_\alpha \le \|AT(1)\|_{B(\mathscr{X},\mathscr{X}_\alpha)}\|T(t-1)Px\| \tag{2.22}$$

$$\le ce^{-\delta t}\|x\|_\beta \quad \text{for } t \ge 1. \tag{2.23}$$

Now for $t \in (0,1]$, by Eq. (2.11), one has

$$\|AT(t)Px\|_\alpha \le ct^{-\alpha-1}\|x\|,$$

and

$$\|AT(t)Px\|_\alpha \le ct^{-\alpha}\|Ax\|,$$

for each $x \in D(A)$. Thus, by the Reiteration Theorem (see [137]), it follows that

$$\|AT(t)Px\|_\alpha \le ct^{\beta-\alpha-1}\|x\|_\beta$$

for every $x \in \mathscr{X}_\beta$ and $0 < \beta < 1$, and hence, there exist constants $M(\alpha) > 0$ and $\gamma > 0$ such that

$$\|T(t)Px\|_\alpha \leq M(\alpha)t^{\beta-\alpha-1}e^{-\gamma t}\|x\|_\beta \quad \text{for } t > 0.$$

2.3 Evolution Families

2.3.1 Evolution Families and Their Estimates

In this section we study time-dependent linear operators of the form $(A(t), D(A(t))_{t\in\mathbb{R}}$ as well as their associated evolution families. For more on evolution families and related topics, we refer to [137].

Definition 2.75. A family of bounded linear operators $\{U(t,s) : t,s \in \mathbb{R},\ t \geq s\}$ on $\mathscr{X}$ is called an evolution family whether the following hold:

(a) $U(t,s)U(s,r) = U(t,r)$ for $t,s,r \in \mathbb{R}$ such that $t \geq s \geq r$;
(b) $U(t,t) = I$ for $t \in \mathbb{R}$; and
(c) for each $x \in \mathscr{X}$, the function $(t,s) \mapsto U(t,s)x$ is continuous for $t \geq s$.

Evolution families (also called "evolution systems," "evolution operators," "evolution processes," "propagators," or "fundamental solutions") play a crucial role especially when it comes to studying some partial differential equations.

Classical examples of evolution families include U defined by $U(t,s) = T(t-s)$ for all $t,s \in \mathbb{R}$ with $t \geq s$, where $(T(t))_{t\geq 0}$ is a strongly continuous semigroup.

Another example of an evolution family consists of $U(t,s) = f(t)f^{-1}(s)$ for all $t,s \in \mathbb{R}$ with $t \geq s$, where $f : \mathbb{R} \mapsto \mathbb{R}$ is a function satisfying, $f(t) \neq 0$ for all $t \in \mathbb{R}$.

Definition 2.76. An evolution family $\{U(t,s) : t,s \in \mathbb{R},\ t \geq s\}$ on $\mathscr{X}$ is said to be exponentially bounded if there exist $M, \omega > 0$ such that

$$\|U(t,s)\| \leq Me^{\omega(t-s)},\ \ t,s \in \mathbb{R}, \quad t \geq s. \tag{2.24}$$

It can be easily seen that the evolution equation $U(t,s) = T(t-s)$ for all $t,s \in \mathbb{R}$ with $t \geq s$, where $(T(t))_{t\geq 0}$ is a strongly continuous semigroup, is an example of an exponentially bounded evolution family.

Definition 2.77. An evolution family $\{U(t,s) : t,s \in \mathbb{R},\ t \geq s\}$ on $\mathscr{X}$ is said to be exponentially stable if there exists $M, \omega > 0$ such that

$$\|U(t,s)\| \leq Me^{-\omega(t-s)},\ \ t,s \in \mathbb{R}, \quad t \geq s. \tag{2.25}$$

It can be easily seen that the evolution equation $U(t,s) = T(t-s)$ for all $t,s \in \mathbb{R}$ with $t \geq s$, where $(T(t))_{t\geq 0}$ is an exponentially stable C_0-semigroup, is an example of an exponentially stable evolution family.

