

## Chapter 2

# Introductory Concepts of Wave Propagation Analysis in Structures

Wave propagation is a transient dynamic phenomenon resulting from short duration loading. Such transient loadings have high frequency content. The main difference between the structural dynamics and wave propagation in structures arises due to high frequency excitations in the later case. Structures very often experience such loadings in the forms of impact and blast loadings like gust, bird hit, tool drops etc. Apart from these, the wave propagation studies are also important to understand the dynamic characteristics of a structure at higher frequencies due to their various real-world applications. Structural health monitoring or detection of damage is one such important application. As wave propagation deals with higher frequencies, diagnostic waves can be used to predict the presence of even minute defects, which occur at initiation of damage and propagate them till the failure of the structure. In many aircraft structures, the undesired vibration and noise transmit from the source to the other parts in form of wave propagation and this requires control or reduction, which is again an important application of wave propagation studies.

A structure, when subjected to dynamic loads, will experience stresses of varying degree of severity depending upon the load magnitude and its duration. If the temporal variation of load is of large duration (of the order of seconds), the intensity of the load felt by the structure will usually be of lower severity and such problems fall under the category of structural dynamics. For such problems, there are two parameters which are of paramount importance in the determination of its response, namely the *natural frequency* of the system and its *normal modes* (mode shapes). The total response of structure is obtained by the superposition of first few normal modes. Large duration of the load makes it low on the frequency content, and hence the load will make only the first few modes to get excited. Hence, the structure could be idealized with fewer unknowns (which we call as degrees of freedom). However, when the duration of load is small (of the order of microseconds), the stress waves are set up, which starts propagating in the medium with certain velocity. Hence, the response is necessarily transient in nature and in the process, many normal modes will get excited. Hence, the model sizes will be many orders bigger than what is required for the structural dynamics problem. Such problems come under the category of wave propagation.

In Sect. 1.3, the need to study wave propagation in structures was highlighted. It is quite well-known that nanostructures can propagate waves of the order of terahertz. Propagation of waves in such structures can be studied using various modeling techniques highlighted in Chap. 3. For example, modeling technique such as ab initio atomistic modeling or molecular dynamics methods can be employed to study wave propagation in such nanostructures. However, the main limitation of these methods is the computational time that these methods will take. Some researchers have used finite element method coupled with molecular mechanics model to solve such problems. However, if the frequency content of the problem is of the terahertz level, then for FE modeling, one needs a FE mesh compatible with its wavelength, which is extremal small at the terahertz frequency. These limitations can be overcome to some extent by using Spectral Finite Element Method (SFEM), which is the FE method in the frequency domain. It uses the spectral analysis to understand the physics of wave propagation, whose results will be directly used in the SFEM formulation. This chapter mainly deals with the spectral analysis of motion, which will tell about the nature of waves that is propagating in the medium and the speed with which the waves are propagating.

## 2.1 Introduction to Wave Propagation

The key factor in the wave propagation is the propagating velocity, level of attenuation of the response and its wavelengths. Hence, phase information of the response is one of the important parameters, which is of no concern in the structural dynamic problems [1].

The wave propagation is a multi-modal phenomenon, and hence the analysis becomes quite complex when the problem is solved in the time domain. This is because, the problem by its nature is a high frequency content problem. Hence, the analysis methods based on the frequency domain is normally preferred for wave propagation problems. That is, all the governing equations, boundary conditions, and the variables are transformed into the frequency domain using any of the integral transforms available. The most common transformation for transforming the problem to the frequency domain is the Fourier Transforms. This transform has the discrete representation and hence amenable for numerical implementation, which makes it very attractive for its usage in the wave propagation problems. By transforming the problem into frequency domain, the complexity of the governing partial differential equation is reduced by removing the time variable out of picture, making the solution simpler than the original equation. In wave propagation problems, we are concerned about two parameters, namely the wavenumber and the speeds of the propagation (normally referred to as group speeds).

There are many types of waves that can be generated in structure. Wavenumber reveals the type of waves that are generated. Hence, in wave propagation problems, two important relations are very important, namely the spectrum relations, which is plot of the wavenumber with the frequency and the dispersion relations, which is a

plot of wave velocity with the frequency. These relations reveal the characteristics of different waves that are generated in a given structure.

