Chapter 2
Solutions to the Linear Stochastic Heat and Wave Equation

In this chapter we analyze the basic properties of some self-similar Gaussian processes that are solutions to stochastic partial differential equations with additive Gaussian noise. We will see that some of these processes are closely related to the fractional-type processes discussed in Chap. 1. The noise of the equation will be defined in various ways: white (meaning that is behaves as a Brownian motion) or corelared ("colored") in time and/or in space. The general context is as follows: consider the equation

$$Lu(t,x) = \Delta u(t,x) + \dot{W}(t,x) \tag{2.1}$$

with $t \in [0,T]$ and $x \in \mathbb{R}^d$ and with vanishing initial conditions. Here Δ is the Laplacian on $\mathbb{R}^d$

$$\Delta u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2},$$

W is the noise of the equation and L is a first or second order operator with constant coefficients. In our analysis, we will consider the heat equation and then

$$Lu(t,x) = \frac{\partial u}{\partial t}(t,x), \quad t \in [0,T], x \in \mathbb{R}^d$$

or the wave equation and in this case

$$Lu(t,x) = \frac{\partial^2 u}{\partial t^2}(t,x), \quad t \in [0,T], x \in \mathbb{R}^d.$$

Usually, the solution to (2.1) is defined through its mild form

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) dW(s,y)$$

where G is the solution of $Lu - \Delta u = 0$ and the above integral is a Wiener integral with respect to W. This Wiener integral can be understood in the sense of

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Appendix C. Essentially the solution to (2.1) exists when this Wiener integral is well-defined and this happens when the integrand G belongs to the Hilbert space associated to the Gaussian noise W. In order to study the existence and the properties of the solution to (2.1), an important fact is the structure of the canonical Hilbert spaces associated with the noise and this depends on the covariance structure of the noise.

We denote by $C_0^\infty(\mathbb{R}^{d+1})$ the space of infinitely differentiable functions on $\mathbb{R}^{d+1}$ with compact support, and $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing C^∞ functions on $\mathbb{R}^d$ and by $\mathcal{S}'(\mathbb{R}^d)$ its dual. For $\varphi \in L^1(\mathbb{R}^d)$, we let $\mathcal{F}\varphi$ be the Fourier transform of φ:

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x}\varphi(x)dx.$$

2.1 The Solution to the Stochastic Heat Equation with Space-Time White Noise

We will first discuss the properties of the solution to the stochastic heat equation with additive Gaussian noise that behaves as a Wiener process both in time and in space.

2.1.1 The Noise

Let us first introduce the noise of the equation. Consider a centered Gaussian field $W = \{W(t, A); t \in [0, T], A \in \mathcal{B}_b(\mathbb{R}^d)\}$ with covariance

$$\mathbf{E}W(t, A)W(s, B) = (t \wedge s)\lambda(A \cap B), \quad t \in [0, T], A \in \mathcal{B}_b\big(\mathbb{R}^d\big) \tag{2.2}$$

where λ denotes the Lebesgue measure. Also consider the stochastic partial differential equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\Delta u + \dot{W}, \quad t \in [0, T],\ x \in \mathbb{R}^d \\ u(0, x) &= 0, \quad x \in \mathbb{R}^d, \end{aligned} \tag{2.3}$$

where the noise W is defined by (2.2). The noise W is usually referred to as a *space-time white noise* because it behaves as a Brownian motion both with respect to both the time and the space variable.

The canonical Hilbert space associated with the Gaussian process W is defined as the closure of the linear span generated by the indicator functions $1_{[0,t]\times A}$, $t \in [0, T]$, $A \in \mathcal{B}_b(\mathbb{R}^d)$ with respect to the inner product

$$\langle 1_{[0,t]\times A}, 1_{[0,s]\times B}\rangle_{\mathcal{H}} = (t \wedge s)\lambda(A \cap B).$$

In our case the space $\mathcal{H}$ is $L^2([0, T] \times \mathbb{R}^d)$.

2.1.2 The Solution

This mild solution is defined as

$$u(t,x) = \int_0^T \int_{\mathbb{R}^d} G(t-s, x-y) W(ds, dy), \quad t \in [0,T], x \in \mathbb{R}^d \tag{2.4}$$

where the above integral is a Wiener integral with respect to the Gaussian process W (see e.g. [13] for details) and G is the Green kernel of the heat equation given by

$$G(t,x) = \begin{cases} (2\pi t)^{-d/2} \exp(-\frac{|x|^2}{2t}) & \text{if } t > 0, x \in \mathbb{R}^d \\ 0 & \text{if } t \leq 0, x \in \mathbb{R}^d. \end{cases} \tag{2.5}$$

The Wiener integral in (2.4) is well-defined whenever the function $(s,y) \to G(t-s, x-y)$ belongs to $L^2([0,T] \times \mathbb{R}^d)$. As we will see in the sequel, this is not always the case and it depends on the spatial dimension d. Consequently the process $(u(t,x), t \in [0,T], x \in \mathbb{R})$, when it exists, is a centered Gaussian process. We also need the following expression of the Fourier transform of the Green kernel

$$\mathcal{F}G(t,\cdot)(\xi) = \exp\left(-\frac{t|\xi|^2}{2}\right), \quad t > 0, \xi \in \mathbb{R}^d \tag{2.6}$$

where $\mathcal{F}G(t,\cdot)$ denotes the Fourier transform of the function $y \to G(t,y)$.

Proposition 2.1 *The solution* (2.4) *exists if and only if $d = 1$. Moreover, the covariance of the solution* (2.4) *satisfies the following: for every $x \in \mathbb{R}$ we have*

$$\mathbf{E}\big(u(t,x)u(s,x)\big) = \frac{1}{\sqrt{2\pi}}(\sqrt{t+s} - \sqrt{|t-s|}), \quad \textit{for every } s,t \in [0,T]. \tag{2.7}$$

Proof Fix $x \in \mathbb{R}^d$. For every $s,t \in [0,T]$, using that for any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \varphi(x)\psi(x)dx = (2\pi)^{-d} \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}d\xi \tag{2.8}$$

we get

$$\begin{aligned} \mathbf{E}u(t,x)u(s,x) &= \int_0^{t\wedge s} du \int_{\mathbb{R}^d} G(t-u, x-y)^2 dy \\ &= (2\pi)^{-d} \int_0^s du \int_{\mathbb{R}^d} d\xi \mathcal{F}G(t-u, x-\cdot)(\xi)\overline{\mathcal{F}G(s-u, x-\cdot)}(\xi) \\ &= (2\pi)^{-d} \int_0^s du \int_{\mathbb{R}^d} d\xi e^{-\frac{1}{2}(t-u)|\xi|^2} e^{-\frac{1}{2}(s-u)|\xi|^2} \end{aligned} \tag{2.9}$$

and then, if $s \leq t$

$$\mathbf{E}u(t,x)u(s,x) = (2\pi)^{-d} \int_0^s du(t+s-2u)^{-\frac{d}{2}} \int_{\mathbb{R}^d} d\xi e^{-\frac{1}{2}|\xi|^2}$$

$$= (2\pi)^{-d/2} \int_0^s du (t+s-2u)^{-\frac{d}{2}}.$$

Take $t = s$. Then

$$\mathbf{E}u(t,x)^2 = (2\pi)^{-d/2} \int_0^t du (t-u)^{-\frac{d}{2}}$$

and it is obvious that the integral above is finite if and only if $d = 1$. In that case, from (2.9)

$$\mathbf{E}u(t,x)u(s,x) = (2\pi)^{-1/2}\big((t+s)^{\frac{1}{2}} - (t-s)^{\frac{1}{2}}\big). \qquad \square$$

This fact establishes an interesting connection between the law of the solution (2.4) and the bifractional Brownian motion from Sect. 1.2.

Corollary 2.1 *Let $(u(t,x), t \in [0,T], x \in \mathbb{R}^d)$ be given by (2.4). Then for every $x \in \mathbb{R}$*

$$\big(u(t,x), t \in [0,T]\big) =^{(d)} \big(\sqrt{C} B_t^{\frac{1}{2},\frac{1}{2}}, t \in [0,T]\big)$$

where $B^{\frac{1}{2},\frac{1}{2}}$ is a bifractional Brownian motion with parameters $H = K = \frac{1}{2}$ and $C := 2^{-K}\frac{1}{\sqrt{2\pi}}$. Here $=^{(d)}$ means equivalence of finite dimensional distributions.

Proof The assertion follows from relation (2.7) and Definition 1.2. $\square$

Remark 2.1 From (2.7), it follows that the stochastic process defined by (2.4) is self-similar of order $\frac{1}{4}$ with respect to the variable t.

2.2 The Spatial Covariance

The restriction $d = 1$ for the existence of the solution with space-time white noise is not convenient because we need to consider such models in higher dimensions. This has led researchers in the last few decades to investigate other types of noise that would allow such consideration of higher dimensions.

We begin by introducing the framework. Let μ be a non-negative tempered measure on $\mathbb{R}^d$, i.e. a non-negative measure which satisfies:

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)^l \mu(d\xi) < \infty, \quad \text{for some } l > 0.$$

Since the integrand is non-increasing in l, we may assume that $l \geq 1$ is an integer. Note that $1 + |\xi|^2$ behaves as a constant near 0, and as $|\xi|^2$ at ∞, and hence (2.10) is equivalent to:

$$\int_{|\xi|\leq 1} \mu(d\xi) < \infty, \quad \text{and} \quad \int_{|\xi|\geq 1} \mu(d\xi)\frac{1}{|\xi|^{2l}} < \infty, \quad \text{for some integer } l \geq 1. \tag{2.10}$$

Let $f : \mathbb{R}^d \to \mathbb{R}_+$ be the Fourier transform of μ in $\mathcal{S}'(\mathbb{R}^d)$, i.e.

$$\int_{\mathbb{R}^d} f(x)\varphi(x)dx = \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\mu(d\xi), \quad \forall\varphi \in \mathcal{S}\big(\mathbb{R}^d\big).$$

Simple properties of the Fourier transform show that for any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x)f(x-y)\psi(y)dxdy = (2\pi)^{-d}\int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}\mu(d\xi). \tag{2.11}$$

2.3 The Solution to the Linear Heat Equation with White-Colored Noise

2.3.1 The Noise

Consider the so-called *white-colored noise*, meaning a Gaussian process $W = \{W(t, A); t \in [0, T], A \in \mathcal{B}_b(\mathbb{R}^d)\}$ with zero mean and covariance

$$\mathbf{E}W(t, A)W(t, B) = (t \wedge s)\int_A\int_B f(x-y)dxdy. \tag{2.12}$$

The noise W behaves as a Brownian motion with respect to the time variable and it has a correlated spatial covariance. Here the kernel f should be the Fourier transform of a tempered non-negative measure μ on $\mathbb{R}^d$ as described in the previous paragraph.

Under this assumption the right-hand side of (2.12) is a covariance function. There are several examples of such kernels f.

Example 2.1 The Riesz kernel of order α:

$$f(x) = R_\alpha(x) := \gamma_{\alpha,d}|x|^{-d+\alpha}, \quad 0 < \alpha < d,$$

where $\gamma_{\alpha,d} = 2^{d-\alpha}\pi^{d/2}\Gamma((d-\alpha)/2)/\Gamma(\alpha/2)$. In this case, $\mu(d\xi) = |\xi|^{-\alpha}d\xi$.

Example 2.2 The Bessel kernel of order α:

$$f(x) = B_\alpha(x) := \gamma'_\alpha\int_0^\infty w^{(\alpha-d)/2-1}e^{-w}e^{-|x|^2/(4w)}dw, \quad \alpha > 0,$$

where $\gamma'_\alpha = (4\pi)^{\alpha/2}\Gamma(\alpha/2)$. In this case, $\mu(d\xi) = (1+|\xi|^2)^{-\alpha/2}d\xi$.

Example 2.3 The Poisson kernel

$$f(x) = P_\alpha(x) := \gamma'''_{\alpha,d}\big(|x|^2 + \alpha^2\big)^{-(d+1)/2}, \quad \alpha > 0,$$

where $\gamma'''_{\alpha,d} = \pi^{-(d+1)/2}\Gamma((d+1)/2)\alpha$. In this case, $\mu(d\xi) = e^{-4\pi^2\alpha|\xi|}d\xi$.

Example 2.4 The heat kernel

$$f(x) = G_\alpha(x) := \gamma''_{\alpha,d}e^{-|x|^2/(4\alpha)}, \quad \alpha > 0,$$

where $\gamma''_{\alpha,d} = (4\pi\alpha)^{-d/2}$. In this case, $\mu(d\xi) = e^{-\pi^2\alpha|\xi|^2}d\xi$.

With the Gaussian process W we can associated a canonical Hilbert space $\mathcal{P}$. The space $\mathcal{P}$ defined as the completion of $\mathcal{D}((0,T)\times\mathbb{R}^d)$ (or the completion of $\mathcal{E}$, the linear space generated by the indicator functions $1_{[0,t]\times A}$, $t\in[0,T]$, $A\subset\mathcal{B}(\mathbb{R}^d)$) with respect to the inner product

$$\langle\varphi,\psi\rangle_{\mathcal{P}} = \int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(t,x)f(x-y)\psi(t,y)dydxdt$$

has been studied by several authors in connection with a Gaussian noise which is white in time and colored in space. In particular this space may contain distributions.

2.3.2 The Solution

The solution is defined again by (2.4) with W given by (2.12). The necessary and sufficient condition for (2.4) to exist has been proven in [59].

Proposition 2.2 *The stochastic heat equation with white-colored noise given by* (2.12) *admits a unique solution if and only if*

$$\int_{\mathbb{R}^d}\frac{1}{1+|\xi|^2}\mu(d\xi) < \infty.$$

Proof For every $t\in[0,T]$ and $x\in\mathbb{R}^d$, using (2.6) and (2.11)

$$\begin{aligned}
\mathbf{E}u(t,x)^2 &= \int_0^t du\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}G(t-u,x-y)G\big(t-u,x-y'\big)f\big(y-y'\big)dydy'\\
&= (2\pi)^{-d}\int_0^t du\int_{\mathbb{R}^d}\mu(d\xi)\mathcal{F}G(t-u,x-\cdot)(\xi)\overline{\mathcal{F}G(t-u,x-\cdot)}(\xi)\\
&= (2\pi)^{-d}\int_0^t du\int_{\mathbb{R}^d}\mu(d\xi)e^{-\frac{1}{2}(t-u)|\xi|^2}e^{-\frac{1}{2}(t-u)|\xi|^2}
\end{aligned}$$

$$= (2\pi)^{-d} \int_0^t du \int_{\mathbb{R}^d} \mu(d\xi) e^{-(t-u)|\xi|^2}$$
$$= (2\pi)^{-d} \int_{\mathbb{R}^d} \mu(d\xi) \frac{1}{|\xi|^2} \big(1 - e^{-t|\xi|^2}\big).$$

One can prove that

$$c_{1,t} \frac{1}{1+|\xi|^2} \le \frac{1}{|\xi|^2} \big(1 - e^{-t|\xi|^2}\big) \le c_{2,t} \frac{1}{1+|\xi|^2}$$

with $c_{1,t}, c_{2,t}$ strictly positive constants that may be dependent on t. It can also be checked that the Green kernel belongs to the space $\mathcal{P}$ and the desired result is obtained. □

Remark 2.2 It has been proved in [59] that even in the non-linear case the stochastic heat equation $u_t = \frac{1}{2}\Delta u + g(u)\dot{W}$ (with standard assumptions on g) with white-colored noise admits a unique solution if and only if

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2} \right) \mu(d\xi) < \infty.$$

Obviously, this condition is also meaningful in higher dimensions. For example in the case of the Riesz or Bessel kernels, we have the following.

Corollary 2.2 *Suppose that the spatial covariance is given by the Riesz kernel* (*Example* 2.1) *or by the Bessel kernel* (*Example* 2.2). *Then the stochastic heat equation with white-colored noise admits a unique solution if and only if*

$$d < 2 + \alpha.$$

This implies that one can consider every dimension $d \ge 1$.

It is possible to compute the covariance of the solution with respect to the time variable; actually for fixed $x \in \mathbb{R}^d$, $d \ne 2$ and for every $s \le t$ we have

$$\begin{aligned}
&\mathbf{E} u(t,x)u(s,x) \\
&\quad = \int_0^{t\wedge s} du \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(t-u, x-y) G\big(s-u, x-y'\big) f\big(y-y'\big) dy dy' \\
&\quad = (2\pi)^{-d} \int_0^s du \int_{\mathbb{R}^d} \mu(d\xi) \mathcal{F}G(t-u, x-\cdot)(\xi) \overline{\mathcal{F}G(s-u, x-\cdot)}(\xi) \\
&\quad = (2\pi)^{-d} \int_0^s du \int_{\mathbb{R}^d} \mu(d\xi) e^{-\frac{1}{2}(t-u)|\xi|^2} e^{-\frac{1}{2}(s-u)|\xi|^2}. \qquad (2.13)
\end{aligned}$$

For the Riesz kernel, this gives

Proposition 2.3 *Suppose we are in the case of the Riesz kernel f of order α (see Example 2.1). Then for every $x \in \mathbb{R}^d$ and for every $s, t \in [0, T]$*

$$\mathbf{E}u(t,x)u(s,x) = C_0^2\left((t+s)^{-\frac{d-\alpha}{2}+1} - (t-s)^{-\frac{d-\alpha}{2}+1}\right)$$

where

$$C_0 = \left[(2\pi)^{-d}\int_{\mathbb{R}^d}\mu(d\xi)e^{-\frac{1}{2}|\xi|^2}\frac{1}{-\frac{d-\alpha}{2}+1}2^{-K}\right]^{\frac{1}{2}}. \tag{2.14}$$

Proof Consider $s \le t$. From (2.13), by the change of variables $\tilde{\xi} = \sqrt{t+s-2u}\xi$

$$\begin{aligned}\mathbf{E}u(t,x)u(s,x) &= (2\pi)^{-d}\int_0^s du(t+s-2u)^{-\frac{d}{2}}\int_{\mathbb{R}^d}\mu(d\xi)e^{-\frac{1}{2}|\xi|^2}\\ &= (2\pi)^{-d}\int_{\mathbb{R}^d}\mu(d\xi)e^{-\frac{1}{2}|\xi|^2}\frac{1}{-\frac{d-\alpha}{2}+1}\\ &\quad\times\left((t+s)^{-\frac{d-\alpha}{2}+1} - (t-s)^{-\frac{d-\alpha}{2}+1}\right).\end{aligned}$$ □

As a consequence, in the case of the spatial covariance given by the Riesz kernel, the solution of the heat equation with white noise in time coincides in distribution with, modulo a constant, a bifractional Brownian motion.