2.3.2 Acquistapace–Terreni Conditions

Definition 2.78. A family of linear operators $(A(t))_{t\in\mathbb{R}}$ is said to satisfy the Acquistapace–Terreni conditions whether there exist $\lambda_0 \geq 0$ and the constants $\phi \in (\frac{\pi}{2}, \pi)$, $L, K \geq 0$, and $\mu, \nu \in (0,1]$ with $\mu + \nu > 1$ such that

$$\Sigma_\phi \cup \{0\} \subseteq \rho(A(t) - \lambda_0) \ni \lambda, \quad \|R(\lambda, A(t) - \lambda_0)\| \leq \frac{K}{1+|\lambda|} \tag{2.26}$$

and

$$\|(A(t) - \lambda_0)R(\lambda_0, A(t) - \lambda_0)\,[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\,\| \leq L|t-s|^\mu\,|\lambda|^{-\nu} \tag{2.27}$$

for $t,s \in \mathbb{R}$, $\lambda \in \Sigma_\phi := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| \leq \phi\}$.

For a given family of closed linear operators $\{A(t) : \ t \in \mathbb{R}\}$ on $\mathscr{X}$, the existence of an evolution family associated with it is not always guaranteed. However, if the family $A(t)$ satisfies Acquistapace–Terreni conditions, then the family of linear operators $A(t)$ has an evolution family associated with it such that $U(t,s)\mathscr{X} \subseteq D(A(t))$ for all $t,s \in \mathbb{R}$. Moreover, the following hold: for $t > s$, the mapping $(t,s) \mapsto U(t,s) \in B(\mathscr{X})$ is continuously differentiable in t with $\partial_t U(t,s) = A(t)U(t,s)$. Furthermore, there exists a constant $C' > 0$ which depends on constants in Eqs. (2.26) and (2.27) such that

$$\|A^k(t)U(t,s)\| \leq C'(t-s)^{-k} \tag{2.28}$$

for $0 < t-s \leq 1$ and $k = 0,1$.

Remark 2.79. (a) In the particular case of a constant domain $D(A(t))$, one can replace Eq. (2.27) (see for instance [148]) with the following: there exist constants L and $0 < \mu \leq 1$ such that

$$\|\,(A(t) - A(s))R(\lambda_0, A(r))\| \leq L|t-s|^\mu, \ s,t,r \in \mathbb{R}. \tag{2.29}$$

(b) The Acquistapace–Terreni conditions were introduced in the literature by Acquistapace and Terreni in [1,2] for $\lambda_0 = 0$.

Definition 2.80. An evolution family $\{U(t,s) : t \geq s \ \text{ with } \ t,s \in \mathbb{R}\} \subset B(\mathscr{X})$ is said to have an *exponential dichotomy* (*or is hyperbolic*) if there are projections $P(t)$ $(t \in \mathbb{R})$ that are uniformly bounded and strongly continuous in t (and we then let $Q(t) = I - P(t)$) and constants $\delta > 0$ and $N \geq 1$ such that

(a) (i) $U(t,s)P(s) = P(t)U(t,s)$ for all $t \geq s$;
(b) the restriction $U_Q(t,s) : Q(s)\mathscr{X} \to Q(t)\mathscr{X}$ of $U(t,s)$ is invertible for all $t \geq s$ (and we then set $U_Q(s,t) := U_Q(t,s)^{-1}$); and
(c) $\|U(t,s)P(s)\| \leq Ne^{-\delta(t-s)}$ and $\|U_Q(s,t)Q(t)\| \leq Ne^{-\delta(t-s)}$ for $t \geq s$ and $t,s \in \mathbb{R}$.

2.3.2.1 Estimates for the Evolution Family $U(t,s)$

This subsection is devoted to the study of some estimates related to $U(t,s)$. For that, we introduce the corresponding interpolation spaces for $A(t)$. We refer the reader to [90, 137] for proofs and further information on these spaces.

Let A be a sectorial operator on $\mathscr{X}$ and let $\alpha \in (0,1)$. Define the real interpolation space

$$\mathscr{X}_\alpha^A := \left\{x \in \mathscr{X} : \|x\|_\alpha^A := \sup_{r>0} \|r^\alpha (A-\zeta)R(r,A-\zeta)x\| < \infty\right\},$$

which, by the way, is a Banach space when endowed with the norm $\|\cdot\|_\alpha^A$. For convenience we further write

$$\mathscr{X}_0^A := \mathscr{X},\ \|x\|_0^A := \|x\|,\ \ \mathscr{X}_1^A := D(A),\ \text{ and }\ \|x\|_1^A := \|(\zeta - A)x\|.$$

We will need the space $\hat{\mathscr{X}}^A := \overline{D(A)}$. In particular, we will frequently be using the following continuous embedding:

$$D(A) \hookrightarrow \mathscr{X}_\beta^A \hookrightarrow D((\zeta - A)^\alpha) \hookrightarrow \mathscr{X}_\alpha^A \hookrightarrow \hat{\mathscr{X}}^A \subset \mathscr{X}, \tag{2.30}$$

for all $0 < \alpha < \beta < 1$, where the fractional powers are defined in the usual way.