## 2.2 Spectral Analysis

Spectral analysis determines the local wave behavior for different waveguides and hence the wave characteristics, namely the *spectrum* and the *dispersion* relation. These local characteristics are synthesized over large number of frequencies to get the global wave behavior. Spectral analysis uses discrete Fourier transform to represent a field variable (say displacement) as a finite series involving a set of coefficients, which requires to be determined based on the boundary conditions of the problem [1–3].

Spectral analysis enables the determination of two important wave parameters, namely the wavenumbers and the group speeds (discussed in next section). These parameters are not only required for spectral element formulation [1], but also to understand the wave mechanics in a given waveguide. These parameters enable us to know whether the wave mode is a propagating mode or a damping mode or a combination of these two (propagation as well as wave amplitude attenuation). If the wave is propagating, the wavenumber expression will let us know whether it is nondispersive (that is, the wave retains its shape as it propagates) or dispersive (when the wave changes its shape as it propagates). More details on spectral analysis and applications to several problems can be found in Refs. [1, 4, 5]. In the next subsection, we will define some of the commonly used wave propagation terminologies.

## 2.3 Wave Propagation Terminologies

### 1. Waveguide

Any structural element is called a waveguide as it guides the wave in a particular manner. For example, a bulk or a nano rod essentially supports only the axial motion and hence it is called axial or longitudinal waveguide. In the case of a beam, only bending motion is possible and hence the beam is called the flexural waveguide. In the case of shafts, the only possible motion is the twist and hence they are called torsional waveguide. In case of laminated composite beam, due to stiffness coupling both axial and flexural motions are possible. In general, if there are  $n$  highly coupled governing partial differential equations, then such a waveguide can support  $n$  different motions.

### 2. Wavenumber

This is a frequency-dependent parameter that determines the following:

- Whether the wave is propagating or nonpropagating or will propagate after certain frequency.

- It also determines the type of wave, namely dispersive or nondispersive wave.

Nondispersive waves are those that retain its shape as it propagates, while the dispersive waves are those that completely change its shape as it propagates. That is if the wavenumber ( $k$ ) is expressed as a linear function of frequency ( $\omega$ ) as  $k = c_1\omega$ , ( $c_1$  is a constant) then the waves will be nondispersive in nature. Wavenumber in rods and in general for most second-order system, will be of this form and hence the waves will be nondispersive in nature. However, if the wave number is of the form  $k = c_2\omega^n$ , ( $c_2$  is a constant) the waves will essentially be dispersive. Such a behavior can be seen in higher order systems such as beams and plates. In such cases, the wave speeds will change with the frequencies. The plot of wavenumber with the frequency is usually referred to as spectrum relations.

### 3. Phase speed

These are the speeds of the individual particles that propagate in the structure. They are related to the wavenumber through the relation

$$C_p = \frac{\omega}{\text{Real}(k)} \quad (2.1)$$

If the waves are nondispersive in nature (that is  $k = a\omega$ ), then the phase speeds are constant and independent of frequency. Conversely, if the phase speeds are constant, then such a system is nondispersive system. Phase speed is not associated with transfer of any physical quantity (e.g., mass, momentum or energy) in a waveguide.

### 4. Group speed

Group velocity is associated with the propagation of a group of waves of similar frequency. During the propagation of waves, groups of particles, travel in bundles. The speeds of each of these bundle are called the group speed of the wave. They are mathematically expressed as

$$C_g = \frac{\partial \omega}{\partial k} \quad (2.2)$$

Again here, for nondispersive system, the group speeds are constant and independent of frequency. Hence, the time of arrival of all waves will be based on this frequency. The plot of phase/group speeds with the frequency is called the dispersion relations. This is a velocity of the energy transportation and it must be bounded. Monograph [1] derives the expression for the wavenumbers and group speeds for some commonly used metallic and composite waveguides in engineering.

