

Chapter 2

Analytical Treatment of Laser Forming and Welding Processes

The analytical treatment of laser forming and welding is categorized into two groups. In the first group, laser heating and melting is presented and in the second group laser solid phase heating, melting, and phase change is included. Due to the difficulties associated with the surface evaporation, stationary source consideration will be incorporated in the second group. The study is extended to include the analytical treatment of the thermal stress development in the laser irradiated region.

2.1 Analytical Formulation of Heating and Stress Field Involving Melting and Solid Heating

The analytical treatment of laser heating of sheet metal surfaces can extend to include the moving heat source consideration. Since the heating problem involves with melting and the heating time is much longer than the thermalization time of the substrate material, the Fourier heating law can be used to formulate the heating situation. The heating process is formulated below in line with the previous studies [1].

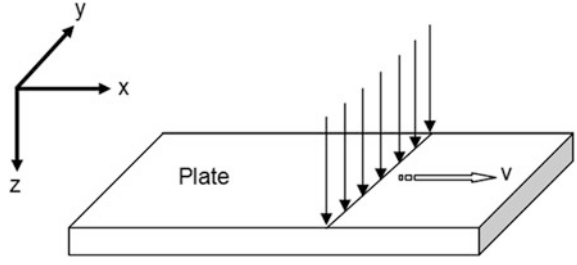
The transient heat transfer equation, which is appropriate to a line heat source moving at a constant speed v along x -direction on a thin plate surface (Fig. 2.1), can be written as:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 2\lambda \frac{\partial T}{\partial t} \quad (2.1)$$

where

$$2\lambda = \frac{\rho C_p}{k} \quad (2.2)$$

Fig. 2.1 Schematic view of laser heat source



Consider the variable $\xi = x - vt$ and introducing into Eq. 2.1 results in:

$$\frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} - 2\lambda v \frac{\partial T}{\partial \xi} + 2\lambda \frac{\partial T}{\partial t} = 0 \quad (2.3)$$

If a solid is of sufficient length, temperature distribution around the heat source becomes constant with time, i.e. $\frac{\partial T}{\partial t} \approx 0$. Equation 2.3, then, becomes:

$$\frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = -2\lambda v \frac{\partial T}{\partial \xi} \quad (2.4)$$

Let the solution of Eq. 2.4 is:

$$T = e^{-(\lambda v \xi)} \cdot F(\xi, y, z) \quad (2.5)$$

Substituting Eq. 2.5 into Eq. 2.4 results in:

$$\nabla^2 F - (\lambda v)^2 F = 0 \quad (2.6)$$

Since the thickness of the plate is small and it is, therefore, acceptable to consider no heat flow in the z -direction (i.e. temperature in the z -direction does not alter), then:

$$\frac{\partial T}{\partial z} = 0 \quad (2.7)$$

The relevant boundary conditions for Eq. (2.4) are as follows.

At infinitely long distance away from the laser source, no heat transfer takes place, i.e. temperature gradient in both ξ and y directions become zero, i.e.:

$$\frac{\partial T}{\partial \xi} = 0 \quad \text{at } \xi = \pm\infty \quad (2.8)$$

and

$$\frac{\partial T}{\partial y} = 0 \quad \text{at } y = \pm\infty \quad (2.9)$$

Now consider the cylindrical coordinates. The heat source with its radius r can be written as:

$$Q = -k \frac{\partial T}{\partial r} 2\pi r \quad (2.10)$$

where Q is the rate of heat per unit length of source.

Moreover, $r = \sqrt{\xi^2 + y^2}$, then Eq. 2.6 becomes:

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r} \frac{\partial F}{\partial r} - (\lambda v)^2 F = 0 \quad (2.11)$$

The solution of Eq. 2.11 satisfies the Bessel's function of the second kind, zero order i.e.:

$$T = A e^{-(\lambda v \xi)} \cdot K_o[\lambda v r] \quad (2.12)$$

when r approaches to 0 then:

$$\frac{\partial K_o[\lambda v r]}{\partial r} = \frac{1}{r} \quad (2.13)$$

Hence the solution with condition at $Q = -k \frac{\partial T}{\partial r} 2\pi r$ is satisfied for:

$$T = \frac{Q}{2\pi k} e^{-(\lambda v \xi)} \cdot K_o[\lambda v r] \quad (2.14)$$

where k is the thermal conductivity.

Equation 2.14 is the solution of the temperature distribution in an infinite solid due to quantity of heat Q instantaneously generated at $t = 0$ at a point (x', y', z') . However, Eq. 2.14 can be reduced to an instantaneous source of heat in a substance lying in the plane $z = 0$. In this case, in the absence of radiation heat loss from the surface, Eq. 2.14 yields:

$$T = \frac{Q}{4\pi k D t} \exp\left[-\frac{x^2 + y^2}{4\alpha t}\right] \quad (2.15)$$

where D is the melt depth, which is equal to the workpiece thickness.

When the substance lies in the x - y plane ($z = 0$) and moves with a velocity u in the direction of x axis, then, the steady temperature at the point x - y due to heat flux q per unit time at the region of origin can be written as:

$$T = \frac{q}{2\pi k D} \exp\left(\frac{ux}{2\alpha}\right) K_o\left[\frac{ur}{2\alpha}\right] \quad (2.16)$$

where r is the radial distance to the point being considered (satisfying $r^2 = x^2 + y^2$), k is the thermal conductivity, D is the thickness of the workpiece, and K_o is the modified Bessel function of second kind zero order. When laser heating is considered, the heat flux per unit time becomes:

$$q = (1 - R_f)I_o \quad (2.17)$$

where R_f is the surface reflectivity and I_o is the incident laser power .

As temperature reaches the melting temperature, melting isotherm develops at the surface. The temperature of the melting isotherm can be expressed as:

$$T_m = \frac{(1 - R_f)I_o}{2\pi kD} \exp\left(\frac{ux}{2\alpha}\right) K_o\left[\frac{ur}{2\alpha}\right] \quad (2.18)$$

Introducing dimensionless coordinates as:

$$X = \frac{ux}{2\alpha}; \quad Y = \frac{uy}{2\alpha}; \quad R = \frac{ur}{2\alpha} \quad (2.19)$$

Equation 2.18 becomes:

$$T_m = \frac{(1 - R_f)I_o}{2\pi kD} \exp(X) K_o[R] \quad (2.20)$$

To determine the maximum melt width, re-arrangement of Eq. 2.20 is necessary, i.e. it yields:

$$\exp(X) = \frac{2\pi kDT_m}{(1 - R_f)I_o} \frac{1}{K_o[R]} \quad (2.21)$$

or

$$K_o\left[\sqrt{X^2 + Y^2}\right] = \frac{2\pi kDT_m}{(1 - R_f)I_o} \exp(-X) \quad (2.22)$$

where $R = \sqrt{X^2 + Y^2}$.

Differentiating with respect to X gives:

$$-K_1\left[\sqrt{X^2 + Y^2}\right] \left(X + Y \frac{dY}{dX}\right) (X^2 + Y^2)^{-1/2} = -\frac{2\pi kDT_m}{(1 - R_f)I_o} \exp(-X) \quad (2.23)$$

where $dY/dX = 0$ then:

$$\frac{XK_1[R]}{R} = \frac{2\pi kDT_m}{(1 - R_f)I_o} \exp(-X) \quad (2.24)$$

where $K_1[R]$ is the modified Bessel function of second kind first order. Using Eq. 2.24, it results:

$$\frac{XK_1[R]}{R} = K_o[R] \quad (2.25)$$

Therefore, for maximum Y :

$$X = \frac{RK_o[R]}{K_1[R]} \quad (2.26)$$

and

$$K_o[R] = \frac{2\pi kDT_m}{(1 - R_f)I_o} \exp\left[-\frac{RK_o[R]}{K_1[R]}\right] \quad (2.27)$$

Solution of Eq. 2.27 gives the values of R for melt at the point of maximum a width. Y can be found from:

$$Y = R \sin \varphi \quad (2.28)$$

where

$$\varphi = \cos^{-1}\left(\frac{K_o[R]}{K_1[R]}\right) \quad (2.29)$$

A computer program was developed to compute the relevant melting parameters.

The stress analysis associated with the laser forming process can be briskly described in line with the previous study [2]. Laser bending angle can be formulated for the plane sheet metal after considering the equivalent nodal forces and the change of the momentum of the plate during the bending process due to laser heating. Consider the plane sheet, which undergoes bending due to the thermal load. In line with the previous study [2], the transverse bending moment per unit length can be written as:

$$M_y = \int_{-h/2}^{d-h/2} \sigma_y z dz \quad (2.30)$$

where $\sigma_y = E\varepsilon(z)$. The solution of Eq. 2.30 is straight forward and it yields:

$$M_y = E \varepsilon_{\max} \left(\frac{(x)h(x)}{8} - \frac{d(x)^2}{3} \right) \quad (2.31)$$

The transverse shrinkage force per unit length is:

$$f_y = \int_{-h/2}^{d-h/2} \sigma_y dz \quad (2.32)$$

The solution of Eq. 2.32 yields:

$$f_y = \frac{\pi E \varepsilon_{\max}}{4} d(x) \quad (2.33)$$

The variables in Eqs. 2.31–2.33 are d is the depth of heat affected zone (HAZ). $\varepsilon_{\max}$ is the maximum plastic strain occurring at the heated surface

($\varepsilon_{\max} = \alpha_{th}T_{\max} - \sigma_y/E$, where α_{th} is thermal expansion coefficient, $T_{\max}$ is the maximum temperature increase, E is the elastic modules, and σ_y is the yield stress).

Since the bending angle varies along the x -axis, the bending curvature at a location with $h(x)$ thickness can be used to calculate the bending angle, i.e.:

$$\frac{1}{\rho} = \frac{M_y}{D} = \frac{12(1 - \nu^2)}{Eh(x)^3} M_y \quad (2.34)$$

where the bending rigidity per unit length (D) is:

$$D = \frac{E}{1 - \nu^2} \int_0^{h(x)} z^2 dz = \frac{Eh(x)^3}{12(1 - \nu^2)} \quad (2.35)$$

Therefore, the bending angle (α_B) of the plate subjected to a laser heating can be written as [2]:

$$\alpha_B = b(1 - \nu^2)\varepsilon_{\max} \left(\frac{3d\pi}{2h^2} - \frac{4d^2}{h^3} \right) \quad (2.36)$$

where b is the width of laser scanned area at the workpiece surface, ν is the Poisson's ratio, $\varepsilon_{\max}$ is the maximum plastic strain ($\varepsilon_{\max} = \alpha_{th}T_{\max} - \sigma/E$, where α_{th} is the thermal expansion coefficient, $T_{\max}$ is the maximum temperature, which can be obtained from Eq. 2.16, σ is the yield stress, and E is the elastic modulus), d is the depth of heat affected zone, h is the thickness of the plate. Equation 2.36 can be used to predict the bending angle.

2.2 Analytical Formulation of Heating and Stress Field Involving Evaporation and Solidification

Laser deep penetration welding involves with surface evaporation. In this case, material subjected to the laser irradiation evaporates at velocity equals to the vapor front velocity. However, the melt surface recesses towards the solid bulk with a recession velocity, which is much lower than the vapor front velocity. Since the heating situation is complex, the analytical solution to evaporation process with a moving heat source becomes difficult. Consequently, surface evaporation is treated here after considering the stationary volumetric heat source in line with the previous study [3].

The transient heat transfer equation pertinent to the evaporation at the surface including the volumetric heat source term can be written as:

$$k \frac{\partial^2 T}{\partial x^2} + \rho C_p V \frac{\partial T}{\partial x} + I_0 \delta \exp(-\delta x) = \rho C_p \frac{\partial T}{\partial t} \quad (2.37)$$

where V is the vapor front velocity and can be written as [4]:

$$V = \left(\frac{k_B T_s}{2\pi m} \right)^{1/2} \exp \left(-\frac{L}{k_B T_s} \right) \quad (2.38)$$

It can be observed that the transient heating problem is non-linear, since the velocity is surface temperature (T_s) dependent. Consequently, closed form solution for Eq. 2.37 is extremely difficult; however, the quasi-steady solution can be possible. Since the evaporation takes place at the surface, energy dissipated via evaporation can be considered as the boundary condition at the surface. As the depth below the surface increases, temperature is assumed to reduce the initial temperature, which is 0. In addition, initially the substrate material is assumed to be at uniform constant temperature. Therefore, the boundary and initial conditions for Eq. 2.37 are:

$$k \frac{\partial T}{\partial x} \Big|_{x=0} = \rho V L; \quad T(\infty, t) = 0 \quad \text{and} \quad T(x, 0) = 0. \quad (2.39)$$

The solution of Eq. 2.37 with the appropriate boundary conditions can be obtain using a Laplace transformation method. In this case, the Laplace transformation of Eq. 2.37 yields:

$$\frac{d^2 \bar{T}(x, p)}{dx^2} + \frac{V}{\alpha} \frac{d\bar{T}(x, p)}{dx} - \frac{p}{\alpha} \bar{T}(x, p) = -\frac{I_0 \delta}{kp} \exp(-\delta x) \quad (2.40)$$

The Laplace transformation of boundary and initial conditions becomes:

$$T(x, 0) = 0; \quad \frac{dT(x, p)}{dx} \Big|_{x=0} = \frac{\rho V L}{kp}; \quad T(\infty, p) = 0 \quad (2.41)$$

where p is the Laplace transform variable.

