Chapter 2
Wide Output Dynamic Range Gaussian Function Synthesizers

2.1 Introduction

The Gaussian function is intensively used in many domains of analog signal processing: neural networks and algorithms, wavelet transform and pattern recognition, neuro-fuzzy and classification applications, on-chip unsupervised learning, back-propagation neural networks. Similar requirements with exponential circuits can be considered also for the design of performance Gaussian function synthesizer structures: improved accuracy, good frequency response, low-power operation, reasonable complexity. Additionally, in order to increase the domain of applications for the developed Gaussian computational structures, the range of their input variable must be as large as possible. Convenient variable changing can be used for improving the operation of Gaussian function synthesizers. From this point of view, it is possible to reduce the order of approximation as a consequence of the accuracy increasing that could be obtained by using previously mentioned variable changing.

The complexity of CMOS implementations for Gaussian function synthesizer circuits can be strongly reduced by developing the approximation functions in such a way that they require relatively simple computational components. From this perspective, the most convenient choice is represented by the utilization of current-mode squaring or multiplier/divider circuits.

The chapter will analyze a multitude of possible realizations of Gaussian function synthesizer circuits, based on particular superior-order approximation functions: fourth-order, sixth-order, and eighth-order approximation functions. In order to improve the area of operation of the developed Gaussian circuits, convenient variable changing will be considered for each analyzed approximation function. Analytical and graphical analysis will be performed for determining the performances of these superior-order approximation functions.

The Gaussian function can be expressed as follows:

$$f(x) = A\exp\left(-\frac{x^2}{2\sigma^2}\right), \tag{2.1}$$

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A and σ being constants that define the amplitude and the width of the Gaussian function, respectively. The expansion in the Taylor series of the Gaussian function has the following general expression:

$$f(x) = 1 - \frac{x^2}{2\sigma^2} + \frac{x^4}{8\sigma^4} - \frac{x^6}{48\sigma^6} + \cdots. \tag{2.2}$$

In order to increase the output dynamic range of the circuits those generate the Gaussian function, the reduction in the range of the input variable can be used, implementing a variable changing expressed by

$$x \to \frac{x}{\alpha}. \tag{2.3}$$

So, a Gaussian function synthesizer based on the previous variable changing will exploit the following mathematical relation:

$$\exp\left(-x^2\right) = \exp\left[-\left(\frac{x}{\alpha}\right)^2\right]^{\alpha^2}. \tag{2.4}$$

In order to maintain the complexity of the Gaussian computational circuit at a reasonable level, the usual value of α parameter is 2, resulting

$$\exp\left(-x^2\right) = \exp\left[-\left(\frac{x}{2}\right)^2\right]^4. \tag{2.5}$$

The method presented in this chapter for generating the Gaussian function is based on the utilization of superior-order approximation functions. Because the Gaussian function is an even-order function, the odd-order terms from its Taylor series expansions will be zero.

2.2 Fourth-Order Approximation of Gaussian Function

In order to generate the Gaussian function with a reasonable complexity of the function synthesizer circuit, a fourth-order approximation function can be used. It is necessary to make a trade-off between the accuracy of the approximation and the complexity of the Gaussian function synthesizer. From this point of view, the fourth-order approximation represents a convenient choice.

2.2.1 Approximation Function Without Variable Changing

2.2.1.1 Implementation of the Particular Gaussian Function

Based on the previous considerations, a possible fourth-order approximation function [1] could be generally expressed as follows:

$$g_{1a}(x) = \frac{b}{1+ax} + \frac{c}{1+x} + dx + e, \tag{2.6}$$

a, b, c, d, and e being constant coefficients having the values imposed by the condition that $g_{1a}(x)$ approximation function should match, in a fourth-order approximation, the Gaussian function.

The superior-order Taylor series of the approximation function has the following expression [1]:

$$\begin{aligned} g_{1a}(x) &= (b+c+e) + (d-c-ab)x + \left(c+a^2b\right)x^2 \\ &\quad - \left(c+a^3b\right)x^3 + \left(c+a^4b\right)x^4 - \left(c+a^5b\right)x^5 + \cdots \end{aligned} \tag{2.7}$$

Because $f(x)$ is an even-order function, all the odd-order terms from its series expansion are zero. The fourth-order identity between the previous functions is equivalent with the necessity of simultaneously fulfilling the following five mathematical relations [1]:

$$b+c+e=1, \tag{2.8}$$

$$d-c-ab=0, \tag{2.9}$$

$$c+a^2b=-\frac{1}{2\sigma^2}, \tag{2.10}$$

$$c+a^3b=0, \tag{2.11}$$

$$c+a^4b=\frac{1}{8\sigma^4}. \tag{2.12}$$

Solving this system, it results the following values for $a \div e$ coefficients from the general expression of the proposed $g_{1a}(x)$ approximation function [1]:

$$a=\frac{1}{4\sigma^2}, \tag{2.13}$$

$$b=\frac{32\sigma^4}{1-4\sigma^2}, \tag{2.14}$$

$$c=\frac{1}{2\sigma^2(4\sigma^2-1)}, \tag{2.15}$$

$$d = \frac{16\sigma^4 - 1}{2\sigma^2(1 - 4\sigma^2)} \tag{2.16}$$

and

$$e = \frac{64\sigma^6 - 1}{2\sigma^2(4\sigma^2 - 1)} + 1. \tag{2.17}$$

As a result, the $g_{1a}(x)$ function can be expressed replacing (2.13)–(2.17) in (2.6).

The expression of $g_{1a}(x)$ function that fourth-order approximates the particular $f(x) = \exp(-x^2)$ Gaussian function can be obtained for $A = 1$ and $\sigma = 1/\sqrt{2}$, resulting [1]:

$$g_{1a}(x) = -\frac{8}{1 + \frac{x}{2}} + \frac{1}{1 + x} - 3x + 8. \tag{2.18}$$

As the fifth-order term of the Taylor series expansion is zero, the approximation error is mainly given by the sixth-order term from the same expansion [1]:

$$\varepsilon_{f(x)}^{g_{1a}(x)} \cong \frac{1}{48\sigma^6 \exp\left(-\frac{x^2}{2\sigma^2}\right)} = \frac{1}{6\exp(-x^2)}. \tag{2.19}$$

A comparison between $f(x) - \exp(-x^2)$ and $g_{1a}(x)$ approximation functions is shown in Table 2.1.

The graphical representations of $f(x) = \exp(-x^2)$ and $g_{1a}(x)$ functions are shown in Fig. 2.1, while the graphical representation of the approximation error, $\varepsilon(x)$, defined as the difference between $g_{1a}(x)$ and $f(x) = \exp(-x^2)$ functions, is presented in Fig. 2.2.

The current-mode squaring circuit used for implementing the Gauusian function synthesizers is presented in Fig. 2.3 [2], and its symbolical representation is shown in Fig. 2.4.

The expression of the output current is

$$I_G = \frac{I_E^2}{16 I_F}. \tag{2.20}$$

Table 2.1 Comparison between $f(x) - \exp(-x^2)$ and $g_{1a}(x)$ approximation functions

x	$f(x)$	$g_{1a}(x)$	ε
−0.4	0.852	0.867	0.015
−0.2	0.961	0.961	0.000
0.0	1.000	1.000	0.000
0.2	0.961	0.961	0.000
0.4	0.852	0.848	−0.004
0.6	0.698	0.671	−0.027
0.8	0.527	0.441	−0.086

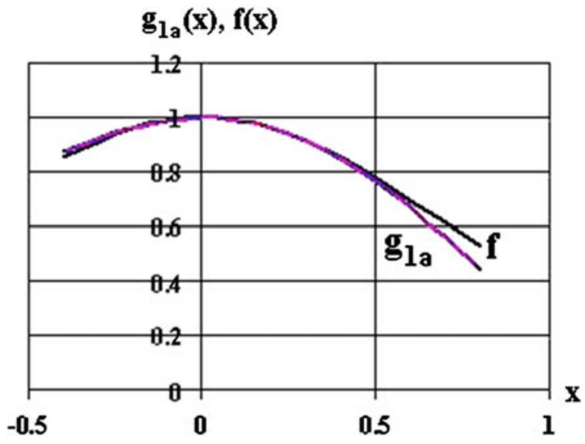


Fig. 2.1 Graphical representation of $f(x) = \exp(-x^2)$ and $g_{1a}(x)$ functions

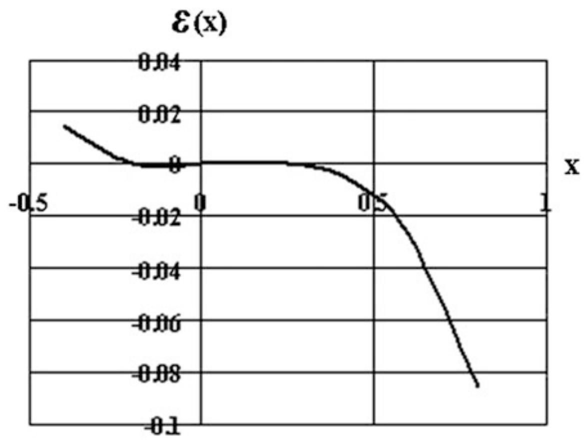


Fig. 2.2 Graphical representation of the approximation error, $\varepsilon(x)$

The block diagram of the particular Gaussian function synthesizer circuit that uses the $g_{1a}(x)$ approximation function and the implementation of the squaring circuit from Fig. 2.3 is presented in Fig. 2.5.

The expressions of I_{Ga} and I_{Gb} currents are:

$$I_{Ga} = \frac{8I_O^2}{I_O + \frac{I_{IN}}{2}} \tag{2.21}$$

and

$$I_{Gb} = \frac{I_O^2}{I_O + I_{IN}}, \tag{2.22}$$

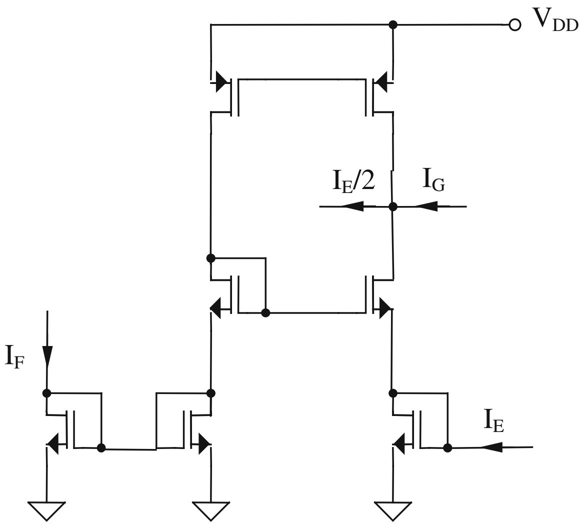


Fig. 2.3 Implementation of the current-mode squaring circuit [2]

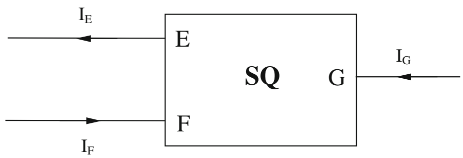


Fig. 2.4 Symbolical representation of the squaring circuit (*SQ*)