### 5. Relation between group and phase speeds

Since from Eq. (2.1) we have

$$k = \frac{\omega}{C_p} \quad (2.3)$$

Substituting Eq. (2.1) into Eq. (2.2) gives

$$C_g = d\omega \left[ d \left( \frac{\omega}{C_p} \right) \right]^{-1} = d\omega \left[ \frac{d\omega}{C_p} - \omega \frac{dC_p}{C_p^2} \right]^{-1} = C_p^2 \left[ C_p - \omega \frac{dC_p}{d\omega} \right]^{-1} \quad (2.4)$$

Using  $\omega = 2\pi f$ ,

$$C_g = C_p^2 \left[ C_p - (ft) \frac{dC_p}{d(ft)} \right]^{-1} \quad (2.5)$$

where  $ft$  denotes the frequency times thickness.

- When the derivative of  $C_p$  with respect to  $ft$  becomes zero,  $C_g = C_p$ .
- As the derivative of  $C_p$  with respect to  $ft$  approaches infinity (that is, cut-off frequency),  $C_g$  approaches zero.

## 6. Cut-off Frequency

In some waveguides, some waves will start propagating only after certain frequency called the cut-off frequency. The wavenumber and group speeds before this frequency will be imaginary and zero, respectively. In the next section, we will outline a procedure to compute the cut-off frequency for the second- and fourth-order systems.

## 2.4 Spectrum and Dispersion Relations

Here, two important frequency-dependent wave characteristics, namely, spectrum and dispersion relations, are obtained for a generalized system defined by the second- and fourth-order partial differential equations (PDE). These relations are the frequency variation of the wave parameters termed as wavenumbers and wave speeds, respectively. These parameters are essential to understand the wave mechanics in a given waveguide and are also required for SFEM formulation. These parameters provide information like whether the wave mode is a propagating mode or a damping mode or a combination of these two (propagation as well as wave amplitude attenuation). Next, for a propagating mode, the nature of frequency variation of wavenumbers gives information whether the mode is nondispersive, i.e., the wave retains its shape as it propagates or dispersive where the shape changes with propagation. In this section, these parameters are explained using the example of a generalized one-dimensional second- and fourth-order systems.

### 2.4.1 Second-Order PDE

The spectral analysis starts with the partial differential equation governing the waveguide. Considering a generalized second-order partial differential equation given by

$$p \frac{\partial^2 u}{\partial x^2} + q \frac{\partial u}{\partial x} = r \frac{\partial^2 u}{\partial t^2} \quad (2.6)$$

where  $p$ ,  $q$ , and  $r$  are known constants depending on the material properties and geometry of the waveguide.  $u(x, t)$  is the field variable to be solved for with  $x$  being the spatial dimension and  $t$  the temporal dimension. First,  $u(x, t)$  is transformed to frequency domain using discrete Fourier transform (DFT) as

$$u(x, t) = \sum_{n=1}^{N-1} \hat{u}_n(x, \omega_n) e^{j\omega_n t} \quad (2.7)$$

where  $\omega_n$  is the discrete circular frequency in rad/sec and  $N$  is the total number of frequency points used in the transformation. The  $\omega_n$  is related to the time window by

$$\omega_n = n\Delta\omega = \frac{n\omega_f}{N} = \frac{n}{N\Delta t} = \frac{n}{T} \quad (2.8)$$

where  $\Delta t$  is the time sampling rate and  $\omega_f$  is the highest frequency captured by  $\Delta t$ . The frequency content of the load decides  $N$  and consideration of the wrap around and aliasing problem decides  $\Delta\omega$ . More details and associated problems are given in Ref. [1].

Here,  $\hat{u}_n$  is the  $n$ th DFT coefficient and can also be referred to as the coefficient at frequency  $\omega_n$ .  $\hat{u}_n$  varies only with  $x$ . Substituting Eq. (2.7) into Eq. (2.6), we get

$$p \frac{d^2 \hat{u}_n}{dx^2} + q \frac{d\hat{u}_n}{dx} + r\omega_n^2 \hat{u}_n = 0, \quad n = 0, 1, \dots, N-1 \quad (2.9)$$