The solution to Eq. 2.40 yields the result in the Laplace domain:

$$\begin{aligned} \bar{T}(x, p) = & A \exp \left[-\frac{x}{\sqrt{\alpha}} (b - \sqrt{b^2 + p}) \right] + B \exp \left[-\frac{x}{\sqrt{\alpha}} (b + \sqrt{b^2 + p}) \right] \\ & - \frac{I_0 \delta \alpha}{kp} \left[\frac{\exp(-\delta x)}{c^2 - (b^2 + p)} \right] \end{aligned} \quad (2.42)$$

where $b = \frac{V}{2\sqrt{\alpha}}$, $c = b - \delta\sqrt{\alpha}$ and A and B are constants of integration. Using the boundary condition (Eq. 2.41), it becomes:

$$A = 0 \quad (2.43)$$

The substitution of boundary condition into Eq. 2.42 results:

$$B = \frac{\sqrt{\alpha}}{b + \sqrt{b^2 + p}} \left[\frac{I_0 \alpha \delta^2}{kp(c^2 - b^2 - p)} - \frac{\rho V L}{kp} \right] \quad (2.44)$$

Therefore, the complete solution to the transformed equation (Eq. 2.42) becomes:

$$\begin{aligned} \bar{T}(x, p) = & \frac{\sqrt{\alpha}}{b + \sqrt{b^2 + p}} \left[\frac{I_0 \alpha \delta^2}{kp(c^2 - b^2 - p)} - \frac{\rho VL}{kp} \right] \\ & \times \exp \left[-\frac{x}{\sqrt{\alpha}}(b + \sqrt{b^2 + p}) \right] - \frac{I_0 \delta \alpha}{kp} \left[\frac{\exp(-\delta x)}{c^2 - (b^2 + p)} \right] \end{aligned} \quad (2.45)$$

The difficulty arises due to the first term in Eq. 2.45 during the inversion of the solution in the transformed plane. However, the problem may be tackled after writing the first term in the indefinite integral form, hence:

$$f(x, p) = - \int_0^x \left[\frac{I_0 \alpha \delta^2}{kp(c^2 - b^2 - p)} - \frac{\rho VL}{kp} \right] \exp \left[-\frac{x}{\sqrt{\alpha}}(b + \sqrt{b^2 + p}) \right] dx \quad (2.46)$$

or

$$f(x, p) = - \int_0^x g(x, p) dx \quad (2.47)$$

The inverse transformation of $f(x, p)$ function may be carried out in the following manner:

$$L^{-1}[f(x, p)] = -L^{-1} \left[\int_0^x g(x, p) dx \right] = - \int_0^x L^{-1}[g(x, p)] dx \quad (2.48)$$

where L^{-1} is the inverse Laplace transformation. The result can be obtained after introducing the method of expansion into partial fractions. Using the relationship:

$$L^{-1}[\varphi(p + a)] = e^{-at} L^{-1}[\varphi(p)] \quad (2.49)$$

it yields:

$$\begin{aligned} L^{-1} \left[\frac{\sqrt{\alpha}}{b + \sqrt{b^2 + p}} \frac{I_0 \alpha \delta^2}{kp(c^2 - b^2 - p)} \exp \left[-\frac{x}{\sqrt{\alpha}}(b + \sqrt{b^2 + p}) \right] \right] \\ = - \exp \left[-\left(\frac{bx}{\sqrt{\alpha}} + b^2 t \right) \right] L^{-1} \left[\frac{I_0 \alpha \delta^2 \exp(-qx)}{k(p - b^2)(\rho - c^2)(q + b/\sqrt{\alpha})} \right] \end{aligned} \quad (2.50)$$

where $q^2 = p/\alpha$, $b = V/(2\sqrt{\alpha})$ and $c = b - \delta\sqrt{\alpha}$.

This expression may be expanded into partial fractions using the residual theorem:

$$-\frac{I_0 \delta^2}{k\alpha} \exp\left[-\left(\frac{bx}{\sqrt{\alpha}} + b^2 t\right)\right] L^{-1} \left\{ \begin{aligned} & -\frac{\alpha\sqrt{\alpha}}{2b(b^2 - c^2)} \frac{\exp(-qx)}{(q + b/\sqrt{\alpha})^2} - \frac{\alpha^2(5b^2 - c^2)}{4b^2(b^2 - c^2)} \frac{\exp(-qx)}{(q + b/\sqrt{\alpha})} \\ & + \frac{\alpha^2 \exp(-qx)}{4b^2(b^2 - c^2)(q + b/\sqrt{\alpha})} + \frac{\alpha^2 \exp(-qx)}{4b^2(b^2 - c^2)(q - b/\sqrt{\alpha})} \\ & + \frac{\alpha^2 \exp(-qx)}{2c(b + c)(b^2 - c^2)(q - c/\sqrt{\alpha})} + \frac{\alpha^2 \exp(-qx)}{2c(b - c)(b^2 - c^2)(q + c/\sqrt{\alpha})} \end{aligned} \right\} \quad (2.51)$$

which gives on inversion and after much algebraic manipulations:

$$\frac{I_0 \delta \sqrt{\alpha}}{2\rho C_p (\alpha \delta - V)} \left\{ \begin{aligned} & 4t \operatorname{ierfc}\left(\frac{x}{2\sqrt{\alpha t}} + b\sqrt{t}\right) + \frac{3b^2 + c^2}{2b(b^2 - c^2)} \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + b\sqrt{t}\right) \\ & + \frac{1}{2b} \exp\left(-\frac{2bx}{\sqrt{\alpha}}\right) \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}} - b\sqrt{t}\right) \\ & - \frac{1}{b - c} \exp[-(\delta x + b^2 - c^2)t] \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + c\sqrt{t}\right) \\ & - \frac{1}{b + c} \exp\left[\frac{x}{\sqrt{\alpha}}(b + c) + (b^2 - c^2)t\right] \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}} - c\sqrt{t}\right) \end{aligned} \right\} \quad (2.52)$$

The second part of the term in the transformed solution may be inverted in a similar manner:

$$\begin{aligned} & L^{-1} \left\{ \frac{\rho VL \sqrt{\alpha}}{kp} \frac{\exp\left[-(x/\sqrt{\alpha})(b + \sqrt{p + b^2})\right]}{b + \sqrt{p + b^2}} \right\} \\ & = \frac{\rho VL}{4kb^2} \exp\left[-\left(b^2 t + \frac{bx}{\sqrt{\alpha}}\right)\right] L^{-1} \left\{ \frac{\exp(-qx)}{q - b/\sqrt{\alpha}} - \frac{\exp(-qx)}{q + b/\sqrt{\alpha}} - \frac{2b}{\sqrt{\alpha}} \frac{\exp(-qx)}{(q + b/\sqrt{\alpha})^2} \right\} \end{aligned} \quad (2.53)$$

which after transformation gives:

$$\frac{\rho VL}{4bk} \left[\begin{aligned} & 4b\sqrt{\alpha t} \operatorname{ierfc}\left(\frac{x}{2\sqrt{\alpha t}} + b\sqrt{t}\right) - \sqrt{\alpha} \left(\frac{x}{2\sqrt{\alpha t}} + b\sqrt{t}\right) \\ & + \sqrt{\alpha} \exp\left(-\frac{2bx}{\sqrt{\alpha}}\right) \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}} - b\sqrt{t}\right) \end{aligned} \right] \quad (2.54)$$

Finally the term:

$$L^{-1} \left[\frac{I_0 \alpha \delta}{k} \frac{\exp(-\delta x)}{p(p + b^2 - c^2)} \right] = \frac{I_0 \delta \sqrt{\alpha}}{\rho C_p (\alpha \delta - V)} \left\{ \frac{\exp[-(\delta x + (b^2 - c^2)t)]}{b - c} - \frac{\exp(-\delta x)}{b - c} \right\} \quad (2.55)$$

where $b - c = \delta \sqrt{\alpha}$.

Substitution of all the terms above into Eq. 2.45 gives the analytic solution to the heating problem associated with the evaporation at the surface. Therefore, the solution becomes:

$$T(x, t) = \frac{I_0 \delta \sqrt{\alpha}}{2 \rho C_p (\alpha \delta - V)} \left\{ \begin{aligned} &4 \operatorname{ierfc} \left(\frac{x}{2\sqrt{\alpha t}} + b\sqrt{t} \right) + \frac{3b^2 + c^2}{2b(b^2 - c^2)} \operatorname{erfc} \left(\frac{x}{2\sqrt{\alpha t}} + b\sqrt{t} \right) \\ &+ \frac{1}{2b} \exp \left(-\frac{2bx}{\alpha} \right) \operatorname{erfc} \left(\frac{x}{2\sqrt{\alpha t}} - b\sqrt{t} \right) \\ &+ \frac{1}{(b-c)} \exp[-(\delta x + (b^2 - c^2)t)] \operatorname{erfc} \left[-\left(\frac{x}{2\sqrt{\alpha t}} + c\sqrt{t} \right) \right] \\ &- \frac{1}{(b+c)} \exp \left[\frac{x}{\sqrt{\alpha}} (b+c) + (b^2 - c^2)t \right] \operatorname{erfc} \left(\frac{x}{2\sqrt{\alpha t}} - c\sqrt{t} \right) \\ &- \frac{2}{(b-c)} \exp(-\delta x) \end{aligned} \right\} \\ - \frac{\rho VL}{4bk} \left\{ \begin{aligned} &4b\sqrt{\alpha t} \operatorname{ierfc} \left(\frac{x}{2\sqrt{\alpha t}} + b\sqrt{t} \right) - \sqrt{\alpha} \operatorname{erfc} \left(\frac{x}{2\sqrt{\alpha t}} + b\sqrt{t} \right) \\ &+ \sqrt{\alpha} \exp \left(-\frac{2bx}{\alpha} \right) \operatorname{erfc} \left(\frac{x}{2\sqrt{\alpha t}} - b\sqrt{t} \right) \end{aligned} \right\} \quad (2.56)$$

Setting $x = 0$ in Eq. 2.56 results in the surface temperature, i.e.:

$$T(0, t) = \frac{I_0 \delta \sqrt{\alpha}}{2 \rho C_p (\alpha \delta - V)} \left[\begin{aligned} &4\sqrt{t} \operatorname{ierfc}(b\sqrt{t}) + \frac{(b^2 + c^2)}{b(b^2 - c^2)} \operatorname{erfc}(b\sqrt{t}) + \frac{1}{b} \\ &+ \frac{\exp[-(b^2 - c^2)t]}{(b-c)} \operatorname{erfc}(-c\sqrt{t}) \\ &- \frac{\exp[-(b^2 - c^2)t]}{(b+c)} \operatorname{erfc}(-c\sqrt{t}) - \frac{2}{b-c} \end{aligned} \right] \\ - \frac{\rho VL}{4bk} [4b\sqrt{\alpha t} \operatorname{ierfc}(b\sqrt{t}) - \sqrt{\alpha} \operatorname{erfc}(b\sqrt{t}) + \sqrt{\alpha} (2 - \operatorname{erfc}(b\sqrt{t}))] \quad (2.57)$$

After rearrangement it becomes:

$$T(0, t) = \frac{I_0 \delta \sqrt{\alpha}}{2 \rho C_p (\alpha \delta - V)} \left[\begin{aligned} &4\sqrt{t} \operatorname{ierfc}(b\sqrt{t}) + \frac{(b^2 + c^2)}{b(b^2 - c^2)} \operatorname{erfc}(b\sqrt{t}) \\ &+ \frac{2c}{b^2 - c^2} \exp[-(b^2 - c^2)t] \operatorname{erfc}(-c\sqrt{t}) - \frac{(b+c)}{b(b-c)} \end{aligned} \right] \\ - \frac{\rho VL}{2bk} [2b\sqrt{\alpha t} \operatorname{ierfc}(b\sqrt{t}) + \sqrt{\alpha} \operatorname{erfc}(b\sqrt{t})] \quad (2.58)$$

Equations 2.56 and 2.58 give the complete quasi-steady solution of the heat transfer equation. However, Eqs. 2.56 and 2.58 can be used to acquire a more

accurate solution which can be obtained by an iterative procedure. In this case, the values for the velocity and the surface temperature can be obtained by stepping forward in time using time steps which are small enough so that the change in the surface velocity between steps is small and therefore the velocity derived in the previous step can be used directly in Eqs. 2.56 and 2.58. With this new value of the surface temperature, an improved estimate of the surface velocity (V) can be obtained and the iteration repeated to give a convergent solution.

