Chapter 2
Direct Problem: The Euler Equation Approach

This chapter concerns deterministic and stochastic nonstationary discrete-time optimal control problems (OCPs) with an infinite horizon. We show, using Gâteaux differentials, that the so-called Euler equation (EE) and a transversality condition (TC) are necessary conditions for optimality. In particular, the TC is obtained in a more general form and under milder hypotheses than in previous works. Sufficient conditions are also provided. We find closed-form solutions to several (discounted) stationary and nonstationary control problems. The results in this chapter come from González–Sánchez and Hernández–Lerma [37].

2.1 Introduction

This chapter is about discrete-time nonstationary (or time-varying) deterministic and stochastic dynamic optimization problems in infinite horizon. Dynamic optimization problems are also known as OCPs. First, we concentrate on deterministic problems described as follows.

Let $X \subseteq \mathbb{R}^n$ and $U \subseteq \mathbb{R}^m$ be the *state space* and the *control set*, respectively. Consider a sequence $\{X_t \subseteq X \mid t = 0,1,2,\ldots\}$ of nonempty subsets of the state space, and a family $\{U_t(x) \subseteq U \mid x \in X_t,\ t = 0,1,2,\ldots\}$ of so-called *feasible control sets*. For each $t = 0,1,\ldots$, $x \in X_t$, and $u \in U_t(x)$, we denote by $f_t(x,u)$ the corresponding state in X_{t+1}. Hence, given an initial state x_0, the state of the system evolves according to

$$x_{t+1} = f_t(x_t,u_t), \quad t = 0,1,2,\ldots. \tag{2.1}$$

We consider the optimality criterion (or performance index)

$$\sum_{t=0}^{\infty} r_t(x_t,u_t), \tag{2.2}$$

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where $r_t : X \times U \to \mathbb{R} \cup \{-\infty\}$ is called the *reward function* at time t. A sequence $\{u_t\}$ is called a *policy* (from x_0) whenever $u_t \in U_t(x_t)$ and $x_{t+1} = f_t(x_t, u_t)$ for each $t = 0, 1, 2, \ldots$. The *OCP* is to find a policy $\{u_t\}$ that maximizes (2.2). To illustrate these concepts we next present a nonstationary version of Example 1.3.

Example 2.1. As in Example 1.3, denote the state variable by k_t and the control variable by c_t. The dynamics is now given by

$$k_{t+1} = A_t k_t^{\alpha} - c_t, \quad t = 0, 1, 2, \ldots, \tag{2.3}$$

where $0 < \alpha < 1$, $A_t > 0$, and k_0 is given. Consider $X_t = [0, \infty)$, $U_t(k) = [0, A_t k^{\alpha}]$, and the same performance index

$$\sum_{t=0}^{\infty} \beta^t \log(c_t), \tag{2.4}$$

Thus, the function r_t in (2.2) is $r_t(k, c) = \beta^t \log(c)$ for each $t = 0, 1, \ldots$. $\Diamond$

If $A_t = A$ for each $t = 0, 1, \ldots$, then Example 2.1 is reduced to the discounted stationary problem (DSP), as in Example 1.3. Note that in DSPs, $X_t = X$, $U_t(x) = U(x)$, $f_t = f$, and $r_t(x, u) = \beta^t r(x, u)$ for each $t = 0, 1, \ldots$, with discount factor $0 < \beta < 1$.

The EE (2.14), below, is a *necessary* condition for a policy to be optimal. For the stationary case, the EE is typically obtained from the *Bellman equation*, also known as the *dynamic programming equation*; this requires, in particular, the differentiability of the value function. See, for instance, Acemoglu [3, Sect. 6.6]. Another way to obtain the EE (2.14) is by means of variational arguments.

On the other hand, under certain hypotheses, the EE (2.14) and the TC (2.18) are sufficient conditions for an optimal plan; see Acemoglu [3, Theorem 6.10] or Stokey and Lucas [70, Theorem 4.15] for details. One of these hypotheses requires that every state x be a vector with nonnegative entries; thus, such a result cannot be applied to unconstrained OCPs such as a linear–quadratic problem (Sect. 2.3.5, below). Although the TC (2.18) is often used as a sufficient condition, Ekeland and Scheinkman [26] and Kamihigashi [43] prove that the TC (2.18) is also a necessary condition.