In general, $D(A)$ is not dense in the spaces $\mathscr{X}_\alpha^A$ and $\mathscr{X}$. However, we have the following continuous injection:

$$\mathscr{X}_\beta^A \hookrightarrow \overline{D(A)}^{\|\cdot\|_\alpha^A} \tag{2.31}$$

for $0 < \alpha < \beta < 1$.

Given the operators $A(t)$ for $t \in \mathbb{R}$, satisfying Acquistapace–Terreni conditions, we set

$$\mathscr{X}_\alpha^t := \mathscr{X}_\alpha^{A(t)}, \quad \hat{\mathscr{X}}^t := \hat{\mathscr{X}}^{A(t)}$$

for $0 \leq \alpha \leq 1$ and $t \in \mathbb{R}$, with the corresponding norms. Then the embedding in Eq. (2.30) holds with constants independent of $t \in \mathbb{R}$. These interpolation spaces are of class $\mathscr{J}_\alpha$ and hence there is a constant $l(\alpha)$ such that

$$\|y\|_\alpha^t \leq l(\alpha)\|y\|^{1-\alpha}\|A(t)y\|^\alpha, \quad y \in D(A(t)). \tag{2.32}$$

We will need the following estimates to establish some of the results of this book.

Proposition 2.81 (Baroun et al. [15]). *For $x \in \mathscr{X}$, $0 \le \alpha \le 1$, and all $t > s$, the following hold:*

(a) There is a constant $c(\alpha)$, such that

$$\|U(t,s)P(s)x\|_\alpha^t \le c(\alpha) e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|. \tag{2.33}$$

(b) There is a constant $m(\alpha)$, such that

$$\|\widetilde{U}_Q(s,t)Q(t)x\|_\alpha^s \le m(\alpha)e^{-\delta(t-s)}\|x\|. \tag{2.34}$$

Proof. (a) Using Eq. (2.32) we obtain

$$\begin{aligned}
\|U(t,s)P(s)x\|_\alpha^t &\le l(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U(t,s)P(s)x\|^\alpha \\
&\le l(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U(t,t-1)U(t-1,s)P(s)x\|^\alpha \\
&\le l(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U(t,t-1)\|^\alpha\|U(t-1,s)P(s)x\|^\alpha \\
&\le l(\alpha)N'\, e^{-\delta(t-s)(1-\alpha)}e^{-\delta(t-s-1)\alpha}\|x\| \\
&\le c'(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}(t-s)^\alpha e^{-\frac{\delta}{2}(t-s)}\|x\|
\end{aligned}$$

for $t-s \ge 1$ and $x \in \mathscr{X}$.

Since $(t-s)^\alpha e^{-\frac{\delta}{2}(t-s)} \to 0$ as $t \to \infty$ it easily follows that

$$\|U(t,s)P(s)x\|_\alpha^t \le c_1(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\|x\|.$$

If $0 < t-s \le 1$, we have

$$\begin{aligned}
\|U(t,s)P(s)x\|_\alpha^t &\le l(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U(t,s)P(s)x\|^\alpha \\
&\le l(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\left\|A(t)U\left(t,\frac{t+s}{2}\right)U\left(\frac{t+s}{2},s\right)P(s)x\right\|^\alpha \\
&\le l(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\left\|A(t)U\left(t,\frac{t+s}{2}\right)\right\|^\alpha\left\|U\left(\frac{t+s}{2},s\right)P(s)x\right\|^\alpha \\
&\le l(\alpha)Ne^{-\delta(t-s)(1-\alpha)}2^\alpha(t-s)^{-\alpha}e^{-\frac{\delta\alpha}{2}(t-s)}\|x\| \\
&\le l(\alpha)Ne^{-\frac{\delta}{2}(t-s)(1-\alpha)}2^\alpha(t-s)^{-\alpha}e^{-\frac{\delta\alpha}{2}(t-s)}\|x\| \\
&\le c_2(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|,
\end{aligned}$$

and hence

$$\|U(t,s)P(s)x\|_\alpha^t \le c(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\|x\| \;\text{ for }\; t>s.$$