Equation 2.58 can be simplified assuming $\alpha\delta \gg V$, which is true for high power intensities, it may be further assumed that $t \gg L/(\alpha\delta^2)$, which in practice for metals means well in excess of 1 ps, then Eq. 2.58 reduces to:

$$T(0, t) = \frac{I_o \rho V L}{k} \left[\sqrt{\alpha t} \operatorname{ierfc} \left(\frac{V}{2} \sqrt{\frac{\alpha}{t}} \right) + \frac{\alpha}{V} \operatorname{erfc} \left(\frac{V}{2} \sqrt{\frac{t}{\alpha}} \right) \right] \quad (2.59)$$

After re-arranging Eqs. 2.58 and 2.59, it results in:

$$\frac{T(0, t)}{T(0, \infty)} \frac{V(t)}{V(\infty)} \frac{\left[1 - \frac{VL}{I_o} \right]}{\left[1 - \frac{VL}{I_o} \right]_{t=\infty}} = 2x \operatorname{ierfc}(x) + \operatorname{erf}(x) \quad (2.60)$$

where

$$x = \frac{V}{2} \sqrt{\frac{t}{\alpha}} \quad (2.61)$$

Introducing of the fraction, r , of the steady temperature which has been reached, allows Eq. 2.60 to be written as:

$$2x \operatorname{ierfc}(x) + \operatorname{erf}(x) = \frac{r^{2/3} \exp(q)}{1 + \frac{L_o}{c_p T_c} [1 - Tr^2(\infty)]^{1/2} - r^{2/3} \left[\frac{1}{r^2} - Tr^2(\infty) \right]^{1/2} - \exp(q)} \quad (2.62)$$

where

$$q = \frac{m L_o}{k T_c} \frac{1}{Tr(\infty)} [1 - Tr^2(\infty)]^{1/2} - \left[\frac{1}{r^2} - Tr^2(\infty) \right]^{1/2} \quad (2.63)$$

and

$$Tr(t) = \frac{T(0, t)}{T_c} \quad (2.64)$$

Equation 2.64 can be used to estimate the times for the surface to reach 90 % ($r = 0.90$) of its steady state temperature.

The thermal stress analysis for the heating situation associated with the evaporation surface can be reduced to the stress analysis in the solid phase. This is because of the fact that thermal stress developed in the liquid phase is almost zero.

Since the liquid layer thickness is small, the hydrostatic pressure generated in the liquid phase is negligibly small. Moreover, the surface evaporation results in convection boundary at the surface. In this case, the analytic solution for the heat transfer equation in the solid phase needs to be obtained after incorporating the convective boundary at the surface. The resulting equation is, then, used to determine the stress filed due to laser heating and surface evaporation situation. The thermal stress analysis is given below in the line of the previous study [5].

Since the solution of the thermal stress equation requires the temperature distribution in the solid, the solution of the Fourier heat transfer equation is provided below briskly for a laser pulse intensity decaying exponentially with time. The governing equation of heat transfer can be written as:

$$\frac{\partial^2 T}{\partial x^2} + \frac{I_1 \delta}{k} \exp(-\beta t) \exp(-\delta x) = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (2.65)$$

where:

$$I_1 = (1 - r_f) I_o \quad (2.66)$$

The substrate material is considered as a semi-infinite body and heated by a laser beam on the surface. The convective boundary condition is assumed be on the substrate surface. In addition, as the depth is considered to extend to infinity and the temperature to go down to zero. Heating occurs in the surface region during the laser pulse. Therefore, the corresponding boundary conditions are:

$$\begin{aligned} \text{At } x = 0 &\Rightarrow \left[\frac{\partial T}{\partial x} \right]_{x=0} = \frac{h}{k} (T(0, t) - T_0) \\ \text{At } x = \infty &\Rightarrow T(\infty, t) = 0 \end{aligned} \quad (2.67)$$

Initially substrate material is assumed to be at uniform temperature. Therefore, the initial condition is:

$$\text{At } t = 0 \Rightarrow T(x, 0) = 0 \quad (2.68)$$

The Laplace transformation of Eq. (2.65) with respect to t , results in:

$$\frac{\partial^2 \bar{T}}{\partial x^2} + \frac{I_1 \delta \exp(-\delta x)}{k(s + \beta)} = \frac{1}{\alpha} [s \bar{T}(x, s) - T(x, 0)] \quad (2.69)$$

Introducing the initial condition and rearranging the Eq. 2.69 yields:

$$\frac{\partial^2 \bar{T}}{\partial x^2} - g^2 \bar{T} = -\frac{I_1 \delta \exp(-\delta x)}{k(s + \beta)} \quad (2.70)$$

where $g^2 = s/\alpha$ and s is the Laplace transform variable. Equation 2.70 has the solution:

$$\bar{T}(x, s) = A_1 \exp(gx) + A_2 \exp(-gx) - \frac{I_1 \delta \exp(-\delta x)}{k(s + \beta)(\delta^2 - g^2)} \quad (2.71)$$

where A_1 and A_2 are constants. Introducing the boundary conditions will enable calculation of the constants A_1 and A_2 , i.e.:

$$A_1 = 0 \quad \text{and} \quad A_2 = \frac{I_1 \delta (h + \delta k)}{k(s + \beta)(\delta^2 - g^2)(h + kg)} + \frac{hT_0}{s(h + kg)} \quad (2.72)$$

After substituting the values of A_1 and A_2 in Eq. 2.71, it yields:

$$\bar{T}(x, s) = \frac{I_1 \delta (h + \delta k) e^{-gx}}{k(s + \beta)(\delta^2 - g^2)(h + kg)} + \frac{hT_0 e^{-gx}}{s(h + kg)} - \frac{I_1 \delta e^{-\delta x}}{k(\delta^2 - g^2)(s + \beta)} \quad (2.73)$$

which gives the solution for the temperature distribution in the Laplace domain.

The inverse Laplace Transform of Eq. 2.73 provides the temperature distribution within the substrate material in space (x) and time (t). The mathematical arrangements of the Laplace inversion of Eq. 2.73 are given in the previous study [6]. Therefore, the equation after the Laplace inversion is given below as:

$$T(x, t) = \frac{I_1 \delta \alpha^{3/2} (h + k\delta)}{k^2} \left\{ \begin{aligned} & - \frac{e^{(\alpha\delta^2 t - \delta x)} \operatorname{Erfc}\left(\frac{x}{2\sqrt{\alpha t}} - \delta\sqrt{\alpha t}\right)}{2(\beta + \alpha\delta^2) \left(\frac{h\sqrt{\alpha}}{k} + \delta\sqrt{\alpha}\right)} \\ & - \frac{e^{(\alpha\delta^2 t + \delta x)} \operatorname{Erfc}\left(\frac{x}{2\sqrt{\alpha t}} - \delta\sqrt{\alpha t}\right)}{2(\beta + \alpha\delta^2) \left(\frac{h\sqrt{\alpha}}{k} + \delta\sqrt{\alpha}\right)} \\ & + \frac{e^{-\beta t} e^{\sqrt{(\beta/\alpha)x}} \operatorname{Erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \sqrt{\beta t}\right)}{2(\beta + \alpha\delta^2) \left(\frac{h\sqrt{\alpha}}{k} - \sqrt{\beta t}\right)} \\ & + \frac{e^{-\beta t} e^{-\sqrt{(\beta/\alpha)x}} \operatorname{Erfc}\left(\frac{x}{2\sqrt{\alpha t}} - \sqrt{\beta t}\right)}{2(\beta + \alpha\delta^2) \left(\frac{h\sqrt{\alpha}}{k} + \sqrt{\beta t}\right)} \\ & + \frac{h\sqrt{\alpha}}{k} \frac{e^{hx/k} e^{(h^2/k^2)\alpha t} \operatorname{Erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h}{k}\sqrt{\alpha t}\right)}{\left(\frac{h^2\alpha}{k^2} + \beta\right) \left(\frac{h^2\alpha}{k^2} - \alpha\delta^2\right)} \\ & - \frac{ke^{-\delta x} (e^{-\alpha\delta^2 t} - e^{-\beta t})}{\sqrt{\alpha}(\beta + \alpha\delta^2)(h + k\delta)} \end{aligned} \right\} \\ + T_0 \left[\operatorname{Erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - e^{hx/k} e^{(h^2/k^2)\alpha t} \operatorname{Erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h}{k}\sqrt{\alpha t}\right) \right] \quad (2.74)$$

where Erfc is the complementary error function. Equation 2.74 is the closed form solution for temperature distribution. The temperature distribution can be expressed in a non-dimensional form by introducing dimensionless quantities and substituting in Eq. 2.74. The dimensionless quantities are:

$$x^* = x\delta; \quad t^* = \alpha\delta^2 t; \quad T^* = \frac{Tk\delta}{I_1}; \quad h^* = \frac{h}{\delta k}; \quad \beta^* = \beta t \quad (2.75)$$

The dimensionless temperature distribution for a full pulse is then:

$$T^*(x^*, t^*) = (h^* + 1) \left\{ \begin{aligned} & \left[-\frac{e^{-x^*} e^{t^*} \operatorname{Erfc}\left(\frac{x^*}{2\sqrt{t^*}} - \sqrt{t^*}\right)}{2(\beta^* + 1)(h^* + 1)} - \frac{e^{-x^*} e^{t^*} \operatorname{Erfc}\left(\frac{x^*}{2\sqrt{t^*}} - \sqrt{t^*}\right)}{2(\beta^* + 1)(h^* - 1)} \right. \\ & + \frac{e^{-\beta^*} e^{\sqrt{\beta^*} x^* i} \operatorname{Erfc}\left(\frac{x^*}{2\sqrt{t^*}} - \sqrt{\beta^*} t^* i\right)}{2(\beta^* + 1)(h^* - \sqrt{\beta^*} i)} + \frac{e^{-\beta^*} e^{t^*} e^{-\sqrt{\beta^*} x^* i} \operatorname{Erfc}\left(\frac{x^*}{2\sqrt{t^*}} - \sqrt{\beta^*} t^* i\right)}{2(\beta^* + 1)(h^* + \sqrt{\beta^*} i)} \\ & \left. + \frac{h^* e^{h^* x^*} e^{(h^*)^2 t^*} \operatorname{Erfc}\left(\frac{x^*}{2\sqrt{t^*}} - h^* \sqrt{t^*}\right)}{2((h^*)^2 + \beta^*)((h^*)^2 - 1)} - \frac{e^{-x^*} (e^{-t^*} - e^{-\beta^* t^*})}{(\beta^* + 1)(h^* + 1)} \right] \\ & - \left[-\frac{e^{-x^*} e^{t^*} \operatorname{Erfc}\left(\frac{x^*}{2\sqrt{t^*}} - \sqrt{t^*}\right)}{2(\gamma^* + 1)(h^* + 1)} - \frac{e^{-x^*} e^{t^*} \operatorname{Erfc}\left(\frac{x^*}{2\sqrt{t^*}} - \sqrt{t^*}\right)}{2(\gamma^* + 1)(h^* - 1)} \right. \\ & + \frac{e^{-\gamma^*} e^{\sqrt{\gamma^*} x^* i} \operatorname{Erfc}\left(\frac{x^*}{2\sqrt{t^*}} - \sqrt{\gamma^*} t^* i\right)}{2(\gamma^* + 1)(h^* - \sqrt{\gamma^*} i)} + \frac{e^{-\gamma^*} e^{t^*} e^{-\sqrt{\gamma^*} x^* i} \operatorname{Erfc}\left(\frac{x^*}{2\sqrt{t^*}} - \sqrt{\gamma^*} t^* i\right)}{2(\gamma^* + 1)(h^* + \sqrt{\gamma^*} i)} \\ & \left. + \frac{h^* e^{h^* x^*} e^{(h^*)^2 t^*} \operatorname{Erfc}\left(\frac{x^*}{2\sqrt{t^*}} - h^* \sqrt{t^*}\right)}{2((h^*)^2 + \gamma^*)((h^*)^2 - 1)} - \frac{e^{-x^*} (e^{-t^*} - e^{-\gamma^* t^*})}{(\gamma^* + 1)(h^* + 1)} \right] \end{aligned} \right\} \quad (2.76)$$

In order to solve for the stress distribution within the substrate it is possible to consider the equation governing the momentum in a one-dimensional solid for the linear elastic case, i.e.:

$$\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{1}{c_1^2} \frac{\partial^2 \sigma_x}{\partial t^2} = c_2 \frac{\partial^2 T}{\partial t^2} \quad (2.77)$$

where c_1 is the wave speed in the solid $c_1 = \sqrt{E/\rho}$ and

$$c_2 = \frac{1 + \nu}{1 - \nu} \rho \alpha_T \quad (2.78)$$

where ν is Poisson's ratio, ρ is the density of the solid and α_T is the thermal expansion coefficient of the solid.