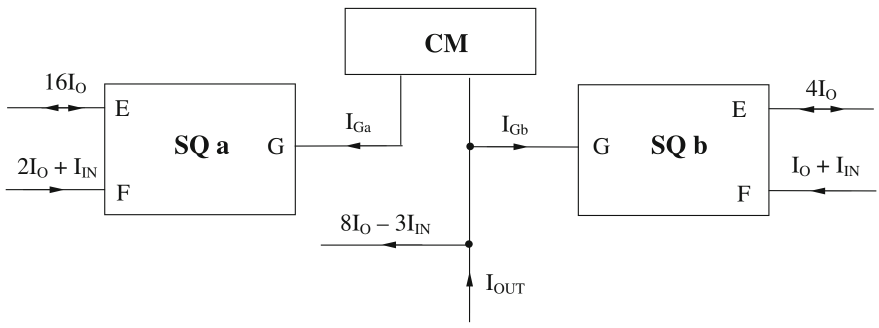


Fig. 2.5 Block diagram of the particular Gaussian function synthesizer circuit that uses the $g_{1a}(x)$ approximation function

resulting the following expression of I_{OUT} current:

$$I_{\text{OUT}} = I_{\text{Gb}} - I_{\text{Ga}} + 8I_O - 3I_{\text{IN}}, \tag{2.23}$$

equivalent with

$$I_{\text{OUT}} = I_O\left[-\frac{8}{1+\frac{1}{2}\left(\frac{I_{\text{IN}}}{I_O}\right)} + \frac{1}{1+\left(\frac{I_{\text{IN}}}{I_O}\right)} - 3\left(\frac{I_{\text{IN}}}{I_O}\right) + 8\right]. \tag{2.24}$$

So, I_{OUT} current approximates the particular Gaussian function using $g_{1a}(x)$ approximation function:

$$I_{\text{OUT}} = I_O g_{1a}\left(\frac{I_{\text{IN}}}{I_O}\right) \cong I_O \exp\left[-\left(\frac{I_{\text{IN}}}{I_O}\right)^2\right]. \tag{2.25}$$

2.2.1.2 Implementation of the General Gaussian Function

In order to implement the general Gaussian function $f(x) = A\exp\left(-x^2/2\sigma^2\right)$, the $x \to x/\sigma\sqrt{2}$ variable changing can be used, resulting the block diagram of the general Gaussian function synthesizer presented in Fig. 2.6.

The expressions of I_{Ga} and I_{Gb} currents are:

$$I_{\text{Ga}} = \frac{8I_O^2}{I_O + \frac{I_{\text{IN}}}{2\sigma\sqrt{2}}} \tag{2.26}$$

and

$$I_{\text{Gb}} = \frac{I_O^2}{I_O + \frac{I_{\text{IN}}}{\sigma\sqrt{2}}}, \tag{2.27}$$

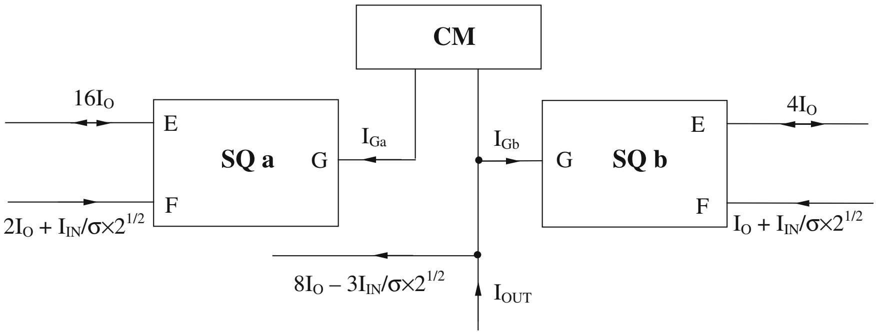


Fig. 2.6 Block diagram of the general Gaussian function synthesizer circuit that uses the $g_{1a}(x)$ approximation function

resulting the following expression of I_{OUT} current

$$I_{\mathrm{OUT}} = I_{\mathrm{Gb}} - I_{\mathrm{Ga}} + 8I_O - \frac{3I_{\mathrm{IN}}}{\sigma\sqrt{2}}, \tag{2.28}$$

equivalent with

$$I_{\mathrm{OUT}} = I_O\left[-\frac{8}{1+\frac{1}{2\sigma\sqrt{2}}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)} + \frac{1}{1+\frac{1}{\sigma\sqrt{2}}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)} - \frac{3}{\sigma\sqrt{2}}\left(\frac{I_{\mathrm{IN}}}{I_O}\right) + 8\right]. \tag{2.29}$$

So, I_{OUT} current approximates the general Gaussian function using $g_{1a}(x)$ approximation function:

$$I_{\mathrm{OUT}} = I_O g_{1a}\left[\frac{1}{\sigma\sqrt{2}}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)\right] \cong I_O \exp\left[-\frac{1}{2\sigma^2}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2\right]. \tag{2.30}$$

2.2.2 Approximation Function with Variable Changing

2.2.2.1 Implementation of the Particular Gaussian Function

A possible method for increasing the output dynamic range of the Gaussian function synthesizer based on $g_{1a}(x)$ approximation function uses the $x \to x/2$ variable changing. The resulted approximation function, $g_{1b}(x)$, allows a relatively facile CMOS implementation, using two additional current-mode squaring circuits.

$$g_{1b}(x) = \left(-\frac{8}{1+\frac{x}{4}} + \frac{1}{1+\frac{x}{2}} - \frac{3x}{2} + 8\right)^4. \tag{2.31}$$

A comparison between $f(x) = \exp(-x^2)$ and $g_{1b}(x)$ approximation functions is shown in Table 2.2.

The graphical representations of $f(x) = \exp(-x^2)$ and $g_{1b}(x)$ functions are shown in Fig. 2.7, while the graphical representation of the approximation error, $\varepsilon(x)$, defined as the difference between $g_{1b}(x)$ and $f(x) = \exp(-x^2)$ functions, is presented in Fig. 2.8.

The block diagram of the particular Gaussian function synthesizer circuit that uses the $g_{1b}(x)$ approximation function is presented in Fig. 2.9.

The expressions of I_{Ga}, I_{Gb}, and I_{Gc} currents are

$$I_{\mathrm{Ga}} = \frac{8I_O^2}{I_O + \frac{I_{\mathrm{IN}}}{4}}, \tag{2.32}$$

Table 2.2 Comparison between $f(x) = \exp(-x^2)$ and $g_{1b}(x)$ approximation functions

x	$f(x)$	$g_{1b}(x)$	ε
−0.8	0.527	0.564	0.037
−0.6	0.698	0.706	0.008
−0.4	0.852	0.853	0.001
−0.2	0.961	0.961	0.000
0.0	1.000	1.000	0.000
0.2	0.961	0.961	0.000
0.4	0.852	0.851	−0.001
0.6	0.698	0.694	−0.004
0.8	0.527	0.516	−0.011
1.0	0.368	0.345	−0.023
1.2	0.237	0.203	−0.034

Fig. 2.7 Graphical representation of $f(x) = \exp(-x^2)$ and $g_{1b}(x)$ functions

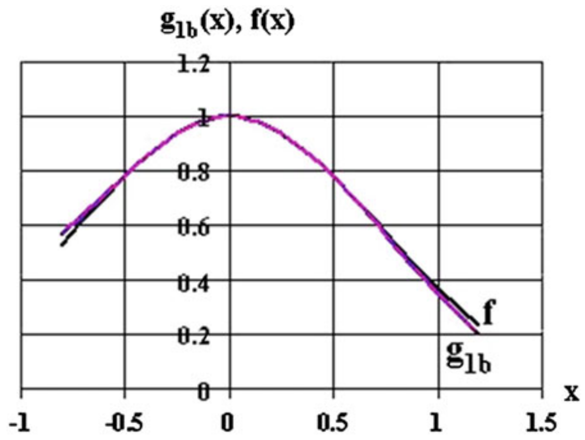
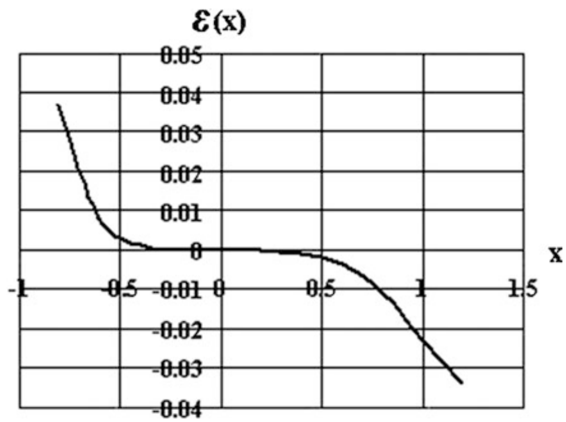


Fig. 2.8 Graphical representation of the approximation error, $\varepsilon(x)$



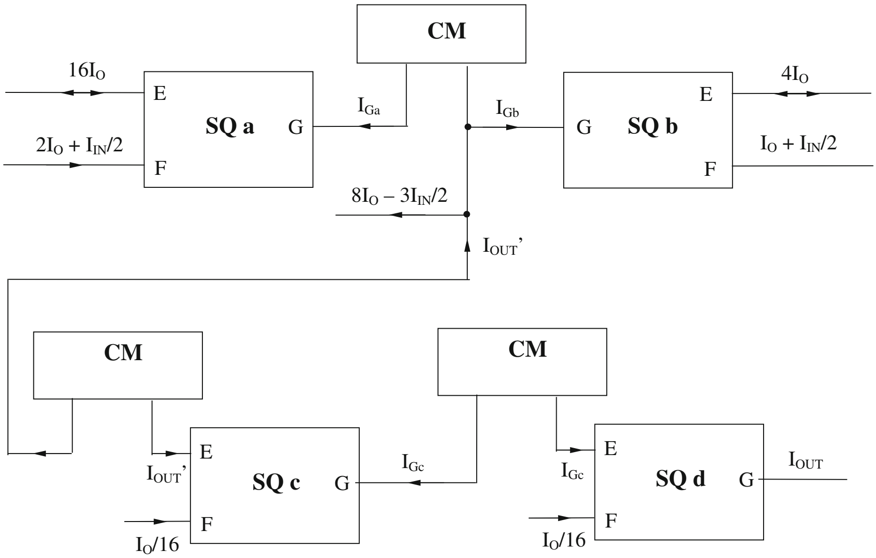


Fig. 2.9 Block diagram of the particular Gaussian function synthesizer circuit that uses the $g_{1b}(x)$ approximation function

$$I_{\mathrm{Gb}} = \frac{I_O^2}{I_O + \frac{I_{\mathrm{IN}}}{2}} \tag{2.33}$$

and

$$I_{\mathrm{Gc}} = \frac{(I_{\mathrm{OUT'}})^2}{I_O} = \frac{\left(I_{\mathrm{Gb}} - I_{\mathrm{Ga}} - \frac{3I_{\mathrm{IN}}}{2} + 8I_O\right)^2}{I_O}, \tag{2.34}$$

resulting

$$I_{\mathrm{Gc}} = I_O\left[-\frac{8}{1+\frac{1}{4}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)} + \frac{1}{1+\frac{1}{2}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)} - \frac{3}{2}\left(\frac{I_{\mathrm{IN}}}{I_O}\right) + 8\right]^2. \tag{2.35}$$

The I_{OUT} current can be expressed as follows:

$$I_{\mathrm{OUT}} = \frac{I_{\mathrm{Gc}}^2}{I_O} = I_O\left[-\frac{8}{1+\frac{1}{4}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)} + \frac{1}{1+\frac{1}{2}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)} - \frac{3}{2}\left(\frac{I_{\mathrm{IN}}}{I_O}\right) + 8\right]^4. \tag{2.36}$$

So, I_{OUT} current approximates the particular Gaussian function using $g_{1b}(x)$ approximation function:

$$I_{OUT} = I_O g_{1b}\left(\frac{I_{IN}}{I_O}\right) \cong I_O \exp\left[-\left(\frac{I_{IN}}{I_O}\right)^2\right]. \tag{2.37}$$

2.2.2.2 Implementation of the General Gaussian Function

Derived from the previous block diagram, presented for implementing the particular Gaussian function and using the previous variable changing, the block diagram of the general Gaussian function synthesizer circuit that uses the $g_{1b}(x)$ approximation function is presented in Fig. 2.10.