(ii) We have,

$$\begin{aligned}
\|\tilde{U}_Q(s,t)Q(t)x\|_\alpha^s &\le l(\alpha)\|\tilde{U}_Q(s,t)Q(t)x\|^{1-\alpha}\|A(s)\tilde{U}_Q(s,t)Q(t)x\|^\alpha\\
&\le l(\alpha)\|\tilde{U}_Q(s,t)Q(t)x\|^{1-\alpha}\|A(s)Q(s)\tilde{U}_Q(s,t)Q(t)x\|^\alpha\\
&\le l(\alpha)\|\tilde{U}_Q(s,t)Q(t)x\|^{1-\alpha}\|A(s)Q(s)\|^\alpha\|\tilde{U}_Q(s,t)Q(t)x\|^\alpha\\
&\le l(\alpha)Ne^{-\delta(t-s)(1-\alpha)}\|A(s)Q(s)\|^\alpha e^{-\delta(t-s)\alpha}\|x\|\\
&\le m(\alpha)e^{-\delta(t-s)}\|x\|.
\end{aligned}$$

In the last inequality we made use of the fact that $\|A(s)Q(s)\| \le c$ for some constant $c \ge 0$ (see [153, Proposition 3.18]).

Remark 2.82. It should be mentioned that if $U(t,s)$ is exponentially stable, then $P(t) = I$ and $Q(t) = I - P(t) = 0$ for all $t \in \mathbb{R}$. In that case, Eq. (2.33) still holds and can be rewritten as follows: for all $x \in \mathscr{X}$,

$$\|U(t,s)x\|_\alpha^t \le c(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|. \tag{2.35}$$

We will need the following technical lemma later on.

Lemma 2.83 (Diagana [76]). *Let $\alpha \in (0,1)$. Suppose that the evolution family $(U(t,s))_{t\ge s}$ not only satisfies Acquistapace–Terreni conditions but also is exponentially stable. Then, there exists a function $H_{\alpha,\delta} : (0,\infty) \mapsto (0,\infty)$ with $H_{\alpha,\delta} \in L^1(0,\infty)$ such that*

$$\|A(t)U(t,s)x\| \le H_{\alpha,\delta}(t-s)\|x\|_\alpha, \quad t > s \tag{2.36}$$

for all $x \in \mathscr{X}_\alpha$.

Proof. First of all, note that there exists a constant $C > 0$ such that

$$\|A(t)U(t,s)\|_{B(\mathscr{X}_\alpha,\mathscr{X})} \le C(t-s)^{\alpha-1} \tag{2.37}$$

for all $t,s \in \mathbb{R}$ such that $0 < t-s \le 1$ (see [152]).

Now for $t-s \ge 1$, using Eq. (2.28) in the case when $k=1$, we obtain that

$$\begin{aligned}
\|A(t)U(t,s)x\| &= \|A(t)U(t,t-1)U(t-1,s)x\|\\
&\le \|A(t)U(t,t-1)\|_{B(\mathscr{X})}\|U(t-1,s)x\|\\
&\le C'\|U(t-1,s)x\|\\
&\le C'Ne^\delta e^{-\delta(t-s)}\|x\|\\
&\le M'C'Ne^\delta e^{-\delta(t-s)}\|x\|_\alpha
\end{aligned}$$

for all $x \in \mathscr{X}_\alpha$, where M' is the bound of the continuous injection $\mathscr{X}_\alpha \hookrightarrow \mathscr{X}$.

Choosing the function $H_{\alpha,\delta} : (0,\infty) \mapsto (0,\infty)$ as

$$H_{\alpha,\delta}(t) = \begin{cases} Ct^{\alpha-1} & \text{if } \ t \in (0,1] \\ \\ N'e^{-\delta t} & \text{if } \ t \in (1,\infty) \end{cases}$$

where $N' = M'C'Ne^{\delta}$, one can easily check that $H_{\alpha,\delta} \in L^1(0,\infty)$ and that

$$\|A(t)U(t,s)x\| \leq H_{\alpha,\delta}(t-s)\|x\|_\alpha, \quad t > s$$

for all $x \in \mathscr{X}_\alpha$.

Bibliographical Notes

The material of Sect. 2.1 is taken from the following sources: Diagana [57], and Bezandry and Diagana [18]. Additional relevant references include the following: Conway [40], Diagana [56], Naylar and Sell [141], Pazy [148], Rudin [151], Weidmann [160], Locker [136], Lunardi [137], and Engel and Nagel [90].

The material of Sects. 2.2 and 2.3 is taken from the following sources: Baroun et al. [15], Bezandry and Diagana [18], Diagana [76], Diagana and Nelson [75], and Lunardi [137]. Additional relevant references include Engel and Nagel [90], Schnaubelt [152, 153], and Chicone and Latushkin [38].

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