Corollary 2.3 *For fixed $x \in \mathbb{R}^d$, the solution to the white-colored heat equation coincides in distribution with*

$$\left(C_0 B_t^{H,K}\right)_{t\in[0,T]}$$

where $B^{H,K}$ is a bifractional Brownian motion with parameters $H = \frac{1}{2}$ and $K = 1 - \frac{d-\alpha}{2}$ and C_0 is defined in (2.14).

Proof This follows from Proposition 2.3 and the expression of the covariance of the bi-fBm in Definition 1.2. □

Remark 2.3 In the case $\alpha = 0$ and $d = 1$ (corresponding to the space-time white noise case) we retrieve the formula (2.7) because $\int_{\mathbb{R}}\mu(d\xi)e^{-\frac{1}{2}|\xi|^2} = \sqrt{2\pi}$.

Corollary 2.4 *The solution of the heat equation with additive white-colored noise and with the spatial covariance given by the Riesz kernel of order α is self-similar of order $\frac{1}{2}(1 - \frac{d-\alpha}{2})$.*

Proof This is a consequence of Corollary 2.3 and of the self-similarity property of the bi-fBm (Proposition 1.6). □

Remark 2.4 Note that $1 - \frac{d-\alpha}{2} > 0$ because $d < \alpha + 2$ and $1 - \frac{d-\alpha}{2} < 1$ because $\alpha < d$. When $\alpha = 0$ and $d = 1$ (the space-time white noise case), the self-similarity order is $\frac{1}{4}$.

2.4 The Solution to the Fractional-White Heat Equation

In the sequel, the driving noise of the equation will behave as a fractional Brownian motion with respect to its time variable.

2.4.1 The Noise

On a complete probability space $(\Omega, \mathcal{F}, P)$, we consider a zero-mean Gaussian process $W^H = \{W^H(t, A); t \in [0, T], A \in \mathcal{B}_b(\mathbb{R}^d)\}$ with covariance:

$$\mathbf{E}(W^H(t, A)W^H(s, B) = R_H(t, s)\lambda(A \cap B) =: \langle 1_{[0,t]\times A}, 1_{[0,s]\times B}\rangle_{\mathcal{H}} \qquad (2.15)$$

where λ is the Lebesgue measure. This noise is usually called "fractional-white" because it behaves as a fBm in time and as a Wiener process ("white") in space.

We will assume throughout that the Hurst parameter H is contained in the interval $(\frac{1}{2}, 1)$.

We introduce now the canonical Hilbert space associated to the noise. Let $\mathcal{E}$ be the set of linear combinations of elementary functions $1_{[0,t]\times A}$, $t \in [0, T]$, $A \in \mathcal{B}_b(\mathbb{R}^d)$, and $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$.

We have, for $f, h \in \mathcal{H}$ smooth enough

$$\langle f, g\rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T dudv \int_{\mathbb{R}^d} dy|u - v|^{2H-2} f(y, u)g(y, v) \qquad (2.16)$$

where $\alpha_H = H(2H - 1)$.

The map $1_{[0,t]\times A} \mapsto W_t^H(A)$ is an isometry between $\mathcal{E}$ and the Gaussian space of W^H, which can be extended by density to $\mathcal{H}$. We denote this extension by:

$$\varphi \mapsto W(\varphi) = \int_0^T \int_{\mathbb{R}^d} \varphi(t, x)W^H(dt, dx).$$

The above integral is a Wiener integral with respect to the Gaussian process W^H. This Wiener integral can be expressed as a Wiener integral with respect to the space-time white noise W which has a covariance given by (2.15). Actually, we will use the following transfer formula (see [112]).

Proposition 2.4 *If $f \in \mathcal{H}$ then*

$$\int_0^T \int_{\mathbb{R}^d} f(s,y) dW^H(s,y) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}} 1_{(0,T)}(u) f(u,x)(u-s)_+^{H-\frac{3}{2}} du \right) dW(s,y) \tag{2.17}$$

where W is a space-time white noise with covariance (2.2).

The representation (2.17) is obtained using the moving average expression of the fractional Brownian motion (1.7). See also Sect. 3.1.3 in the next chapter. Notice that a similar transfer formula can be written using the representation of the fractional Brownian motion as a Wiener integral on a finite interval (see e.g. [136]).

2.4.2 The Solution

Let us consider the linear stochastic heat equation

$$u_t = \frac{1}{2}\Delta u + \dot{W}^H, \quad t \in [0,T], x \in \mathbb{R}^d \tag{2.18}$$

with $u(.,0) = 0$, where $(W^H(t,x))_{t\in[0,T],x\in\mathbb{R}^d}$ is a centered Gaussian noise with covariance (2.15). The solution of (2.18) can be written in mild form as

$$U(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) W^H(ds, dy), \quad t \in [0,T], x \in \mathbb{R}^d \tag{2.19}$$

where the above integral is a Wiener integral with respect to the noise W^H and G is given by (2.5).

Theorem 2.1 *The process $(U(t,x))_{t\in[0,T],x\in\mathbb{R}^d}$ exists and satisfies*

$$\sup_{t\in[0,T],x\in\mathbb{R}^d} \mathbf{E}\big(U(t,x)^2\big) < +\infty$$

if and only if $d < 4H$.

Proof We have, as in the case of white noise, using (2.16) and using the expression of the Fourier transform of the Green kernel (2.6),

$$\begin{aligned}
\mathbf{E}|U(t,x)|^2 &= (2\pi)^{-d}\alpha_H \int_0^t \int_0^t du dv |u-v|^{2H-2} \int_{\mathbb{R}^d} e^{-\frac{1}{2}|\xi|^2(2t-u-v)} \\
&= (2\pi)^{-d/2}\alpha_H \int_0^t \int_0^t du dv |u-v|^{2H-2} (2t-u-v)^{-\frac{d}{2}} \\
&= (2\pi)^{-d/2}\alpha_H \int_0^t \int_0^t du dv |u-v|^{2H-2} (u+v)^{-\frac{d}{2}}
\end{aligned}$$

and the last integral is finite if and only if $2H > \frac{d}{2}$. □

Remark 2.5 This implies that, in contrast to the white-noise case, we are allowed to consider the spatial dimension d to be 1, 2 or 3.

Suppose that $s, t \in [0, T]$ and let

$$R(t, s) = \mathbf{E}\big(U(t, x)U(s, x)\big)$$

where $x \in \mathbb{R}^d$ is fixed. We will see that R does not depend on x.

Proposition 2.5 *For* $s, t \in [0, T]$

$$R(t, s) = \alpha_H (2\pi)^{-d/2} \int_0^t \int_0^s |u - v|^{2H-2} \big((t + s) - (u + v)\big)^{-\frac{d}{2}} dv du. \tag{2.20}$$

Proof The following holds

$$\begin{aligned} R(t, s) &= (2\pi)^{-d} \alpha_H \int_0^t \int_0^s du dv |u - v|^{2H-2} \int_{\mathbb{R}^d} e^{-\frac{1}{2}|\xi|^2(t+s-u-v)} \\ &= \alpha_H (2\pi)^{-d/2} \int_0^t \int_0^s |u - v|^{2H-2} \big((t + s) - (u + v)\big)^{-\frac{d}{2}} dv du. \end{aligned}$$ □

Proposition 2.6 *The process* U *is self-similar* (*with respect to* t) *of order* $H - \frac{d}{4}$.

Proof This is an immediate consequence of relation (2.20). Indeed, for every $c > 0$,

$$\begin{aligned} R(ct, cs) &= \alpha_H (2\pi)^{-d/2} \int_0^{ct} \int_0^{cs} |u - v|^{2H-2} \big((ct + cs) - (u + v)\big)^{-\frac{d}{2}} dv du \\ &= c^{2H-\frac{d}{2}} R(t, s) \end{aligned}$$

by the change of variables $\tilde{u} = \frac{u}{c}$, $\tilde{v} = \frac{v}{c}$. □

In this part we will focus our attention on the behavior of the increments of the solution $U(t, x)$ to (2.18) with respect to the variable t. We will give sharp upper and lower bounds for the L^2-norm of this increment. We will assume in the sequel that $T = 1$. Concretely, we prove the following result.

Theorem 2.2 *There exists two strictly positive constants* C_1, C_2 *such that for any* $t, s \in [0, 1]$ *and for any* $x \in \mathbb{R}^d$

$$C_1 |t - s|^{2H-\frac{d}{2}} \leq \mathbf{E}\big|U(t, x) - U(s, x)\big|^2 \leq C_2 |t - s|^{2H-\frac{d}{2}}. \tag{2.21}$$

Proof By $c, c(H)\ldots$ we will denote generic constants. We can write, for every $x \in \mathbb{R}^d$, $s, t \in [0, 1]$

$$\begin{aligned}
&\mathbf{E}\big|U(t,x) - U(s,x)\big|^2 \\
&= R(t,t) - 2R(t,s) + R(s,s) \\
&= c(H)\int_0^s \int_0^s dvdu|u-v|^{2H-2}\big[\big(2t-(u+v)\big)^{-\frac{d}{2}} \\
&\quad - 2\big((t+s)-(u+v)\big)^{-\frac{d}{2}} + \big(2s-(u+v)\big)^{-\frac{d}{2}}\big] \\
&\quad + \int_s^t \int_s^t dvdu|u-v|^{2H-2}\big(2t-(u+v)\big)^{-\frac{d}{2}} \\
&\quad - 2\int_s^t du \int_0^s dv|u-v|^{2H-2}\big[\big((t+s)-(u+v)\big)^{-\frac{d}{2}} - \big(2t-(u+v)\big)^{-\frac{d}{2}}\big] \\
&= A + B - C.
\end{aligned}$$

Since the term C is positive, we clearly have

$$\mathbf{E}\big|U(t,x) - U(s,x)\big|^2 \le A + B.$$

The term B can easily be estimated. Indeed, by the change of variables $\tilde{u} = s-u, \tilde{v} = v - s$ and then $\tilde{u} = \frac{u}{t-s}, \tilde{v} = \frac{v}{t-s}$,

$$B = c(H)(t-s)^{2H-\frac{d}{2}}. \tag{2.22}$$

Let us now consider the term A. By the change of variables $\tilde{u} = s-u, \tilde{v} = v-s$ and then $\tilde{u} = \frac{u}{t-s}, \tilde{v} = \frac{v}{t-s}$ we have

$$\begin{aligned}
A &= \int_0^s \int_0^s dudv|u-v|^{2H-2}\big[(2t-2s+u+v)^{-\frac{d}{2}} \\
&\quad - 2(t-s+u+v)^{-\frac{d}{2}} + (u+v)^{-\frac{d}{2}}\big] \\
&= (t-s)^{2H-\frac{d}{2}} \int_0^{\frac{s}{t-s}} \int_0^{\frac{s}{t-s}} dudv|u-v|^{2H-2}\big[(2+u+v)^{-\frac{d}{2}} \\
&\quad - (1+u+v)^{-\frac{d}{2}} + (u+v)^{-\frac{d}{2}}\big] \\
&\le (t-s)^{2H-\frac{d}{2}} \int_0^{\infty} \int_0^{\infty} dudv|u-v|^{2H-2}\big[(2+u+v)^{-\frac{d}{2}} \\
&\quad - (1+u+v)^{-\frac{d}{2}} + (u+v)^{-\frac{d}{2}}\big].
\end{aligned} \tag{2.23}$$

Note that the integral $\int_0^\infty \int_0^\infty dudv|u-v|^{2H-2}[(2+u+v)^{-\frac{d}{2}} - (1+u+v)^{-\frac{d}{2}} + (u+v)^{-\frac{d}{2}}]$ is finite: it is finite for u, v close to zero since $2H - \frac{d}{2} > 0$ and it is also finite for u, v close to infinity because

$$\left[(2+u+v)^{-\frac{d}{2}} - (1+u+v)^{-\frac{d}{2}} + (u+v)^{-\frac{d}{2}}\right] \le c(u+v)^{-\frac{d}{2}-2}$$

(this can be seen by analyzing the asymptotic behavior of the function $(2+x)^{-\frac{d}{2}} - 2(1+x)^{-\frac{d}{2}} + x^{-\frac{d}{2}}$). By (2.22) and (2.23) we obtain the right-hand side of (2.21).

Let us now consider the lower bound. Using the Wiener integral representation (2.19) of the solution $U(t,x)$, we can write, for every $x \in \mathbb{R}^d$

$$\begin{aligned} U(t,x) - U(s,y) = \int_0^1 \int_{\mathbb{R}^d} \big(&G(t-a, x-y)1_{(0,t)}(a) \\ &- G(s-a, x-y)1_{(0,s)}(a)\big) dW^H(s,y) \end{aligned}$$

and by the transfer rule (2.17)

$$\begin{aligned} U(t,x) - U(s,y) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} dW(a,y) \bigg(&\int_{\mathbb{R}} du G(t-u, x-y) 1_{(0,t)}(u)(u-a)_+^{H-\frac{3}{2}} \\ &- \int_{\mathbb{R}} du G(s-u, x-y) 1_{(0,s)}(u)(u-a)_+^{H-\frac{3}{2}} \bigg) \end{aligned}$$

where W is a space time white noise given by (2.2).

Now, by the isometry of the Brownian motion W we get

$$\begin{aligned} \mathbf{E}\left|U(t,x) - U(s,x)\right|^2 &= \int_{\mathbb{R}} \int_{\mathbb{R}^d} dady \bigg(\int_{\mathbb{R}} du G(t-u, x-y) 1_{(0,t)}(u)(u-a)_+^{H-\frac{3}{2}} \\ &\qquad - \int_{\mathbb{R}} du G(s-u, x-y) 1_{(0,s)}(u)(u-a)_+^{H-\frac{3}{2}} \bigg)^2 \\ &\ge \int_s^t \int_{\mathbb{R}^d} dady \bigg(\int_{\mathbb{R}} du G(t-u, x-y) 1_{(0,t)}(u)(u-a)_+^{H-\frac{3}{2}} \\ &\qquad - \int_{\mathbb{R}} du G(s-u, x-y) 1_{(0,s)}(u)(u-a)_+^{H-\frac{3}{2}} \bigg)^2 \\ &= \int_s^t da \int_{\mathbb{R}^d} dy \bigg(\int_a^t du G(t-u, x-y)(u-a)^{H-\frac{3}{2}} \bigg)^2 \end{aligned}$$

because the part on the interval $(0,s)$ vanishes. By interchanging the order of integration,

$$\begin{aligned} &\mathbf{E}\left|U(t,x) - U(s,x)\right|^2 \\ &\quad \ge \int_s^t da \int_{\mathbb{R}^d} dy \int_a^t \int_a^t dvdu \\ &\qquad \times G(t-u, x-y)(u-a)^{H-\frac{3}{2}} G(t-v, x-y)(v-a)^{H-\frac{3}{2}} \end{aligned}$$

$$
\begin{aligned}
&= \int_s^t du \int_s^t dv \int_{\mathbb{R}^d} dy G(t-u, x-y) G(t-v, x-y) \\
&\quad \times \int_s^{u \wedge v} (u-a)^{H-\frac{3}{2}} (v-a)^{H-\frac{3}{2}} da.
\end{aligned}
$$

We recall that (see e.g. [13]), for every $x \in \mathbb{R}^d$

$$
\int_{\mathbb{R}^d} dy G(t-u, x-y) G(t-v, x-y) = c\big(2t - (u+v)\big)^{-\frac{d}{2}} \tag{2.24}
$$

and, when $v < u$, by the change of variable $z = \frac{v-a}{u-a}$, we have

$$
\int_s^{u \wedge v} da (u-a)^{H-\frac{3}{2}} (v-a)^{H-\frac{3}{2}} = (u-v)^{2H-2} \int_0^{\frac{v-s}{u-s}} z^{H-\frac{3}{2}} (1-z)^{1-2H} dz. \tag{2.25}
$$

Therefore, by (2.24) and (2.25)

$$
\begin{aligned}
&\mathbf{E}\big|U(t,x) - U(s,x)\big|^2 \\
&\geq \int_s^t du \int_s^t dv \big(2t - (u+v)\big)^{-\frac{d}{2}} |u-v|^{2H-2} \int_0^{\frac{(v-s)\wedge(u-s)}{(u-s)\vee(v-s)}} z^{H-\frac{3}{2}} (1-z)^{1-2H} dz \\
&= \int_0^{t-s} du dv (u+v)^{-\frac{d}{2}} |u-v|^{2H-2} \int_0^{\frac{v \wedge u}{u \vee v}} z^{H-\frac{3}{2}} (1-z)^{1-2H} dz \\
&= (t-s)^{2H-\frac{d}{2}} \int_0^1 \int_0^1 du dv (u+v)^{-\frac{d}{2}} |u-v|^{2H-2} \int_0^{\frac{v \wedge u}{u \vee v}} z^{H-\frac{3}{2}} (1-z)^{1-2H} dz \\
&= C(t-s)^{2H-\frac{d}{2}},
\end{aligned}
$$

where in the third and fourth lines we used successively the change of variables $u - s = \tilde{u}$, $v - s = \tilde{v}$ and $\frac{u}{t-s} = \tilde{u}$, $\frac{v}{t-s} = \tilde{v}$. The proof of the lower bound follows since the integral $\int_0^1 \int_0^1 du dv (u+v)^{-\frac{d}{2}} |u-v|^{2H-2} \int_0^{\frac{v \wedge u}{u \vee v}} z^{H-\frac{3}{2}} (1-z)^{1-2H} dz$ is clearly finite when $H > \frac{1}{2}$. □

Remark 2.6 The above result implies that the process U is Hölder continuous of order $H - \frac{d}{4}$ in time (this coincides with the self-similarity order, see Proposition 2.6). This extends the case of the space-time white noise in dimension $d = 1$ (recall that the solution of the heat equation with space-time time white noise is Hölder continuous of order $\frac{1}{4}$). Note also that in the case $d = 1$ the upper bound has also been obtained in [108] or [36].