Our contributions and related literature. In this chapter, we use Gâteaux differentials to show that the EE (2.14) and the TC (2.15) are necessary conditions for optimality (Theorem 2.1). The EE has been also studied by Cadzow [15] and Bar-Ness [6] following similar variational arguments. However, the former is restricted to finite horizon problems, which are a particular case of our model. On the other hand, Bar-Ness [6, Theorem 3.2] derives the EE in a normed linear space of sequences; in particular, an optimal policy needs to be a sequence converging to zero. Our results do not require the assumption made by Bar-Ness, that is, we allow policies to be more general sequences. In fact, a norm in the linear space of sequences is not needed since we work with Gâteaux differentials; see Luenberger [54, Sect. 7.2]. On the other hand, the TC (2.18), which is well-known in the literature, is less general than ours. Ekeland and Scheinkman [26] prove the necessity of the

TC (2.18) for a class of problems arising in economics. They consider finite-horizon problems in which an approximation result [26, Theorem 3.1] is applied to obtain solutions converging to the solution of the infinite-horizon problem. Kamihigashi [43] obtains the same conclusion on (2.18) through a perturbation argument; his proof is simpler than the Ekeland and Scheinkman's. However, Kamihigashi [43] assumes concavity of the reward functions; this assumption is not required in [26]. A further comparison between [26, 43], and some related works can be found in Kamihigashi [43, Sect. 4]. In contrast, to prove Theorem 2.1, we do not assume concavity of the reward functions. Further, we use neither the approximation result [26, Theorem 3.1] nor finite-horizon approximations.

We also show (Theorem 2.3) that, under appropriate convexity hypotheses, the EE (2.14) and the TC (2.15) are sufficient for an optimal plan. In Theorems 2.1 and 2.3 nonnegativity of the state vectors is not required; this allows us to apply our results to unconstrained OCPs, for instance, a linear-quadratic problem. The EE approach allows us to see the class of nonstationary OCPs, described in Sect. 2.2, as problems of classical optimization.

Theorems 2.1 and 2.3 give the same conclusions that Theorem 6.12 in Acemoglu [3]; nonetheless, there are some important differences. Our results concern the more general TC (2.15) whereas Acemoglu deals with (2.18). In addition, we require fewer assumptions; for instance, Assumption 2.4(c)–(d) is not needed.

The remainder of the chapter is organized as follows. We describe in Sect. 2.2 the deterministic control model we are concerned with; our main results, Theorems 2.1 and 2.3, are also stated. Section 2.3 is devoted to solving, by the EE approach, some well-known DSPs, the solutions of which have been found by dynamic programming. In addition, we explicitly solve two nonstationary examples. In Sect. 2.4 we explain how to extend the main results of Sect. 2.2 to stochastic models. The stochastic case is illustrated with two examples.

2.2 Deterministic Control Problems

Let us now go back to the OCP (2.1)–(2.2). We will assume that this OCP can be stated in terms of the state sequence $\{x_t\}$ only. To this end, recall that the control variable u_t and the state variables x_t, x_{t+1} are coupled according to (2.1). Hence, we are assuming that u_t can be written in terms of x_t and x_{t+1}, say $u_t = h_t(x_t, x_{t+1})$, and so the performance index takes the form

$$\sum_{t=0}^{\infty} \beta^t g_t(x_t, x_{t+1}), \tag{2.5}$$

where $g_t(x_t, x_{t+1}) := r_t(x_t, h_t(x_t, x_{t+1}))$ for each $t = 0, 1, \ldots$. If there is more than one value of u such that $x_{t+1} = f_t(x_t, u)$ (see Sect. 2.3.4, below, for an example), then we define

$$g_t(x_t, x_{t+1}) := \max_{u \in U_t(x_t)} \{r_t(x_t, u) \mid x_{t+1} = f_t(x_t, u)\}.$$

Note that each feasible control set $U_t(x_t)$ defines a feasible state set $\Gamma_t(x_t)$ for x_{t+1}, where

$$\Gamma_t(x) := \{f_t(x,u) \mid u \in U_t(x)\} \quad \text{for all } \ x \in X_t,\ t = 0,1,\ldots.$$

For instance, in the nonstationary Brock and Mirman model (2.3)–(2.4) we have

$$g_t(k_t,k_{t+1}) = \beta^t \log(A_t k_t^\alpha - k_{t+1}), \quad \Gamma_t(k) = [0, A_t k^\alpha], \quad t = 0,1,\ldots. \tag{2.6}$$

This model is studied in further detail in Sect. 2.3.1.