In order to solve the momentum equation (Eq. 2.76), it is necessary to establish the initial conditions for stress and temperature fields. In this case, the substrate material is assumed to be free from stresses initially (at time = 0) and as the time extends to infinity, the stress-free state must apply in the substrate. The same initial condition for the temperature is applied as in Eq. 2.77, provided that as time approaches infinity, the temperature in the substrate material reduces to zero. This is due to the fact that the laser pulse decays exponentially with time; therefore, as time approaches infinity, the laser pulse intensity becomes zero. Therefore, the initial and boundary conditions for the stress field are:

$$\begin{aligned}
\text{At } t = 0 &\Rightarrow \sigma_x = 0 \\
\text{At } t = \infty &\Rightarrow \sigma_x = 0 \\
\text{At } x = 0 &\Rightarrow \sigma_x = 0 \\
\text{At } x = \infty &\Rightarrow \sigma_x = 0
\end{aligned} \tag{2.79}$$

Taking the Laplace Transformation of Eq. 2.76 with respect to time yields:

$$\frac{\partial^2 \bar{\sigma}_x}{\partial x^2} - \frac{1}{c_1^2} [s^2 \bar{\sigma}_x(x, s) - s\sigma_x(x, 0) - \sigma_x(x, 0)] = c_2 \left[s^2 \bar{T}(x, s) - sT(x, 0) - \dot{T}(x, 0) \right] \tag{2.80}$$

where $\bar{\sigma}_x(x, s)$ and $\bar{T}(x, s)$ are the Laplace transforms of thermal stress and temperature respectively in the x and s domains.

By substituting the initial conditions, Eq. 2.80 reduces to:

$$\frac{\partial^2 \bar{\sigma}_x}{\partial x^2} - \frac{s^2}{c_1^2} \bar{\sigma}_x(x, s) = c_2 s^2 \bar{T}(x, s) \tag{2.81}$$

Considering the temperature distribution in a Laplace domain for an exponentially decaying pulse with time, Eq. 2.73, and substituting it into Eq. 2.81, and solving for the stress field, yields:

$$\frac{\partial^2 \bar{\sigma}_x}{\partial x^2} - \frac{s^2}{c_1^2} \bar{\sigma}_x(x, s) = c_2 s^2 \left[\frac{I_1 \delta (h + \delta k) e^{-gx}}{k(s + \beta)(\delta^2 - g^2)(h + kg)} + \frac{hT_0 e^{-gx}}{s(h + kg)} - \frac{I_1 \delta e^{-\delta x}}{k(\delta^2 - g^2)(s + \beta)} \right] \tag{2.82}$$

Now let M_1 and M_2 be defined as:

$$M_1 = \frac{I_1 \delta (h + \delta k) c_2 s^2}{k(s + \beta)(\delta^2 - g^2)(h + kg)} + \frac{hT_0 c_2 s}{(h + kg)} \tag{2.83}$$

and

$$M_2 = \frac{I_1 \delta c_2 s^2}{k(\delta^2 - g^2)(s + \beta)} \tag{2.84}$$

Then Eq. 2.82 becomes:

$$\frac{\partial^2 \bar{\sigma}_x}{\partial x^2} - \frac{s^2}{c_1^2} \bar{\sigma}_x(x, s) = M_1 e^{-gx} + M_2 e^{-\delta x} \tag{2.85}$$

The complementary and the particular solutions of Eq. 2.85 are:

$$(\bar{\sigma}_x)_h = A_3 e^{\frac{sx}{c_1}} + A_4 e^{\frac{-sx}{c_1}} \tag{2.86}$$

While the particular solution has two parts, the first part is:

$$(\bar{\sigma}_x)_{p_1} = G_1 e^{-\sqrt{\frac{s}{\alpha}}x} \quad (2.87)$$

Substituting Eq. 2.87 in Eq. 2.85 yields:

$$G_1 = \frac{M_1}{g^2 - \frac{s^2}{c_1^2}} \quad (2.88)$$

The second part of the particular solution is:

$$(\bar{\sigma}_x)_{p_2} = G_2 e^{-\delta x} \quad (2.89)$$

Substituting Eq. 2.89 in Eq. 2.85 yields:

$$G_2 = \frac{M_2}{\delta^2 - \frac{s^2}{c_1^2}} \quad (2.90)$$

So, the general solution for the stress field becomes:

$$(\bar{\sigma}_x)_g = A_3 e^{\frac{sx}{c_1}} + A_4 e^{\frac{-sx}{c_1}} + G_1 e^{-gx} + G_2 e^{-\delta x} \quad (2.91)$$

From the boundary condition ($x = \infty \Rightarrow \sigma_x = 0$), this yields $A_3 = 0$.

Then, Eq. 2.91 reduces to:

$$(\bar{\sigma}_x)_g = A_4 e^{\frac{-sx}{c_1}} + G_1 e^{-\sqrt{\frac{s}{\alpha}}x} + G_2 e^{-\delta x} \quad (2.92)$$

Consider the boundary condition of the stress field at the surface, where at $x = 0 \Rightarrow \partial \bar{\sigma}_x = 0$, the constant in Eq. 2.92 becomes:

$$A_4 = -\frac{c_1}{s} \left[\sqrt{\frac{s}{\alpha}} G_1(s) + \delta G_2(s) \right] \quad (2.93)$$

Therefore, Eq. 2.92 becomes:

$$\bar{\sigma}_x(x, s) = G_1(s) e^{-\sqrt{\frac{s}{\alpha}}x} - G_1(s) \frac{c_1}{\sqrt{s\alpha}} e^{\frac{-sx}{c_1}} + G_2(s) e^{-\delta x} - G_2(s) \frac{c_1 \delta}{s} e^{\frac{-sx}{c_1}} \quad (2.94)$$

Finding the solution for σ_x in the x and t domain, one should take the inverse Laplace Transform for each term in Eq. 2.94. To do this, the following terms are introduced:

$$\begin{aligned} (\bar{\sigma}_x)_1 &= G_1(s) e^{-\sqrt{\frac{s}{\alpha}}x} & (\bar{\sigma}_x)_2 &= -G_1(s) \frac{c_1}{\sqrt{s\alpha}} e^{\frac{-sx}{c_1}} \\ (\bar{\sigma}_x)_3 &= G_2(s) e^{-\delta x} & (\bar{\sigma}_x)_4 &= -G_2(s) \frac{c_1 \delta}{s} e^{\frac{-sx}{c_1}} \end{aligned} \quad (2.95)$$

Consequently, the solution for the stress distribution is the summation of the inverse Laplace Transforms of the above terms.

Therefore, the Laplace inversion of Terms $(\bar{\sigma}_x)_1, (\bar{\sigma}_x)_2, (\bar{\sigma}_x)_3, (\bar{\sigma}_x)_4$ can be stated as follows.

$(\bar{\sigma}_x)_1$ is composed of the terms:

$$(\bar{\sigma}_x)_1 = (\bar{\sigma}_x)_{11} + (\bar{\sigma}_x)_{12} \quad (2.96)$$

where

$$(\bar{\sigma}_x)_{11} = \frac{I_1 \delta(h + \delta k) c_2}{k} \left[\frac{s e^{-\sqrt{\frac{\delta}{\alpha}} x}}{(\delta^2 - s/\alpha)(s + \beta)(h + k\sqrt{s/\alpha})(1/\alpha - s/c_1^2)} \right] \quad (2.97)$$

and

$$(\bar{\sigma}_x)_{12} = h T_0 c_2 \left[\frac{e^{-\sqrt{\frac{\delta}{\alpha}} x}}{(h + k\sqrt{s/\alpha})(1/\alpha - s/c_1^2)} \right] \quad (2.98)$$

Let

$$C_{10} = \frac{I_1 \delta(h + \delta k) c_2}{k} \quad (2.99)$$

and

$$C_{20} = h T_0 c_2 \quad (2.100)$$

Then, the inverse Laplace transformation of $(\bar{\sigma}_x)_1$ can be written as:

$$L^{-1}[(\bar{\sigma}_x)_1] = (\sigma_x)_{11} + (\sigma_x)_{12} \quad (2.101)$$

where

$$(\sigma_x)_{11} = (\sigma_x)_{111} + (\sigma_x)_{112} + (\sigma_x)_{113} + (\sigma_x)_{114} + (\sigma_x)_{115} + (\sigma_x)_{116} + (\sigma_x)_{117} + (\sigma_x)_{118} \quad (2.102)$$

These terms are:

$$(\sigma_x)_{111} = -\frac{c_1^2 h^2 k^3 \alpha^2 \sqrt{\alpha} C_{10}}{2(\alpha^2 h^2 - c_1^2 k^2)(h^2 \alpha + k^2 \beta)(h^2 - k^2 \delta^2)} \left[\frac{h\sqrt{\alpha}}{k} e^{\frac{h^2 \alpha t}{k^2} - \frac{h x}{k}} \text{Erfc}\left(-\frac{h\sqrt{\alpha t}}{k} + \frac{x}{2\sqrt{\alpha t}}\right) + \frac{h\sqrt{\alpha}}{k} e^{\frac{h^2 \alpha t}{k^2} + \frac{h x}{k}} \text{Erfc}\left(\frac{h\sqrt{\alpha t}}{k} + \frac{x}{2\sqrt{\alpha t}}\right) \right] \quad (2.103)$$

and

$$(\sigma_x)_{112} = -\frac{c_1^2 h \alpha^3 \sqrt{\beta} C_{10}}{2(\beta + \alpha \delta^2)(h^2 \alpha + k^2 \beta)(c_1^2 + \alpha \beta)} \left[\begin{aligned} &\sqrt{\beta} e^{\beta t - \sqrt{\frac{\beta}{\alpha}} x} \text{Erfc} \left(-\sqrt{\beta t} + \frac{x}{2\sqrt{\alpha t}} \right) \\ &+ \sqrt{\beta} e^{\sqrt{\frac{\beta}{\alpha}} x + \beta t} \text{Erfc} \left(\sqrt{\beta t} + \frac{x}{2\sqrt{\alpha t}} \right) \end{aligned} \right] \quad (2.104)$$

$$(\sigma_x)_{113} = -\frac{c_1^3 h \alpha^4 \sqrt{\alpha} C_{10}}{2(\alpha^2 h^2 - c_1^2 k^2)(c_1^2 + \alpha \beta)(\delta^2 \alpha^2 - c_1^2)} \left[\begin{aligned} &\frac{c_1}{\sqrt{\alpha}} e^{\frac{c_1^2 t}{\alpha} - \frac{c_1 x}{\alpha}} \text{Erfc} \left(-c_1 \sqrt{\frac{t}{\alpha}} + \frac{x}{2\sqrt{\alpha t}} \right) \\ &+ \frac{c_1}{\sqrt{\alpha}} e^{\frac{c_1^2 t}{\alpha} + \frac{c_1 x}{\alpha}} \text{Erfc} \left(c_1 \sqrt{\frac{t}{\alpha}} + \frac{x}{2\sqrt{\alpha t}} \right) \end{aligned} \right] \quad (2.105)$$

and

$$(\sigma_x)_{114} = \frac{c_1^2 h \alpha^2 \sqrt{\alpha} \delta C_{10}}{2(\beta + \alpha \delta^2)(\delta^2 \alpha^2 - c_1^2)(h^2 - k^2 \delta^2)} \left[\begin{aligned} &\sqrt{\alpha} \delta e^{\alpha \delta^2 t - \delta x} \text{Erfc} \left(-\delta \sqrt{\alpha t} + \frac{x}{2\sqrt{\alpha t}} \right) \\ &+ \sqrt{\alpha} \delta e^{\alpha \delta^2 t + \delta x} \text{Erfc} \left(\delta \sqrt{\alpha t} + \frac{x}{2\sqrt{\alpha t}} \right) \end{aligned} \right] \quad (2.106)$$

and

$$(\sigma_x)_{115} = \frac{c_1^2 h^2 k^3 \alpha^2 \sqrt{\alpha} C_{10}}{2(\alpha^2 h^2 - c_1^2 k^2)(h^2 \alpha + k^2 \beta)(h^2 - k^2 \delta^2)} \left[\begin{aligned} &\frac{2}{\sqrt{\pi t}} \exp \left(\frac{-x^2}{4\alpha t} \right) \\ &+ \frac{h\sqrt{\alpha}}{k} e^{\frac{h^2 \alpha t}{k^2} - \frac{hx}{k}} \text{Erfc} \left(-\frac{h\sqrt{\alpha t}}{k} + \frac{x}{2\sqrt{\alpha t}} \right) \\ &- \frac{h\sqrt{\alpha}}{k} e^{\frac{h^2 \alpha t}{k^2} + \frac{hx}{k}} \text{Erfc} \left(\frac{h\sqrt{\alpha t}}{k} + \frac{x}{2\sqrt{\alpha t}} \right) \end{aligned} \right] \quad (2.107)$$

and

$$(\sigma_x)_{116} = \frac{c_1^2 k \alpha^2 \sqrt{\alpha} \beta C_{10}}{2(c_1^2 + \alpha \beta)(h^2 \alpha + k^2 \beta)(\beta + \alpha \delta^2)} \left[\begin{aligned} &\frac{2}{\sqrt{\pi t}} \exp \left(\frac{-x^2}{4\alpha t} \right) \\ &+ \sqrt{\beta} e^{\beta t - x \sqrt{\frac{\beta}{\alpha}}} \text{Erfc} \left(-\sqrt{\beta t} + \frac{x}{2\sqrt{\alpha t}} \right) \\ &- \sqrt{\beta} e^{\beta t + x \sqrt{\frac{\beta}{\alpha}}} \text{Erfc} \left(\sqrt{\beta t} + \frac{x}{2\sqrt{\alpha t}} \right) \end{aligned} \right] \quad (2.108)$$

and

$$(\sigma_x)_{117} = \frac{c_1^2 k \alpha^3 \sqrt{\alpha} C_{10}}{2(\alpha^2 h^2 - c_1^2 k^2)(c_1^2 + \alpha\beta)(\delta^2 \alpha^2 - c_1^2)} \left[\begin{aligned} & \frac{2}{\sqrt{\pi t}} \exp\left(\frac{-x^2}{4\alpha t}\right) \\ & + \frac{c_1}{\sqrt{\alpha}} e^{\frac{c_1^2 t}{\alpha} - \frac{c_1 x}{\alpha}} \text{Erfc}\left(-c_1 \sqrt{\frac{t}{\alpha}} + \frac{x}{2\sqrt{\alpha t}}\right) \\ & - \frac{c_1}{\sqrt{\alpha}} e^{\frac{c_1^2 t}{\alpha} + \frac{c_1 x}{\alpha}} \text{Erfc}\left(c_1 \sqrt{\frac{t}{\alpha}} + \frac{x}{2\sqrt{\alpha t}}\right) \end{aligned} \right] \quad (2.109)$$

$$(\sigma_x)_{118} = \frac{c_1^2 k \alpha^2 \sqrt{\alpha} \delta^2 C_{10}}{2(\beta + \alpha \delta^2)(\delta^2 \alpha^2 - c_1^2)(k^2 \delta^2 - h^2)} \left[\begin{aligned} & \frac{2}{\sqrt{\pi t}} \exp\left(\frac{-x^2}{4\alpha t}\right) \\ & + \sqrt{\alpha} \delta e^{\alpha \delta^2 t - \delta x} \text{Erfc}\left(-\delta \sqrt{\alpha t} + \frac{x}{2\sqrt{\alpha t}}\right) \\ & - \sqrt{\alpha} \delta e^{\alpha \delta^2 t + \delta x} \text{Erfc}\left(\delta \sqrt{\alpha t} + \frac{x}{2\sqrt{\alpha t}}\right) \end{aligned} \right] \quad (2.110)$$