2.4.3 On the Law of the Solution

Consider the process U given by (2.19). Suppose that $s \le t$ and recall the notation

$$R(t,s) = \mathbf{E}\big(U(t,x)U(s,x)\big)$$

where $x \in \mathbb{R}^d$ is fixed. Also recall the formula (2.20)

$$R(t,s) = \alpha_H (2\pi)^{-\frac{d}{2}} \int_0^t \int_0^s |u-v|^{2H-2} \big((t+s)-(u+v)\big)^{-\frac{d}{2}} dv du$$

with $\alpha_H = H(2H-1)$.

The purpose of this section is to analyze the covariance of the solution $U(t,x)$ and to understand its relation with bifractional Brownian motion. Corollaries 2.1 and 2.3 say that, when the noise is white in time, the solution coincides in distribution with a bi-fBm. Proposition 2.2 shows that its increments have a similar behavior as those of the bi-fBm. But we will see that the situation is different if the noise is no longer white in time.

The following proposition gives a decomposition of the covariance function of $U(t,.)$ in the case $d \ne 2$ i.e. $d=1$ or $d=3$ since the solution exists for $d<4H$. The lines of the below proof will explain why the case $d=2$ has to be excluded.

Proposition 2.7 *Suppose $d \ne 2$. The covariance function $R(t,s)$ can be decomposed as follows*

$$R(t,s) = (2\pi)^{-\frac{d}{2}} \alpha_H C_d \beta\left(2H-1, -\frac{d}{2}+2\right)\left[(t+s)^{2H-\frac{d}{2}} - (t-s)^{2H-\frac{d}{2}}\right] + R_1^{(d)}(t,s)$$

where $C_d = \frac{2}{2-d}$, $\beta(x,y)$ is the Beta function defined for $x,y>0$ by $\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$ and

$$R_1^{(d)}(t,s) = (2\pi)^{-\frac{d}{2}} \alpha_H C_d \left[\int_0^s da a^{2H-2}\left[\big((t+s)-a\big)^{-\frac{d}{2}+1} - \big((t-s)+a\big)^{-\frac{d}{2}+1}\right] - \int_0^s da (s-a)^{-\frac{d}{2}+1}\left[(t-a)^{2H-2} + (t+a)^{2H-2}\right]\right].$$

Proof Fix $t > s$. By performing the change of variables $u - v = a$ and $u + v = b$ with $a + b = 2u \in (0, 2t)$ and $b - a = 2v \in (0, 2s)$ in (2.20), we get

$$\begin{aligned} R(t,s) &= (2\pi)^{-\frac{d}{2}} \alpha_H \int_{-s}^t |a|^{2H-2} \int_{a\vee(-a)}^{(2t-a)\wedge(2s+a)} \big((t+s)-b\big)^{-\frac{d}{2}} db da \\ &= (2\pi)^{-\frac{d}{2}} \alpha_H \Bigg[\int_{-s}^0 (-a)^{2H-2} \int_{-a}^{2s+a} \big((t+s)-b\big)^{-\frac{d}{2}} db da \end{aligned}$$

$$+\int_0^{t-s} a^{2H-2}\int_a^{2s+a}\big((t+s)-b\big)^{-\frac{d}{2}}dbda$$
$$+\int_{t-s}^{t} a^{2H-2}\int_a^{2t-a}\big((t+s)-b\big)^{-\frac{d}{2}}dbda\Bigg].$$

By performing the change of variables $a\mapsto(-a)$ in the first summand, we get

$$R(t,s)=(2\pi)^{-\frac{d}{2}}\alpha_H\Bigg[\int_0^{s} a^{2H-2}\int_a^{2s-a}\big((t+s)-b\big)^{-\frac{d}{2}}dbda$$
$$+\int_0^{t-s} a^{2H-2}\int_a^{2s+a}\big((t+s)-b\big)^{-\frac{d}{2}}dbda$$
$$+\int_{t-s}^{t} a^{2H-2}\int_a^{2t-a}\big((t+s)-b\big)^{-\frac{d}{2}}dbda\Bigg].\qquad\square$$

Remark 2.7 We can see why the case $d=2$ must be treated separately in the latter equation. The integral with respect to db involves logarithms and it cannot lead to the covariance of the bifractional Brownian motion.

By explicitly computing the inner integrals, we obtain

$$R(t,s)=(2\pi)^{-\frac{d}{2}}\alpha_H C_d\Bigg[\int_0^{s} a^{2H-2}\Big[-\big((t+s)-b\big)^{-\frac{d}{2}+1}\Big]_{b=a}^{b=2s-a}da$$
$$+\int_0^{t-s} a^{2H-2}\Big[-\big((t+s)-b\big)^{-\frac{d}{2}+1}\Big]_{b=a}^{b=2s+a}da$$
$$+\int_{t-s}^{t} a^{2H-2}\Big[-\big((t+s)-b\big)^{-\frac{d}{2}+1}\Big]_{b=a}^{b=2t-a}da\Bigg]$$
$$=(2\pi)^{-\frac{d}{2}}\alpha_H C_d\Bigg[\int_0^{s} a^{2H-2}\big((t+s)-a\big)^{-\frac{d}{2}+1}da$$
$$-\int_0^{s} a^{2H-2}\big((t-s)+a\big)^{-\frac{d}{2}+1}da\Bigg]$$
$$+\alpha_H C_d\Bigg[\int_0^{t-s} a^{2H-2}\big((t+s)-a\big)^{-\frac{d}{2}+1}da$$
$$-\int_0^{t-s} a^{2H-2}\big((t-s)-a\big)^{-\frac{d}{2}+1}da\Bigg]$$
$$+\alpha_H C_d\Bigg[\int_{t-s}^{t} a^{2H-2}\big((t+s)-a\big)^{-\frac{d}{2}+1}da$$
$$-\int_{t-s}^{t} a^{2H-2}\big(a-(t-s)\big)^{-\frac{d}{2}+1}da\Bigg]$$

$$
\begin{aligned}
&= (2\pi)^{-\frac{d}{2}}\alpha_H C_d\Bigg[\int_0^{t+s} a^{2H-2}\big((t+s)-a\big)^{-\frac{d}{2}+1}da \\
&\quad - \int_0^{t-s} a^{2H-2}\big((t-s)-a\big)^{-\frac{d}{2}+1}da\Bigg] \\
&\quad + R_1^{(d)}(t,s)
\end{aligned}
$$

where

$$
\begin{aligned}
&R_1^{(d)}(t,s) \\
&= (2\pi)^{-\frac{d}{2}}\alpha_H C_d\Bigg[\int_0^{s} a^{2H-2}\big((t+s)-a\big)^{-\frac{d}{2}+1}da \\
&\quad - \int_0^{s} a^{2H-2}\big((t-s)+a\big)^{-\frac{d}{2}+1}da \\
&\quad - \int_{t-s}^{t} a^{2H-2}\big(a-(t-s)\big)^{-\frac{d}{2}+1}da - \int_t^{t+s} a^{2H-2}\big((t+s)-a\big)^{-\frac{d}{2}+1}da\Bigg].
\end{aligned}
\tag{2.26}
$$

At this point, we perform the change of variable $a \mapsto \frac{a}{t+s}$ and we obtain

$$
\begin{aligned}
\int_0^{t+s} a^{2H-2}\big((t+s)-a\big)^{-\frac{d}{2}+1}da &= (t+s)^{2H-\frac{d}{2}}\int_0^1 a^{2H-2}(1-a)^{-\frac{d}{2}+1}da \\
&= \beta\left(2H-1, -\frac{d}{2}+2\right)(t+s)^{2H-\frac{d}{2}}
\end{aligned}
$$

and in the same way, with the change of variable $a \mapsto \frac{a}{t-s}$, we obtain

$$
\begin{aligned}
\int_0^{t-s} a^{2H-2}\big((t-s)-a\big)^{-\frac{d}{2}+1}da &= (t-s)^{2H-\frac{d}{2}}\int_0^1 a^{2H-2}(1-a)^{-\frac{d}{2}+1}da \\
&= \beta\left(2H-1, -\frac{d}{2}+2\right)(t-s)^{2H-\frac{d}{2}}.
\end{aligned}
$$

As a consequence, we obtain

$$
R(t,s) = \alpha_H(2\pi)^{-\frac{d}{2}}C_d\beta\left(2H-1, -\frac{d}{2}+2\right)\left[(t+s)^{2H-\frac{d}{2}} - (t-s)^{2H-\frac{d}{2}}\right] + R_1^{(d)}(t,s)
$$

with $R_1^{(d)}$ given by (2.26). Let us further analyze the function denoted by $R_1^{(d)}(t,s)$. Note that for every $s, t \in [0, T]$

$$
(2\pi)^{\frac{d}{2}}R_1^{(d)}(t,s) = A(t,s) + B(t,s)
$$

where

$$A(t,s) = \alpha_H C_d \Big[\int_0^s a^{2H-2} \big((t+s) - a \big)^{-\frac{d}{2}+1} da \\ - \int_0^s a^{2H-2} \big((t-s) + a \big)^{-\frac{d}{2}+1} da \Big]$$

and

$$B(t,s) = \alpha_H C_d \Big[- \int_{t-s}^t a^{2H-2} \big(a - (t-s) \big)^{-\frac{d}{2}+1} da \\ - \int_t^{t+s} a^{2H-2} \big((t+s) - a \big)^{-\frac{d}{2}+1} da \Big].$$

By the change of variables $a - t = \tilde{a}$, we can express B as

$$\begin{aligned} B(t,s) &= \alpha_H C_d \Big[- \int_{-s}^0 (a+t)^{2H-2} (a+s)^{-\frac{d}{2}+1} da \\ &\quad - \int_0^s (a+t)^{2H-2} (s-a)^{-\frac{d}{2}+1} da \Big] \\ &= -\alpha_H C_d \int_0^s da (s-a)^{-\frac{d}{2}+1} \big[(t-a)^{2H-2} + (t+a)^{2H-2} \big] \end{aligned}$$

and the desired conclusion is obtained.

Let us point out that the constant C_d is positive for $d = 1$ and negative for $d = 3$. This partially explains why different decompositions holds in these two cases. Thanks to the decomposition in Proposition 2.7, we have the following.

Theorem 2.3 *Assume $d = 1$ and let U be the solution to the heat equation (2.18) with fractional-white noise (2.15). Let $B^{\frac{1}{2}, 2H-\frac{1}{2}}$ be a bifractional Brownian motion with parameters $H = \frac{1}{2}$ and $K = 2H - \frac{1}{2}$. Let $(X_t^H)_{t \in [0,T]}$ be a centered Gaussian process with covariance, for $s, t \in [0, T]$*

$$\begin{aligned} R^{X^H}(t,s) &= 2 \frac{1}{\sqrt{2\pi}} \alpha_H \int_0^s (s-a)^{2H-2} \big[(t+a)^{\frac{1}{2}} - (t-a)^{\frac{1}{2}} \big] da \\ &= H \frac{1}{\sqrt{2\pi}} \int_0^s (s-a)^{2H-1} \big[(t+a)^{-\frac{1}{2}} + (t-a)^{-\frac{1}{2}} \big] da, \end{aligned} \tag{2.27}$$

and let $(Y_t^H)_{t \in [0,T]}$ be a centered Gaussian process with covariance

$$\begin{aligned} R^{Y^H}(t,s) &= 2 \frac{1}{\sqrt{2\pi}} \alpha_H \int_0^s (s-a)^{\frac{1}{2}} \big[(t+a)^{2H-2} + (t-a)^{2H-2} \big] da \\ &= H \frac{1}{\sqrt{2\pi}} \int_0^s (s-a)^{-\frac{1}{2}} \big[(t+a)^{2H-1} - (t-a)^{2H-1} \big] da. \end{aligned} \tag{2.28}$$

Suppose that U, X^H and Y^H are independent. Then for every $x \in \mathbb{R}^d$,

$$\big(U(t,x) + Y^H, t \in [0,T]\big) \overset{\text{Law}}{=} \big(C_0 B_t^{\frac{1}{2},\frac{1}{2}} + X_t^H, t \in [0,T]\big),$$

where $C_0^2 = \frac{2}{\sqrt{2\pi}}\alpha_H \beta(2H-1, -\frac{d}{2}+2)$.

Remark 2.8 As it is assumed that $H > 1/2$, the function R^{Y^H} always remains positive.

Proof Let us first verify that R^{X^H} is a covariance function. Clearly, it is symmetric and it can be written, for every $s, t \in [0,T]$, as

$$\begin{aligned}
&\sqrt{2\pi} R^{X^H}(t,s) \\
&\quad = H \int_0^{s\wedge t} (t\wedge s - a)^{2H-1}\big[\big((t\vee s)+a\big)^{-\frac{1}{2}} + \big((t\vee s)-a\big)^{-\frac{1}{2}}\big] \\
&\quad = H \int_0^{\infty} 1_{[0,t]}(a) 1_{[0,s]}(a)(t\wedge s - a)^{2H-1}\big((t+a)^{-\frac{1}{2}} \wedge (s+a)^{-\frac{1}{2}}\big) da \\
&\qquad + H \int_0^{\infty} 1_{[0,t]}(a) 1_{[0,s]}(a)(t\wedge s - a)^{2H-1}\big((t-a)^{-\frac{1}{2}} \wedge (s-a)^{-\frac{1}{2}}\big) da
\end{aligned}$$

and both summands above are positive definite (the same argument is used in [32], in the proof of Theorem 2.1). Similarly, the function R^{Y^H} is a covariance. If $d = 1$, we have $C_d = 2$ and

$$R(t,s) = 2\alpha_H (2\pi)^{-\frac{1}{2}} \beta\left(2H-1, \frac{3}{2}\right)\big[(t+s)^{\frac{1}{2}} - (t-s)^{\frac{1}{2}}\big] + R_1^{(1)}(t,s)$$

with

$$\begin{aligned}
\sqrt{2\pi} R_1^{(1)}(t,s) &= 2\alpha_H \int_0^s (s-a)^{2H-2}\big[(t+a)^{\frac{1}{2}} - (t-a)^{\frac{1}{2}}\big] da \\
&\quad - 2\alpha_H \int_0^s (s-a)^{\frac{1}{2}}\big[(t+a)^{2H-2} + (t-a)^{2H-2}\big] da \\
&= H \int_0^s (s-a)^{2H-1}\big[(t+a)^{-\frac{1}{2}} + (t-a)^{-\frac{1}{2}}\big] da \\
&\quad - 2\alpha_H \int_0^s (s-a)^{\frac{1}{2}}\big[(t+a)^{2H-2} + (t-a)^{2H-2}\big] da
\end{aligned}$$

where we used integration by parts in the first integral. □

In the case $d = 3$ we have the following.

Theorem 2.4 *Assume $d = 3$. Let $B^{\frac{1}{2}, 2H-\frac{3}{2}}$ be a bifractional Brownian motion with $H = \frac{1}{2}$ and $K = 2H - \frac{3}{2}$ and let $(Z_t^H)_{t\in[0,T]}$ be a centered Gaussian process with covariance $R_1^{(3)}(t,s)$. Then*

$$\left(U(t,x) + C_0 B^{\frac{1}{2}, 2H-\frac{3}{2}}, t \in [0,T]\right) \stackrel{\text{Law}}{=} \left(Z_t^H, t \in [0,T]\right),$$

with C_0 defined as in Theorem 2.3.

Proof We have $C_3 = -2$. In this case we can write

$$R(t,s) + 2\alpha_H \beta\left(2H-1, \frac{1}{2}\right)\left[(t+s)^{2H-\frac{3}{2}} - (t-s)^{2H-\frac{3}{2}}\right] = R_1^{(3)}(t,s)$$

with

$$\begin{aligned}(2\pi)^{\frac{3}{2}} R_1^{(3)}(t,s) = &-2\alpha_H \int_0^s (s-a)^{2H-2}\left[(t+a)^{-\frac{1}{2}} - (t-a)^{-\frac{1}{2}}\right]da \\ &+ 2\alpha_H \int_0^s (s-a)^{-\frac{1}{2}}\left[(t+a)^{2H-2} + (t-a)^{2H-2}\right].\end{aligned}$$

Note that $R_1^{(3)}$ is a covariance function because it is the sum of two covariance functions. □

Remark 2.9 Let us understand what happens with the decompositions in Theorems 2.3 and 2.4 when H is close to $\frac{1}{2}$. We focus on the case $d = 1$. The phenomenon is interesting. We first notice that the process Y^H vanishes in this case. The covariance of the process X^H becomes

$$R^{X^{\frac{1}{2}}}(t,s) = \frac{1}{2\sqrt{2\pi}}\left((t+s)^{\frac{1}{2}} - (t-s)^{\frac{1}{2}}\right).$$

The constant $C_0^2 = \frac{2}{\sqrt{2\pi}}\alpha_H \beta(2H-1, \frac{3}{2})$ is not defined for $H = \frac{1}{2}$ because of the presence of $2H-1$ in the argument of the beta function. But the following happens: since $2\alpha_H 1_{(0,1)}(u)(1-u)^{2H-2}$ is an approximation of unity, it follows that $\alpha_H \beta(2H-1, \frac{3}{2})$ converges to $\frac{1}{2}$ as H tends to $\frac{1}{2}$. Therefore C_0^2 becomes $\frac{1}{2\sqrt{2\pi}}$. Therefore we retrieve the result established in [166] and recalled in relation (2.7). In other words, in the fractional case $H \neq \frac{1}{2}$ the solution "retains" half of the bifractional Brownian motion $B^{\frac{1}{2}, \frac{1}{2}}$ while the other half "spreads" into two parts.