2.2.1 The Deterministic Control Model

A sequence $\{x_1,x_2,\ldots\} \subseteq \mathbb{R}^n$ is a *feasible plan* (or *feasible path*) *from* x_0 if $x_{t+1} \in \Gamma_t(x_t)$ for each $t = 0,1,\ldots$. The set of all feasible plans from x_0 is denoted by $\Phi(x_0)$.

In reduced form, a nonstationary OCP can be described by the three-tuple

$$(\{X_t\},\{g_t\},\Phi(x_0)) \tag{2.7}$$

of sequences, where $g_t : X_t \times X_{t+1} \to \mathbb{R} \cup \{-\infty\}$ for $t = 0,1,\ldots$. The three-tuple (2.7) is also called a *control model*.

The following assumption is supposed to hold throughout the remainder of this chapter.

Assumption 2.1 The control model in (2.7) satisfies the following for each $x_0 \in X_0$:

(a) The set $\Phi(x_0)$ is nonempty;
(b) There is a sequence $\{m_t(x_0)\}$ of nonnegative real numbers such that

$$g_t(x_t,x_{t+1}) \le m_t(x_0), \quad t = 0,1,\ldots,$$

for each $(x_1,x_2,\ldots) \in \Phi(x_0)$, and $\sum_{t=0}^\infty m_t(x_0) < \infty$;
(c) For each $(x_1,x_2,\ldots) \in \Phi(x_0)$, the limit $\lim_{T\to\infty} \sum_{t=0}^T g_t(x_t,x_{t+1})$ exists (it may be $-\infty$);
(d) There exists $(x_1,x_2,\ldots) \in \Phi(x_0)$ such that $\sum_{t=0}^\infty g_t(x_t,x_{t+1}) > -\infty$;
(e) For each $t = 0,1,\ldots$, the function g_t is differentiable in the interior of $X_t \times X_{t+1}$ (with the usual topology of $\mathbb{R}^{n\times n}$).

For $x_0 \in X_0$, define $v : \Phi(x_0) \to \mathbb{R} \cup \{-\infty\}$ by

$$v(\varphi) := \lim_{T\to\infty} \sum_{t=0}^T g_t(x_t,x_{t+1}), \tag{2.8}$$

where $\varphi = (x_1,x_2,\ldots)$, and $v^* : X_0 \to \mathbb{R}$ by

$$v^*(x_0) := \sup\{v(\varphi) \mid \varphi \in \Phi(x_0)\}. \tag{2.9}$$

Assumption 2.1(a)–(d) ensures that the functions v, v^* are well defined.

For the three-tuple (2.7) and $x_0 \in X_0$ given, the OCP we are concerned with is to find $\hat{\varphi} \in \Phi(x_0)$ such that

$$v(\hat{\varphi}) = v^*(x_0). \tag{2.10}$$

In such a case, we say that $\hat{\varphi}$ is an *optimal plan*, also known as an *optimal policy* or *optimal strategy*.

There are some results about the existence of optimal plans for nonstationary OCPs. For instance, Acemoglu [3, Theorem 6.11, p. 212] supposes, among other hypotheses, the continuity of g_t and the compactness of $\Gamma_t(x)$ $(t = 0, 1, \ldots)$. Then he proves that the set $\Phi(x_0)$ is compact (with a certain topology) and the function v is continuous; therefore, the existence of an optimal plan follows. Guo et al. [39, Theorem 3.3] suppose, for minimization problems, lower semicontinuity of each g_t and do not assume compactness of $\Gamma_t(x)$. Similar assumptions are made by Ekeland and Scheinkman [26, Proposition 4.1]. In contrast, our sufficiency conditions combine the EE (2.14) and the TC (2.15) along with suitable concavity–convexity conditions—see Theorems 2.2 and 2.3. Instead of the EE–TC approach we propose here sufficiency conditions based on Gâteaux differentials; see Theorem 2.4.