Also $(\sigma_x)_{12}$ consists of the terms:

$$(\sigma_x)_{12} = (\sigma_x)_{121} + (\sigma_x)_{122} + (\sigma_x)_{123} + (\sigma_x)_{124} \quad (2.111)$$

These terms are:

$$(\sigma_x)_{121} = \frac{c_1^2 h k \alpha \sqrt{\alpha} C_{20}}{2h(\alpha^2 h^2 - c_1^2 k^2)} \left[\begin{aligned} & \frac{h\sqrt{\alpha}}{k} e^{\frac{h^2 \alpha t}{k^2} - \frac{hx}{k}} \text{Erfc}\left(-\frac{h\sqrt{\alpha t}}{k} + \frac{x}{2\sqrt{\alpha t}}\right) \\ & + \frac{h\sqrt{\alpha}}{k} e^{\frac{h^2 \alpha t}{k^2} + \frac{hx}{k}} \text{Erfc}\left(\frac{h\sqrt{\alpha t}}{k} + \frac{x}{2\sqrt{\alpha t}}\right) \end{aligned} \right] \quad (2.112)$$

and

$$(\sigma_x)_{122} = -\frac{c_1 h \alpha^2 \sqrt{\alpha} C_{20}}{2(\alpha^2 h^2 - c_1^2 k^2)} \left[\begin{aligned} & \frac{c_1}{\sqrt{\alpha}} e^{\frac{c_1^2 t}{\alpha} - \frac{c_1 x}{\alpha}} \text{Erfc}\left(-c_1 \sqrt{\frac{t}{\alpha}} + \frac{x}{2\sqrt{\alpha t}}\right) \\ & + \frac{c_1}{\sqrt{\alpha}} e^{\frac{c_1^2 t}{\alpha} + \frac{c_1 x}{\alpha}} \text{Erfc}\left(c_1 \sqrt{\frac{t}{\alpha}} + \frac{x}{2\sqrt{\alpha t}}\right) \end{aligned} \right] \quad (2.113)$$

and

$$(\sigma_x)_{123} = -\frac{c_1^2 k \alpha \sqrt{\alpha} C_{20}}{2(\alpha^2 h^2 - c_1^2 k^2)} \left[\begin{aligned} & \frac{2}{\sqrt{\pi t}} \exp\left(\frac{-x^2}{4\alpha t}\right) \\ & + \frac{h\sqrt{\alpha}}{k} e^{\frac{h^2 \alpha t}{k^2} - \frac{hx}{k}} \text{Erfc}\left(-\frac{h\sqrt{\alpha t}}{k} + \frac{x}{2\sqrt{\alpha t}}\right) \\ & - \frac{h\sqrt{\alpha}}{k} e^{\frac{h^2 \alpha t}{k^2} + \frac{hx}{k}} \text{Erfc}\left(\frac{h\sqrt{\alpha t}}{k} + \frac{x}{2\sqrt{\alpha t}}\right) \end{aligned} \right] \quad (2.114)$$

and

$$(\sigma_x)_{124} = \frac{c_1^2 k \alpha \sqrt{\alpha} C_{20}}{2(\alpha^2 h^2 - c_1^2 k^2)} \left[\begin{aligned} & \frac{2}{\sqrt{\pi t}} \exp\left(\frac{-x^2}{4\alpha t}\right) \\ & + \frac{c_1}{\sqrt{\alpha}} e^{\frac{c_1^2 t}{\alpha} - \frac{c_1 x}{\alpha}} \text{Erfc}\left(-c_1 \sqrt{\frac{t}{\alpha}} + \frac{x}{2\sqrt{\alpha t}}\right) \\ & - \frac{c_1}{\sqrt{\alpha}} e^{\frac{c_1^2 t}{\alpha} + \frac{c_1 x}{\alpha}} \text{Erfc}\left(c_1 \sqrt{\frac{t}{\alpha}} + \frac{x}{2\sqrt{\alpha t}}\right) \end{aligned} \right] \quad (2.115)$$

The Laplace transform of $(\bar{\sigma}_x)_2$ can be written as:

$$L^{-1}[(\bar{\sigma}_x)_2] = (\sigma_x)_{21} + (\sigma_x)_{22} \quad (2.116)$$

where $(\sigma_x)_{21}$ consists of:

$$(\sigma_x)_{21} = (\sigma_x)_{211} + (\sigma_x)_{212} + (\sigma_x)_{213} + (\sigma_x)_{214} + (\sigma_x)_{215} + (\sigma_x)_{216} + (\sigma_x)_{217} + (\sigma_x)_{218} \quad (2.117)$$

Now let:

$$C_{30} = -\frac{I_1 \delta(h + \delta k) c_2 c_1}{\sqrt{\alpha} k} \quad (2.118)$$

and

$$C_{40} = -\frac{h T_0 c_2 c_1}{\sqrt{\alpha}} \quad C_{40} = -\frac{h T_0 c_2 c_1}{\sqrt{\alpha}} \quad (2.119)$$

then

$$(\sigma_x)_{211} = -\frac{c_1^2 h k^6 \alpha^2 C_{30}}{k^2 (c_1^2 k^2 - h^2 \alpha^2) (h^2 \alpha + k^2 \beta) (h^2 - k^2 \delta^2)} \left[\frac{1}{\sqrt{\pi(t-x/c_1)}} + \frac{h\sqrt{\alpha}}{k} e^{\frac{h^2 \alpha t}{k^2}} \text{Erf}\left(\frac{h\sqrt{\alpha}}{k} \sqrt{t-x/c_1}\right) \right] \cdot U\left(t - \frac{x}{c_1}\right) \quad (2.120)$$

and

$$(\sigma_x)_{212} = \frac{c_1^2 h \alpha^3 C_{30}}{(\beta + \alpha \delta^2) (h^2 \alpha + k^2 \beta) (c_1^2 + \alpha \beta)} \left[\frac{1}{\sqrt{\pi(t-x/c_1)}} + \frac{1}{\sqrt{-\beta} e^{-\beta t}} \text{Erf}\left(\sqrt{-\beta(t-x/c_1)}\right) \right] \cdot U\left(t - \frac{x}{c_1}\right) \quad (2.121)$$

and

$$(\sigma_x)_{213} = -\frac{c_1^2 h \alpha^5 C_{30}}{(\alpha^2 h^2 - c_1^2 k^2)(c_1^2 + \alpha\beta)(\delta^2 \alpha^2 - c_1^2)} \left[\frac{1}{\sqrt{\pi(t-x/c_1)}} + \frac{c_1}{\sqrt{\alpha}} e^{\frac{c_1^2}{\alpha} t} \text{Erf}\left(\frac{c_1}{\sqrt{\alpha}} \sqrt{t-x/c_1}\right) \right] \cdot U\left(t - \frac{x}{c_1}\right) \quad (2.122)$$

and

$$(\sigma_x)_{214} = \frac{c_1^2 h \alpha^2 C_{30}}{(\beta + \alpha\delta^2)(\delta^2 \alpha^2 - c_1^2)(h^2 - k^2 \delta^2)} \left[\frac{1}{\sqrt{\pi(t-x/c_1)}} + \frac{1}{\sqrt{\alpha\delta}} e^{\alpha\delta^2 t} \text{Erf}\left(\sqrt{\alpha\delta} \sqrt{t-x/c_1}\right) \right] \cdot U\left(t - \frac{x}{c_1}\right) \quad (2.123)$$

and

$$(\sigma_x)_{215} = \frac{c_1^2 k^5 \alpha^2 C_{30}}{(\alpha^2 h^2 - c_1^2 k^2)(h^2 \alpha + k^2 \beta)(h^2 - k^2 \delta^2)} \left[D\left(t - \frac{x}{c_1}\right) + \frac{h^2 \alpha}{k^2} e^{\frac{h^2 \alpha}{k^2}(t-x/c_1)} \right] \cdot U\left(t - \frac{x}{c_1}\right) \quad (2.124)$$

and

$$(\sigma_x)_{216} = -\frac{c_1^2 k \alpha^2 \sqrt{\alpha} C_{30}}{(c_1^2 + \alpha\beta)(h^2 \alpha + k^2 \beta)(\beta + \alpha\delta^2)} \left[D\left(t - \frac{x}{c_1}\right) - \beta e^{-\beta(t-x/c_1)} \right] \cdot U\left(t - \frac{x}{c_1}\right) \quad (2.125)$$

and

$$(\sigma_x)_{217} = -\frac{c_1^2 k \alpha^4 \sqrt{\alpha} C_{30}}{(\alpha^2 h^2 - c_1^2 k^2)(c_1^2 + \alpha\beta)(\delta^2 \alpha^2 - c_1^2)} \left[D\left(t - \frac{x}{c_1}\right) + \frac{c_1^2}{\alpha} e^{\frac{c_1^2}{\alpha}(t-x/c_1)} \right] \cdot U\left(t - \frac{x}{c_1}\right) \quad (2.126)$$

and

$$(\sigma_x)_{218} = \frac{c_1^2 k \alpha \sqrt{\alpha} C_{30}}{(\beta + \alpha \delta^2)(\delta^2 \alpha^2 - c_1^2)(k^2 \delta^2 - h^2)} \left[D\left(t - \frac{x}{c_1}\right) + \alpha \delta^2 e^{2\delta^2(t-x/c_1)} \right] \cdot U\left(t - \frac{x}{c_1}\right) \quad (2.127)$$

Also $(\sigma_x)_{22}$ consists of the terms:

$$(\sigma_x)_{22} = (\sigma_x)_{221} + (\sigma_x)_{222} + (\sigma_x)_{223} + (\sigma_x)_{224} + (\sigma_x)_{225} + (\sigma_x)_{226} \quad (2.128)$$

These terms are:

$$(\sigma_x)_{221} = \frac{\alpha C_{40}}{h \sqrt{\pi(t-x/c_1)}} \cdot U\left(t - \frac{x}{c_1}\right) \quad (2.129)$$

and

$$(\sigma_x)_{222} = \frac{c_1^2 k^2 \alpha C_{40}}{h(\alpha^2 h^2 - c_1^2 k^2)} \left[\frac{1}{\sqrt{\pi(t-x/c_1)}} + \frac{h \sqrt{\alpha}}{k} e^{\frac{h^2 \alpha}{k^2} t} \text{Erf}\left(\frac{h \sqrt{\alpha}}{k} \sqrt{t - \frac{x}{c_1}}\right) \right] \cdot U\left(t - \frac{x}{c_1}\right) \quad (2.130)$$

and

$$(\sigma_x)_{223} = -\frac{h \alpha^3 C_{40}}{(\alpha^2 h^2 - c_1^2 k^2)} \left[\frac{1}{\sqrt{\pi(t-x/c_1)}} + \frac{c_1}{\sqrt{\alpha}} e^{\frac{c_1^2}{\alpha} t} \text{Erf}\left(\frac{c_1}{\sqrt{\alpha}} \sqrt{t - \frac{x}{c_1}}\right) \right] \cdot U\left(t - \frac{x}{c_1}\right) \quad (2.131)$$

and

$$(\sigma_x)_{224} = -\frac{k \sqrt{\alpha} C_{40}}{h^2} D\left(t - \frac{x}{c_1}\right) \cdot U\left(t - \frac{x}{c_1}\right) \quad (2.132)$$

and

$$(\sigma_x)_{225} = \frac{c_1^2 k^3 \sqrt{\alpha} C_{40}}{h^2(c_1^2 k^2 - h^2 \alpha^2)} \left[D\left(t - \frac{x}{c_1}\right) + \frac{h^2 \alpha}{k^2} e^{\frac{h^2 \alpha}{k^2}(t-x/c_1)} \right] \cdot U\left(t - \frac{x}{c_1}\right) \quad (2.133)$$

and

$$(\sigma_x)_{226} = -\frac{k \alpha^2 \sqrt{\alpha} C_{40}}{(c_1^2 k^2 - h^2 \alpha^2)} \left[D\left(t - \frac{x}{c_1}\right) + \frac{c_1^2}{\alpha} e^{\frac{c_1^2}{\alpha}(t-x/c_1)} \right] \cdot U\left(t - \frac{x}{c_1}\right) \quad (2.134)$$