2.5 The Solution to the Heat Equation with Fractional-Colored Noise

2.5.1 The Noise

The next step is to consider a noise with correlation structure both in time and in space. Consider the so-called *fractional-colored noise*, meaning a centered Gaussian process $W^H = \{W^H(t, A); t \in [0, T], A \in \mathcal{B}_b(\mathbb{R}^d)\}$ with covariance:

$$\mathbf{E}\big(W^H(t, A)W^H(s, B)\big) = R_H(t, s)\int_A\int_B f\big(y - y'\big)dydy' =: \langle 1_{[0,t]\times A}, 1_{[0,s]\times B}\rangle_{\mathcal{HP}} \tag{2.29}$$

where f is the spatial covariance kernel and R_H denotes the covariance of the fractional Brownian motion (1.1). Recall that f is the Fourier of a tempered nonnegative measure μ on $\mathbb{R}^d$.

To this Gaussian process we will associate a canonical Hilbert space whose structure is important in obtaining the existence and the properties of the solution. Let $\mathcal{E}$ be the set of linear combinations of elementary functions $1_{[0,t]\times A}$, $t \in [0, T]$, $A \in \mathcal{B}_b(\mathbb{R}^d)$, and $\mathcal{HP}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathcal{HP}}$. (Alternatively, $\mathcal{HP}$ can be defined as the completion of $C_0^\infty(\mathbb{R}^{d+1})$, with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathcal{HP}}$; see [13].)

The map $1_{[0,t]\times A} \mapsto W(t, A)$ is an isometry between $\mathcal{E}$ and the Gaussian space H^W of W, which can be extended to $\mathcal{HP}$. We denote this extension by:

$$\varphi \mapsto W(\varphi) = \int_0^T \int_{\mathbb{R}^d} \varphi(t, x)W(dt, dx).$$

We assume that $H > 1/2$. From (1.9) and (2.29), it follows that for any $\varphi, \psi \in \mathcal{E}$,

$$\begin{aligned}
&\langle\varphi, \psi\rangle_{\mathcal{HP}} \\
&\quad= \alpha_H \int_0^T\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(u, x)\psi(v, y)f(x - y)|u - v|^{2H-2}dxdydudv \\
&\quad= \alpha_H(2\pi)^{-d}\int_0^T\int_0^T\int_{\mathbb{R}^d} \mathcal{F}\varphi(u,\cdot)(\xi)\overline{\mathcal{F}\psi(v,\cdot)(\xi)}|u - v|^{2H-2}\mu(d\xi)dudv.
\end{aligned}$$

Moreover, we can interchange the order of the integrals $dudv$ and $\mu(d\xi)$, since for indicator functions φ and ψ, the integrand is a product of a function of (u, v) and a function of ξ. Hence, for $\varphi, \psi \in \mathcal{E}$, we have:

$$\begin{aligned}
\langle\varphi, \psi\rangle_{\mathcal{HP}} &= \alpha_H(2\pi)^{-d} \\
&\quad\times \int_{\mathbb{R}^d}\int_0^T\int_0^T \mathcal{F}\varphi(u,\cdot)(\xi)\overline{\mathcal{F}\psi(v,\cdot)(\xi)}|u - v|^{2H-2}dudv\mu(d\xi).
\end{aligned} \tag{2.30}$$

The space $\mathcal{HP}$ may contain distributions, but contains the space $|\mathcal{HP}|$ of measurable functions $\varphi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ such that

$$\|\varphi\|^2_{|\mathcal{HP}|} := \alpha_H \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \big|\varphi(u,x)\big|\big|\varphi(v,y)\big| f(x-y) \\ \times |u-v|^{2H-2} dxdydudv < \infty.$$

2.5.2 The Solution

Let us consider the equation (2.18) with the covariance of the noise W^H given by (2.12) and recall that the solution can be written in the mild form

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t-u, x-y) W^H(ds,dy), \quad t \in [0,T], x \in \mathbb{R}^d.$$

We have the transfer formula

$$u(t,x) = \int_{-\infty}^t \int_{\mathbb{R}^d} \left(\int_a^t G(t-u, x-y)(u-a)^{H-\frac{3}{2}} \right) dW(a,y) \tag{2.31}$$

where W is a centered Gaussian process with covariance given by (2.12).

Relation (2.31) follows from relation (2.17) using the moving average representation of the fBm (1.7). See also Sect. 3.1.3.

Remark 2.10 The process W behaves as a Wiener process with respect to the time variable and it has spatial covariance given by the Riesz kernel. In particular the increments of W with respect to the time variable are independent, meaning that $W(t,x) - W(s,x)$ is independent.

We have

Theorem 2.5 *The process $(u(t,x))_{t \in [0,T], x \in \mathbb{R}^d}$ given by (2.31) exists and satisfies*

$$\sup_{t \in [0,T], x \in \mathbb{R}^d} \mathbf{E}\big(u(t,x)^2\big) < +\infty$$

if and only if

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2} \right)^{2H} \mu(d\xi) < \infty.$$

Proof Note that $g_{tx} = G(t-\cdot, x-\cdot)$ is non-negative. Hence, $g_{tx} \in \mathcal{HP}$ if and only if $g_{tx} \in |\mathcal{HP}|$. This is equivalent to saying that $J_t := \|g_{tx}\|^2_{|\mathcal{HP}|} < \infty$ for all $t > 0$.

Note that

$$
\begin{aligned}
J_t &= \alpha_H \int_0^t \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g_{tx}(u,y) g_{tx}(v,z) f(y-z) |u-v|^{2H-2} dy dz du dv \\
&= (2\pi)^{-d} \alpha_H \int_0^t \int_0^t \int_{\mathbb{R}^d} \mathcal{F} g_{tx}(u,\cdot)(\xi) \overline{\mathcal{F} g_{tx}(v,\cdot)(\xi)} |u-v|^{2H-2} \mu(d\xi) du dv \\
&= (2\pi)^{-d} \alpha_H \int_0^t \int_0^t \int_{\mathbb{R}^d} \mathcal{F} G(t-u,\cdot)(\xi) \\
&\quad \times \overline{\mathcal{F} G(t-v,\cdot)(\xi)} |u-v|^{2H-2} \mu(d\xi) du dv.
\end{aligned}
$$

Using (2.6) and Fubini's theorem (whose application is justified since the integrand is non-negative), we obtain:

$$
J_t = \alpha_H (2\pi)^{-d} \int_{\mathbb{R}^d} \int_0^t \int_0^t \exp\left(-\frac{u|\xi|^2}{2}\right) \exp\left(-\frac{v|\xi|^2}{2}\right) |u-v|^{2H-2} du dv \mu(d\xi).
$$

The existence of the solution follows from Proposition 2.8 below, which also gives estimates for $J_t = \mathbf{E}|u(t,x)|^2$. □

Let $\mathcal{H}(0,t)$ denote the canonical Hilbert space of the fBm on the interval $(0,t)$ and let

$$
B_t(\xi) = \int_0^t \int_0^t \exp\left(-\frac{u|\xi|^2}{2}\right) \exp\left(-\frac{v|\xi|^2}{2}\right) |u-v|^{2H-2} du dv.
$$

Proposition 2.8 *For any $t > 0, \xi \in \mathbb{R}^d$,*

$$
\frac{1}{4}\left(t^{2H} \wedge 1\right) \left(\frac{1}{1+|\xi|^2}\right)^{2H} \le B_t(\xi) \le C'_H \left(t^{2H}+1\right) \left(\frac{1}{1+|\xi|^2}\right)^{2H},
$$

where $C'_H = b_H^2 (4H)^{2H}$.

Proof Suppose that $|\xi| \le 1$. Using the fact that $\|\varphi\|^2_{\mathcal{H}(0,t)} \le b_H^2 t^{2H-1} \|\varphi\|^2_{L^2(0,t)}$ (see Exercise 1.12) for all $\varphi \in L^2(0,t)$, $e^{-x} \le 1$ for any $x > 0$, and $\frac{1}{2} \le \frac{1}{1+|\xi|^2}$ if $|\xi| \le 1$,

$$
B_t(\xi) \le b_H^2 t^{2H-1} \int_0^t \exp\left(-u|\xi|^2\right) du \le b_H^2 t^{2H} \le b_H^2 2^{2H} t^{2H} \left(\frac{1}{1+|\xi|^2}\right)^{2H}.
$$

Suppose that $|\xi| \ge 1$. Using the fact that

$$
\|\varphi\|^2_{\mathcal{H}(0,t)} \le b_H^2 \|\varphi\|^2_{L^{1/H}(0,t)}
$$

for any $\varphi \in L^{1/H}(0,t)$ (see Chap. 1), $1-e^{-x} \le 1$ for all $x>0$, and $\frac{1}{|\xi|^2} \le \frac{2}{1+|\xi|^2}$, we obtain:

$$B_t(\xi) \le b_H^2 \left[\int_0^t \exp\left(-\frac{u|\xi|^2}{2H}\right)du\right]^{2H} = b_H^2\left(\frac{2H}{|\xi|^2}\right)^{2H}\left[1-\exp\left(-\frac{t|\xi|^2}{2H}\right)\right]^{2H}$$
$$\le b_H^2 (4H)^{2H}\left(\frac{1}{1+|\xi|^2}\right)^{2H}.$$

This proves the upper bound.

Next, we establish the lower bound. Suppose that $t|\xi|^2 \le 1$. For any $u \in [0,t]$, $\frac{u|\xi|^2}{2} \le \frac{t|\xi|^2}{2} \le \frac{1}{2}$. Using the fact that $e^{-x} \ge 1-x$ for all $x>0$, we conclude that:

$$\exp\left(-\frac{u|\xi|^2}{2}\right) \ge 1-\frac{u|\xi|^2}{2} \ge \frac{1}{2}, \quad \forall u \in [0,t].$$

Hence

$$B_t(\xi) \ge \alpha_H \left(\frac{1}{2}\right)^2 \int_0^t \int_0^t |u-v|^{2H-2}dudv = \frac{1}{4}t^{2H} \ge \frac{1}{4}t^{2H}\left(\frac{1}{1+|\xi|^2}\right)^{2H}.$$

For the last inequality, we used the fact that $1 \ge \frac{1}{1+|\xi|^2}$.

Suppose that $t|\xi|^2 \ge 1$. Using the change of variables $u' = u|\xi|^2/2$, $v' = v|\xi|^2/2$, we obtain:

$$B_t(\xi) = \alpha_H \frac{2^{2H}}{|\xi|^{4H}} \int_0^{t|\xi|^2/2} \int_0^{t|\xi|^2/2} e^{-u'}e^{-v'}\left|u'-v'\right|^{2H-2}du'dv'.$$

Since the integrand is non-negative,

$$B_t(\xi) \ge \alpha_H \frac{2^{2H}}{|\xi|^{4H}} \int_0^{1/2} \int_0^{1/2} e^{-u}e^{-v}|u-v|^{2H-2}dudv$$
$$= 2^{2H}\left\|e^{-u}\right\|^2_{\mathcal{H}(0,1/2)} \frac{1}{|\xi|^{4H}} \ge 2^{2H}\left(\frac{1}{2}\right)^{2H+2}\left(\frac{1}{1+|\xi|^2}\right)^{2H},$$

where for the last inequality we used the fact that $\frac{1}{|\xi|^2} \ge \frac{1}{1+|\xi|^2}$, and $\|e^{-u}\|^2_{\mathcal{H}(0,1/2)} \ge (\frac{1}{2})^{2H+2}$. This follows since $e^{-u} \ge 1-u \ge \frac{1}{2}$ for all $u \in [0, \frac{1}{2}]$. □

Corollary 2.5 *For the covariance given by Riesz kernels (Example 2.1) and Bessel kernels (Example 2.2) of order α, the solution exists if and only if*

$$d < 4H + \alpha.$$

Proof This follows from Theorem 2.5 using the fact that the integral

$$\int_{\mathbb{R}^d} d\xi |\xi|^{-\alpha} \left(\frac{1}{1+|\xi|^2} \right)^{2H}$$

converges at zero if $\alpha < d$ and at infinity if $\alpha + 4H > d$. □

The covariance of the process can be written as (here $x \in \mathbb{R}^d$ is fixed)

$$\begin{aligned} \mathbf{E}u(t,x)u(s,y) = \alpha_H (2\pi)^{-d} \int_0^t \int_0^s dudv |u-v|^{2H-2} \\ \times \int_{\mathbb{R}^d} \mu(d\xi) e^{-\frac{(t-u)|\xi|^2}{2}} e^{-\frac{(s-v)|\xi|^2}{2}} \end{aligned} \tag{2.32}$$

with $\alpha_H = H(2H-1)$.

The particular case of the Riesz kernel leads to some nice scaling properties.

Proposition 2.9 *Assume f is the Riesz kernel from Example 2.1. Then*

$$\begin{aligned} \mathbf{E}u(t,x)u(s,y) = \alpha_H (2\pi)^{-d} \int_{\mathbb{R}^d} \mu(d\xi) e^{-\frac{|\xi|^2}{2}} \\ \times \int_0^t \int_0^s dudv |u-v|^{2H-2} \big((t+s) - (u+v) \big)^{-\frac{d-\alpha}{2}}. \end{aligned}$$

Proof It suffices to make the change of variables $\tilde{\xi} = \sqrt{t+s-u-v}\,\xi$ in (2.32). □

Proposition 2.10 *When the spatial covariance is given by the Riesz kernel, the process u is self-similar with parameter*

$$H - \frac{d-\alpha}{2}.$$

Proof Taking into account the expression of the measure μ in Example 2.1, for every s, t

$$\begin{aligned} R(t,s) = \mathbf{E}u(t,x)u(s,y) = \alpha_H (2\pi)^{-d} \int_0^t \int_0^s dudv |u-v|^{2H-2} \\ \times \int_{\mathbb{R}^d} d\xi |\xi|^{-\alpha} e^{-\frac{(t-u)|\xi|^2}{2}} e^{-\frac{(s-v)|\xi|^2}{2}}. \end{aligned}$$

Let $c > 0$. By the change of variables $\tilde{u} = \frac{u}{c}, \tilde{v} = \frac{v}{c}$ in the integral $dudv$ and then by the change of variables $\tilde{\xi} = \sqrt{c}\xi$ in the integral $d\xi$ we get

$$R(ct, cs) = c^{2H - \frac{d-\alpha}{2}} R(t,s).$$ □

Let us analyze the behavior of the square mean of the increment of the solution to (2.18), that is,

$$\mathbf{E}\big| u(t,x) - u(s,y) \big|^2.$$

We will make the following assumption:

$$\mu(\xi) \sim c|\xi|^{-\alpha} d\xi, \quad \text{with } 0 < \alpha < d. \tag{2.33}$$

This means that for every function h such that the below integrals are finite, there exists two strictly positive constants c and c' such that

$$c' \int_{\mathbb{R}^d} h(\xi)|\xi|^{-\alpha} d\xi \le \int_{\mathbb{R}^d} h(\xi)\mu(d\xi) \le c \int_{\mathbb{R}^d} h(\xi)|\xi|^{-\alpha} d\xi.$$

Remark 2.11 The Riesz kernel and the Bessel kernel (with $\alpha < d$) satisfy (2.33).

Theorem 2.6 *Assume* (2.33). *There exists two strictly positive constants* C_1, C_2 *such that for any* $t, s \in [0, 1]$ *and for any* $x \in \mathbb{R}^d$

$$C_1|t-s|^{2H-\frac{d-\alpha}{2}} \le \mathbf{E}\big|u(t,x) - u(s,x)\big|^2 \le C_2|t-s|^{2H-\frac{d-\alpha}{2}}. \tag{2.34}$$

Remark 2.12 In the case $\alpha = 0$ (corresponding to fractional-white noise) we retrieve the result in Theorem 2.2.