The expressions of I_{Ga}, I_{Gb}, and I_{Gc} currents are

$$I_{Ga} = \frac{8I_O^2}{I_O + \frac{I_{IN}}{4\sigma\sqrt{2}}}, \tag{2.38}$$

$$I_{Gb} = \frac{I_O^2}{I_O + \frac{I_{IN}}{2\sigma\sqrt{2}}} \tag{2.39}$$

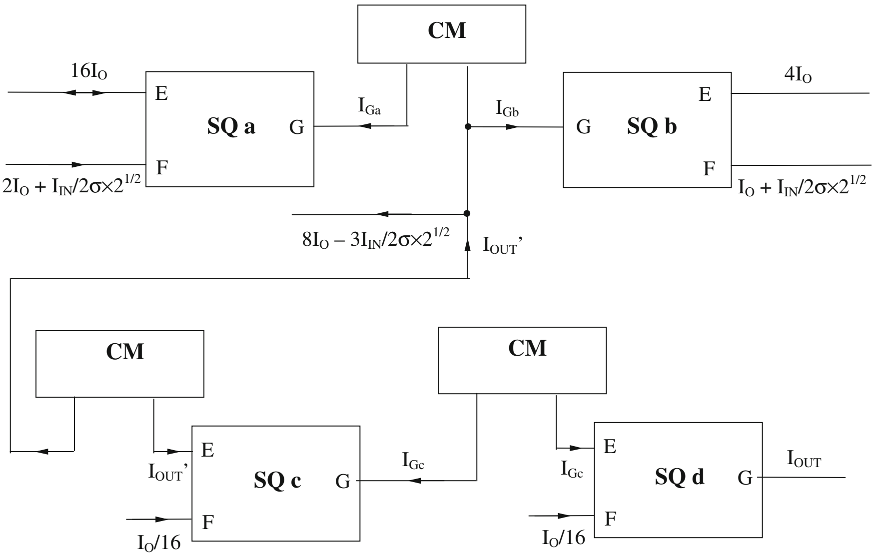


Fig. 2.10 Block diagram of the particular Gaussian function synthesizer circuit that uses the $g_{1b}(x)$ approximation function

and

$$I_{\mathrm{Gc}} = \frac{(I_{\mathrm{OUT}'})^2}{I_O} = \frac{\left(I_{\mathrm{Gb}} - I_{\mathrm{Ga}} - \frac{3I_{\mathrm{IN}}}{2\sigma\sqrt{2}} + 8I_O\right)^2}{I_O}, \tag{2.40}$$

resulting

$$I_{\mathrm{Gc}} = I_O\left[-\frac{8}{1+\frac{1}{4\sigma\sqrt{2}}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)} + \frac{1}{1+\frac{1}{2\sigma\sqrt{2}}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)} - \frac{3}{2\sigma\sqrt{2}}\left(\frac{I_{\mathrm{IN}}}{I_O}\right) + 8\right]^2. \tag{2.41}$$

The expression of I_{OUT} current will be

$$I_{\mathrm{OUT}} = \frac{I_{\mathrm{Gc}}^2}{I_O} = I_O\left[-\frac{8}{1+\frac{1}{4\sigma\sqrt{2}}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)} + \frac{1}{1+\frac{1}{2\sigma\sqrt{2}}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)} - \frac{3}{2\sigma\sqrt{2}}\left(\frac{I_{\mathrm{IN}}}{I_O}\right) + 8\right]^4. \tag{2.42}$$

Thus, I_{OUT} current approximates the particular Gaussian function using $g_{1b}(x)$ approximation function:

$$I_{\mathrm{OUT}} = I_O g_{1b}\left[\frac{1}{\sigma\sqrt{2}}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)\right] \cong I_O \exp\left[-\frac{1}{2\sigma^2}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2\right]. \tag{2.43}$$

2.3 Sixth-Order Approximation of Gaussian Function Using Approximation Functions

In order to improve the accuracy of the Gaussian function synthesizer circuits, the order of approximation can be increased. This paragraph will present sixth-order Gaussian function synthesizer structures, developed in two cases: without or including variable changing for improving the circuit performances.

2.3.1 Approximation Function Without Variable Changing

2.3.1.1 Implementation of the Particular Gaussian Function

Based on the previous considerations, a possible general form of a sixth-order approximation function could be expressed as follows:

$$g_3(x) = \frac{b}{1+ax^2} + cx^2 + d, \tag{2.44}$$

Table 2.3 Comparison between $f(x) - \exp(-x^2)$ and $g_{2a}(x)$ approximation functions

x	$f(x)$	$g_{2a}(x)$	ε
−1.2	0.237	0.261	0.024
−1.0	0.368	0.375	0.007
−0.8	0.527	0.529	0.002
−0.6	0.698	0.698	0.000
−0.4	0.852	0.852	0.000
−0.2	0.961	0.961	0.000
0.0	1.000	1.000	0.000
0.2	0.961	0.961	0.000
0.4	0.852	0.852	0.000
0.6	0.698	0.698	0.000
0.8	0.527	0.529	0.002
1.0	0.368	0.375	0.007
1.2	0.237	0.261	0.024

a, b, c, and d being constant coefficients having the values imposed by the condition that $g_{2a}(x)$ approximation function should match, in a sixth-order approximation, the Gaussian function:

$$g_{2a}(x) = \frac{9}{2}\frac{1}{1+\frac{x^2}{3}} + \frac{x^2}{2} - \frac{7}{2}. \tag{2.45}$$

A comparison between $f(x) - \exp(-x^2)$ and $g_{2a}(x)$ approximation functions is shown in Table 2.3.

The graphical representations of $f(x) = \exp(-x^2)$ and $g_{2a}(x)$ functions are shown in Fig. 2.11, while the graphical representation of the approximation error, $\varepsilon(x)$, defined as the difference between $g_{2a}(x)$ and $f(x) = \exp(-x^2)$ functions, is presented in Fig. 2.12.

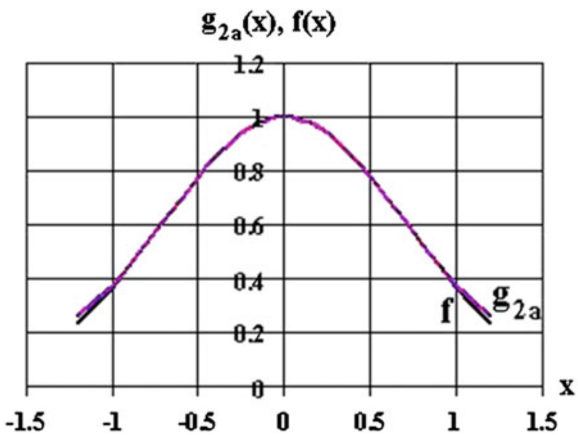
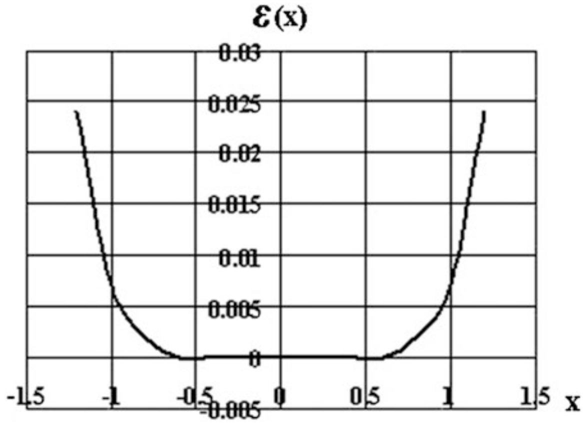


Fig. 2.11 Graphical representation of $f(x) = \exp(-x^2)$ and $g_{2a}(x)$ functions

Fig. 2.12 Graphical representation of the approximation error, $\varepsilon(x)$



The block diagram of the particular Gaussian function synthesizer circuit that uses the $g_{2a}(x)$ approximation function is presented in Fig. 2.13.

The expressions of I_{Ga} and I_{Gb} currents are

$$I_{\mathrm{Ga}} = \frac{I_{\mathrm{IN}}^2}{3I_O}, \tag{2.46}$$

$$I_{\mathrm{Gb}} = \frac{72I_O^2}{16I_{\mathrm{OUT'}}} = \frac{9}{2}\frac{I_O^2}{I_{\mathrm{Ga}} + I_O} = \frac{9}{2}\frac{I_O^2}{\frac{I_{\mathrm{IN}}^2}{3I_O} + I_O}. \tag{2.47}$$

The I_{OUT} current can be expressed as follows:

$$I_{\mathrm{OUT}} = \frac{3}{2}I_{\mathrm{Ga}} + I_{\mathrm{Gb}} - \frac{7}{2}I_O, \tag{2.48}$$

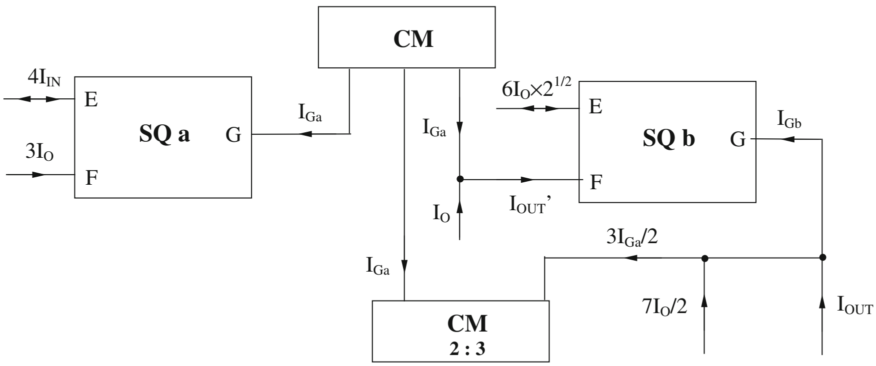


Fig. 2.13 Block diagram of the particular Gaussian function synthesizer circuit that uses the $g_{2a}(x)$ approximation function

resulting

$$I_{\text{OUT}} = I_O \left[\frac{9}{2} \frac{1}{1 + \frac{1}{3}\left(\frac{I_{\text{IN}}}{I_O}\right)^2} + \frac{1}{2}\left(\frac{I_{\text{IN}}}{I_O}\right)^2 - \frac{7}{2} \right]. \tag{2.49}$$

So, I_{OUT} current approximates the particular Gaussian function using $g_{2a}(x)$ approximation function:

$$I_{\text{OUT}} = I_O g_{2a}\left(\frac{I_{\text{IN}}}{I_O}\right) \cong I_O \exp\left[-\left(\frac{I_{\text{IN}}}{I_O}\right)^2\right]. \tag{2.50}$$

2.3.1.2 Implementation of the General Gaussian Function

The block diagram of the general Gaussian function synthesizer circuit derived from the general form of $g_{2a}(x)$ approximation function is presented in Fig. 2.14.