Further

$$(\bar{\sigma}_x)_3 = \frac{-I_1 \delta c_2}{k} \left[\frac{s^2 e^{-\delta x}}{(\delta^2 - s/\alpha)(s + \beta)(\delta^2 - s^2/c_1^2)} \right] \quad (2.135)$$

and

$$L^{-1}(\bar{\sigma}_x)_3 = (\sigma_x)_3 = \frac{-I_1 \delta c_2}{k} \left[\begin{aligned} & \frac{c_1^2 \alpha e^{c_1 \delta t - \delta x}}{2(\beta + c_1 \delta)(c_1 - \alpha \delta)} - \frac{c_1^3 \alpha e^{-(c_1 \delta t + \delta x)}}{2(c_1 \delta - \beta)(c_1 + \alpha \delta)} \\ & - \frac{c_1^2 \alpha \beta^2 e^{-(\beta t + \delta x)}}{(\beta^2 - c_1^2 \delta^2)(\beta + \alpha \delta^2)} + \frac{c_1^2 \alpha^3 \delta^2 e^{\alpha \delta^2 t - \delta x}}{(\alpha^2 \delta^2 - c_1^2)(\beta + \alpha \delta^2)} \end{aligned} \right] \quad (2.136)$$

also

$$(\bar{\sigma}_x)_4 = \frac{I_1 \delta^2 c_2 c_1}{k} \left[\frac{s e^{-(s/c_1)x}}{(\delta^2 - s/\alpha)(s + \beta)(\delta^2 - s^2/c_1^2)} \right] \quad (2.137)$$

and

$$L^{-1}(\bar{\sigma}_x)_4 = (\sigma_x)_4 = \frac{I_1 \delta^2 c_2 c_1}{k} \left[\begin{aligned} & \frac{c_1^2 \alpha e^{c_1 \delta(t-x/c_1)}}{2\delta(\beta + c_1 \delta)(c_1 - \alpha \delta)} + \frac{c_1^2 \alpha e^{-c_1 \delta(t-x/c_1)}}{2\delta(c_1 \delta - \beta)(c_1 + \alpha \delta)} \\ & - \frac{c_1^2 \alpha \beta e^{-\beta(t-x/c_1)}}{(c_1^2 \delta^2 - \beta^2)(\beta + \alpha \delta^2)} - \frac{c_1^2 \alpha^2 e^{\alpha \delta^2(t-x/c_1)}}{(c_1^2 - \alpha^2 \delta^2)(\beta + \alpha \delta^2)} \end{aligned} \right] \cdot U\left(t - \frac{x}{c_1}\right) \quad (2.138)$$

where $U(t - x/c_1)$ is a unit step function, $\text{Erf}(y)$ is the error function of the variable y and $D(t - x/c_1)$ is the Dirac delta function. The unit step function has the values of 0 for $t \leq x/c_1$ and 1 for $t > x/c_1$.

The closed form solution of the stress distribution can be written as:

$$\sigma_x(x, t) = (\sigma_x)_1 + (\sigma_x)_2 + (\sigma_x)_3 + (\sigma_x)_4 \quad (2.139)$$

Some additional dimensionless quantities are defined to present the stress distribution in the dimensionless form, these are:

$$c_1^\bullet = \frac{c_1}{\alpha \delta}; \quad \sigma_x^\bullet = \frac{k \sigma_x}{I_1 \delta \alpha^2 c_2} \quad \text{and} \quad U^\bullet = \left(t^\bullet - \frac{x^\bullet}{c_1^\bullet} \right) \quad (2.140)$$

where $U^\bullet$ is the dimensionless unit step function.

Therefore, for the dimensionless stress distribution, the followings are resulted:

$$(\sigma_x^\bullet)_1 = (\sigma_x^\bullet)_{11} + (\sigma_x^\bullet)_{21} \quad (2.141)$$

where

$$(\sigma_x^\bullet)_{11} = (\sigma_x^\bullet)_{111} + (\sigma_x^\bullet)_{112} + (\sigma_x^\bullet)_{113} + (\sigma_x^\bullet)_{114} + (\sigma_x^\bullet)_{115} + (\sigma_x^\bullet)_{116} + (\sigma_x^\bullet)_{117} + (\sigma_x^\bullet)_{118} \quad (2.142)$$

and each of these terms are given as:

$$(\sigma_x^\bullet)_{111} = -\frac{(h^\bullet + 1)h^{\bullet 2}c_1^{\bullet 2}}{2(h^{\bullet 2} - 1)(h^{\bullet 2} - c_1^{\bullet 2})(h^{\bullet 2} + \beta^\bullet/t^\bullet)} \left[\begin{aligned} &h^\bullet e^{h^{\bullet 2}t^\bullet - h^\bullet x^\bullet} \text{Erfc}\left(-h^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) + \\ &h^\bullet e^{h^{\bullet 2}t^\bullet + h^\bullet x^\bullet} \text{Erfc}\left(h^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \end{aligned} \right] \quad (2.143)$$

and

$$(\sigma_x^\bullet)_{112} = -\frac{(h^\bullet + 1)h^\bullet c_1^{\bullet 2} \beta^\bullet}{2(\beta^\bullet/t^\bullet + 1)(h^{\bullet 2} + \beta^\bullet/t^\bullet)(c_1^{\bullet 2} + \beta^\bullet/t^\bullet)} \left[\begin{aligned} &e^{\beta^\bullet - \sqrt{\frac{\beta^\bullet}{t^\bullet}} x^\bullet} \text{Erfc}\left(-\sqrt{\beta^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \\ &+ e^{\beta^\bullet + \sqrt{\frac{\beta^\bullet}{t^\bullet}} x^\bullet} \text{Erfc}\left(\sqrt{\beta^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \end{aligned} \right] \quad (2.144)$$

and

$$(\sigma_x^\bullet)_{113} = -\frac{(h^\bullet + 1)h^\bullet c_1^{\bullet 3}}{2(h^{\bullet 2} - c_1^{\bullet 2})(1 - c_1^{\bullet 2})(c_1^{\bullet 2} + \beta^\bullet/t^\bullet)} \left[\begin{aligned} &c_1^{\bullet 2} e^{c_1^{\bullet 2}t^\bullet - c_1^\bullet x^\bullet} \text{Erfc}\left(-c_1^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) + \\ &c_1^{\bullet 2} e^{c_1^{\bullet 2}t^\bullet + c_1^\bullet x^\bullet} \text{Erfc}\left(c_1^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \end{aligned} \right] \quad (2.145)$$

and

$$(\sigma_x^\bullet)_{114} = \frac{(h^\bullet + 1)h^\bullet c_1^{\bullet 2}}{2(h^{\bullet 2} - 1)(1 - c_1^{\bullet 2})(1 + \beta^\bullet/t^\bullet)} \left[\begin{aligned} &e^{t^\bullet - x^\bullet} \text{Erfc}\left(-\sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) + \\ &e^{t^\bullet + x^\bullet} \text{Erfc}\left(\sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \end{aligned} \right] \quad (2.146)$$

and

$$(\sigma_x^\bullet)_{115} = \frac{(h^\bullet + 1)h^{\bullet 2}c_1^{\bullet 2}}{2(h^{\bullet 2} - 1)(h^{\bullet 2} - c_1^{\bullet 2})(h^{\bullet 2} + \beta^\bullet/t^\bullet)} \left[\begin{aligned} &\frac{2}{\sqrt{\pi t^\bullet}} \exp\left(\frac{-x^{\bullet 2}}{4t^\bullet}\right) \\ &+ h^\bullet e^{h^{\bullet 2}t^\bullet - h^\bullet x^\bullet} \text{Erfc}\left(-h^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \\ &- h^\bullet e^{h^{\bullet 2}t^\bullet + h^\bullet x^\bullet} \text{Erfc}\left(h^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \end{aligned} \right] \quad (2.147)$$

and

$$(\sigma_x^\bullet)_{116} = \frac{(h^\bullet + 1)c_1^{\bullet 2}\beta^\bullet}{2t^\bullet(\beta^\bullet/t^\bullet + 1)(h^{\bullet 2} + \beta^\bullet/t^\bullet)(c_1^{\bullet 2} + \beta^\bullet/t^\bullet)} \left[\begin{aligned} & \frac{2}{\sqrt{\pi t^\bullet}} \exp\left(\frac{-x^{\bullet 2}}{4t^\bullet}\right) \\ & + \sqrt{\frac{\beta^\bullet}{t^\bullet}} e^{\beta^\bullet} - \sqrt{\frac{\beta^\bullet}{t^\bullet}} x^\bullet \operatorname{Erfc}\left(-\sqrt{\beta^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \\ & - \sqrt{\frac{\beta^\bullet}{t^\bullet}} e^{\beta^\bullet} + \sqrt{\frac{\beta^\bullet}{t^\bullet}} x^\bullet \operatorname{Erfc}\left(\sqrt{\beta^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \end{aligned} \right] \quad (2.148)$$

and

$$(\sigma_x^\bullet)_{117} = \frac{(h^\bullet + 1)c_1^{\bullet 4}}{2(h^{\bullet 2} - c_1^{\bullet 2})(1 - c_1^{\bullet 2})(c_1^{\bullet 2} + \beta^\bullet/t^\bullet)} \left[\begin{aligned} & \frac{2}{\sqrt{\pi t^\bullet}} \exp\left(\frac{-x^{\bullet 2}}{4t^\bullet}\right) \\ & + c_1^\bullet e^{c_1^{\bullet 2}t^\bullet - c_1^\bullet x^\bullet} \operatorname{Erfc}\left(-c_1^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \\ & - c_1^\bullet e^{c_1^{\bullet 2}t^\bullet + c_1^\bullet x^\bullet} \operatorname{Erfc}\left(c_1^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \end{aligned} \right] \quad (2.149)$$

and

$$(\sigma_x^\bullet)_{118} = \frac{(h^\bullet + 1)c_1^{\bullet 2}}{2(1 - h^{\bullet 2})(1 - c_1^{\bullet 2})(1 + \beta^\bullet/t^\bullet)} \left[\begin{aligned} & \frac{2}{\sqrt{\pi t^\bullet}} \exp\left(\frac{-x^{\bullet 2}}{4t^\bullet}\right) \\ & + e^{t^\bullet - x^\bullet} \operatorname{Erfc}\left(-\sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \\ & - e^{t^\bullet + x^\bullet} \operatorname{Erfc}\left(\sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \end{aligned} \right] \quad (2.150)$$

$(\sigma_x^\bullet)_{12}$ in dimensionless form is:

$$(\sigma_x^\bullet)_{12} = (\sigma_x^\bullet)_{121} + (\sigma_x^\bullet)_{122} + (\sigma_x^\bullet)_{123} + (\sigma_x^\bullet)_{124} \quad (2.151)$$

where

$$(\sigma_x^\bullet)_{121} = \frac{T_0^\bullet h^\bullet c_1^{\bullet 2}}{2(h^{\bullet 2} - c_1^{\bullet 2})} \left[\begin{aligned} & h^\bullet e^{h^{\bullet 2}t^\bullet - h^\bullet x^\bullet} \operatorname{Erfc}\left(-h^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \\ & + h^\bullet e^{h^{\bullet 2}t^\bullet + h^\bullet x^\bullet} \operatorname{Erfc}\left(h^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \end{aligned} \right] \quad (2.152)$$

and

$$(\sigma_x^\bullet)_{122} = -\frac{T_0^\bullet h^{\bullet 2} c_1^\bullet}{2(h^{\bullet 2} - c_1^{\bullet 2})} \left[\begin{aligned} & c_1^\bullet e^{c_1^{\bullet 2}t^\bullet - c_1^\bullet x^\bullet} \operatorname{Erfc}\left(-c_1^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \\ & + c_1^\bullet e^{c_1^{\bullet 2}t^\bullet + c_1^\bullet x^\bullet} \operatorname{Erfc}\left(c_1^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \end{aligned} \right] \quad (2.153)$$

and

$$(\sigma_x^\bullet)_{123} = -\frac{T_0^\bullet h^\bullet c_1^{\bullet 2}}{2(h^{\bullet 2} - c_1^{\bullet 2})} \left[\frac{2}{\sqrt{\pi t^\bullet}} \exp\left(\frac{-x^{\bullet 2}}{4t^\bullet}\right) + h^\bullet e^{h^{\bullet 2} t^\bullet - h^\bullet x^\bullet} \text{Erfc}\left(-h^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \right. \\ \left. - h^\bullet e^{h^{\bullet 2} t^\bullet + h^\bullet x^\bullet} \text{Erfc}\left(h^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \right] \quad (2.154)$$

and

$$(\sigma_x^\bullet)_{124} = \frac{T_0^\bullet c_1^{\bullet 2} h^\bullet}{2(h^{\bullet 2} - c_1^{\bullet 2})} \left[\frac{2}{\sqrt{\pi t^\bullet}} \exp\left(\frac{-x^{\bullet 2}}{4t^\bullet}\right) + c_1^\bullet e^{c_1^{\bullet 2} t^\bullet - c_1^\bullet x^\bullet} \text{Erfc}\left(-c_1^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \right. \\ \left. - c_1^\bullet e^{c_1^{\bullet 2} t^\bullet + c_1^\bullet x^\bullet} \text{Erfc}\left(c_1^\bullet \sqrt{t^\bullet} + \frac{x^\bullet}{2\sqrt{t^\bullet}}\right) \right] \quad (2.155)$$