Proof We will first prove the upper bound. Take $s \le t$, $s, t \in [0, 1]$.

$$\begin{aligned}
&\mathbf{E}\big|u(t,x) - u(s,x)\big|^2 \\
&= \alpha_H (2\pi)^{-d} \int_0^t \int_0^t dudv|u-v|^{2H-2} \int_{\mathbb{R}^d} \mu(d\xi) e^{-\frac{(t-u)|\xi|^2}{2}} e^{-\frac{(t-v)|\xi|^2}{2}} \\
&\quad - 2\alpha_H (2\pi)^{-d} \int_0^t \int_0^s dudv|u-v|^{2H-2} \int_{\mathbb{R}^d} \mu(d\xi) e^{-\frac{(t-u)|\xi|^2}{2}} e^{-\frac{(s-v)|\xi|^2}{2}} \\
&\quad + \alpha_H (2\pi)^{-d} \int_0^s \int_0^s dudv|u-v|^{2H-2} \int_{\mathbb{R}^d} \mu(d\xi) e^{-\frac{(s-u)|\xi|^2}{2}} e^{-\frac{(s-v)|\xi|^2}{2}} \\
&= \alpha_H (2\pi)^{-d} \int_s^t du \int_s^t dv|u-v|^{2H-2} \int_{\mathbb{R}^d} \mu(d\xi) e^{-\frac{(t-u)|\xi|^2}{2}} e^{-\frac{(t-v)|\xi|^2}{2}} \\
&\quad + \alpha_H (2\pi)^{-d} \int_0^s du \int_0^s dv|u-v|^{2H-2} \int_{\mathbb{R}^d} \mu(d\xi) \\
&\quad \times \left(e^{-\frac{(t-u)|\xi|^2}{2}} e^{-\frac{(t-v)|\xi|^2}{2}} - 2e^{-\frac{(t-u)|\xi|^2}{2}} e^{-\frac{(s-v)|\xi|^2}{2}} + e^{-\frac{(s-u)|\xi|^2}{2}} e^{-\frac{(s-v)|\xi|^2}{2}}\right) \\
&\quad + 2\alpha_H (2\pi)^{-d} \int_s^t du \int_0^s dv|u-v|^{2H-2} \\
&\quad \times \int_{\mathbb{R}^d} \mu(d\xi)\big(e^{-\frac{(t-u)|\xi|^2}{2}} e^{-\frac{(t-v)|\xi|^2}{2}} - 2e^{-\frac{(t-u)|\xi|^2}{2}} e^{-\frac{(s-v)|\xi|^2}{2}}\big) \\
&:= A(t,s) + B(t,s) + C(t,s).
\end{aligned}$$

Let us first note that

$$C(t,s) = 2\alpha_H (2\pi)^{-d} \int_s^t du \int_0^s dv |u-v|^{2H-2}$$
$$\times \int_{\mathbb{R}^d} d\xi e^{-\frac{(t-u)|\xi|^2}{2}} \left(e^{-\frac{(t-v)|\xi|^2}{2}} - e^{-\frac{(s-v)|\xi|^2}{2}}\right)$$

is negative and therefore it can be neglected for the proof of the upper bound.

Concerning the first term above (denoted by $A(t,s)$) we can write, by the change of variables $\tilde{u} = u - s$, $\tilde{v} = v - s$ and then $\tilde{u} = \frac{u}{t-s}$, $\tilde{v} = \frac{v}{t-s}$ and using (2.33)

$$A(t,s) \le c\alpha_H (2\pi)^{-d} |t-s|^{2H} \int_0^1 \int_0^1 dudv |u-v|^{2H-2}$$
$$\times \int_{\mathbb{R}^d} de\xi |\xi|^{-\alpha} e^{-\frac{1}{2}(t-s)u|\xi|^2} e^{-\frac{1}{2}(t-s)v|\xi|^2}$$

and then, by the change of variables $\tilde{\xi} = \sqrt{t-s}\xi$ (meaning that $\tilde{\xi}_i = \sqrt{t-s}\xi_i$ for every $i = 1, \ldots, d$) we obtain

$$A(t,s) \le |t-s|^{2H - \frac{d-\alpha}{2}} C_0$$

with

$$C_0 = c\alpha_H (2\pi)^{-d} \int_0^1 \int_0^1 dudv |u-v|^{2H-2} \int_{\mathbb{R}^d} d\xi |\xi|^{-\alpha} e^{-\frac{1}{2}u|\xi|^2} e^{-\frac{1}{2}v|\xi|^2}$$
$$= c\alpha_H (2\pi)^{-d} \int_0^1 \int_0^1 dudv |u-v|^{2H-2} (u+v)^{-\frac{d-\alpha}{2}} \int_{\mathbb{R}^d} d\xi |\xi|^{-\alpha} e^{-\frac{1}{2}|\xi|^2}.$$

Note that the integral above is finite since $d < 4H + \alpha$.

It remains to analyze the term $B(t,s)$. Recall that

$$B(t,s) = \alpha_H (2\pi)^{-d} \int_0^s du \int_0^s dv |u-v|^{2H-2} \int_{\mathbb{R}^d} d\mu(\xi)$$
$$\times \left(e^{-\frac{(t-u)|\xi|^2}{2}} e^{-\frac{(t-v)|\xi|^2}{2}} - 2e^{-\frac{(t-u)|\xi|^2}{2}} e^{-\frac{(sv)|\xi|^2}{2}} + e^{-\frac{(s-u)|\xi|^2}{2}} e^{-\frac{(s-v)|\xi|^2}{2}}\right)$$

and with the change of variables $\tilde{u} = \frac{s-u}{t-s}$, $\tilde{v} = \frac{s-v}{t-s}$ and (2.33)

$$B(t,s) \le c\alpha_H (2\pi)^{-d} (t-s)^{2H} \int_0^s du \int_0^s dv |u-v|^{2H-2} \int_{\mathbb{R}^d} d\xi |\xi|^{-\alpha}$$
$$\times \left(e^{-\frac{(t-s)(2+u+v)|\xi|^2}{2}} - 2e^{-\frac{(t-s)(1+u+v)|\xi|^2}{2}} + e^{-\frac{(t-s)(u+v)|\xi|^2}{2}}\right)$$

and using $\tilde{\xi} = \sqrt{t-s}\xi$

$$B(t,s) \le c\alpha_H (2\pi)^{-d} (t-s)^{2H - \frac{d-\alpha}{2}} \int_0^{\frac{s}{t-s}} du \int_0^{\frac{s}{t-s}} dv |u-v|^{2H-2} \int_{\mathbb{R}^d} d\xi |\xi|^{-\alpha}$$

$$\times\left(e^{-\frac{(2+u+v)|\xi|^2}{2}}-2e^{-\frac{(1+u+v)|\xi|^2}{2}}+e^{-\frac{(u+v)|\xi|^2}{2}}\right)$$

$$\leq c\alpha_H(2\pi)^{-d}(t-s)^{2H-\frac{d-\alpha}{2}}\int_0^\infty du\int_0^\infty dv|u-v|^{2H-2}\int_{\mathbb{R}^d}d\xi|\xi|^{-\alpha}$$

$$\times\left(e^{-\frac{(2+u+v)|\xi|^2}{2}}-2e^{-\frac{(1+u+v)|\xi|^2}{2}}+e^{-\frac{(u+v)|\xi|^2}{2}}\right).$$

Now, using the changes of variables $\tilde{\xi}=(2+u+v)\xi,\tilde{\xi}=(1+u+v)\xi$ and $\tilde{\xi}=(u+v)\xi$ respectively, we can write (with C_H a generic positive constant)

$$B(t,s)\leq C_H(t-s)^{2H-\frac{d-\alpha}{2}}\int_{\mathbb{R}^d}d\xi|\xi|^{-\alpha}e^{-\frac{|\xi|^2}{2}}\int_0^\infty du\int_0^\infty dv|u-v|^{2H-2}$$

$$\times\left[(2+u+v)^{-\frac{d-\alpha}{2}}-2(1+u+v)^{-\frac{d-\alpha}{2}}+(u+v)^{-\frac{d-\alpha}{2}}\right].$$

The integral $\int_0^\infty\int_0^\infty dudv|uv|^{2H-2}[(2+u+v)^{-\frac{d}{2}}-(1+u+v)^{-\frac{d}{2}}+(u+v)^{-\frac{d}{2}}]$ is finite: it is finite for u,v close to zero since $2H-\frac{d}{2}>0$ and it is also finite for u,v close to infinitely because

$$\left[(2+u+v)^{-\frac{d}{2}}-(1+u+v)^{-\frac{d}{2}}+(u+v)^{-\frac{d}{2}}\right]\leq c(u+v)^{-\frac{d}{2}-2}$$

(this can be seen by analyzing the asymptotic behavior of the function $(2+x)^{-\frac{d}{2}}-2(1+x)^{-\frac{d}{2}}+x^{-\frac{d}{2}}$). The proof of the lower bound follows the lines of the proof of the lower bound in Theorem 2.2, using the transfer formula (2.31) and the lower bound in (2.33). □

2.6 The Solution to the Wave Equation with White Noise in Time

The solutions to the linear wave equation with additive Gaussian noise constitute another interesting class of self-similar processes. In contrast to the other examples treated earlier in this monograph, they have the following interesting property: the self-similarity order is not the same as the Hölder regularity order. We first analyze the case of noise white in time and then we will discuss the situation when the noise behaves as a fractional Brownian motion with respect to the time variable.

2.6.1 The Equation

Consider the linear stochastic wave equation driven by a white-colored noise W. That is,

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + \dot{W}(t,x), \quad t \in [0,T], x \in \mathbb{R}^d \\
&u(0,x) = 0, \quad x \in \mathbb{R}^d \\
&\frac{\partial u}{\partial t}(0,x) = 0, \quad x \in \mathbb{R}^d.
\end{aligned} \tag{2.35}$$

Here Δ is the Laplacian on $\mathbb{R}^d$ and $W = \{W(t,A); t \in [0,T], A \in \mathcal{B}_b(\mathbb{R}^d)\}$ is a centered Gaussian field with covariance

$$\mathbf{E}\big(W(t,A)W(s,B)\big) = (t \wedge s) \int_A \int_B f(x-y)dxdy \tag{2.36}$$

where f is the Fourier transform of a tempered measure μ on $\mathbb{R}^d$ (see Sect. 2.2). This is the so-called white-colored noise defined in Sect. 2.3.

Let G_1 be the fundamental solution of $u_{tt} - \Delta u = 0$. It is known that $G_1(t,\cdot)$ is a distribution in $\mathcal{S}'(\mathbb{R}^d)$ with rapid decrease. The easiest way to define G_1 is via its Fourier transform

$$\mathcal{F}G_1(t,\cdot)(\xi) = \frac{\sin(t|\xi|)}{|\xi|}, \tag{2.37}$$

for any $\xi \in \mathbb{R}^d, t > 0, d \geq 1$ (see e.g. [173]). In particular,

$$\begin{aligned}
&G_1(t,x) = \frac{1}{2} 1_{\{|x|<t\}}, \quad \text{if } d = 1 \\
&G_1(t,x) = \frac{1}{2\pi} \frac{1}{\sqrt{t^2 - |x|^2}} 1_{\{|x|<t\}}, \quad \text{if } d = 2 \\
&G_1(t,x) = c_d \frac{1}{t} \sigma_t, \quad \text{if } d = 3,
\end{aligned}$$

where σ_t denotes the surface measure on the 3-dimensional sphere of radius t.

2.6.2 The Solution

The solution of (2.35) is a square-integrable process $u = (u(t,x); t \in [0,T], x \in \mathbb{R}^d)$ defined by the Wiener integral representation with respect to the noise (2.36)

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} G_1(t-s, x-y) W(ds,dy). \tag{2.38}$$

The solution exists when the above integral is well-defined. As for the heat equation, it depends on the dimension d and on the spatial covariance of the noise. For example, when the noise is white both in time and in space the solution exists if and only if $d = 1$.

The necessary and sufficient condition for the existence of the solution follows from [59].

Theorem 2.7 *The stochastic wave equation* (2.35) *admits a unique mild solution* $(u(t,x))_{t\in[0,T],x\in\mathbb{R}^d}$ *if and only if*

$$\int_{\mathbb{R}^d}\left(\frac{1}{1+|\xi|^2}\right)\mu(d\xi) < \infty. \tag{2.39}$$

Remark 2.13 Recall that the same condition holds in the case of the heat equation with white-colored noise (Proposition 2.2).

Remark 2.14 When f is the Riesz kernel, condition (2.39) is equivalent to

$$d < 2+\alpha.$$

When the noise is space-time white noise (corresponding to the case $\alpha = 0$) the solution exists if and only if $d = 1$.

Fix $x \in \mathbb{R}^d$. Then the covariance of the solution u (viewed as a process with respect to t) is

$$\begin{aligned}
&\mathbf{E}u(t,x)u(s,x)\\
&= (2\pi)^{-d}\int_0^{t\wedge s} du \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dy' G_1(t-u,x-y)G_1\big(t-u,x-y'\big)f\big(y-y'\big)\\
&= \int_0^{t\wedge s} du \int_{\mathbb{R}^d} d\xi \mathcal{F}G_1(t-u,\cdot)(\xi)\overline{\mathcal{F}G_1(s-u,\cdot)}(\xi)\mu(d\xi)
\end{aligned}$$

where we used (2.11) and (2.37).

We will assume from now on that the spatial covariance of the noise W is given by the Riesz kernel. We make the change of notation $\alpha = d-\beta$ in the expression given in Example 2.1 and assume that the measure μ is

$$d\mu(\xi) = |\xi|^{-d+\beta}d\xi \quad \text{with } \beta \in (0,d).$$

In this case the kernel f is given by

$$f(\xi) = c_{\beta,d}|\xi|^{-\beta} \quad \text{with } \beta \in (0,d). \tag{2.40}$$

If f is as above, then

$$\begin{aligned}
\mathbf{E}u(t,x)u(s,x) = (2\pi)-d\int_0^{t\wedge s} du\\
\times \int_{\mathbb{R}^d} d\xi \frac{\sin((t-u)|\xi|)}{|\xi|}\frac{\sin((s-u)|\xi|)}{|\xi|}|\xi|^{-d+\beta}d\xi.
\end{aligned}$$

Proposition 2.11 *Suppose f is defined by (2.40). Then the process $(u(t,x), t \geq 0)$ given by (2.38) is self-similar of order $\frac{3-\beta}{2}$.*

Proof Let $c > 0$ and let R be the covariance of the process $t \to u(t,x)$. Then, with $a = (2\pi)^{-d}$, for every $s, t \geq 0$

$$\begin{aligned} R(ct, cs) &= a \int_0^{ct \wedge cs} du \int_{\mathbb{R}^d} d\xi \frac{\sin((ct-u)|\xi|)}{|\xi|} \frac{\sin((cs-u)|\xi|)}{|\xi|} |\xi|^{-d+\beta} d\xi \\ &= ac \int_0^{t \wedge s} du \int_{\mathbb{R}^d} d\xi \frac{\sin((ct-cu)|\xi|)}{|\xi|} \frac{\sin((cs-cu)|\xi|)}{|\xi|} |\xi|^{-d+\beta} d\xi \\ &= c^{3-\beta} R(t,s) \end{aligned}$$

where we made successively the change of variable $\tilde{u} = \frac{u}{c}$ and $\tilde{\xi} = c\xi$. ☐

The solution has the following time regularity (see [63, 64]).

Proposition 2.12 *Assume that*

$$\beta \in (0, d \wedge 2). \tag{2.41}$$

Let $t_0, M > 0$ and fix $x \in [-M, M]^d$. Then there exist positive constants c_1, c_2 such that for every $s, t \in [t_0, T]$

$$c_1 |t-s|^{2-\beta} \leq \mathbf{E} \left| u(t,x) - u(s,x) \right|^2 \leq c_2 |t-s|^{2-\beta}.$$

Remark 2.15 Let us highlight an interesting fact: the order of self-similarity and the order of Hölder continuity do not coincide in this case. This is the first example among the Gaussian processes discussed in this chapter when this phenomenon occurs.

Proposition 2.12 implies the following Hölder property for the solution to (2.35).

Corollary 2.6 *Assume (2.41). Then for every $x \in \mathbb{R}^d$ the application*

$$t \to u(t,x)$$

is almost surely Hölder continuous of order $\delta \in (0, \frac{2-\beta}{2})$.

Proof This follows easily from Proposition 2.12 and from the fact that u is Gaussian. ☐

Remark 2.16 The proof of Theorem 5.1 in [63] implies that the mapping $t \to u(t,x)$ is not Hölder continuous of order $\frac{2-\beta}{2}$.

2.7 The Stochastic Wave Equation with Linear Fractional-Colored Noise

For an interval $(a, b) \subset \mathbb{R}$, we define the restricted Fourier transform of a function $\varphi \in L^1(a, b)$:

$$\mathcal{F}_{a,b}\varphi(\tau) := \int_a^b e^{-i\tau x}\varphi(x)dx = \mathcal{F}(\varphi 1_{[a,b]})(\tau).$$

One can prove that $\mathcal{F}\varphi \in L^2(\mathbb{R})$, for any $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. By Plancharel's identity (2.8), for any $\varphi, \psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we have:

$$\int_{\mathbb{R}} \varphi(x)\psi(x)dx = (2\pi)^{-1}\int_{\mathbb{R}} \mathcal{F}\varphi(\tau)\overline{\mathcal{F}\psi(\tau)}d\xi.$$

In particular, for any $\varphi, \psi \in L^2(a, b)$, we have:

$$\int_a^b \varphi(x)\psi(x)dx = (2\pi)^{-1}\int_{\mathbb{R}} \mathcal{F}_{a,b}\varphi(\tau)\overline{\mathcal{F}_{a,b}\psi(\tau)}d\xi. \tag{2.42}$$

2.7.1 *The Equation*

Consider the linear stochastic wave equation driven by a fractional colored noise W with Hurst parameter $H \in (\frac{1}{2}, 1)$. That is

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(t,x) &= \Delta u(t,x) + \dot{W}(t,x), \quad t \in [0,T], x \in \mathbb{R}^d \\ u(0,x) &= 0, \quad x \in \mathbb{R}^d \\ \frac{\partial u}{\partial t}(0,x) &= 0, \quad x \in \mathbb{R}^d. \end{aligned} \tag{2.43}$$

Here Δ is the Laplacian on $\mathbb{R}^d$ and $W = \{W(t, A); t \in [0, T], A \in \mathcal{B}_b(\mathbb{R}^d)\}$ is a centered Gaussian field with covariance

$$\mathbf{E}\big(W(t,A)W(s,B)\big) = R_H(t,s)\int_A\int_B f(x-y)dxdy \tag{2.44}$$

where R_H is the covariance of the fractional Brownian motion (1.1) and f is the Fourier transform of a tempered measure μ on $\mathbb{R}^d$ (see Sect. 2.2).