For each integer $T \geq 1$, denote by $\Phi_T(x_0)$ the set of all truncated plans φ_T, that is, φ_T consists of the first T entries of the feasible plan $\varphi \in \Phi(x_0)$. Define $v_T : \Phi_T(x_0) \to \mathbb{R} \cup \{-\infty\}$ by

$$v_T(\varphi_T) := \sum_{t=0}^{T-1} g_t(x_t, x_{t+1}), \tag{2.11}$$

where $\varphi_T = (x_1, x_2, \ldots, x_T)$, and $v_T^* : X_0 \to \mathbb{R}$ as

$$v_T^*(x_0) := \sup\{v_T(\varphi_T) \mid \varphi_T \in \Phi_T(x_0)\}. \tag{2.12}$$

Remark 2.1. Under suitable assumptions, it can be shown that the truncated optimal values $v_T^*(x_0)$ converge to the infinite horizon optimal value $v^*(x_0)$, that is,

$$\lim_{T\to\infty} v_T^*(x_0) = v^*(x_0).$$

See Ekeland and Scheinkman [26, Proposition 4.1] and Guo et al. [39, Theorem 4.5]. Other references on approximation results for infinite horizon problems are, for instance, Flåm and Fougères [31] and Schochetman and Smith [68]. $\Diamond$

2.2.2 Necessary Conditions

In this section we find the EE (2.14) and the TC (2.15), below, as necessary conditions for the existence of an optimal plan. To this end we consider the vector space Λ of all sequences in $\mathbb{R}^n$ with the standard addition and scalar multiplication.

We suppose that the initial state $x_0 \in X_0$ is fixed. Recall that Assumption 2.1 holds. We also require the following.

Definition 2.1. Let $\hat{\varphi} \in \Phi(x_0)$ and $\varphi \in \Lambda$. We say that $\hat{\varphi}$ is an *internal plan* in the direction φ if there exists a real number $\varepsilon_0 > 0$ such that

$$\hat{\varphi} + \varepsilon\varphi \in \Phi(x_0) \quad \forall \varepsilon \in (-\varepsilon_0, \varepsilon_0).$$

Assumption 2.2 Let $\hat{\varphi} = \{\hat{x}_t\} \in \Phi(x_0)$ be some internal plan in the direction $\varphi = \{x_t\} \in \Lambda$. Define

$$h_0(\varepsilon) := g_0(\hat{x}_0, \hat{x}_1 + \varepsilon x_1), \quad h_t(\varepsilon) := g_t(\hat{x}_t + \varepsilon x_t, \hat{x}_{t+1} + \varepsilon x_{t+1}),$$

for $t = 1, 2, \ldots$. There exist $\varepsilon_0 > 0$ such that the series $\sum_{t=0}^{\infty} h_t'(\varepsilon)$ converges uniformly on the interval $(-\varepsilon_0, \varepsilon_0)$.

Lemma 2.1. *Consider the function v in (2.8), and let $\hat{\varphi}$ and φ be as in Assumption 2.2. Then there exists the Gâteaux differential of the function v at $\hat{\varphi}$ in the direction φ, which is defined as*

$$\delta v(\hat{\varphi}; \varphi) := \frac{\mathrm{d}v}{\mathrm{d}\varepsilon}(\hat{\varphi} + \varepsilon\varphi)\Big|_{\varepsilon=0}.$$

In fact,

$$\delta v(\hat{\varphi}; \varphi) = \frac{\partial g_0}{\partial y}(x_0, \hat{x}_1) \cdot x_1 + \sum_{t=1}^{\infty} \left[\frac{\partial g_t}{\partial x}(\hat{x}_t, \hat{x}_{t+1}) \cdot x_t + \frac{\partial g_t}{\partial y}(\hat{x}_t, \hat{x}_{t+1}) \cdot x_{t+1} \right], \tag{2.13}$$

where $\partial/\partial x$ and $\partial/\partial y$ denote the gradients with respect to the first and the second variables, respectively.