Thus, through DFT, the governing PDE given by Eq. (2.6) is reduced to  $N$  ODE. Equation (2.9) being constant coefficient ODEs, have a solution of the form  $\hat{u}_n(x) = A_n e^{jk_n x}$ , where  $A_n$  are the unknown constants, which will be computed from the boundary values and  $k_n$  is called the wavenumbers corresponding to the frequency  $\omega_n$ . Substituting the above solution into Eq. (2.9), we get the following characteristic equation to determine  $k_n$ ,

$$\left( -pk_n^2 + jqk_n + r\omega_n^2 \right) A_n = 0 \quad (2.10)$$

The subscript  $n$  is dropped hereafter for simplified notations. The above equation is quadratic in  $k$  and has two roots corresponding to the incident and reflected waves. If the wavenumbers are real, then the wave is called propagating mode. On the

other hand, if the wavenumbers are purely imaginary, then the wave damps out as it propagates and hence is called evanescent mode. If wavenumbers are complex having both the real and imaginary parts, then the wave with such a wavenumber will attenuate as they propagate. These waves are normally referred to as inhomogeneous waves. The set of the wavenumbers obtained by solving the characteristic Eq. (2.10) is given as

$$\begin{aligned} k_1 &= j \frac{q}{2p} + \sqrt{-\frac{q^2}{4p^2} + \frac{r\omega^2}{p}} \\ k_2 &= j \frac{q}{2p} - \sqrt{-\frac{q^2}{4p^2} + \frac{r\omega^2}{p}} \end{aligned} \quad (2.11)$$

Equation (2.11) is the generalized expression for the determination of the wavenumbers. Different wave behaviors are possible depending upon the values of the radical  $\frac{r\omega^2}{p} - \frac{q^2}{4p^2}$ . As an example, for a case with  $q = 0$ , the wavenumbers are given as

$$\begin{aligned} k_1 &= \omega \sqrt{\frac{r}{p}} \\ k_2 &= -\omega \sqrt{\frac{r}{p}} \end{aligned} \quad (2.12)$$

For such a case, the wavenumbers are real and hence the corresponding waves are propagating. When  $r\omega^2/p < q^2/4p^2$ , then the wavenumber is purely imaginary and the system will not allow any way to propagate. However, when  $r\omega^2/p > q^2/4p^2$ , the wavenumber will be complex and in this case, the waves will attenuate as they propagate.

Next, two other important wave parameters, namely, phase speed ( $C_p$ ) and group speed ( $C_g$ ), are briefly explained.

Let us consider the previous example where wavenumbers vary linearly with frequency as given by Eq. (2.12). Correspondingly, the wave speeds are obtained as follows:

$$\begin{aligned} C_p &= \frac{\omega}{k} = \sqrt{\frac{p}{r}} \\ C_g &= \frac{d\omega}{dk} = \sqrt{\frac{p}{r}} \end{aligned} \quad (2.13)$$

We find that both group and phase speeds are constant and equal. Hence, when wavenumbers vary linearly with frequency  $\omega$ , the wave retains its shape as it propagates. Such waves are called nondispersive waves. When wavenumbers have a nonlinear variation with frequency, the phase and group speeds will not be constant but will be function of frequency. As a result, each frequency component will travel

with different speeds and the wave shape will not be preserved with wave propagation. Such waves are called dispersive waves. For nonzero values of  $p$ ,  $q$  and  $r$ , the expression for phase and group speeds become

$$\begin{aligned} C_p &= \text{Re} \left( \frac{\omega}{\sqrt{\frac{r\omega^2}{p} - \frac{q^2}{4p^2}}} \right) \\ C_g &= \text{Re} \left( \frac{p\sqrt{\frac{r\omega^2}{p} - \frac{q^2}{4p^2}}}{r\omega} \right) \end{aligned} \quad (2.14)$$

Here “Re” represents *real* part of the expression. Thus, it can be seen that the wave speeds  $C_p$  and  $C_g$  are not the same and hence the waves are dispersive in nature. The value of the radical, however, depends on frequency and there can be a frequency after which the wavenumbers transit from being purely imaginary to complex or real wavenumbers resulting in propagation of the wave mode. This transition frequency is called the *cut-off frequency* ( $\omega_c$ ) and can be derived by equating the radical to zero. The expression for the cut-off frequency for this second-order system is given as

$$\omega_c = \frac{q}{2\sqrt{pr}} \quad (2.15)$$

Once the wavenumbers are determined the solution of the transformed ODEs given by Eq. (2.9) can be written as