The I_{Ga} and I_{Gb} currents can be expressed as follows:

$$I_{\text{Ga}} = \frac{I_{\text{IN}}^2}{6\sigma^2 I_O} \tag{2.51}$$

and

$$I_{\text{Gb}} = \frac{72 I_O^2}{16 I_{\text{OUT}'}} = \frac{9}{2}\frac{I_O^2}{I_{\text{Ga}} + I_O} = \frac{9}{2}\frac{I_O^2}{\frac{I_{\text{IN}}^2}{6\sigma^2 I_O} + I_O}. \tag{2.52}$$

The expression of I_{OUT} current will be

$$I_{\text{OUT}} = \frac{3}{2} I_{\text{Ga}} + I_{\text{Gb}} - \frac{7}{2} I_O, \tag{2.53}$$

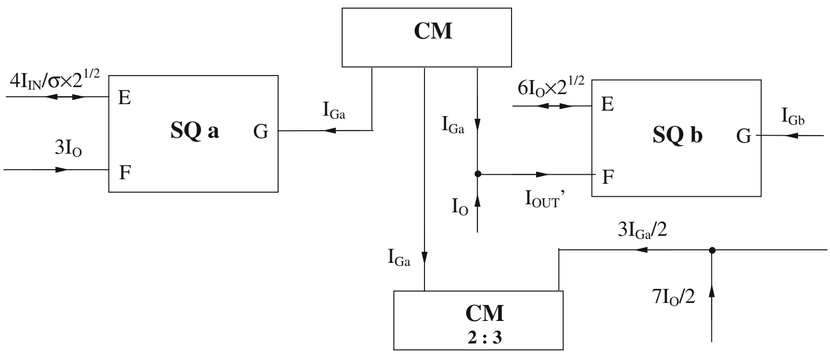


Fig. 2.14 Block diagram of the general Gaussian function synthesizer circuit that uses the $g_{2a}(x)$ approximation function

resulting

$$I_{\rm OUT} = I_O\left[\frac{9}{2}\frac{1}{1+\frac{1}{6\sigma^2}\left(\frac{I_{\rm IN}}{I_O}\right)^2} + \frac{1}{4\sigma^2}\left(\frac{I_{\rm IN}}{I_O}\right)^2 - \frac{7}{2}\right]. \tag{2.54}$$

It results that $I_{\rm OUT}$ current approximates the general Gaussian function using $g_{2a}(x)$ approximation function:

$$I_{\rm OUT} = I_O g_{2a}\left[\frac{1}{\sigma\sqrt{2}}\left(\frac{I_{\rm IN}}{I_O}\right)\right] \cong I_O \exp\left[-\frac{1}{2\sigma^2}\left(\frac{I_{\rm IN}}{I_O}\right)^2\right]. \tag{2.55}$$

2.3.2 Approximation Function with Variable Changing

2.3.2.1 Implementation of the Particular Gaussian Function

In order to improve the accuracy of the approximation and, in consequence, to increase the output dynamic range of the Gaussian function synthesizer based on $g_{2a}(x)$ approximation function, the $x \rightarrow x/2$ variable changing can be used. The resulted approximaton function, $g_{2b}(x)$, permits a relatively facile CMOS implementation, using only two additional current-mode squaring circuits.

$$g_{2b}(x) = \left(\frac{9}{2}\frac{1}{1+\frac{x^2}{12}} + \frac{x^2}{8} - \frac{7}{2}\right)^4. \tag{2.56}$$

A comparison between $f(x) = \exp(-x^2)$ and $g_{2b}(x)$ approximation functions is shown in Table 2.4.

The graphical representations of $f(x) = \exp(-x^2)$ and $g_{2b}(x)$ functions are shown in Fig. 2.15. The graphical representation of the approximation error, $\varepsilon(x)$, defined as the difference between $g_{2b}(x)$ and $f(x) = \exp(-x^2)$ functions, is presented in Fig. 2.16.

The block diagram of the particular Gaussian function synthesizer circuit that uses the $g_{2b}(x)$ approximation function is presented in Fig. 2.17.

The expressions of $I_{\rm Ga}$ and $I_{\rm Gb}$ currents are

$$I_{\rm Ga} = \frac{I_{\rm IN}^2}{12 I_O} \tag{2.57}$$

and

$$I_{\rm Gb} = \frac{72 I_O^2}{16 I_{\rm OUT'}} = \frac{9}{2}\frac{I_O^2}{I_{\rm Ga}+I_O} = \frac{9}{2}\frac{I_O^2}{\frac{I_{IN}^2}{12 I_O}+I_O}. \tag{2.58}$$

Table 2.4 Comparison between $f(x) = \exp(-x^2)$ and $g_{2b}(x)$ approximation functions

x	$f(x)$	$g_{2b}(x)$	ε
−2.4	0.003	0.005	0.002
−2.0	0.018	0.020	0.002
−1.6	0.077	0.078	0.001
−1.2	0.237	0.237	0.000
−0.8	0.527	0.527	0.000
−0.4	0.852	0.852	0.000
0.0	1.000	1.000	0.000
0.4	0.852	0.852	0.000
0.8	0.527	0.527	0.000
1.2	0.237	0.237	0.000
1.6	0.077	0.078	0.001
2.0	0.018	0.020	0.002
2.4	0.003	0.005	0.002

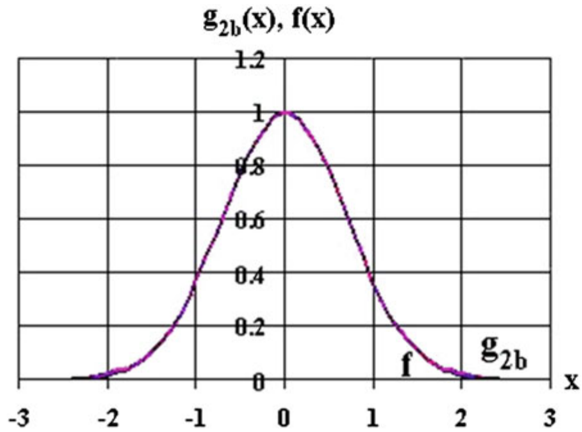


Fig. 2.15 Graphical representation of $f(x) = \exp(-x^2)$ and $g_{2b}(x)$ functions

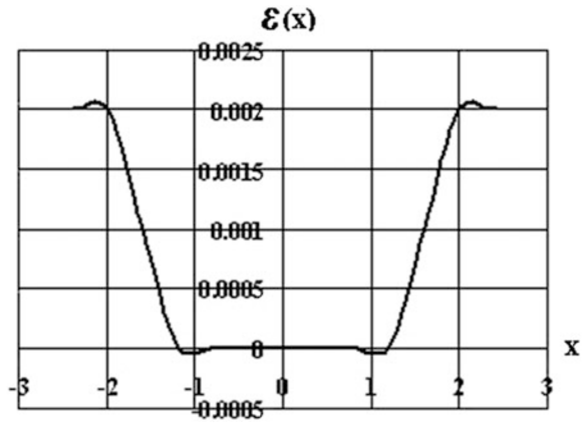


Fig. 2.16 Graphical representation of the approximation error, $\varepsilon(x)$

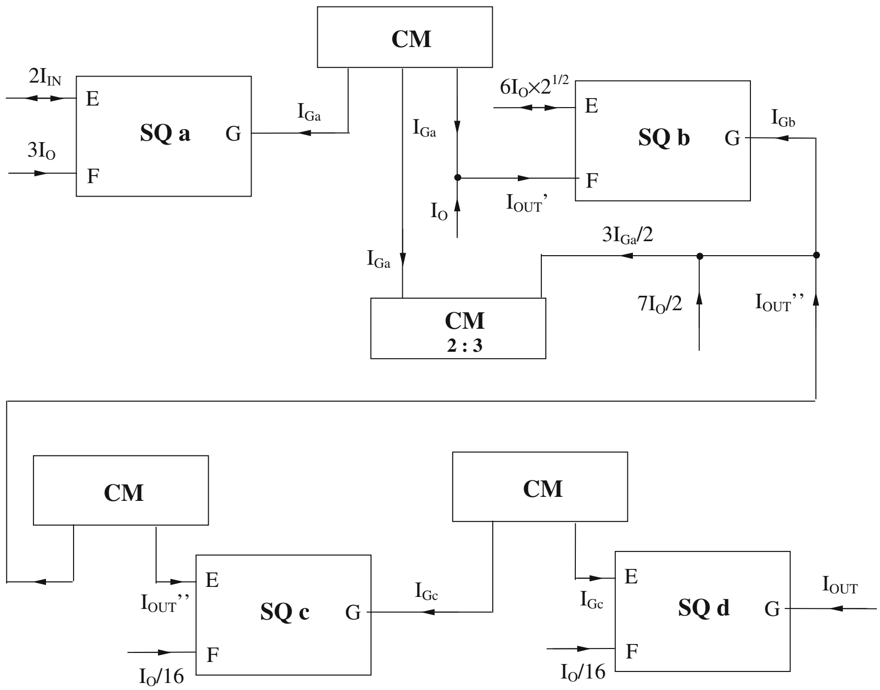


Fig. 2.17 Block diagram of the particular Gaussian function synthesizer circuit that uses the $g_{2b}(x)$ approximation function

The $I_{\mathrm{OUT}''}$ current can be expressed as follows:

$$I_{\mathrm{OUT}''} = \frac{3}{2} I_{\mathrm{Ga}} + I_{\mathrm{Gb}} - \frac{7}{2} I_O. \tag{2.59}$$

It results

$$I_{\mathrm{OUT}''} = I_O \left[\frac{9}{2} \frac{1}{1 + \frac{1}{12}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2} + \frac{1}{8}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2 - \frac{7}{2} \right]. \tag{2.60}$$

The expression of I_{Gc} current is

$$I_{\mathrm{Gc}} = \frac{(I_{\mathrm{OUT}''})^2}{I_O} = I_O \left[\frac{9}{2} \frac{1}{1 + \frac{1}{12}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2} + \frac{1}{8}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2 - \frac{7}{2} \right]^2, \tag{2.61}$$

resulting the following expression of the output current:

$$I_{\text{OUT}} = \frac{I_{\text{Gc}}^2}{I_O} = I_O \left[\frac{9}{2} \frac{1}{1 + \frac{1}{12}\left(\frac{I_{\text{IN}}}{I_O}\right)^2} + \frac{1}{8}\left(\frac{I_{\text{IN}}}{I_O}\right)^2 - \frac{7}{2} \right]^4. \tag{2.62}$$

So, I_{OUT} current approximates the particular Gaussian function using $g_{2b}(x)$ approximation function:

$$I_{\text{OUT}} = I_O g_{2b}\left(\frac{I_{\text{IN}}}{I_O}\right) \cong I_O \exp\left[-\left(\frac{I_{\text{IN}}}{I_O}\right)^2\right]. \tag{2.63}$$

2.3.2.2 Implementation of the General Gaussian Function

The block diagram of the general Gaussian function synthesizer circuit that uses the $g_{2b}(x)$ approximation function can be obtained from the previous block diagram using the $x \rightarrow x/\sigma\sqrt{2}$ variable changing (Fig. 2.18).