Proof. Let $\hat{\varphi}$ and φ be as in Assumption 2.2. By Theorem 7.17 in Rudin [66, p. 152]

$$\frac{\mathrm{d}v}{\mathrm{d}\varepsilon}(\hat{\varphi} + \varepsilon\varphi) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \sum_{t=0}^{\infty} h_t(\varepsilon) = \sum_{t=0}^{\infty} h_t'(\varepsilon)$$

for ε in some interval $(-\varepsilon_0, \varepsilon_0)$. Note that

$$\begin{aligned} h_0'(\varepsilon) &= \frac{\partial g_0}{\partial y}(x_0, \hat{x}_1 + \varepsilon x_1) \cdot x_1, \\ h_t'(\varepsilon) &= \frac{\partial g_t}{\partial x}(\hat{x}_t + \varepsilon x_t, \hat{x}_{t+1} + \varepsilon x_{t+1}) \cdot x_t \\ &\quad + \frac{\partial g_t}{\partial y}(\hat{x}_t + \varepsilon x_t, \hat{x}_{t+1} + \varepsilon x_{t+1}) \cdot x_{t+1}, \quad t = 1, 2, \ldots. \end{aligned}$$

Making $\varepsilon = 0$ we get (2.13). □

Note that Lemma 2.1 is true for every internal plan $\hat{\varphi}$ (which is not necessarily an optimal plan) in the direction φ. If $\hat{\varphi}$ is an optimal plan, the following assumption guarantees the existence of a direction φ such that $\hat{\varphi}$ and φ satisfy Assumption 2.2.

Assumption 2.3 Let $\hat{\varphi} = \{\hat{x}_t\} \in \Phi(x_0)$ be an optimal plan for the OCP (2.7)–(2.9). For each $t = 0, 1, \ldots$:

(a) $\hat{x}_{t+1}$ is an interior point (with the usual topology of $\mathbb{R}^n$) of the set $\Gamma_t(\hat{x}_t)$;
(b) There exists $\varepsilon_t > 0$ such that $\|x - \hat{x}_t\| < \varepsilon_t$ implies $\hat{x}_{t+1} \in \Gamma_t(x)$.

We can now state one of our main results.

Theorem 2.1. *Let $\hat{\varphi} = \{\hat{x}_t\} \in \Phi(x_0)$ be an optimal plan for the OCP (2.7)–(2.9). Suppose that Assumption 2.3 holds. Then:*

(a) $\hat{\varphi}$ satisfies the so-called EE

$$\frac{\partial g_{t-1}}{\partial y}(\hat{x}_{t-1}, \hat{x}_t) + \frac{\partial g_t}{\partial x}(\hat{x}_t, \hat{x}_{t+1}) = 0, \quad t = 1, 2, \ldots. \tag{2.14}$$

(b) Suppose that, in addition, $\hat{\varphi}$ is an internal plan in the direction $\varphi = \{x_t\} \in \Lambda$ and, moreover, Assumption 2.2 holds. Then $\hat{\varphi}$ and φ satisfy the TC

$$\lim_{t\to\infty} \frac{\partial g_{t-1}}{\partial y}(\hat{x}_{t-1}, \hat{x}_t) \cdot x_t = 0. \tag{2.15}$$

Proof. Pick $x \in \mathbb{R}^n$ and an integer $\tau \geq 1$. Define the plan $\varphi_\tau(x) = \{x_t\}$ by $x_\tau = x$ and $x_t = 0$ for every $t \neq \tau$. By Assumption 2.3, there exists $\varepsilon_\tau > 0$ such that

$$\hat{x}_\tau + \varepsilon x \in \Gamma_{\tau-1}(\hat{x}_{\tau-1}), \quad \hat{x}_{\tau+1} \in \Gamma_\tau(\hat{x}_\tau + \varepsilon x),$$

for all $\varepsilon \in (-\varepsilon_\tau, \varepsilon_\tau)$. That is, $\hat{\varphi}$ is an internal plan in the direction $\varphi_\tau(x)$. Moreover, $\hat{\varphi}$ and $\varphi_\tau(x)$ verify Assumption 2.2.

(a) By Lemma 2.1, above, and Theorem 2.1 in Fleming and Rishel [32, p. 4], we have

$$\delta v(\hat{\varphi}; \varphi_\tau(x)) = \left[\frac{\partial g_{\tau-1}}{\partial y}(\hat{x}_{\tau-1}, \hat{x}_\tau) + \frac{\partial g_\tau}{\partial x}(\hat{x}_\tau, \hat{x}_{\tau+1})\right] \cdot x = 0. \tag{2.16}$$

Since (2.16) is true for every $x \in \mathbb{R}^n$ and every integer $\tau \geq 1$, the EE (2.14) follows.