$$\hat{u}(x, \omega) = A_1 e^{-jk_1 x} + A_2 e^{-jk_2 x} \quad (2.16)$$

The unknown constants  $A_1$  and  $A_2$  can be evaluated in terms of the physical boundary conditions of the one-dimensional waveguide. This can be done in a formal manner using the spectral finite element technique which will be explained in details later. For  $q = 0$  the above equation is of the form,

$$\hat{u}(x, \omega) = A_1 e^{-jkx} + A_2 e^{jkx}, \quad k = \omega \sqrt{\frac{r}{p}} \quad (2.17)$$

where  $A_1$  represent the incident wave coefficient while  $A_2$  stands for the reflected wave coefficient.

### 2.4.2 Fourth Order PDE

Next, let us consider a fourth-order system and study its wave behavior. Consider the following governing PDE

$$p \frac{\partial^4 w}{\partial x^4} + qw + r \frac{\partial^2 w}{\partial t^2} = 0 \quad (2.18)$$

where  $w(x, t)$  is the field variable and  $p$ ,  $q$ , and  $r$  are arbitrary known constants depending on the material and geometric properties of the waveguide as in the case of the second-order system. The above equation is similar to the equation of motion of a Euler–Bernoulli beam on elastic foundation. The DFT of  $w(x, t)$  can be written in a similar form as Eq. (2.7),

$$w(x, t) = \sum_{n=1}^{N-1} \hat{w}_n(x, \omega_n) e^{j\omega_n t} \quad (2.19)$$

Substituting Eq. (2.19) into the governing PDE given by Eq. (2.18) we get the reduced ODEs as

$$p \frac{d^4 \hat{w}_n}{dx^4} + (q - r\omega_n^2) \hat{w}_n = 0 \quad (2.20)$$

The above ODEs have constant coefficients and hence the solution will be of the form  $\hat{w}_n(x) = A_n e^{jk_n x}$ . Substituting this solution into Eq. (2.20) we get the characteristic equation for solution of the wavenumbers. Again the subscript  $n$  is dropped hereafter for simplified notations and all the following equations have to be derived for  $n$  varying from 0 to  $N-1$ . The characteristic equation is of the form

$$pk^4 + q - r\omega^2 = 0 \quad \text{or} \quad k^4 - \left( \frac{r}{p} \omega^2 - \frac{q}{p} \right) = 0 \quad (2.21)$$

This is a fourth-order equation and will give two sets of wavenumbers. The type of wave is dependent upon the numerical value of  $\frac{r}{p} \omega^2 - \frac{q}{p}$ . For  $\frac{r}{p} \omega^2 > \frac{q}{p}$ , the solution of Eq. (2.21) will give the following wavenumbers,

$$\begin{aligned} k_1 &= +\alpha, & k_2 &= -\alpha \\ k_3 &= +j\alpha, & k_4 &= -j\alpha \end{aligned} \quad (2.22)$$

where  $\alpha = \left( \frac{r}{p} \omega^2 - \frac{q}{p} \right)^{1/4}$ . In the above equation,  $k_1$  and  $k_2$  represent the propagating wave modes while  $k_3$  and  $k_4$  are the damping or evanescent modes. From the above equations, we find that the wavenumbers are nonlinear functions of the frequency, and hence the corresponding waves are expected to be highly dispersive in nature. Also, using the above expression we can find the phase and group speeds for the propagating mode from Eq. (2.13).

Next, consider the case when  $\frac{r}{p} \omega^2 < \frac{q}{p}$ . For such conditions, the wavenumbers are given by

$$\begin{aligned}
k_1 &= +\frac{1+j}{\sqrt{2}}\alpha, & k_2 &= -\frac{1+j}{\sqrt{2}}\alpha \\
k_3 &= +\frac{-1+j}{\sqrt{2}}\alpha, & k_4 &= -\frac{-1+j}{\sqrt{2}}\alpha
\end{aligned} \tag{2.23}$$

From the above equation, we see that the change of sign of  $\frac{r}{p}\omega^2 - \frac{q}{p}$  has completely changed the wave behavior. Now, all the wavenumbers have both real and imaginary parts. Hence, all the wave modes are propagating as well as attenuating. The initial evanescent mode also becomes a propagating mode after the cut-off frequency  $\omega_c$ .