2.7.2 The Solution

The solution of (2.35) is a square-integrable process $u = \{u(t,x); t \geq 0, x \in \mathbb{R}^d\}$ defined by:

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} G_1(t-s, x-y) W(ds, dy). \tag{2.45}$$

By definition, $u(t,x)$ exists if and only if the stochastic integral above is well-defined, i.e. $g_{tx} := G_1(t-\cdot, x-\cdot) \in \mathcal{HP}$ (this space was introduced in Sect. 2.5). In this case, $\mathbf{E}|u(t,x)|^2 = \|g_{tx}\|_{\mathcal{HP}}^2$.

We begin with an auxiliary result. To simplify the notation, we introduce the following functions: for $\lambda > 0$, $\tau > 0$, let

$$f_t(\lambda,\tau) = \sin \tau\lambda t - \tau \sin \lambda t, \qquad g_t(\lambda,\tau) = \cos \tau\lambda t - \cos \lambda t. \tag{2.46}$$

Lemma 2.1 *For any $\lambda > 0$ and $t > 0$,*

$$c_t \frac{\lambda^3}{1+\lambda^2} \leq \int_{\mathbb{R}} \frac{1}{(\tau^2-1)^2} \big[f_t^2(\lambda,\tau) + g_t^2(\lambda,\tau)\big] d\tau \leq C_t \frac{\lambda^3}{1+\lambda^2},$$

where c_t, C_t are some positive constants.

Proof Using Exercise 1.15, we have

$$\frac{1}{(\tau^2-1)^2} \big[f_t^2(\lambda,\tau) + g_t^2(\lambda,\tau)\big] = \big|\mathcal{F}_{0,\lambda t}\varphi(\tau)\big|^2,$$

where $\varphi(x) = \sin x$. Using Plancharel's identity (2.42), we obtain:

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{1}{(\tau^2-1)^2} \big[f_t^2(\lambda,\tau) + g_t^2(\lambda,\tau)\big] d\tau \\
&= \int_{\mathbb{R}} \big|\mathcal{F}_{0,\lambda t}\varphi(\tau)\big|^2 d\tau = 2\pi \int_0^{\lambda t} |\sin x|^2 dx \\
&= 2\pi\lambda \int_0^t |\sin \lambda s|^2 ds = 2\pi\lambda^3 \int_0^t \frac{|\sin \lambda s|^2}{\lambda^2} ds.
\end{aligned}$$

It now suffices to use the bound (see e.g. Lemma 6.1.2 of [161])

$$c_t \frac{1}{1+\lambda^2} \leq \int_0^t \frac{|\sin \lambda s|^2}{\lambda^2} ds \leq C_t \frac{1}{1+\lambda^2}. \qquad \square$$

We denote by $N_t(\xi)$ the $\mathcal{H}(0,t)$-norm of $u \mapsto \mathcal{F}G_1(u,\cdot)(\xi)$, i.e.

$$N_t(\xi) = \frac{\alpha_H}{|\xi|^2} \int_0^t \int_0^t \sin\big(u|\xi|\big) \sin\big(v|\xi|\big) |u-v|^{2H-2} du dv.$$

We also recall that (see Exercise 1.12) there exists a constant $b_H > 0$ such that

$$\|\varphi\|^2_{\mathcal{H}(0,t)} \le b_H^2 \|\varphi\|^2_{L^{1/H}(0,t)} \le b_H^2 t^{2H-1} \|\varphi\|^2_{L^2(0,t)} \tag{2.47}$$

for any $\varphi \in L^2(0,t)$.

Proposition 2.13 *For any $t > 0$, $\xi \in \mathbb{R}^d$*

$$N_t(\xi) \le C_{H,t} t^{2H+2} \left(\frac{1}{1+|\xi|^2} \right)^{H+1/2}, \quad \textit{if } |\xi| \le 1$$

$$N_t(\xi) \le c_{H,t} \left(\frac{1}{1+|\xi|^2} \right)^{H+1/2}, \quad \textit{if } |\xi| \ge 1.$$

Proof (a) Suppose that $|\xi| \le 1$. We use (2.47) and $|\sin x| \le x$ for any $x > 0$. Hence,

$$\begin{aligned} N_t(\xi) &\le b_H^2 t^{2H-1} \frac{1}{|\xi|^2} \int_0^t \sin^2\big(u|\xi|\big) du \le b_H^2 t^{2H-1} \int_0^t u^2 du \\ &= b_H^2 t^{2H-1} \frac{t^3}{3} \le \frac{1}{3} b_H^2 t^{2H+2} 2^{H+1/2} \left(\frac{1}{1+|\xi|^2} \right)^{H+1/2}, \end{aligned}$$

where for the last inequality we used the fact that $\frac{1}{2} \le \frac{1}{1+|\xi|^2}$ if $|\xi| \le 1$.

(b) Suppose that $|\xi| \ge 1$. Using the change of variables $u' = u|\xi|$, $v' = v|\xi|$,

$$\begin{aligned} N_t(\xi) &= \frac{\alpha_H}{|\xi|^{2H+2}} \int_0^{t|\xi|} \int_0^{t|\xi|} \sin\big(u'\big) \sin\big(v'\big) \big|u' - v'\big|^{2H-2} du' dv' \\ &= \frac{1}{|\xi|^{2H+2}} \big\|\sin(\cdot)\big\|^2_{\mathcal{H}(0,t|\xi|)}. \end{aligned}$$

Using the expression of the $\mathcal{H}(0, t|\xi|)$-norm of $\sin(\cdot)$ given in Exercise 1.15, we obtain:

$$N_t(\xi) = \frac{c_H}{|\xi|^{2H+2}} \int_{\mathbb{R}} \frac{|\tau|^{-(2H-1)}}{(\tau^2-1)^2} \big[f_t^2\big(|\xi|, \tau\big) + g_t^2\big(|\xi|, \tau\big) \big] d\tau. \tag{2.48}$$

We split the integral into the regions $|\tau| \le 1/2$ and $|\tau| \ge 1/2$, and we denote the two integrals by $N_t^{(1)}(\xi)$ and $N_t^{(2)}(\xi)$.

Since $|f_t(\lambda, \tau)| \le 1 + |\tau|$ and $|g_t(\lambda, \tau)| \le 2$ for any $\lambda > 0$, $\tau > 0$, we have:

$$\begin{aligned} N_t^{(1)}(\xi) &\le c_H \frac{1}{|\xi|^{2H+2}} \int_{|\tau| \le 1/2} \frac{|\tau|^{-(2H-1)}}{(1-\tau^2)^2} \big[\big(1+|\tau|\big)^2 + 4 \big] d\tau \\ &\le c_H \frac{1}{|\xi|^{2H+1}} \int_{|\tau| \le 1/2} C |\tau|^{-(2H-1)} d\tau \end{aligned}$$

$$= C \frac{c_H}{1-H} \left(\frac{1}{2}\right)^{2-2H} \frac{1}{|\xi|^{2H+1}}.$$

We used the fact that $|\xi|^{2H+2} \geq |\xi|^{2H+1}$ if $|\xi| \geq 1$, and $\frac{1}{(1-\tau^2)^2}[(1+|\tau|)^2+4] \leq \frac{1}{(3/4)^2}[(3/2)^2+4] := C$ if $|\tau| \leq 1/2$.

Using the fact that $|\tau|^{-(2H-1)} \leq (\frac{1}{2})^{-(2H-1)}$ if $|\tau| \geq \frac{1}{2}$, Lemma 2.1, and the fact that $|\xi|^2/(1+|\xi|^2) \leq 1$, we obtain:

$$\begin{aligned} N_t^{(2)}(\xi) &\leq \frac{c_H}{2^{-(2H-1)}} \frac{1}{|\xi|^{2H+2}} \int_{|\tau|\geq 1/2} \frac{1}{(\tau^2-1)^2} \left[f_t^2\left(|\xi|,\tau\right) + g_t^2\left(|\xi|,\tau\right)\right] d\tau \\ &\leq \frac{c_H}{2^{-(2H-1)}} \frac{1}{|\xi|^{2H+2}} \int_{\mathbb{R}} \frac{1}{(\tau^2-1)^2} \left[f_t^2\left(|\xi|,\tau\right) + g_t^2\left(|\xi|,\tau\right)\right] d\tau \\ &\leq \frac{c_H}{2^{-(2H-1)}} c_t^{(2)} \frac{1}{|\xi|^{2H+2}} \cdot |\xi| \frac{|\xi|^2}{1+|\xi|^2} \\ &\leq \frac{c_H}{2^{-(2H-1)}} c_t^{(2)} \frac{1}{|\xi|^{2H+1}}. \end{aligned}$$

□

Proposition 2.14

(a) *If* $I_t^{(1)} < \infty$ *for* $t = 1$, *then* $\int_{|\xi|\leq 1} \mu(d\xi) < \infty$.
(b) *Let* $l \geq 1$ *be the integer from* (2.10) *and* $m = 2l - 2$. *For any* $t > 0$,

$$\int_{|\xi|\geq 1} \frac{\mu(d\xi)}{|\xi|^{2H+1}} \leq a_{H,t} \left(\sum_{i=0}^{m} b_t^i\right) I_t^{(2)} + b_t^{m+1} \int_{|\xi|\geq 1} \frac{\mu(d\xi)}{|\xi|^{2H+2+m}}, \qquad (2.49)$$

where $a_{H,t}, b_t, c_t$ *are positive constants.*

In particular, if $I_t^{(2)} < \infty$ *for some* $t > 0$, *then* $\int_{|\xi|\geq 1} |\xi|^{-(2H+1)} \mu(d\xi) < \infty$.

Proof (a) Using the fact that $\sin x/x \geq \sin 1$ for all $x \in [0, 1]$, we have:

$$\begin{aligned} I_1^{(1)} &= \int_{|\xi|\leq 1} \frac{\mu(d\xi)}{|\xi|^2} \int_0^1 \int_0^1 \sin\left(u|\xi|\right) \sin\left(v|\xi|\right) |u-v|^{2H-2} du dv \\ &\geq \sin^2 1 \int_{|\xi|\leq 1} \mu(d\xi) \int_0^1 \int_0^1 uv|u-v|^{2H-2} du dv. \end{aligned}$$

(b) According to (2.48),

$$I_t^{(2)} = c_H \int_{|\xi|\geq 1} \frac{\mu(d\xi)}{|\xi|^{2H+2}} \int_{\mathbb{R}} \frac{|\tau|^{-(2H-1)}}{(\tau^2-1)^2} \left[f_t^2\left(|\xi|,\tau\right) + g_t^2\left(|\xi|,\tau\right)\right] d\tau. \qquad (2.50)$$

For any $k \in \{-1, 0, \ldots, m\}$, let

$$I(k) := \int_{|\xi|\geq 1} \frac{1}{|\xi|^{2H+2+k}} \mu(d\xi).$$

By (2.10), $I(m)=\int_{|\xi|\geq 1}|\xi|^{-(2H+2+m)}\mu(d\xi)\leq \int_{|\xi|\geq 1}|\xi|^{-2l}\mu(d\xi)<\infty$.

We will prove that the integrals $I(k)$ satisfy a certain recursive relation. By reverse induction, this will imply that all integrals $I(k)$ with $k\in\{-1,0,\ldots,m\}$ are finite. For this, for $k\in\{0,1\ldots,m\}$, we let

$$A_t(k):=\int_{|\xi|\geq 1}\frac{\mu(d\xi)}{|\xi|^{2H+2+k}}\int_{\mathbb{R}}\frac{1}{(\tau^2-1)^2}\big[f_t^2\big(|\xi|,\tau\big)+g_t^2\big(|\xi|,\tau\big)\big]d\tau. \qquad (2.51)$$

We consider separately the regions $\{|\tau|\leq 2\}$ and $\{|\tau|\geq 2\}$ and we denote the corresponding integrals by $A_t'(k)$ and $A_t''(k)$. For the region $\{|\tau|\leq 2\}$, we use the expression (2.50) of $I_t^{(2)}$. Using the fact that $|\xi|^{2H+2+k}\geq|\xi|^{2H+2}$ (since $k\geq 0$), and $|\tau|^{-(2H-1)}\geq 2^{-(2H-1)}$ if $|\tau|\leq 2$, we obtain:

$$\begin{aligned}
A_t'(k)&:=\int_{|\xi|\geq 1}\frac{\mu(d\xi)}{|\xi|^{2H+2+k}}\int_{|\tau|\leq 2}\frac{1}{(\tau^2-1)^2}\big[f_t^2\big(|\xi|,\tau\big)+g_t^2\big(|\xi|,\tau\big)\big]d\tau\\
&\leq 2^{2H-1}\int_{|\xi|\geq 1}\frac{\mu(d\xi)}{|\xi|^{2H+2}}\int_{|\tau|\leq 2}\frac{|\tau|^{-(2H-1)}}{(\tau^2-1)^2}\big[f_t^2\big(|\xi|,\tau\big)+g_t^2\big(|\xi|,\tau\big)\big]d\tau\\
&\leq 2^{2H-1}\frac{1}{c_H}I_t^{(2)},\quad \text{by (2.50)}.
\end{aligned}$$

For the region $\{|\tau|\geq 2\}$, we use the fact $|f_t(\lambda,\tau)|\leq 1+|\tau|$ and $|g_t(\lambda,\tau)|\leq 2$ for all $\lambda>0$, $\tau>0$. Hence,

$$\begin{aligned}
A_t''(k)&:=\int_{|\xi|\geq 1}\frac{\mu(d\xi)}{|\xi|^{2H+2+k}}\int_{|\tau|\geq 2}\frac{1}{(\tau^2-1)^2}\big[f_t^2\big(|\xi|,\tau\big)+g_t^2\big(|\xi|,\tau\big)\big]d\tau\\
&\leq \int_{|\xi|\geq 1}\frac{\mu(d\xi)}{|\xi|^{2H+2+k}}\int_{|\tau|\geq 2}\frac{1}{(\tau^2-1)^2}\big[\big(1+|\tau|\big)^2+4\big]d\tau=CI(k).
\end{aligned}$$

Hence, for any $k\in\{0,1,\ldots,m\}$

$$A_t(k)\leq 2^{2H-1}\frac{1}{c_H}I_t^{(2)}+CI(k).$$

Using Lemma 2.1, and the fact that $\frac{|\xi|^2}{1+|\xi|^2}\geq\frac{1}{2}$ if $|\xi|\geq 1$, we obtain:

$$A_t(k)\geq c_t^{(1)}\int_{|\xi|\geq 1}\frac{\mu(d\xi)}{|\xi|^{2H+2+k}}\cdot\frac{|\xi|^3}{1+|\xi|^2}\geq\frac{1}{2}c_t^{(1)}I(k-1),$$

for all $k\in\{0,1,\ldots,m\}$. From the last two relations, we conclude that:

$$\frac{1}{2}c_t^{(1)}I(k-1)\leq 2^{2H-1}\frac{1}{c_H}I_t^{(2)}+CI(k),\quad \forall k\in\{0,1,\ldots,m\}, \qquad (2.52)$$

or equivalently, $I(k-1)\leq a_{H,t}I_t^{(2)}+b_tI(k)$, for all $k\in\{0,1,\ldots,m\}$. Relation (2.49) follows by recursion. □

Remark 2.17 In the previous argument, the recursion relation (2.52) uses the fact that k is *non-negative* (see the estimate of $A'_t(k)$). Therefore, the "last" index k for which this relation remains true (counting downwards from m) is $k = 0$, leading us to the conclusion that $\int_{|\xi|\geq 1} |\xi|^{-(2H+1)}\mu(d\xi) < \infty$, if $I_t^{(2)} < \infty$.

Theorem 2.8 *The stochastic wave equation* (2.35) *admits a unique mild solution* $(u(t,x))_{t\in[0,T],x\in\mathbb{R}^d}$ *if and only if*

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)^{H+\frac{1}{2}} \mu(d\xi) < \infty. \tag{2.53}$$

Proof To have that $g_{tx} \in \mathcal{HP}$ we need in particular to have $I_t < \infty$ for all $t > 0$ (see [14] for more details), where

$$I_t := \alpha_H \int_{\mathbb{R}^d} \int_0^t \int_0^t \mathcal{F}g_{tx}(u,\cdot)(\xi)\overline{\mathcal{F}g_{tx}(v,\cdot)(\xi)}|u-v|^{2H-2}dudv\mu(d\xi),$$

and $\mathbf{E}|u(t,x)|^2 = \|g_{tx}\|^2_{\mathcal{HP}} = I_t$. Since $\mathcal{F}g_{tx}(u,\cdot)(\xi) = e^{-i\xi\cdot x}\overline{\mathcal{F}G_1(t-u,\cdot)(\xi)}$,

$$I_t = \alpha_H \int_{\mathbb{R}^d} \int_0^t \int_0^t \mathcal{F}G_1(u,\cdot)(\xi)\overline{\mathcal{F}G_1(v,\cdot)(\xi)}|u-v|^{2H-2}dudv\mu(d\xi).$$

Using (2.37), we obtain:

$$I_t = \alpha_H \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{|\xi|^2} \int_0^t \int_0^t \sin\big(u|\xi|\big)\sin\big(v|\xi|\big)|u-v|^{2H-2}dudv.$$

We split the integral $\mu(d\xi)$ into two parts, corresponding to the regions $\{|\xi|\leq 1\}$ and $\{|\xi|\geq 1\}$. We denote the respective integrals by $I_t^{(1)}$ and $I_t^{(2)}$. Since the integrand is non-negative $I_t < \infty$ if and only if $I_t^{(1)} < \infty$ and $I_t^{(2)} < \infty$.