(b) Since $\hat{\varphi}$ and φ satisfy Assumption 2.2, Lemma 2.1 ensures the existence of the Gâteaux differential $\delta v(\hat{\varphi}; \varphi)$. Theorem 2.1 in Fleming and Rishel [32, p. 4] implies that $\delta v(\hat{\varphi}; \varphi) = 0$. Then, by (2.13) and the EE (2.14),

$$\frac{\partial g_{t-1}}{\partial y}(\hat{x}_{t-1}, \hat{x}_t) \cdot x_t + \sum_{\tau=t}^{\infty} \left[\frac{\partial g_\tau}{\partial x}(\hat{x}_\tau, \hat{x}_{\tau+1}) \cdot x_\tau + \frac{\partial g_\tau}{\partial y}(\hat{x}_\tau, \hat{x}_{\tau+1}) \cdot x_{\tau+1}\right] = 0,$$

for each $t = 2, 3, \ldots$. If we let $t \to \infty$, (2.15) follows. □

Remark 2.2.

(a) From the EE (2.14) we can see that the TC (2.15) is equivalent to

$$\lim_{t\to\infty} \frac{\partial g_t}{\partial x}(\hat{x}_t, \hat{x}_{t+1}) \cdot x_t = 0. \tag{2.17}$$

(b) If the optimal plan $\hat{\varphi}$ is an internal plan in the direction $\varphi = \hat{\varphi}$, then the TC (2.17) becomes

$$\lim_{t\to\infty} \frac{\partial g_t}{\partial x}(\hat{x}_t, \hat{x}_{t+1}) \cdot \hat{x}_t = 0. \tag{2.18}$$

In fact, (2.18) is precisely the TC known in the literature. We have shown the necessity of (2.17) which is more general than (2.18).

(c) Ekeland and Scheinkman [26, Corollary 5.2] and Kamihigashi [43, Theorem 2.1] prove the necessity of the TC (2.18) assuming that $\partial g_t/\partial x_j \geq 0$ $(j = 1, 2, \ldots, n)$ and that the states have nonnegative entries. However, the proof given here is a direct consequence of the equality $\delta v(\hat{\varphi}; \varphi) = 0$ and we do not require their assumptions. $\Diamond$

The TC (2.18) is more useful than (2.17) when we want to get information about an optimal plan $\hat{\varphi}$; in Sect. 2.3, below, we show how to do it. The TC (2.18) is explained by Kamihigashi [44] for finite horizon problems and the relationship to some problems in dynamic economics.

2.2.3 Sufficient Conditions

We have seen that the EE (2.14) and the TC (2.15) are necessary conditions for optimality. Actually, under suitable convexity assumptions, they are also sufficient; see Theorem 2.2 and Assumption 2.4, below. Theorem 2.2 is well known in the literature; see, for instance, Acemoglu [3, Theorem 6.12, p. 212] for the nonstationary case, or Stokey and Lucas [70, Theorem 4.15, pp. 98–99] for DSPs. However, Assumption 2.4(d) does not hold in some OCPs, for instance, in linear–quadratic problems; see Sect. 2.3.5, below. We present the proof of Theorem 2.2 to see how Assumption 2.4(c)–(d) can be replaced by the TC (2.15); see Theorem 2.3, below.

Assumption 2.4 The control model (2.7) satisfies the following for each $t = 0, 1, \ldots$:

(a) The function g_t is concave and differentiable;
(b) The set of feasible plans $\Phi(x_0)$ is convex;
(c) X_t is a subset of $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \mid x_j \geq 0,\ j = 1, \ldots, n\}$;
(d) For each $j = 1, 2, \ldots, n$, $\partial g_t/\partial x_j \geq 0$.