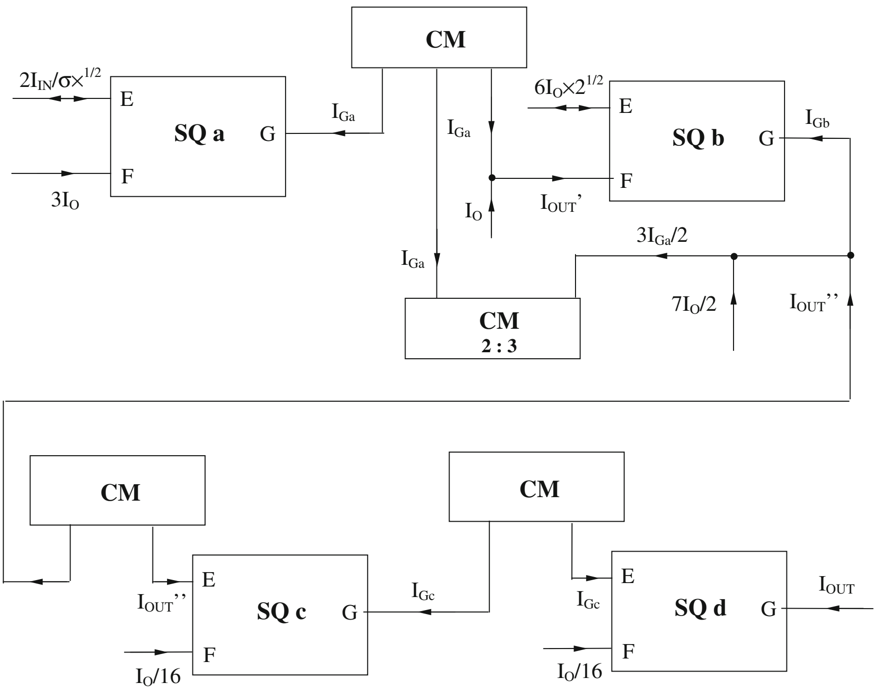


Fig. 2.18 Block diagram of the general Gaussian function synthesizer circuit that uses the $g_{2b}(x)$ approximation function

The I_{Ga} and I_{Gb} currents have the following expressions:

$$I_{\mathrm{Ga}} = \frac{I_{\mathrm{IN}}^2}{24\sigma^2 I_O} \tag{2.64}$$

and

$$I_{\mathrm{Gb}} = \frac{72 I_O^2}{16 I_{\mathrm{OUT}'}} = \frac{9}{2}\frac{I_O^2}{I_{\mathrm{Ga}} + I_O} = \frac{9}{2}\frac{I_O^2}{\frac{I_{\mathrm{IN}}^2}{24\sigma^2 I_O} + I_O}. \tag{2.65}$$

The $I_{\mathrm{OUT}''}$ current can be expressed as follows:

$$I_{\mathrm{OUT}''} = \frac{3}{2} I_{\mathrm{Ga}} + I_{\mathrm{Gb}} - \frac{7}{2} I_O, \tag{2.66}$$

resulting

$$I_{\mathrm{OUT}''} = I_O \left[\frac{9}{2} \frac{1}{1 + \frac{1}{24\sigma^2}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2} + \frac{1}{16\sigma^2}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2 - \frac{7}{2} \right]. \tag{2.67}$$

The expression of I_{Gc} current is

$$I_{\mathrm{Gc}} = \frac{(I_{\mathrm{OUT}''})^2}{I_O} = I_O \left[\frac{9}{2} \frac{1}{1 + \frac{1}{24\sigma^2}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2} + \frac{1}{16\sigma^2}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2 - \frac{7}{2} \right]^2, \tag{2.68}$$

while the output current of the circuit can be expressed as follows:

$$I_{\mathrm{OUT}} = \frac{I_{\mathrm{Gc}}^2}{I_O} = I_O \left[\frac{9}{2} \frac{1}{1 + \frac{1}{24\sigma^2}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2} + \frac{1}{16\sigma^2}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2 - \frac{7}{2} \right]^4. \tag{2.69}$$

In conclusion, I_{OUT} current approximates the particular Gaussian function using $g_{2b}(x)$ approximation function:

$$I_{\mathrm{OUT}} = I_O g_{2b}\left[\frac{1}{\sigma\sqrt{2}}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)\right] \cong I_O \exp\left[-\frac{1}{2\sigma^2}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2\right]. \tag{2.70}$$

2.4 Sixth-Order Approximation of Gaussian Function Using Limited Taylor Series

A possible implementation of a sixth-order approximation function for developing an improved accuracy Gaussian function synthesizer circuit uses the sixth-order Taylor series expansion of the Gaussian function. In order to additionally increase the approximation accuracy, a proper variable changing can be used.

2.4.1 Approximation Function Without Variable Changing

2.4.1.1 Implementation of the Particular Gaussian Function

The sixth-order approximation function that uses the limited Taylor expansion of the Gaussian function can be expressed as follows:

$$g_{3a}(x) = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6}. \tag{2.71}$$

A comparison between $f(x) - \exp(-x^2)$ and $g_{3a}(x)$ approximation functions is shown in Table 2.5.

The graphical representations of $f(x) = \exp(-x^2)$ and $g_{3a}(x)$ functions are shown in Fig. 2.19, while the graphical representation of the approximation error, $\varepsilon(x)$, defined as the difference between $g_{3a}(x)$ and $f(x) = \exp(-x^2)$ functions, is presented in Fig. 2.20.

The block diagram of the particular Gaussian function synthesizer circuit that uses the $g_{3a}(x)$ approximation function is presented in Fig. 2.21.

The I_{Ga}, I_{Gb}, and I_{Gc} currents' expressions are

$$I_{Ga} = \frac{I_{IN}^2}{I_O}, \tag{2.72}$$

Table 2.5 Comparison between $f(x) - \exp(-x^2)$ and $g_{3a}(x)$ approximation functions

x	$f(x)$	$g_{3a}(x)$	ε
−1.0	0.368	0.333	−0.035
−0.8	0.527	0.521	−0.006
−0.6	0.698	0.697	−0.001
−0.4	0.852	0.852	0.000
−0.2	0.961	0.961	0.000
0.0	1.000	1.000	0.000
0.2	0.961	0.961	0.000
0.4	0.852	0.852	0.000
0.6	0.698	0.697	−0.001
0.8	0.527	0.521	−0.006
1.0	0.368	0.333	−0.035

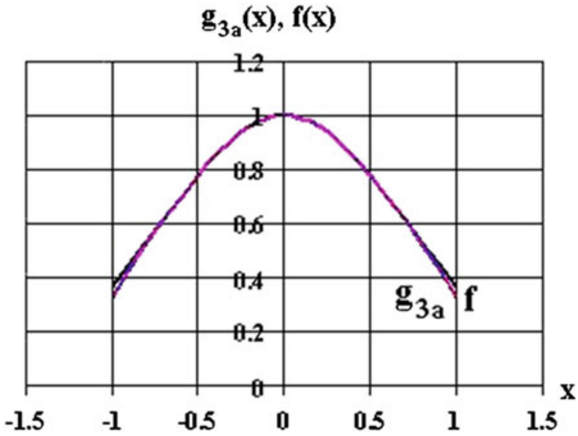


Fig. 2.19 Graphical representation of $f(x) = \exp(-x^2)$ and $g_{3a}(x)$ functions

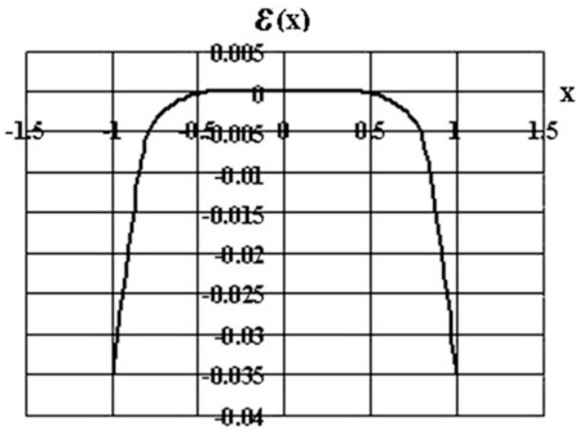


Fig. 2.20 Graphical representation of the approximation error, $\varepsilon(x)$

$$I_{\mathrm{Gb}} = \frac{I_{G1}^2}{2I_O} = \frac{I_{\mathrm{IN}}^4}{2I_O^3} \tag{2.73}$$

and

$$I_{\mathrm{Gc}} = \frac{2}{3}\frac{I_{\mathrm{Gb}}^2}{I_{\mathrm{Ga}}} = \frac{I_{\mathrm{IN}}^6}{6I_O^5}. \tag{2.74}$$

The I_{OUT} current can be expressed as follows:

$$I_{\mathrm{OUT}} = I_{\mathrm{Gb}} - I_{\mathrm{Ga}} - I_{\mathrm{Gc}} + I_O. \tag{2.75}$$

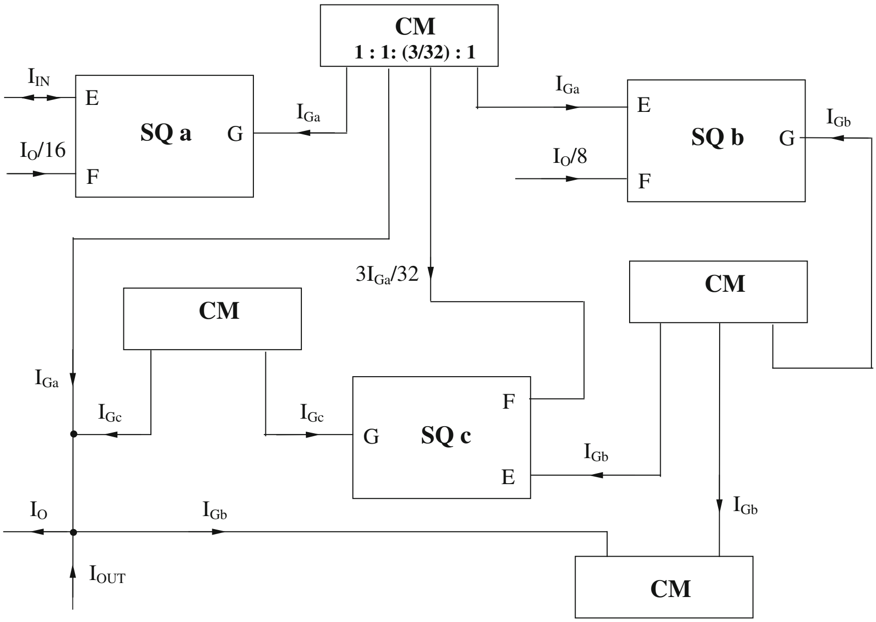


Fig. 2.21 Block diagram of the particular Gaussian function synthesizer circuit that uses the $g_{3a}(x)$ approximation function

It results

$$I_{\mathrm{OUT}} = I_O\left[1 - \left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2 + \frac{1}{2}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^4 - \frac{1}{6}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^6\right]. \quad (2.76)$$

So, I_{OUT} current approximates the particular Gaussian function using $g_{3a}(x)$ approximation function:

$$I_{\mathrm{OUT}} = I_O g_{3a}\left(\frac{I_{\mathrm{IN}}}{I_O}\right) \cong I_O \exp\left[-\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2\right]. \quad (2.77)$$

2.4.1.2 Implementation of the General Gaussian Function

The block diagram of the function synthesizer circuit that implements the general Gaussian function based on $g_{2b}(x)$ approximation function is presented in Fig. 2.22. The same variable changing was used, $x \to x/\sigma\sqrt{2}$.