The fact that condition (2.53) is sufficient for $I_t < \infty$ follows by Proposition 2.13. The necessity follows by Proposition 2.14 (using Remark 2.18). □

Remark 2.18 Condition (2.53) is equivalent to

$$\int_{|\xi|\leq 1} \mu(d\xi) < \infty \quad \text{and} \quad \int_{|\xi|\geq 1} \frac{1}{|\xi|^{2H+1}}\mu(d\xi) < \infty.$$

Corollary 2.7

(i) *Let* $f(x) = \gamma_{\alpha,d}|x|^{-(d-\alpha)}$ *be the Riesz kernel of order* $\alpha \in (0,d)$. *Then* $\mu(d\xi) = |\xi|^{-\alpha}d\xi$ *and* (2.53) *is equivalent to* $\alpha > d - 2H - 1$.

(ii) *Let* $f(x) = \gamma_\alpha \int_0^\infty w^{(\alpha-d)/2-1}e^{-w}e^{-|x|^2/(4w)}dw$ *be the Bessel kernel of order* $\alpha > 0$. *Then* $\mu(d\xi) = (1+|\xi|^2)^{-\alpha/2}$ *and* (2.53) *is equivalent to* $\alpha > d - 2H - 1$.

The solution to the wave equation is also self-similar.

Proposition 2.15 *Fix $x \in \mathbb{R}^d$ and assume* (2.53). *Then the process $(u(t,x), t \geq 0)$ defined by* (2.45) *is self-similar of order*

$$H + 1 - \frac{d-\alpha}{2}.$$

Proof The covariance of u can be expressed as

$$\mathbf{E}u(t,x)u(s,x) = a(H)\int_0^t du \int_0^s dv|u-v|^{2H-2}$$
$$\times \int_{\mathbb{R}^d} d\xi \frac{\sin((t-u)|\xi|)}{|\xi|}\frac{\sin((s-v)|\xi|)}{|\xi|}|\xi|^{-d}d\xi.$$

This easily implies the conclusion by a standard change of variables. □

Remark 2.19 Note that the self-similarity index

$$H + 1 - \frac{d-\alpha}{2}$$

is positive under condition (2.53).

Assume in the sequel that the spatial covariance of the noise W is given by the Riesz kernel under the form (2.40). Note that in this case condition (2.53) is equivalent to

$$\beta \in \big(0, d \wedge (2H+1)\big). \tag{2.54}$$

Remark 2.20 Since $H > \frac{1}{2}$ and so $2H + 1 \in (2, 3)$, for dimension $d = 1, 2$ we have $\beta \in (0, d)$ while for $d \geq 3$ we have $\beta \in (0, 2H + 1)$.

Remark 2.21 As a consequence of Exercise 1.13 we deduce the following:

(i) For any $x > 0$ the quantity $\int_0^x v^{2H-2}\cos(v)(x-v)dv$ is positive (it is the sum of two norms).

(ii) For every $a, b \in \mathbb{R}$, $a < b$

$$\|f 1_{(a,b)}\|^2_{\mathcal{H}} \leq 2\alpha_H \int_0^{b-a} dv \cos(v) v^{2H-2}(b-a-v).$$

(iii) For every $a, b \in \mathbb{R}$, $a < b$

$$\|f 1_{(a,b)}\|^2_{\mathcal{H}} \geq 2\alpha_H \cos(a+b)\int_0^{b-a} dv v^{2H-2}\sin(b-a-v).$$

Proposition 2.16 *Assume that*

$$\beta \in \big(2H-1, d \wedge (2H+1)\big). \tag{2.55}$$

Let $t_0, M > 0$ *and fix* $x \in [-M, M]^d$. *Then there exist positive constants* c_1, c_2 *such that for every* $s, t \in [t_0, T]$

$$c_1|t-s|^{2H+1-\beta} \leq \mathbf{E}\big|u(t,x)-u(s,x)\big|^2 \leq c_2|t-s|^{2H+1-\beta}.$$

Proof Let $h > 0$ and let us estimate the $L^2(\Omega)$-norm of the increment $u(t+h,x)-u(t,x)$. Splitting the interval $[0, t+h]$ into the intervals $[0, t]$ and $[t, t+h]$, and using the inequality $|a+b|^2 \leq 2(a^2+b^2)$, we obtain:

$$\begin{aligned}\mathbf{E}\big|u(t+h,x)-u(t,x)\big|^2 &\leq 2\big\{\big\|(g_{t+h,x}-g_{t,x})1_{[0,t]}\big\|^2_{\mathcal{HP}} + \|g_{t+h,x}1_{[t,t+h]}\|^2_{\mathcal{HP}}\big\} \\ &=: 2\big[E_{1,t}(h)+E_2(h)\big]. \end{aligned} \tag{2.56}$$

The first summand can be handled in the following way.

$$\begin{aligned} E_{1,t}(h) &= \alpha_H(2\pi)^{-d}\int_{\mathbb{R}^d}\mu(d\xi)\int_0^t\int_0^t dvdv|u-v|^{2H-2}\mathcal{F}(g_{t+h,x}-g_{tx})(u,\cdot)(\xi) \\ &\quad\times\overline{\mathcal{F}(g_{t+h,x}-g_{tx})(v,\cdot)(\xi)} \\ &= \alpha_H(2\pi)^{-d}\int_{\mathbb{R}^d}\mu(d\xi) \\ &\quad\times\int_0^t\int_0^t dudv|u-v|^{2H-2}\big[\mathcal{F}G_1(u+h,\cdot)(\xi)-\mathcal{F}G_1(u,\cdot)(\xi)\big] \\ &\quad\times\overline{\mathcal{F}G_1(v+h,\cdot)(\xi)-\mathcal{F}G_1(v,\cdot)(\xi)} \\ &= \alpha_H(2\pi)^{-d}\int_0^t\int_0^t dudv|u-v|^{2H-2}I_h, \end{aligned}$$

where

$$\begin{aligned} I_h &= \int_{\mathbb{R}^d}\mu(d\xi)\big[\mathcal{F}G_1(u+h,\cdot)(\xi)-\mathcal{F}G_1(u,\cdot)(\xi)\big] \\ &\quad\times\big[\overline{\mathcal{F}G_1(v+h,\cdot)(\xi)-\mathcal{F}G_1(v,\cdot)(\xi)}\big] \\ &= \int_{\mathbb{R}^d}\mu(d\xi)\frac{(\sin((u+h)|\xi|)-\sin(u|\xi|))}{|\xi|}\,\frac{(\sin((v+h)|\xi|)-\sin(v|\xi|))}{|\xi|}. \end{aligned}$$

Using trigonometric identities we obtain

$$\begin{aligned} E_{1,t}(h) &= \alpha_H\int_0^t\int_0^t dudv|u-v|^{2H-2}\int_{\mathbb{R}^d}\mu(d\xi)\frac{\sin(\frac{h|\xi|}{2})^2}{|\xi|^2}\cos\left(\frac{(2u+h)|\xi|}{2}\right) \\ &\quad\times\cos\left(\frac{(2v+h)|\xi|}{2}\right) \\ &= c\cdot\alpha_H\int_0^t\int_0^t dudv|u-v|^{2H-2}\int_{\mathbb{R}^d}\frac{d\xi}{|\xi|^{d-\beta+2}}\sin\big(h|\xi|\big)^2 \end{aligned}$$

$$\times \cos\big((2u+h)|\xi|\big)\cos\big((2v+h)|\xi|\big),$$

and by making the change of variables $\tilde{u} = (2u+h)|\xi|$, $\tilde{v} = (2v+h)|\xi|$,

$$\begin{aligned} E_{1,t}(h) &= c\cdot\alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^2 \\ &\quad \times \int_{h|\xi|}^{(2t+h)|\xi|}\int_{h|\xi|}^{(2t+h)|\xi|} dudv|u-v|^{2H-2}\cos u\cos v \\ &= c\int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^2 \big\|\cos(\cdot)1_{(h|\xi|,(2t+h)|\xi|)}(\cdot)\big\|^2_{\mathcal{H}}, \end{aligned} \tag{2.57}$$

and using Exercise 1.13,

$$\begin{aligned} E_{1,t}(h) &= c\cdot\alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^2 \times \Bigg[\int_0^{2t|\xi|} \cos(v)v^{2H-2}\big(2t|\xi|-v\big)dv \\ &\quad + \cos\big(2t|\xi|+2h|\xi|\big)\int_0^{2t|\xi|} v^{2H-2}\big(\sin\big(2t|\xi|-v\big)\big)\Bigg] \\ &= c\cdot\alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^2 \times \Bigg[2t|\xi|\int_0^{2t|\xi|} \cos(v)v^{2H-2}dv \\ &\quad - \sin\big(2t|\xi|\big)\big(2t|\xi|\big)^{2H-1} + (2H-1)\int_0^{2t|\xi|} \sin(v)v^{2H-2}dv \\ &\quad + \cos\big(2t|\xi|+2h|\xi|\big)\int_0^{2t|\xi|} v^{2H-2}\big(\sin\big(2t|\xi|-v\big)\big)\Bigg] \end{aligned} \tag{2.58}$$

where we use integration by parts. By Remark 2.21, point (ii) we have the upper bound

$$\begin{aligned} E_{1,t}(h) &\le c\cdot\alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^2 \\ &\quad \times \Bigg[2t|\xi|\int_0^{2t|\xi|} \cos(v)v^{2H-2}dv - \sin\big(2t|\xi|\big)\big(2t|\xi|\big)^{2H-1} \\ &\quad + (2H-1)\int_0^{2t|\xi|} \sin(v)v^{2H-2}dv\Bigg]. \end{aligned}$$

We will treat the three summands above separately. For the first one,

$$\begin{aligned} &\int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^2 2t|\xi|\int_0^{2t|\xi|} \cos(v)v^{2H-2}dv \\ &\quad = c_{t,H}h^{2H+1-\beta}\left|\int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+1}} \sin\big(|\xi|\big)^2 \int_0^{\frac{2t|\xi|}{h}} \cos(v)v^{2H-2}dv\right| \end{aligned}$$

$$\leq c_{t,H} h^{2H+1-\beta} \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+1}} \sin\big(|\xi|\big)^2 \left| \int_0^{\frac{2t|\xi|}{h}} \cos(v) v^{2H-2} dv \right|$$
$$\leq c_{t,H} h^{2H+1-\beta}$$

using condition (2.54) and the fact that the integral $\int_0^\infty \cos(v) v^{2H-2} dv$ is convergent (this implies that the function $x \in [0,\infty) \to \int_0^x \cos(v) v^{2H-2} dv$ admits a limit at infinity and is therefore bounded). On the other hand

$$\int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^2 \sin\big(2t|\xi|\big)\big(2t|\xi|\big)^{2H-1}$$
$$= c_t h^{3-\beta} \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+3}} \sin\big(|\xi|\big)^2 \sin\left(\frac{2t|\xi|}{h}\right)$$
$$= c_t h^{3-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+3}} \sin\big(|\xi|\big)^2 \sin\left(\frac{2t|\xi|}{h}\right)$$
$$+ c_t h^{3-\beta} \int_{|\xi|>1} \frac{d\xi}{|\xi|^{d-\beta+3}} \sin\big(|\xi|\big)^2 \sin\left(\frac{2t|\xi|}{h}\right).$$

The second part over the region $|\xi| \geq 1$ is bounded by $ch^{3-\beta}$ simply by majorizing sine by one. The second integral has a singularity for $|\xi|$ close to zero. Using the fact that $\sin(x) \leq x$ for all $x \geq 0$, we will bound it above by

$$h^{3-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+3}} \sin\big(|\xi|\big)^2 \sin\left(\frac{2t|\xi|}{h}\right)$$
$$\leq c_t h^{3-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+3}} |\xi|^2 \left| \sin\left(\frac{2t|\xi|}{h}\right) \right|^{2-2H} \left| \sin\left(\frac{2t|\xi|}{h}\right) \right|^{2H-1}$$
$$\leq c_t h^{2H+1-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+2H-1}}$$

where we bounded $|\sin(\frac{2t|\xi|}{h})|^{2-2H}$ by $c_t(|\xi|h^{-1})^{2-2H}$ and $|\sin(\frac{2t|\xi|}{h})|^{2H-1}$ by 1. The last integral is finite since $\beta > 2H-1$ (assumption (2.55)).

Finally

$$\int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^2 \int_0^{2t|\xi|} \sin(v) v^{2H-2} dv$$
$$= h^{2H+2-\beta} \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(|\xi|\big)^2 \int_0^{\frac{2t|\xi|}{h}} \sin(v) v^{2H-2} dv$$
$$= h^{2H+2-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(|\xi|\big)^2 \int_0^{\frac{2t|\xi|}{h}} \sin(v) v^{2H-2} dv$$

$$+ h^{2H+2-\beta} \int_{|\xi|\geq 1} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(|\xi|\big)^2 \int_0^{\frac{2t|\xi|}{h}} \sin(v) v^{2H-2} dv$$

$$\leq h^{2H+2-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} |\xi|^2 \int_0^{\frac{2t|\xi|}{h}} |\sin v| v^{2H-2} dv$$

$$+ h^{2H+2-\beta} \int_{|\xi|\geq 1} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \int_0^{\frac{2t|\xi|}{h}} \sin(v) v^{2H-2} dv. \tag{2.59}$$

Again using the fact that $\int_0^\infty \sin(v) v^{2H-2} dv$ it is convergent it is easy to see that the integral over the region $|\xi| \geq 1$ is bounded by $c_t h^{2H+2-\beta}$. For the integral over $|\xi| \leq 1$ we make the change of variables $\tilde{v} = \frac{vh}{\xi}$ and we get

$$h^{3-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+1}} \int_0^{2t} \left|\sin\left(\frac{v|\xi|}{h}\right)\right| v^{2H-2} dv$$

$$= h^{3-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+1}} \int_0^{2t} \left|\sin\left(\frac{v|\xi|}{h}\right)\right|^{2-2H} \left|\sin\left(\frac{v|\xi|}{h}\right)\right|^{2H-1} v^{2H-2} dv$$

$$\leq c_t h^{2h+1-\beta} \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{d-\beta+2H-1}},$$

where we have made the same considerations as for the second summand in the decomposition of $E_{1,t}(h)$. In this way, we obtain the upper bound for the summand $E_{1,t}(h)$ in (2.56)

$$E_{1,t}(h) \leq C h^{2H+1-\beta}. \tag{2.60}$$

We now study the term $E_2(h)$ in (2.56) (the notation $E_2(h)$ instead of $E_{2,t}(h)$ is due to the fact that it does not depend on t, see below). Using successively the change of variables $\tilde{u} = \frac{u}{h}$, $\tilde{v} = \frac{v}{h}$ in the integral $dudv$ and $\tilde{\xi} = h\xi$ in the integral $d\xi$, the summand $E_2(h)$ can be written as

$$E_2(h) = \alpha_H \int_{\mathbb{R}^d} \int_t^{t+h} \int_t^{t+h} \mathcal{F}G_1(t+h-u, \cdot)(\xi)$$

$$\times \overline{\mathcal{F}G_1(t+h-v, \cdot)(\xi)} |u-v|^{2H-2} dudv \mu(d\xi)$$

$$= \alpha_H \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{|\xi|^2} \int_0^h \int_0^h \sin\big(u|\xi|\big) \sin\big(v|\xi|\big) |u-v|^{2H-2} dudv$$

$$= \alpha_H h^{2H} \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{|\xi|^2} \int_0^1 \int_0^1 \sin\big(u|\xi|h\big) \sin\big(v|\xi|h\big) |u-v|^{2H-2} dudv$$

$$= \alpha_H h^{2H+2-\beta} \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{|\xi|^2} \int_0^1 \int_0^1 \sin\big(u|\xi|\big) \sin\big(v|\xi|\big) |u-v|^{2H-2} dudv.$$

Let us use the following notation:

$$N_t(\xi) = \frac{\alpha_H}{|\xi|^2} \int_0^t \int_0^t \sin\big(u|\xi|\big) \sin\big(v|\xi|\big) |u-v|^{2H-2} du dv, \quad t \in [0,T], \xi \in \mathbb{R}^d. \tag{2.61}$$

By Proposition 2.13 the term

$$N_1(\xi) = \frac{\alpha_H}{|\xi|^2} \int_0^1 \int_0^1 \sin\big(u|\xi|\big) \sin\big(v|\xi|\big) |u-v|^{2H-2} du dv$$

satisfies the inequality

$$N_1(\xi) \le C_H \left(\frac{1}{1+|\xi|^2} \right)^{H+1/2},$$

with C_H a positive constant not depending on h. Consequently the term $E_2(h)$ is bounded by

$$E_2(h) \le C h^{2H+2-\beta} \int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2} \right)^{H+\frac{1}{2}} \mu(d\xi) \tag{2.62}$$

and this is clearly finite due to (2.53). Relations (2.60) and (2.62) give the first part of the conclusion.