$(\sigma_x^\bullet)_2$ in dimensionless form is:

$$(\sigma_x^\bullet)_2 = (\sigma_x^\bullet)_{21} + (\sigma_x^\bullet)_{22} \quad (2.156)$$

where

$$(\sigma_x^\bullet)_{21} = (\sigma_x^\bullet)_{211} + (\sigma_x^\bullet)_{212} + (\sigma_x^\bullet)_{213} + (\sigma_x^\bullet)_{214} + (\sigma_x^\bullet)_{215} + (\sigma_x^\bullet)_{216} + (\sigma_x^\bullet)_{217} + (\sigma_x^\bullet)_{218} \quad (2.157)$$

and

$$(\sigma_x^\bullet)_{211} = \frac{(h^\bullet + 1)h^\bullet c_1^{\bullet 3}}{(h^{\bullet 2} - 1)(h^{\bullet 2} - c_1^{\bullet 2})(h^{\bullet 2} + \beta^\bullet/t^\bullet)} \left[\frac{1}{\sqrt{\pi(t^\bullet - x^\bullet/c_1^\bullet)}} + h^\bullet e^{h^{\bullet 2} t^\bullet} \text{Erf}\left(h^\bullet \sqrt{t^\bullet - \frac{x^\bullet}{c_1^\bullet}}\right) \right] \cdot U^\bullet \quad (2.158)$$

and

$$(\sigma_x^\bullet)_{212} = -\frac{(h^\bullet + 1)c_1^{\bullet 3} h^\bullet}{(\beta^\bullet/t^\bullet + 1)(h^{\bullet 2} + \beta^\bullet/t^\bullet)(c_1^{\bullet 2} + \beta^\bullet/t^\bullet)} \left[\frac{1}{\sqrt{\pi(t^\bullet - x^\bullet/c_1^\bullet)}} + \sqrt{-\frac{\beta^\bullet}{t^\bullet}} e^{-\beta^\bullet} \text{Erf}\left(\sqrt{-\beta^\bullet + \frac{\beta^\bullet x^\bullet}{c_1^\bullet t^\bullet}}\right) \right] \cdot U^\bullet \quad (2.159)$$

and

$$(\sigma_x^\bullet)_{213} = \frac{(h^\bullet + 1)c_1^{\bullet 3} h^\bullet}{(h^{\bullet 2} - c_1^{\bullet 2})(1 - c_1^{\bullet 2})(c_1^{\bullet 2} + \beta^\bullet/t^\bullet)} \left[\frac{1}{\sqrt{\pi(t^\bullet - x^\bullet/c_1^\bullet)}} + c_1^\bullet e^{c_1^{\bullet 2} t^\bullet} \text{Erf}\left(c_1^\bullet \sqrt{t^\bullet - \frac{x^\bullet}{c_1^\bullet}}\right) \right] \cdot U^\bullet \quad (2.160)$$

and

$$(\sigma_x^\bullet)_{214} = \frac{(h^\bullet + 1)c_1^{\bullet 3} h^\bullet}{(1 - h^{\bullet 2})(1 - c_1^{\bullet 2})(1 + \beta^\bullet/t^\bullet)} \left[\frac{1}{\sqrt{\pi(t^\bullet - x^\bullet/c_1^\bullet)}} + e^{t^\bullet} \text{Erf}\left(\sqrt{t^\bullet - \frac{x^\bullet}{c_1^\bullet}}\right) \right] \cdot U^\bullet \quad (2.161)$$

and

$$(\sigma_x^\bullet)_{215} = -\frac{(h^\bullet + 1)c_1^{\bullet 3}}{(h^{\bullet 2} - 1)(h^{\bullet 2} - c_1^{\bullet 2})(h^{\bullet 2} + \beta^\bullet/t^\bullet)} \left[D\left(t^\bullet - \frac{x^\bullet}{c_1^\bullet}\right) + h^{\bullet 2} e^{h^{\bullet 2} t^\bullet - \frac{h^{\bullet 2} x^\bullet}{c_1^\bullet}} \right] \cdot U^\bullet \quad (2.162)$$

and

$$(\sigma_x^\bullet)_{216} = \frac{(h^\bullet + 1)c_1^{\bullet 3}}{(\beta^\bullet/t^\bullet + 1)(h^{\bullet 2} + \beta^\bullet/t^\bullet)(c_1^{\bullet 2} + \beta^\bullet/t^\bullet)} \left[D\left(t^\bullet - \frac{x^\bullet}{c_1^\bullet}\right) - \frac{\beta^\bullet}{t^\bullet} e^{-\beta^\bullet + \frac{\beta^\bullet x^\bullet}{c_1^\bullet}} \right] \cdot U^\bullet \quad (2.163)$$

and

$$(\sigma_x^\bullet)_{217} = \frac{(h^\bullet + 1)c_1^{\bullet 3}}{(h^{\bullet 2} - c_1^{\bullet 2})(1 - c_1^{\bullet 2})(c_1^{\bullet 2} + \beta^\bullet/t^\bullet)} \left[D\left(t^\bullet - \frac{x^\bullet}{c_1^\bullet}\right) + c_1^{\bullet 2} e^{c_1^{\bullet 2} t^\bullet - c_1^\bullet x^\bullet} \right] \cdot U^\bullet \quad (2.164)$$

and

$$(\sigma_x^\bullet)_{218} = -\frac{(h^\bullet + 1)c_1^{\bullet 3}}{(1 - h^{\bullet 2})(1 - c_1^{\bullet 2})(1 + \beta^\bullet/t^\bullet)} \left[D\left(t^\bullet - \frac{x^\bullet}{c_1^\bullet}\right) + e^{t^\bullet - \frac{x^\bullet}{c_1^\bullet}} \right] \cdot U^\bullet \quad (2.165)$$

Also $(\sigma_x^\bullet)_{22}$ is:

$$(\sigma_x^\bullet)_{22} = (\sigma_x^\bullet)_{221} + (\sigma_x^\bullet)_{222} + (\sigma_x^\bullet)_{223} + (\sigma_x^\bullet)_{224} + (\sigma_x^\bullet)_{225} + (\sigma_x^\bullet)_{226} \quad (2.166)$$

where

$$(\sigma_x^\bullet)_{221} = -\frac{T_0^\bullet c_1^\bullet}{\sqrt{\pi(t^\bullet - x^\bullet/c_1^\bullet)}} \cdot U^\bullet \quad (2.167)$$

and

$$(\sigma_x^\bullet)_{222} = -\frac{T_0^\bullet c_1^{\bullet 3}}{(h^{\bullet 2} - c_1^{\bullet 2})} \left[\frac{1}{\sqrt{\pi(t^\bullet - x^\bullet/c_1^\bullet)}} + h^\bullet e^{h^{\bullet 2} t^\bullet} \text{Erf}\left(h^\bullet \sqrt{t^\bullet - \frac{x^\bullet}{c_1^\bullet}}\right) \right] \cdot U^\bullet \quad (2.168)$$

and

$$(\sigma_x^\bullet)_{223} = \frac{T_0^\bullet c_1^\bullet h^{\bullet 2}}{(h^{\bullet 2} - c_1^{\bullet 2})} \left[\frac{1}{\sqrt{\pi(t^\bullet - x^\bullet/c_1^\bullet)}} + c_1^\bullet e^{c_1^{\bullet 2} t^\bullet} \text{Erf}\left(c_1^\bullet \sqrt{t^\bullet - \frac{x^\bullet}{c_1^\bullet}}\right) \right] \cdot U^\bullet \quad (2.169)$$

and

$$(\sigma_x^*)_{224} = \frac{T_0^* c_1^*}{h^*} D \left(t^* - \frac{x^*}{c_1^*} \right) \cdot U^* \quad (2.170)$$

and

$$(\sigma_x^*)_{225} = \frac{T_0^* c_1^{*3}}{h^* (h^{*2} - c_1^{*2})} \left[D \left(t^* - \frac{x^*}{c_1^*} \right) + h^{*2} e^{h^{*2} t^* - \frac{h^{*2} x^*}{c_1^*}} \right] \cdot U^* \quad (2.171)$$

and

$$(\sigma_x^*)_{226} = -\frac{T_0^* c_1^* h^*}{(h^{*2} - c_1^{*2})} \left[D \left(t^* - \frac{x^*}{c_1^*} \right) + c_1^{*2} e^{c_1^{*2} t^* - c_1^* x^*} \right] \cdot U^*$$

Further

$$\begin{aligned} (\sigma_x^*)_3 = & -\frac{c_1^{*3} t^* e^{c_1^* t^* - x^*}}{2(\beta^* + c_1^* t^*)(c_1^* - 1)} + \frac{c_1^{*3} t^* e^{-c_1^* t^* - x^*}}{2(c_1^* t^* - \beta^*)(c_1^* + 1)} + \frac{c_1^{*2} t^* \beta^{*2} e^{-\beta^* - x^*}}{(\beta^* + t^*)(\beta^{*2} - c_1^{*2} t^{*2})} \\ & - \frac{c_1^{*2} t^* e^{t^* - x^*}}{(\beta^* + t^*)(1 - c_1^{*2})} \end{aligned} \quad (2.172)$$

and

$$(\sigma_x^*)_4 = \left[\frac{c_1^{*3} t^* e^{c_1^* t^* - x^*}}{2(\beta^* + c_1^* t^*)(c_1^* - 1)} + \frac{c_1^{*3} t^* e^{-c_1^* t^* + x^*}}{2(c_1^* t^* - \beta^*)(c_1^* + 1)} - \frac{c_1^{*3} t^* \beta^* e^{-\beta^* + \frac{t^*}{c_1^*}}}{(\beta^* + t^*)(c_1^{*2} t^{*2} - \beta^{*2})} - \frac{c_1^{*3} t^* e^{t^* - \frac{x^*}{c_1^*}}}{(\beta^* + t^*)(c_1^{*2} - 1)} \right] \cdot U^* \quad (2.173)$$

Consequently, the dimensionless form of the stress equation is:

$$\sigma_x^* = (\sigma_x^*)_1 + (\sigma_x^*)_2 + (\sigma_x^*)_3 + (\sigma_x^*)_4 \quad (2.174)$$

The dimensionless temperature (Eq. 2.76) and stress distributions (Eq. 2.174) are computed during the heating pulse.

2.3 Findings and Discussions

The findings of the analytical treatment of laser heating process in relation to forming and welding are given in line with the previous studies [7, 8]. In the analytical treatment of the laser heating process, one-dimensional semi-infinite substrate material is assumed. This assumption can be justified after considering the size of the absorption depth with the thickness of the workpiece. In this case, the absorption depth is much smaller than the thickness of the workpiece.

2.3.1 Stationary Source Consideration

The findings of the closed form solution for temperature and stress distributions due to time exponentially decaying laser pulse and the convective boundary condition at the surface due to phase change are considered. Steel is employed to simulate the temperature and stress fields in line with the previous study [7].

Figure 2.2 shows the dimensionless temperature distribution within the substrate material for various dimensionless heating periods. The influence of the heat transfer coefficient on the temperature distribution becomes significant when the dimensionless heat transfer coefficient reaches $h^* \geq 2.02 \times 10^{-2} (\approx 10^8 \text{ W/m}^2\text{K})$. In this case, the temperature and its gradient in the surface region are reduced. The temperature gradient in the surface region is reduced to its minimum. At the point of minimum temperature gradient, the internal energy gain by the substrate from the irradiated area is balanced by the diffusional energy transport from the substrate to the solid bulk. In this case, the depth beyond the point of minimum temperature gradient diffusional energy transport dominates over the internal energy gain of the substrate material due to absorption of irradiated field. The point of minimum temperature gradient changes with the heat transfer coefficient, which is more pronounced for the heating period of 0.021. Moreover, the sharp decay in the temperature gradient in the surface region ($x^* \leq 0.1$) is because of: (i) the absorption process, i.e. the absorbed energy decreases exponentially with increasing depth (Lambert's law), and (ii) the internal energy gain in the surface region is high and diffusional energy transport due to the temperature gradient from this region to the solid bulk is low, i.e. the increase in temperature due to diffusional energy transport in the neighbouring region is low; therefore, the temperature profile is governed by the internal energy gain in this region.