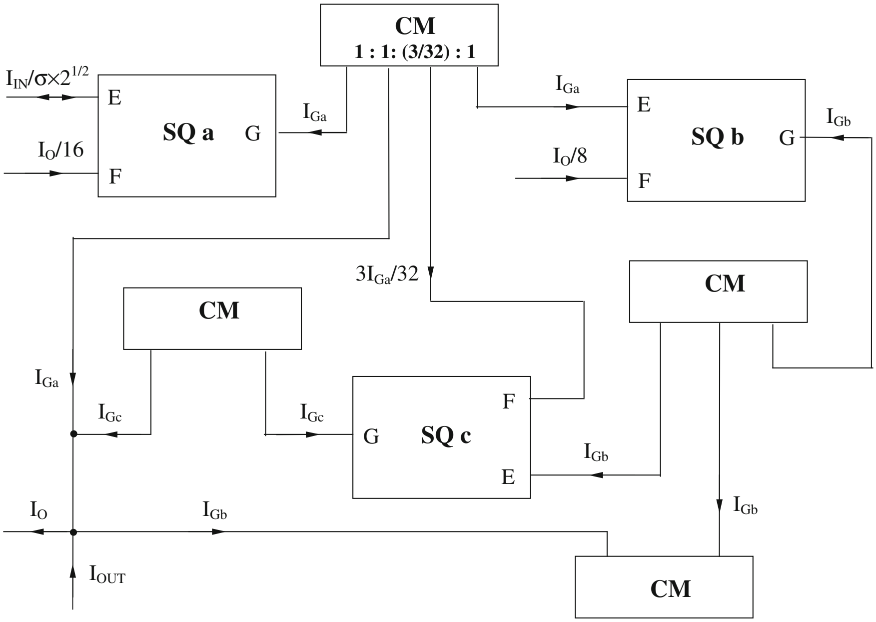


Fig. 2.22 Block diagram of the general Gaussian function synthesizer circuit that uses the $g_{3a}(x)$ approximation function

The expressions of I_{Ga}, I_{Gb}, and I_{Gc} currents are

$$I_{\mathrm{Ga}} = \frac{I_{\mathrm{IN}}^2}{2\sigma^2 I_O}, \tag{2.78}$$

$$I_{\mathrm{Gb}} = \frac{I_{G1}^2}{2I_O} = \frac{I_{\mathrm{IN}}^4}{8\sigma^4 I_O^3} \tag{2.79}$$

and

$$I_{\mathrm{Gc}} = \frac{2}{3}\frac{I_{\mathrm{Gb}}^2}{I_{\mathrm{Ga}}} = \frac{I_{\mathrm{IN}}^6}{48\sigma^6 I_O^5}. \tag{2.80}$$

The I_{OUT} current can be expressed as follows:

$$I_{\mathrm{OUT}} = I_{\mathrm{Gb}} - I_{\mathrm{Ga}} - I_{\mathrm{Gc}} + I_O, \tag{2.81}$$

resulting

$$I_{\mathrm{OUT}} = I_O\left[1 - \frac{1}{2\sigma^2}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2 + \frac{1}{8\sigma^4}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^4 - \frac{1}{48\sigma^6}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^6\right]. \tag{2.82}$$

Thus, I_{OUT} current approximates the particular Gaussian function using $g_{3a}(x)$ approximation function:

$$I_{\mathrm{OUT}} = I_O g_{3a}\left[\frac{1}{\sigma\sqrt{2}}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)\right] \cong I_O \exp\left[-\frac{1}{2\sigma^2}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2\right]. \tag{2.83}$$

2.4.2 Approximation Function with Variable Changing

2.4.2.1 Implementation of the Particular Gaussian Function

The increase in the Gaussian function synthesizer output dynamic range can be obtained, considering the $x \rightarrow x/2$ variable changing for developing the expression of $g_{3b}(x)$ approximation function. Using the $g_{3a}(x)$ function and the previously mentioned variable changing, the new approximation function can be expressed as follows:

$$g_{3b}(x) = \left(1 - \frac{x^2}{4} + \frac{x^4}{32} - \frac{x^6}{384}\right)^4. \tag{2.84}$$

A comparison between $f(x) = \exp(-x^2)$ and $g_{3b}(x)$ approximation functions is shown in Table 2.6.

Table 2.6 Comparison between $f(x) = \exp(-x^2)$ and $g_{3b}(x)$ approximation functions

x	$f(x)$	$g_{3b}(x)$	ε
−1.8	0.039	0.034	−0.005
−1.6	0.077	0.074	−0.003
−1.4	0.141	0.139	−0.002
−1.2	0.237	0.236	−0.001
−1.0	0.368	0.368	0.000
−0.8	0.527	0.527	0.000
−0.6	0.698	0.698	0.000
−0.4	0.852	0.852	0.000
−0.2	0.961	0.961	0.000
0.0	1.000	1.000	0.000
0.2	0.961	0.961	0.000
0.4	0.852	0.852	0.000
0.6	0.698	0.698	0.000
0.8	0.527	0.527	0.000
1.0	0.368	0.368	0.000
1.2	0.237	0.236	−0.001
1.4	0.141	0.139	−0.002
1.6	0.077	0.074	−0.003
1.8	0.039	0.034	−0.005

The graphical representations of $f(x) = \exp(-x^2)$ and $g_{3b}(x)$ functions are shown in Fig. 2.23, while the graphical representation of the approximation error, $\varepsilon(x)$, defined as the difference between $g_{3b}(x)$ and $f(x) = \exp(-x^2)$ functions, is presented in Fig. 2.24.

The block diagram of the particular Gaussian function synthesizer circuit that uses the $g_{3b}(x)$ approximation function is derived from the block diagram of the circuit using $g_{3a}(x)$ approximation function, having two additional current-mode squaring circuits. The output current of this Gaussian function synthesizer circuit will approximate the particular Gaussian function using $g_{3b}(x)$ approximation function:

$$I_{OUT} = I_O g_{3b}\left(\frac{I_{IN}}{I_O}\right) \cong I_O \exp\left[-\left(\frac{I_{IN}}{I_O}\right)^2\right]. \tag{2.85}$$

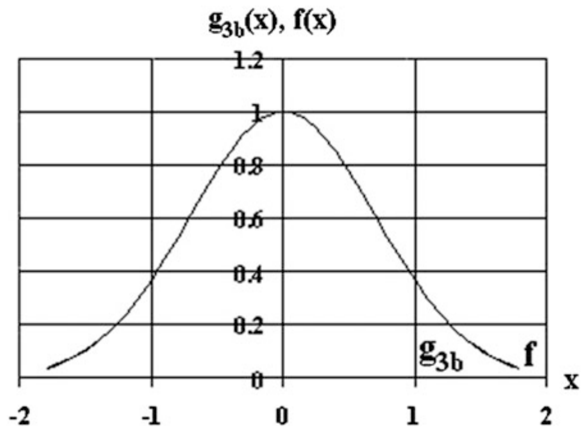


Fig. 2.23 Graphical representation of $f(x) = \exp(-x^2)$ and $g_{3b}(x)$ functions

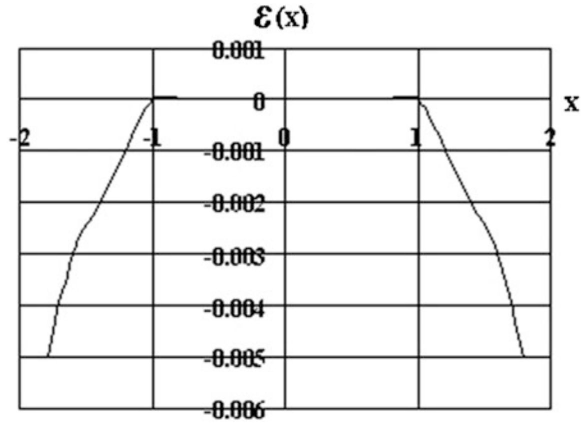


Fig. 2.24 Graphical representation of the approximation error, $\varepsilon(x)$

2.4.2.2 Implementation of the General Gaussian Function

Similar with the previous presented method, the $x \to x/\sigma\sqrt{2}$ variable changing allows to obtain an improved accuracy Gaussian function synthesizer circuit, based on the general form of the $g_{3b}(x)$ approximation function. The output current of the Gaussian function synthesizer structure, using the general form of $g_{3b}(x)$ approximation function, will accurately approximate the general Gaussian function:

$$I_{OUT} = I_O g_{3b}\left[\frac{1}{\sigma\sqrt{2}}\left(\frac{I_{IN}}{I_O}\right)\right] \cong I_O \exp\left[-\frac{1}{2\sigma^2}\left(\frac{I_{IN}}{I_O}\right)^2\right]. \tag{2.86}$$

2.5 Eighth-Order Approximation of Gaussian Function Using Approximation Functions

For applications that require extremely accurate generation of the Gaussian function, eighth-order approximation functions can be developed, taking into account similar restrictions comparing with previously designed computational structures.

2.5.1 Approximation Function Without Variable Changing

2.5.1.1 Implementation of the Particular Gaussian Function

The general form of a possible eighth-order approximation function could be expressed as follows:

$$g_{4a}(x) = \frac{b}{1 + ax^2} + cx^4 + dx^2 + e, \tag{2.87}$$

a, b, c, d, and e being constant coefficients having the values imposed by the condition that $g_{4a}(x)$ approximation function should match, in a sixth-order approximation, the Gaussian function:

$$g_{4a}(x) = \frac{32}{3}\frac{1}{1 + \frac{x^2}{4}} - \frac{x^4}{6} + \frac{5x^2}{3} - \frac{29}{3}. \tag{2.88}$$

A comparison between $f(x) - \exp(-x^2)$ and $g_{4a}(x)$ approximation functions is shown in Table 2.7.

Table 2.7 Comparison between $f(x) - \exp(-x^2)$ and $g_{4a}(x)$ approximation functions

x	$f(x)$	$g_{4a}(x)$	ε
−1.2	0.237	0.231	−0.006
−1.0	0.368	0.367	−0.001
−0.8	0.527	0.527	0.000
−0.6	0.698	0.698	0.000
−0.4	0.852	0.852	0.000
−0.2	0.961	0.961	0.000
0.0	1.000	1.000	0.000
0.2	0.961	0.961	0.000
0.4	0.852	0.852	0.000
0.6	0.698	0.698	0.000
0.8	0.527	0.527	0.000
1.0	0.368	0.367	−0.001
1.2	0.237	0.231	−0.006

The graphical representations of $f(x) = \exp(-x^2)$ and $g_{4a}(x)$ functions are shown in Fig. 2.25, while the graphical representation of the approximation error, $\varepsilon(x)$, defined as the difference between $g_{4a}(x)$ and $f(x) = \exp(-x^2)$ functions, is presented in Fig. 2.26.

The block diagram of the particular Gaussian function synthesizer circuit that uses the $g_{4a}(x)$ approximation function is presented in Fig. 2.27.