The expression for the cut-off frequency obtained by equating  $\frac{r}{p}\omega^2 - \frac{q}{p}$  to zero is

$$\omega_c = \frac{q}{r} \tag{2.24}$$

Again, if  $q = 0$ , the cut-off frequency vanishes and the wave behavior is similar to the first case, i.e., it will have propagating and damping modes. In all cases, however, the waves will be highly dispersive in nature.

The solution of the fourth-order governing Eq. (2.20) can be written as

$$\hat{w}(x, \omega) = A_1 e^{-j\alpha x} + B_1 e^{-\alpha x} + A_2 e^{j\alpha x} + B_2 e^{\alpha x} \tag{2.25}$$

As in the previous case,  $A_1, B_1$  are the incident wave coefficients and  $A_2, B_2$  are the reflected wave coefficients. These unknown constants can be determined in terms of the physical boundary conditions of the beam.

From the above discussion, we see that the spectral analysis gives an insight into the wave mechanics of a system defined by its governing differential equation. Though spectral analysis can be done similarly using wavelet transform, it is not as straightforward as Fourier transform-based analysis. This is because the wavelet basis functions are bounded both in time and frequency unlike the basis for Fourier transform which is unbounded in time. This has been explained in greater detail with application to nanostructures in the other chapters.

In the last section, the parameters, wavenumber, and wave speed were explained with the examples of generalized second- and fourth-order partial differential wave equations. The wavenumbers  $k$  were obtained as a function of frequency by solving second- and fourth-order polynomial equations, respectively. The computation of wavenumbers is, however, not so straightforward for structures with higher complexities (especially for the cases when the characteristic equation for solution of wavenumbers is of the order greater than 3). For one-dimensional structures such cases arise when the governing equation is a set of coupled PDEs and a couple of such examples are Timoshenko beam and other higher order beams. In a Timoshenko beam, the governing equations consist of two coupled PDEs with transverse and shear displacements as the variables. Another common example of structure having a set of coupled PDEs is the governing equations for a composite beam with asymmetric ply lay-up resulting in elastic coupling. In addition to the different one-dimensional structures, computation of wavenumbers for two-dimensional structures is also dif-

ficult primarily, because the wavenumbers here are a function of both frequency and wavenumber in the other direction.

In order to handle such problems, generalized and computationally implementable methods have been proposed to calculate the wavenumbers and associated wave amplitude. The two different approaches to solve the problem are based on singular value decomposition (SVD) and polynomial eigenvalue problem (PEP) methods. The methods are described briefly in refs. [1, 4, 5].

## 2.5 Summary

In this chapter, a brief introduction to wave propagation is given. First, the different terminologies used in wave propagation are defined. This is followed by a brief description of wave characteristics for systems defined by second- and fourth-order PDEs. The wave behavior is described based on wavenumbers and group speeds. Existence of cut-off frequencies in these two systems is also highlighted.

The PDEs that govern the nano waveguides described by nonlocal elasticity (which is the focus of this book) are normally defined by either second- or fourth-order PDEs. However, their form is entirely different than what is described in this chapter. The concepts outlined in this chapter will be very important and necessary for the reader to understand the wave propagation in different nano waveguides that are outlined in the later chapters of the book.

## References

1. S. Gopalakrishnan, A. Chakraborty, D. Roy Mahapatra, *Spectral Finite Element Method* (Springer, London, 2008)
2. J.F. Doyle, *Wave propagation in structures* (Springer, New York, 1997)
3. A. Graff, *Wave Motion in Elastic Solids* (Dover Publications, New York, 1995)
4. S. Gopalakrishnan, M. Mitra, *Wavelet Methods for Dynamical Problems* (Taylor and Francis Group, LLC, Boca Raton, FL, 2010)
5. U. Lee, *Spectral Finite Element Method*, (Cambridge University Press, Cambridge, 2009)

Wave Propagation in Nanostructures

Nonlocal Continuum Mechanics Formulations

Gopalakrishnan, S.; Narendar, S.

2013, XIII, 359 p. 157 illus., 147 illus. in color.,

Hardcover

ISBN: 978-3-319-01031-1