Figure 2.3 shows the dimensionless stress distribution within the substrate material for different dimensionless heat transfer coefficients and times. The thermal stress is zero at the surface as a result of the surface boundary condition used in the analysis and it increases sharply close to the surface. The thermal stress is tensile in this region due to expansion of the surface. As the depth increases ($x^* > 0.06$), the stress becomes compressive, as a result of the thermal strain developed in this region, i.e., at this depth and beyond the material contracts resulting in a compressive thermal stress field. The influence of the heat transfer coefficient on the stress development is considerable, as illustrated by $h^* = 0.0202 (10^9 \text{ W/m}^2\text{K})$. In this case, the stress developed is compressive and with a high magnitude in the vicinity of the surface and decays sharply as the depth increases. However, the compressive stress wave is developed at some point below the surface. The magnitude of the stress wave is lower at this point as time progresses. In addition, the magnitude of the thermal stress levels, corresponding to a heat transfer coefficient other than $h^* = 0.0202$, increases with an increase in time, provided that this increase is less than 10 %.

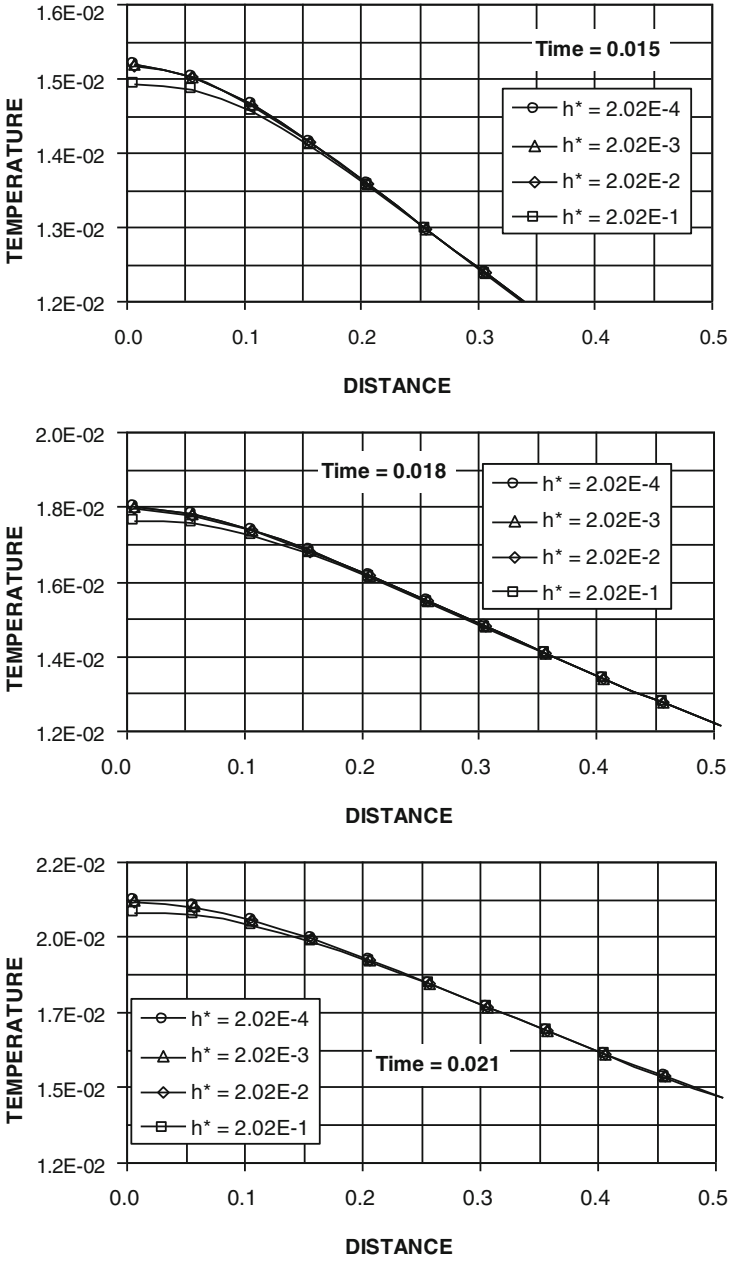


Fig. 2.2 Dimensionless temperature distributions within the substrate material

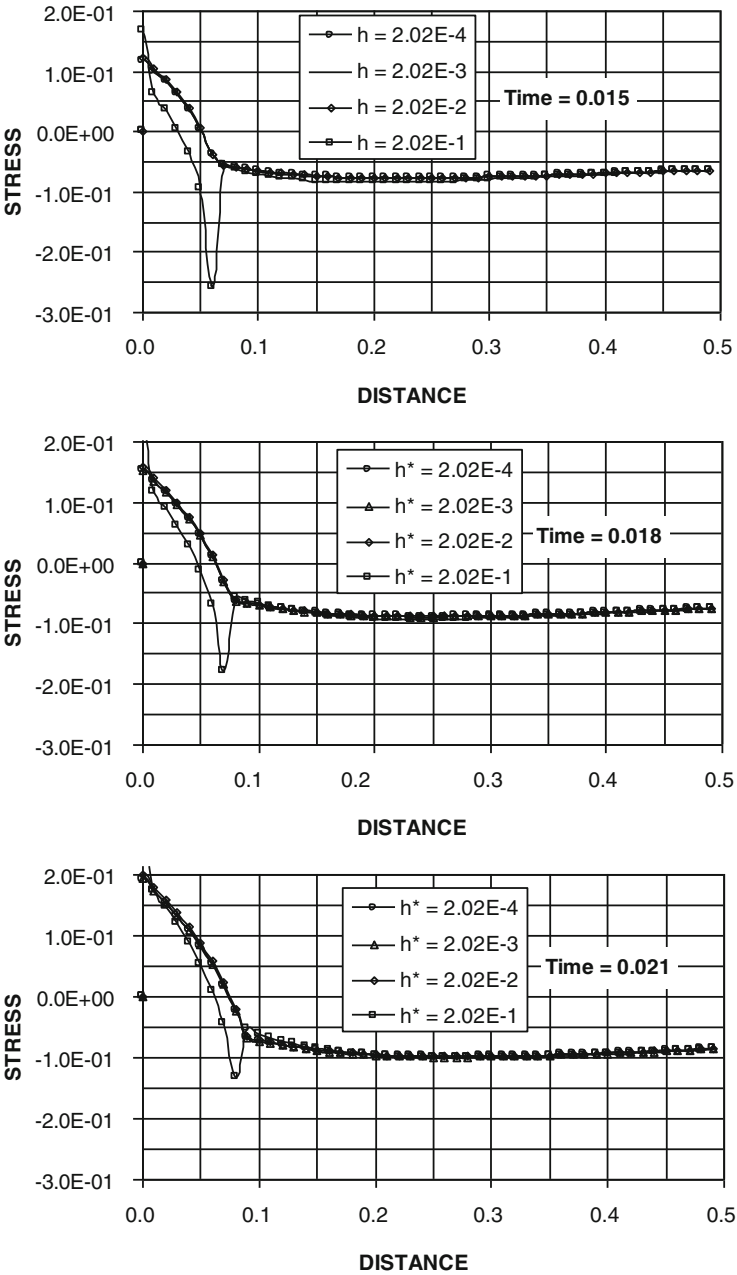


Fig. 2.3 Dimensionless stress distribution within the substrate material

2.3.2 Moving Source Consideration

Temperature field due to laser heating of sheet metal is formulated and thermal stress field is computed using FEM in line with the previous study [9]. Laser beam is considered as a line source scanning the workpiece surface with a constant speed ($v = \text{constant}$). The laser beam diameter is taken as 1 mm.

Figure 2.4 shows temperature contours in the workpiece for two laser beam scanning velocities. Temperature profile in the front region of the laser beam is higher than that corresponding to the back region. This is because of the internal energy gain of the substrate material, which is higher in the front region due to the direction of the motion. Temperature reaches maximum when r/R is low and angular location (θ) is zero. This is the location corresponding to the front of the moving source. Increasing r/R represents the locations departing away from the heat source, i.e. r/R is the radial location from centre of the heat source. As the scanning velocity increases, the magnitude of temperature reduces which is more pronounced in the region close to the heat source front. In this case, increasing scanning velocity reduces the internal energy gain of the substrate material, particularly in the region close to the heat source.

Figure 2.5 shows radial stress contours for two laser scanning speed. Radial stress component is compressive in the region close to the heat source. This is because of the temperature distribution in this region. Radial stress component becomes tensile as the radial distance increases further away from the heat source. The stress behaviour in the substrate material is because of the thermal strain

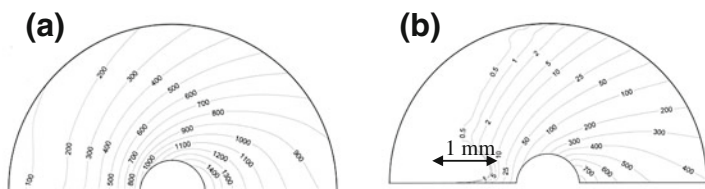


Fig. 2.4 Temperature contours ($^{\circ}\text{C}$) in the workpiece for two laser beam scanning velocities. **a** – $v = 1 \text{ cm/s}$. **b** – $v = 5 \text{ cm/s}$

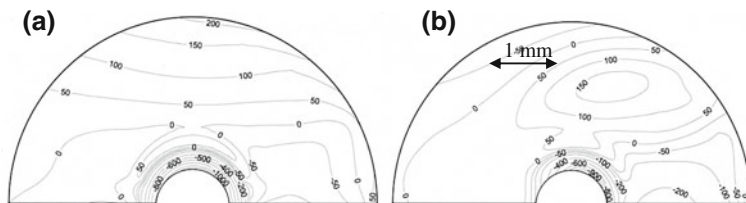


Fig. 2.5 Radial stress contours (MPa) for two laser scanning speed. **a** – $v = 1 \text{ cm/s}$. **b** – $v = 5 \text{ cm/s}$

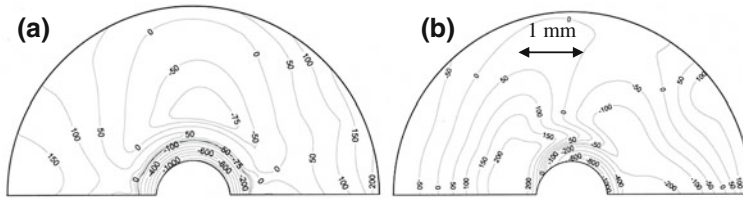


Fig. 2.6 Tangential stress contours (MPa) for two laser scanning speeds. **a** – $v = 1$ cm/s. **b** – $v = 5$ cm/s

developed due to temperature field. The magnitude of radial stress component increases slightly for low scanning speed and the compressive stress behaviour is extended further away from the heat source as scanning speed increases to 5 cm/sec. The compressive and tensile behaviour of the radial stress is because of the radial temperature gradient.

Figure 2.6 shows tangential stress contours for two laser scanning speeds. Tangential stress component is compressive in the region close to the heat source, which is particularly true for high scanning speed. In this case, increasing scanning speed results in extended compressive stress component in front of the heat source while the stress component is tensile behind the heat source. Moreover, stress distribution varies considerably when scanning speed changes. In this case, the locations of zero stress line changes. It should be noted that similar behaviour is observed for the radial stress component (Fig. 2.5), provided that the location of zero stress line differs. Consequently, radial and tangential temperature distribution depends highly on the laser scanning speed, which in turn results in variation in stress distribution.

2.4 Concluding Remarks

The findings from the closed form solution for temperature and stress fields reveal that the influence of the heat transfer coefficient on temperature profiles is significant as the dimensionless heat transfer coefficient at the surface increases. The temperature gradient is reduced to its minimum at some point below the surface. At a depth beyond the point of minimum temperature gradient the diffusional energy transport dominates over the gain in internal energy of the substrate from the irradiated area. The point of minimum temperature gradient changes for high heat transfer coefficient. The thermal stress developed in the vicinity of the surface is tensile and as the depth increases it becomes compressive. This is because the thermal strain developed in the vicinity of the surface, which is positive, and at some point below the surface, it becomes negative due to the compressive effect of the substrate. The thermal stress wave is generated within the substrate material

and the magnitude of stress wave is reduced as the depth increases from the surface towards the bulk solid.

The analytical solution for the moving source consideration shows that laser scanning speed influences the temperature distribution, which in turn modifies the stress field in the substrate material. Temperature distribution in front of the laser heat source attains high values than that corresponding to behind the source. This is because of the internal energy gain of the substrate material, which is high in the region in front of the laser source. Moreover, as the scanning speed increases, the magnitude of temperature away from the laser-heating source reduces. Radial stress component is compressive in the region close to the heat source and as distance increases away from the source, it becomes tensile. This is more pronounced when scanning speed increases. This occurs because of the radial distribution of the temperature gradient. Increasing scanning speed results in extension of compressive stress field in front of the heat source. The similar situation is observed from the tangential stress component.

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