The expressions of I_{Ga}, I_{Gb}, and I_{Gc} currents are

$$I_{Ga} = \frac{I_{IN}^2}{4I_O}, \tag{2.89}$$

$$I_{Gb} = \frac{32I_O^2}{3I_{OUT'}} = \frac{32}{3}\frac{I_O^2}{I_{Ga} + I_O} = \frac{32}{3}\frac{I_O^2}{\frac{I_{IN}^2}{4I_O} + I_O} \tag{2.90}$$

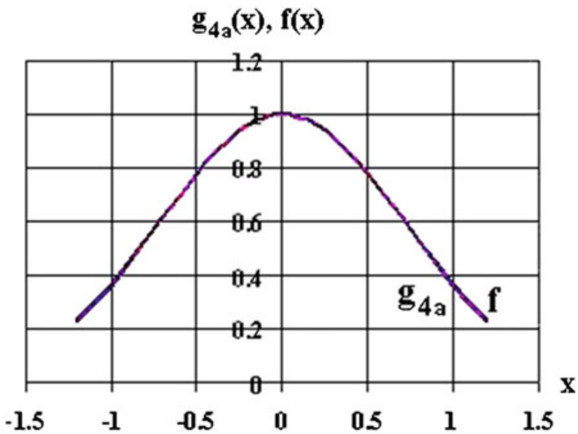


Fig. 2.25 Graphical representation of $f(x) = \exp(-x^2)$ and $g_{4a}(x)$ functions

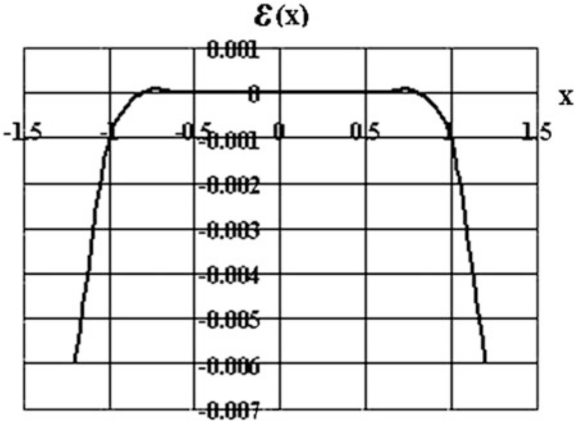


Fig. 2.26 Graphical representation of the approximation error, $\varepsilon(x)$

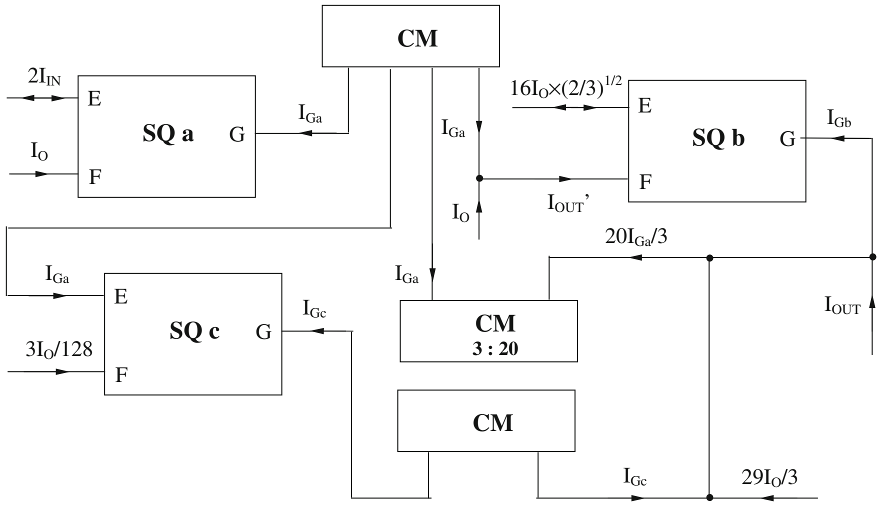


Fig. 2.27 Block diagram of the particular Gaussian function synthesizer circuit that uses the $g_{4a}(x)$ approximation function

and

$$I_{\mathrm{Gc}} = \frac{8}{3}\frac{I_{\mathrm{Ga}}^2}{I_O} = \frac{I_{\mathrm{IN}}^4}{6I_O^3}. \tag{2.91}$$

The I_{OUT} current expression will be

$$I_{\mathrm{OUT}} = I_{\mathrm{Gb}} + \frac{20}{3}I_{\mathrm{Ga}} - I_{\mathrm{Gc}} - \frac{29}{3}I_O, \tag{2.92}$$

resulting

$$I_{\mathrm{OUT}} = I_O\left[\frac{32}{3}\frac{1}{1+\frac{1}{4}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2} + \frac{5}{3}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2 - \frac{1}{6}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^4 - \frac{29}{3}\right]. \tag{2.93}$$

So, I_{OUT} current approximates the particular Gaussian function using $g_{4a}(x)$ approximation function:

$$I_{\mathrm{OUT}} = I_O g_{4a}\left(\frac{I_{\mathrm{IN}}}{I_O}\right) \cong I_O \exp\left[-\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2\right]. \tag{2.94}$$

2.5.1.2 Implementation of the General Gaussian Function

Using the previous function synthesizer circuit, developed for implementing the particular form of the $g_{4a}(x)$ approximation function, it is possible to design another Gaussian function synthesizer for implementing the general form of the same $g_{4a}(x)$ approximation function (Fig. 2.28).

The expressions of I_{Ga}, I_{Gb}, and I_{Gc} currents are

$$I_{\mathrm{Ga}} = \frac{I_{\mathrm{IN}}^2}{8\sigma^2 I_O}, \tag{2.95}$$

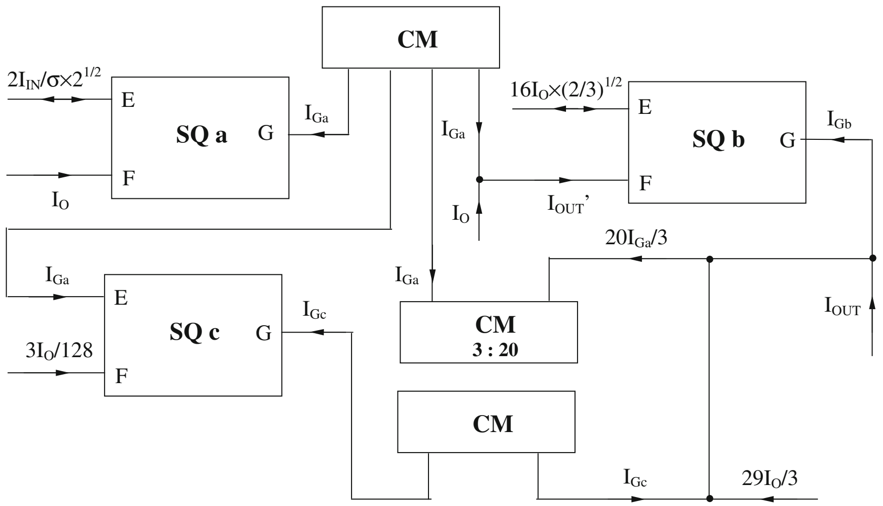


Fig. 2.28 Block diagram of the general Gaussian function synthesizer circuit that uses the $g_{4a}(x)$ approximation function

$$I_{\mathrm{Gb}} = \frac{32 I_O^2}{3 I_{\mathrm{OUT'}}} = \frac{32}{3} \frac{I_O^2}{I_{\mathrm{Ga}} + I_O} = \frac{32}{3} \frac{I_O^2}{\frac{I_{\mathrm{IN}}^2}{8\sigma^2 I_O} + I_O} \tag{2.96}$$

and

$$I_{\mathrm{Gc}} = \frac{8}{3} \frac{I_{\mathrm{Ga}}^2}{I_O} = \frac{I_{\mathrm{IN}}^4}{24\sigma^4 I_O^3}. \tag{2.97}$$

So, the I_{OUT} current will have the following expression:

$$I_{\mathrm{OUT}} = I_{\mathrm{Gb}} + \frac{20}{3} I_{\mathrm{Ga}} - I_{\mathrm{Gc}} - \frac{29}{3} I_O, \tag{2.98}$$

resulting

$$I_{\mathrm{OUT}} = I_O \left[\frac{32}{3} \frac{1}{1 + \frac{1}{8\sigma^2}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2} + \frac{5}{6\sigma^2}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2 - \frac{1}{24\sigma^4}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^4 - \frac{29}{3} \right]. \tag{2.99}$$

Thus, I_{OUT} current approximates the particular Gaussian function using $g_{4a}(x)$ approximation function:

$$I_{\mathrm{OUT}} = I_O g_{4a}\left[\frac{1}{\sigma\sqrt{2}}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)\right] \cong I_O \exp\left[-\frac{1}{2\sigma^2}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2\right]. \tag{2.100}$$

2.5.2 Approximation Function with Variable Changing

2.5.2.1 Implementation of the Particular Gaussian Function

In order to obtain extremely accurate generation of the Gaussian function, additionally to the eighth-order of approximation, the classical $x \to x/2$ variable changing can be used. The increase in complexity for the resulted approximation function, $g_{4b}(x)$, is represented by only two additional current-mode squaring circuits:

$$g_{4b}(x) = \left(\frac{32}{3} \frac{1}{1 + \frac{x^2}{16}} - \frac{x^4}{96} + \frac{5x^2}{12} - \frac{29}{3} \right)^4. \tag{2.101}$$

A comparison between $f(x) = \exp(-x^2)$ and $g_{4b}(x)$ approximation functions is shown in Table 2.8.

The graphical representations of $f(x) = \exp(-x^2)$ and $g_{4b}(x)$ functions are shown in Fig. 2.29, while the graphical representation of the approximation error, $\varepsilon(x)$, defined as the difference between $g_{4b}(x)$ and $f(x) = \exp(-x^2)$ functions, is presented in Fig. 2.30.

Table 2.8 Comparison between $f(x) = \exp(-x^2)$ and $g_{4b}(x)$ approximation functions

x	$f(x)$	$g_{4b}(x)$	ε
−2.8	0.0004	0.0002	0.0002
−2.4	0.003	0.003	0.000
−2.0	0.018	0.018	0.000
−1.6	0.077	0.077	0.000
−1.2	0.237	0.237	0.000
−0.8	0.527	0.527	0.000
−0.4	0.852	0.852	0.000
0.0	1.000	1.000	0.000
0.4	0.852	0.852	0.000
0.8	0.527	0.527	0.000
1.2	0.237	0.237	0.000
1.6	0.077	0.077	0.000
2.0	0.018	0.018	0.000
2.4	0.003	0.003	0.000
2.8	0.0004	0.0002	−0.0002

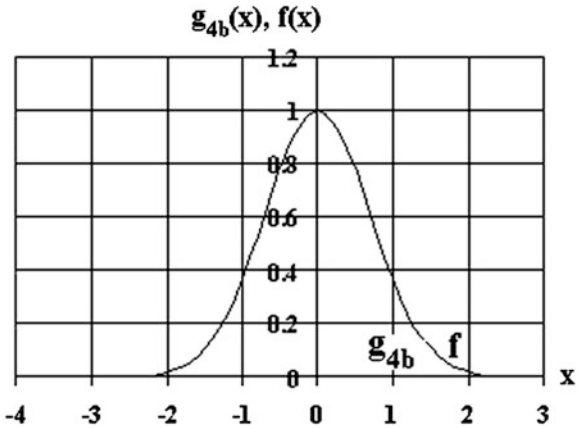


Fig. 2.29 Graphical representation of $f(x) = \exp(-x^2)$ and $g_{4b}(x)$ functions

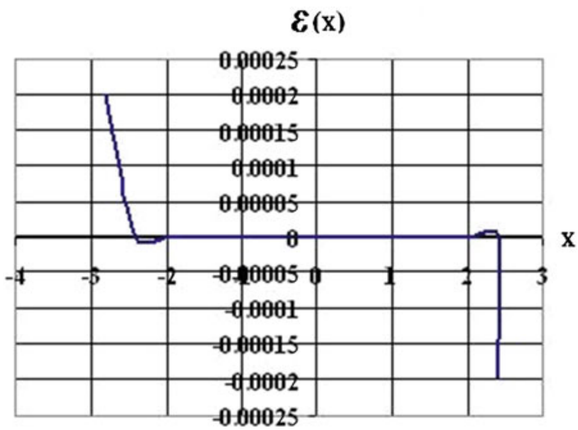


Fig. 2.30 Graphical representation of the approximation error, $\varepsilon(x)$

The block diagram of the particular Gaussian function synthesizer circuit that uses the $g_{4b}(x)$ approximation function is presented in Fig. 2.31.