Let us analyze now the lower bound of the increments of $u(t,x)$ with respect to the variable t. Let $h > 0, x \in [-M,M]^d$ and $t \in [t_0, T]$ such that $t+h \in [t_0, T]$. From the decomposition

$$\begin{aligned} \mathbf{E}\big|u(t+h,x) - u(t,x)\big|^2 = &\ \big\|(g_{t+h,x} - g_{t,x}) 1_{[0,t]}\big\|^2_{\mathcal{HP}} + \|g_{t+h,x} 1_{[t,t+h]}\|^2_{\mathcal{HP}} \\ &+ 2\big\langle (g_{t+h,x} - g_{t,x}) 1_{[0,t]}, g_{t+h,x} 1_{[t,t+h]} \big\rangle_{\mathcal{HP}} \end{aligned}$$

we immediately obtain, since the second summand on the right-hand side is positive,

$$\begin{aligned} \mathbf{E}\big|u(t+h,x) - u(t,x)\big|^2 \ge &\ \big\|(g_{t+h,x} - g_{t,x}) 1_{[0,t]}\big\|^2_{\mathcal{HP}} \\ &+ 2\big\langle (g_{t+h,x} - g_{t,x}) 1_{[0,t]}, g_{t+h,x} 1_{[t,t+h]} \big\rangle_{\mathcal{HP}} \\ := &\ E_{1,t}(h) + E_{3,t}(h). \end{aligned}$$

We can assume, without any loss of the generality, that $t = \frac{1}{2}$. Let $E_{1,\frac{1}{2}}(h) := E_1(h)$. We first prove that

$$E_1(h) \ge c h^{2H+1-\beta} - c' h^{2H+2-\beta} \tag{2.63}$$

for h small enough. Recall that we have an exact expression for $E_1(h)$ (see (2.58)). Indeed,

$$E_1(h) = \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^2 \big\|\cos(\cdot) 1_{(h|\xi|, h|\xi|+|\xi|)}\big\|^2_{\mathcal{H}}$$

$$
\begin{aligned}
&= \alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^2 \\
&\quad \times \int_{h|\xi|}^{(1+h)|\xi|} \int_{h|\xi|}^{(1+h)|\xi|} du dv |u-v|^{2H-2} \cos u \cos v \\
&= \alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^2 \\
&\quad \times \int_0^{|\xi|} \int_0^{|\xi|} du dv \cos\big(u+h|\xi|\big) \cos\big(v+h|\xi|\big) |u-v|^{2H-2}.
\end{aligned}
$$

By the trigonometric formula $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ we have

$$
\begin{aligned}
E_1(h) &= \alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^2 \\
&\quad \times \Big[\cos\big(h|\xi|\big)^2 \int_0^{|\xi|} \int_0^{|\xi|} du dv \cos u \cos v |u-v|^{2H-2} \\
&\quad - 2\sin\big(h|\xi|\big)\cos\big(h|\xi|\big) \int_0^{|\xi|} \int_0^{|\xi|} du dv \sin u \cos v |u-v|^{2H-2} \\
&\quad + \sin\big(h|\xi|\big)^2 \int_0^{|\xi|} \int_0^{|\xi|} du dv \sin u \sin v |u-v|^{2H-2} \Big] \\
&:= A + B + C.
\end{aligned}
$$

We will neglect the first term since it is positive. We will bound the second term above by $ch^{2H+2-\beta}$. Again using trigonometric identities, Exercise 1.14 (used at the third line below), and the change of variables $\tilde{v} = u - v$ we have

$$
\begin{aligned}
&-2\sin\big(h|\xi|\big)\cos\big(h|\xi|\big) \int_0^{|\xi|} \int_0^{|\xi|} du dv \sin u \cos v |u-v|^{2H-2} \\
&\quad = -\sin\big(h|\xi|\big)\cos\big(h|\xi|\big) \int_0^{|\xi|} \int_0^{|\xi|} du dv \big(\sin(u+v) + \sin(u-v)\big) |u-v|^{2H-2} \\
&\quad = -\sin\big(h|\xi|\big)\cos\big(h|\xi|\big) \int_0^{|\xi|} \int_0^{|\xi|} du dv \sin(u+v) |u-v|^{2H-2} \\
&\quad = c \cdot \sin\big(h|\xi|\big)\cos\big(h|\xi|\big) \int_0^{|\xi|} v^{2H-2} \big(\cos\big(2|\xi| - v\big) - \cos(v)\big) dv
\end{aligned}
$$

and thus

$$
\begin{aligned}
B &= c \cdot \alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^2 \sin\big(h|\xi|\big)\cos\big(h|\xi|\big) \\
&\quad \times \int_0^{|\xi|} v^{2H-2} \big(\cos\big(2|\xi| - v\big) - \cos(v)\big) dv
\end{aligned}
$$

$$
\begin{aligned}
&= -c \cdot \alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^3 \cos\big(h|\xi|\big) \sin\big(|\xi|\big) \\
&\quad \times \int_0^{|\xi|} v^{2H-2} \sin\big(|\xi| - v\big) dv \\
&= -c \cdot \alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^3 \cos\big(h|\xi|\big) \sin\big(|\xi|\big) \\
&\quad \times \int_0^{|\xi|} v^{2H-2} \big(\sin\big(|\xi|\big)\cos(v) - \cos\big(|\xi|\big)\sin(v)\big) dv \\
&= -c \cdot \alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^3 \cos\big(h|\xi|\big) \sin\big(|\xi|\big) \\
&\quad \times \left(\sin\big(|\xi|\big) \int_0^{|\xi|} v^{2H-2} \cos(v) dv - \cos\big(|\xi|\big) \int_0^{|\xi|} v^{2H-2} \sin(v) dv \right) \\
&= -c \cdot \alpha_H \int_{|\xi| \le 1} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^3 \cos\big(h|\xi|\big) \sin\big(|\xi|\big) \\
&\quad \times \left(\sin\big(|\xi|\big) \int_0^{|\xi|} v^{2H-2} \cos(v) dv - \cos\big(|\xi|\big) \int_0^{|\xi|} v^{2H-2} \sin(v) dv \right) \\
&\quad - c \cdot \alpha_H \int_{|\xi| \ge 1} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \sin\big(h|\xi|\big)^3 \cos\big(h|\xi|\big) \sin\big(|\xi|\big) \\
&\quad \times \left(\sin\big(|\xi|\big) \int_0^{|\xi|} v^{2H-2} \cos(v) dv - \cos\big(|\xi|\big) \int_0^{|\xi|} v^{2H-2} \sin(v) dv \right).
\end{aligned}
$$

Taking the absolute value we see that the part over the set $|\xi| \le 1$ is bounded by ch^3 simply by majorizing $\sin(h|\xi|)$ by $h|\xi|$, $\cos(h|\xi|)\sin(|\xi|)$ by one, and

$$
\left| \sin\big(|\xi|\big) \int_0^{|\xi|} v^{2H-2} \cos(v) dv - \cos\big(|\xi|\big) \int_0^{|\xi|} v^{2H-2} \sin(v) dv \right|
$$

by a constant. For the part over the region $|\xi| \ge 1$ we again bound the last expression by a constant and we use the change of variables $\tilde{\xi} = h\xi$. This part will by bounded by

$$
\begin{aligned}
& h^{2H+2-\beta} \int_{|\xi| \ge h} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \left| \sin\big(|\xi|\big)^3 \cos\big(|\xi|\big) \sin\big(|\xi|/h\big) \right| \\
&\quad \le h^{2H+2-\beta} \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2H+2}} \left| \sin\big(|\xi|\big)^3 \right| \\
&\quad \le ch^{2H+2-\beta}
\end{aligned}
$$

since the last integral is convergent at infinity by bounding sine by one and at zero by bounding $\sin(x)$ by x and using the assumption $\beta > 2H - 1$. Therefore

$$B \le ch^{2H+2-\beta}. \tag{2.64}$$

We now bound the summand C below. In this summand the $\mathcal{H}$ norm of the sine function appears and this has been analyzed in [14]. We have, after the change of variables $\tilde{u} = \frac{u}{|\xi|}$, $\tilde{v} = \frac{v}{|\xi|}$,

$$\begin{aligned} C &= \alpha_H \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{d-\beta+2}} \sin\big(h|\xi|\big)^4 \int_0^1 \int_0^1 \sin\big(u|\xi|\big) \sin\big(v|\xi|\big) |u-v|^{2H-2} du dv \\ &\ge \alpha_H \int_{|\xi|\ge 1} \frac{d\xi}{|\xi|^{d-\beta+2}} \sin\big(h|\xi|\big)^4 \int_0^1 \int_0^1 \sin\big(u|\xi|\big) \sin\big(v|\xi|\big) |u-v|^{2H-2} du dv. \end{aligned}$$

We will use the proof of Proposition 2.14. For h small, we will have that

$$\begin{aligned} C &\ge \alpha_H \int_{|\xi|\ge 1} \frac{d\xi}{|\xi|^{d-\beta}} \sin\big(h|\xi|\big)^4 \frac{1}{|\xi|^2} \int_0^1 \int_0^1 \sin\big(u|\xi|\big) \sin\big(v|\xi|\big) |u-v|^{2H-2} du dv \\ &\ge \alpha_H \int_{|\xi|\ge 1} \frac{d\xi}{|\xi|^{d-\beta}} \sin\big(h|\xi|\big)^4 \frac{1}{|\xi|^{2H+1}} \\ &= \alpha_H h^{2H+1-\beta} \int_{|\xi|\ge h} \frac{d\xi}{|\xi|^{d-\beta+2H+1}} \sin\big(|\xi|\big)^4 \\ &\ge \alpha_H h^{2H+1-\beta} \int_{|\xi|\ge 1} \frac{d\xi}{|\xi|^{d-\beta+2H+1}} \sin\big(|\xi|\big)^4 \\ &= c \cdot \alpha_H h^{2H+1-\beta}. \end{aligned} \tag{2.65}$$

Relations (2.64) and (2.65) imply (2.63). Now, from relation (2.63), for every $t_0 \le s < t < T$ with s, t close enough

$$E_1(t-s) \ge c(t-s)^{2H+1-\beta} - c'(t-s)^{2H+2-\beta} \ge \frac{c}{2}(t-s)^{2H+1-\beta}$$

if $|t-s| \le \frac{c}{2c'}$. To extend the above inequality to arbitrary values of $|t-s|$, we proceed as in [64], proof of Proposition 4.1. Notice that the function $g(t,s,x,y) := \mathbf{E}|u(t,x) - u(s,x)|^2$ is positive and continuous with respect to all its arguments and therefore it is bounded below on the set $\{(t,s,x,y) \in [t_0,T]^2 \times [-M,M]^{2d};\ |t-s| \ge \varepsilon\}$ by a constant depending on $\varepsilon > 0$. Hence for $|t-s| \ge \frac{c}{2c'}$ it also holds that

$$E_1(t-s) \ge c_1 |t-s|^{2H+1-\beta}.$$

On the other hand, from (2.57) and (2.62) and the Cauchy-Schwarz inequality, we obtain

$$E_{3,t}(h) = \big\langle (g_{t+h,x} - g_{t,x}) 1_{[0,t]}, g_{t+h,x} 1_{[t,t+h]} \big\rangle_{\mathcal{HP}}$$

$$\leq \left\| (g_{t+h,x} - g_{t,x}) 1_{[0,t]} \right\|_{\mathcal{HP}} \| g_{t+h,x} 1_{[t,t+h]} \|_{\mathcal{HP}}$$
$$\leq ch^{\frac{2H+1-\beta}{2} + \frac{2H+2-\beta}{2}}.$$

Consequently,

$$\mathbf{E}\left| u(t+h,x) - u(t,x) \right|^2 \geq Ch^{2H+1-\beta} - C'h^{\frac{2H+1-\beta}{2} + \frac{2H+2-\beta}{2}}$$

and this implies that for every $s, t \in [t_0, T]$ and $x \in [-M, M]^d$

$$\mathbf{E}\left| u(t,x) - u(s,x) \right|^2 \geq \frac{C}{2} |t-s|^{2H+1-\beta} \quad \text{if } |t-s| \leq \left(\frac{C}{2C'} \right)^{\frac{1}{2}}.$$

Similarly as above, the previous inequality can be extended to arbitrary values of $s, t \in [t_0, T]$. □

Proposition 2.16 implies the following Hölder property for the solution to (2.35).

Corollary 2.8 *Assume* (2.55). *Then for every $x \in \mathbb{R}^d$ the application*

$$t \to u(t,x)$$

is almost surely Hölder continuous of order $\delta \in (0, \frac{2H+1-\beta}{2})$.

Proof This is consequence of the relations (2.57) and (2.62) in the proof of Proposition 2.16 and of the fact that u is Gaussian. □

Let us make some remarks on the result in Proposition 2.16.

Remark 2.22

- Following the proof of Theorem 5.1 in [63] we can show that the mapping $t \to u(t,x)$ is not Hölder continuous of order $\frac{2H+1-\beta}{2}$.
- When H is close to $\frac{1}{2}$ we retrieve the regularity in time of the solution to the wave equation with white noise in time (see [63, 64]).

2.8 Bibliographical Notes

The study of stochastic partial differential equations (SPDEs) driven by a Gaussian noise which is white in time and has a non-trivial correlation structure in space (called "color") now constitutes a classical line of research. These equations represent an alternative to the standard SPDEs driven by a space-time white noise. A first step in this direction was made in [60], where the authors identify the necessary and sufficient conditions for the existence of the solution to the stochastic wave equation (in spatial dimension $d = 2$), in the space of real-valued stochastic processes.

The fundamental reference in this area is Dalang's seminal article [59], in which the author gives the necessary and sufficient conditions under which various SPDEs with a white-colored noise (e.g. the wave equation, the damped heat equation, the heat equation) have a process solution, in arbitrary spatial dimension. The methods used in this article exploit the temporal martingale structure of the noise, and cannot be applied when the noise is "colored" in time. Other related references are, among others: [61, 120, 143, 190] and [62]. The development of stochastic calculus with respect to fractional Brownian motion naturally led to the study of SPDEs driven by this Gaussian process. The motivation comes from the wide area of applications of fBm. We refer, among other references, to [84, 119, 139, 150] and [170] for theoretical studies of SPDEs driven by fBm and to [51] or [140] for the sample paths properties of the solution. To list only a few examples of the appearance of fractional noises in practical situations, we mention [103] for biophysics, [25] for financial time series, [66] for electrical engineering, and [42] for physics.

2.9 Exercises

Exercise 2.1 Let u be the solution of the heat equation with space-time white noise. Show that there exist two positive constants C_1, C_2 such that for every $s, t \in [0, T]$

$$C_1|t-s|^{\frac{1}{2}} \le \mathbf{E}\big|u(t,x)-u(s,y)\big|^2 \le C_2|t-s|^{\frac{1}{2}}.$$

Study the variations of this process.

Hint Use the fact that u has the same law as a bi-fBm, modulo a constant.

Exercise 2.2 ([51]) Let u be the solution of the fractional-(Riesz) colored wave equation. Let us denote by Δ the following metric on $[0, T] \times \mathbb{R}^d$

$$\Delta\big((t,x);(s,y)\big) = |t-s|^{2H+1-\beta} + |x-y|^{2H+1-\beta}. \tag{2.66}$$

Fix $M > 0$ and assume (2.55). Prove that for every $t, s \in [t_0, T]$ and $x, y \in [-M, M]^d$ there exist positive constants C_1, C_2 such that

$$C_1\Delta\big((t,x);(s,y)\big) \le \mathbf{E}\big|u(t,x)-u(s,y)\big|^2 \le C_2\Delta\big((t,x);(s,y)\big).$$

Exercise 2.3 ([13]) Consider the linear heat equation with white-colored noise where the spatial covariance is given by the heat kernel (Example 2.4) or by the Poisson kernel (Example 2.3). Give the necessary and sufficient conditions in terms of d and α for the existence of the solution.

Exercise 2.4 Consider the linear heat equation with white-colored noise where the spatial covariance is given by the Riesz or Bessel kernel. Prove that the solution is Hölder continuous with respect to time of order $0 < \delta < \frac{1}{2} - \frac{d-\alpha}{2}$.

Exercise 2.5 Consider the Gaussian processes with covariances given by (2.27) and (2.28) respectively. Prove that these processes are self-similar and give the self-similarity order.

Exercise 2.6 Consider the heat equation with fractional-colored noise and spatial covariance given by a kernel f. If f is the heat kernel of order α, or the Poisson kernel of order α, then prove that the solution exists for any $H > 1/2$ and $d \geq 1$.

Exercise 2.7 Let f be the Riesz kernel of order $\alpha \in (0, d)$, and set

$$I_t = \alpha_H \int_{\mathbb{R}^d} d\xi \, |\xi|^{-\alpha-2H-2} \int_{\mathbb{R}} \frac{|\tau|^{-(2H-1)}}{(\tau^2-1)^2} \left[f_t^2\big(|\xi|, \tau\big) + g_t^2\big(|\xi|, \tau\big) \right] d\tau$$

with f_t, g_t given by (2.46).

1. Show that

$$I_t = 2\alpha_H c_d \int_{\mathbb{R}} \frac{|\tau|^{-(2H-1)}}{(\tau^2-1)^2} \left(\int_0^\infty \frac{(\sin \tau\lambda t - \tau \sin \lambda t)^2}{\lambda^2} \lambda^{-\theta} d\lambda \right.$$
$$\left. + \int_0^\infty \frac{(\cos \tau\lambda t - \cos \lambda t)^2}{\lambda^2} \lambda^{-\theta} d\lambda \right),$$

   where $\theta = \alpha + 1 - d + 2H > 0$.
2. If $\theta < 1$, show that the two integrals $d\lambda$ can be expressed in terms of the covariance functions of the odd and even parts of the fBm (see [70]).

Exercise 2.8 Consider $f(x) = \prod_{i=1}^d (\alpha_{H_i} |x_i|^{2H_i-2})$ with $H_i > 1/2$ for all $i = 1, \ldots, d$.

1. Prove that f is the Fourier transform of the measure $\mu(d\xi) = \prod_{i=1}^d (c_{H_i} \times |\xi_i|^{-(2H_i-1)})$.
2. Prove that (2.53) is equivalent to $\sum_{i=1}^d (2H_i - 1) > d - 2H - 1$.

Hint This can be seen by using the change of variables to polar coordinates.

Exercise 2.9 Prove that the solutions to the heat and wave equation with white or fractional noise in time and with white or colored noise in space are all continuous with respect to the space variable.

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