The expressions of I_{Ga}, I_{Gb}, and I_{Gc} currents are

$$I_{\mathrm{Ga}} = \frac{I_{\mathrm{IN}}^2}{16 I_O}, \tag{2.102}$$

$$I_{\mathrm{Gb}} = \frac{32 I_O^2}{3 I_{\mathrm{OUT'}}} = \frac{32}{3} \frac{I_O^2}{I_{G1} + I_O} = \frac{32}{3} \frac{I_O^2}{\frac{I_{\mathrm{IN}}^2}{16 I_O} + I_O} \tag{2.103}$$

and

$$I_{\mathrm{Gc}} = \frac{8}{3} \frac{I_{G1}^2}{I_O} = \frac{I_{\mathrm{IN}}^4}{96 I_O^3}. \tag{2.104}$$

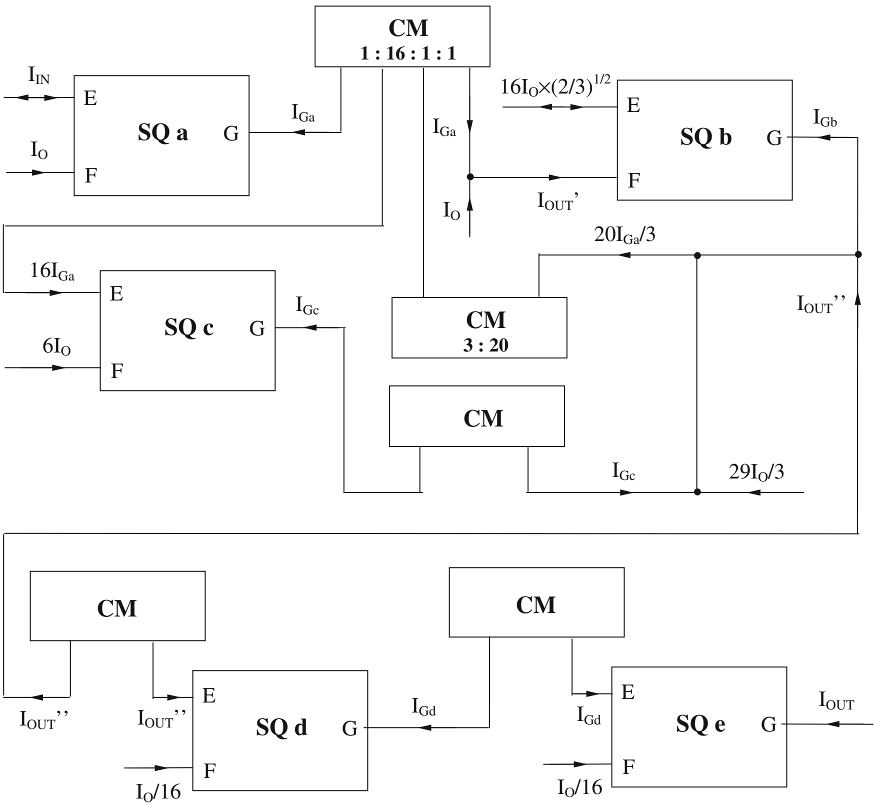


Fig. 2.31 Block diagram of the particular Gaussian function synthesizer circuit that uses the $g_{4b}(x)$ approximation function

The $I_{OUT''}$ current can be expressed as follows:

$$I_{\mathrm{OUT}''} = I_{\mathrm{Gb}} + \frac{20}{3} I_{\mathrm{Ga}} - I_{\mathrm{Gc}} - \frac{29}{3} I_O, \tag{2.105}$$

resulting

$$I_{\mathrm{OUT}''} = I_O \left[\frac{32}{3} \frac{1}{1 + \frac{1}{16}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2} + \frac{5}{12}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2 - \frac{1}{96}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^4 - \frac{29}{3} \right]. \tag{2.106}$$

The I_{Gd} and I_{OUT} currents have the following expressions:

$$I_{\mathrm{Gd}} = \frac{\left(I_{\mathrm{OUT}''}\right)^2}{I_O} \tag{2.107}$$

and

$$I_{\mathrm{OUT}} = \frac{I_{\mathrm{Gd}}^2}{I_O} = \frac{\left(I_{\mathrm{OUT}''}\right)^4}{I_O^3}. \tag{2.108}$$

Thus, the output current of the circuit can be expressed as follows:

$$I_{\mathrm{OUT}} = I_O \left[\frac{32}{3} \frac{1}{1 + \frac{1}{16}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2} + \frac{5}{12}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2 - \frac{1}{96}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^4 - \frac{29}{3} \right]^4. \tag{2.109}$$

In conclusion, I_{OUT} current approximates the Gaussian function using $g_{4b}(x)$ approximation function:

$$I_{\mathrm{OUT}} = I_O g_{4b}\left(\frac{I_{\mathrm{IN}}}{I_O}\right) \cong I_O \exp\left[-\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2 \right]. \tag{2.110}$$

2.5.2.2 Implementation of the General Gaussian Function

The block diagram of the Gaussian function synthesizer circuit that is able to implement the general form of $g_{4b}(x)$ approximation function is derived of the previous block diagram, realizing the $x \to x/\sigma\sqrt{2}$ variable changing (Fig. 2.32).

The expressions of I_{Ga}, I_{Gb}, and I_{Gc} currents are

$$I_{\mathrm{Ga}} = \frac{I_{\mathrm{IN}}^2}{32\sigma^2 I_O}, \tag{2.111}$$

$$I_{\mathrm{Gb}} = \frac{32 I_O^2}{3 I_{\mathrm{OUT}'}} = \frac{32}{3} \frac{I_O^2}{I_{G1} + I_O} = \frac{32}{3} \frac{I_O^2}{\frac{I_{\mathrm{IN}}^2}{32\sigma^2 I_O} + I_O} \tag{2.112}$$

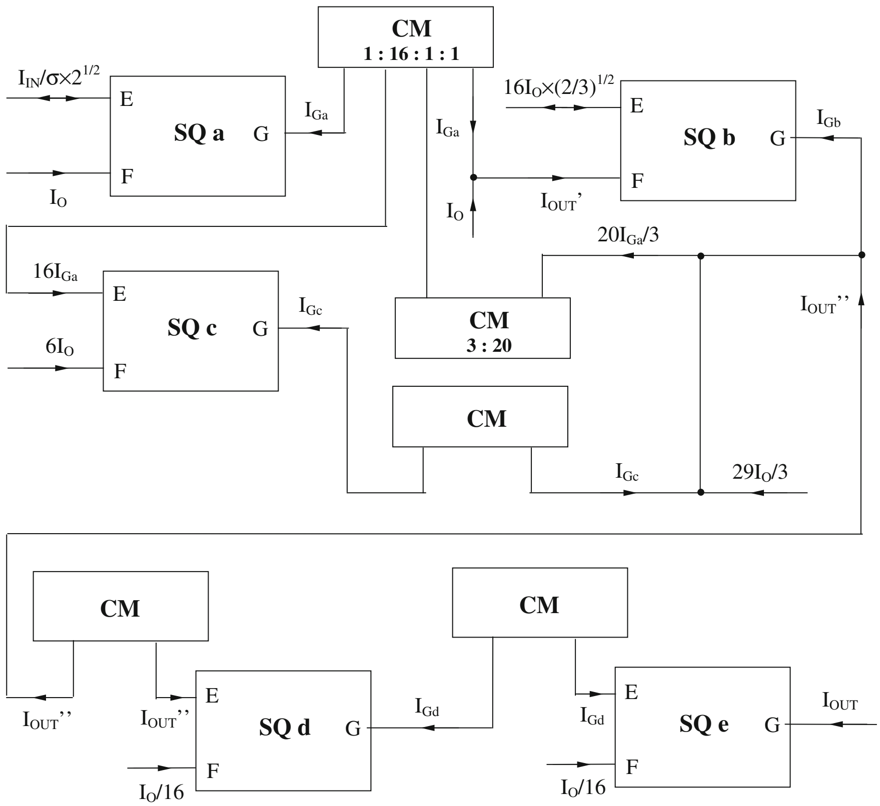


Fig. 2.32 Block diagram of the general Gaussian function synthesizer circuit that uses the $g_{4b}(x)$ approximation function

and

$$I_{\mathrm{Gc}} = \frac{8}{3}\frac{I_{G1}^2}{I_O} = \frac{I_{\mathrm{IN}}^4}{384\sigma^4 I_O^3}. \tag{2.113}$$

The $I_{\mathrm{OUT''}}$ current can be expressed as follows:

$$I_{\mathrm{OUT''}} = I_{\mathrm{Gb}} + \frac{20}{3}I_{\mathrm{Ga}} - I_{\mathrm{Gc}} - \frac{29}{3}I_O, \tag{2.114}$$

resulting

$$I_{\mathrm{OUT''}} = I_O\left[\frac{32}{3}\frac{1}{1+\frac{1}{32\sigma^2}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2} + \frac{5}{24\sigma^2}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2 - \frac{1}{384\sigma^4}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^4 - \frac{29}{3}\right]. \tag{2.115}$$

The I_{Gd} and I_{OUT} currents have the following expressions:

$$I_{\mathrm{Gd}} = \frac{(I_{\mathrm{OUT}''})^2}{I_O} \tag{2.116}$$

and

$$I_{\mathrm{OUT}} = \frac{I_{Gd}^2}{I_O} = \frac{(I_{\mathrm{OUT}''})^4}{I_O^3}. \tag{2.117}$$

Thus, the output current of the circuit can be expressed as follows:

$$I_{\mathrm{OUT}} = I_O \left[\frac{32}{3} \frac{1}{1 + \frac{1}{32\sigma^2}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2} + \frac{5}{24\sigma^2}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2 - \frac{1}{384\sigma^4}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^4 - \frac{29}{3} \right]^4. \tag{2.118}$$

In conclusion, I_{OUT} current approximates the Gaussian function using $g_{4b}(x)$ approximation function:

$$I_{\mathrm{OUT}} = I_O g_{4b}\left[\frac{1}{\sigma\sqrt{2}}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)\right] \cong I_O \exp\left[-\frac{1}{2\sigma^2}\left(\frac{I_{\mathrm{IN}}}{I_O}\right)^2\right]. \tag{2.119}$$

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