Theorem 2.2. *Suppose that a feasible plan $\hat{\varphi} \in \Phi(x_0)$ satisfies the EE (2.14) and the TC (2.15). If Assumption 2.4 holds, then $\hat{\varphi}$ is an optimal plan for the OCP (2.7)–(2.9).*

Proof. Let $\varphi = \{x_t\} \in \Phi(x_0)$ be an arbitrary feasible plan. Since each g_t ($t = 0,1,\ldots$) is concave, we have

$$\begin{aligned}\Delta_\tau(\hat{\varphi},\varphi) &:= g_0(x_0,\hat{x}_1) - g_0(x_0,x_1) + \sum_{t=1}^{\tau-1}[g_t(\hat{x}_t,\hat{x}_{t+1}) - g_t(x_t,x_{t+1})] \\ &\geq \frac{\partial g_0}{\partial y}(x_0,\hat{x}_1)\cdot(\hat{x}_1 - x_1) \\ &\quad + \sum_{t=1}^{\tau-1}\left[\frac{\partial g_t}{\partial x}(\hat{x}_t,\hat{x}_{t+1})\cdot(\hat{x}_t - x_t) + \frac{\partial g_t}{\partial y}(\hat{x}_t,\hat{x}_{t+1})\cdot(\hat{x}_{t+1} - x_{t+1})\right].\end{aligned}$$

The EE (2.14) reduces this inequality to

$$\Delta_\tau(\hat{\varphi},\varphi) \geq \frac{\partial g_{\tau-1}}{\partial y}(\hat{x}_{\tau-1},\hat{x}_\tau)\cdot(\hat{x}_\tau - x_\tau). \tag{2.19}$$

By (2.14) with $t = \tau$, and Assumption 2.4(c)–(d), we obtain

$$\begin{aligned}\Delta_\tau(\hat{\varphi},\varphi) &\geq -\frac{\partial g_\tau}{\partial x}(\hat{x}_\tau,\hat{x}_{\tau+1})\cdot(\hat{x}_\tau - x_\tau) \\ &\geq -\frac{\partial g_\tau}{\partial x}(\hat{x}_\tau,\hat{x}_{\tau+1})\cdot\hat{x}_\tau.\end{aligned}$$

Because $\hat{\varphi}$ satisfies the TC (2.18), $\lim_{\tau\to\infty}\Delta_\tau(\hat{\varphi},\varphi) \geq 0$. That is, $\hat{\varphi}$ is an optimal plan. □

Theorem 2.3. *Suppose that a feasible plan $\hat{\varphi} \in \Phi(x_0)$ satisfies the EE* (2.14) *and the TC* (2.15) *for every $\varphi \in \Phi(x_0)$. If Assumption* 2.4*(a)–(b) holds, then $\hat{\varphi}$ is an optimal plan for the OCP* (2.7)–(2.9)*.*

Proof. Under Assumption 2.4(a)–(b) we have (2.19). Since $\hat{\varphi}$ satisfies the TC (2.15) for every $\varphi \in \Phi(x_0)$, the right-hand side of (2.19) converges to zero as $\tau \to \infty$. Thus, $\hat{\varphi}$ is an optimal plan. □

Another kind of sufficient condition can be given if all the Gâteaux differentials $\delta v(\hat{\varphi};\varphi)$ exist; see Theorem 2.4, below. However, as we will see in Sect. 2.3, it is easier to use Theorems 2.2 or 2.3.

Theorem 2.4. *Suppose that $\delta v(\hat{\varphi};\varphi) \leq 0$ for all $\varphi \in \Lambda$ such that $\hat{\varphi} + \varphi \in \Phi(x_0)$. If Assumption* 2.4*(a)–(b) holds, then $\hat{\varphi}$ is an optimal plan for the OCP* (2.7)–(2.9)*.*

Proof. Let $\varphi = (x_1,x_2,\ldots)$ and $\varphi' = (x'_1,x'_2,\ldots)$ be feasible plans from x_0, and let $\alpha \in [0,1]$. Thus, by the concavity of g_t ($t = 0,1,\ldots$),

$$\begin{aligned}
v(\alpha\varphi+(1-\alpha)\varphi') &= g_0(\alpha x_0+(1-\alpha)x_0, \alpha x_1+(1-\alpha)x_1') \\
&\quad + \sum_{t=1}^{\infty} g_t(\alpha x_t+(1-\alpha)x_t', \alpha x_{t+1}+(1-\alpha)x_{t+1}') \\
&\geq \alpha g_0(x_0,x_1)+(1-\alpha)g_0(x_0,x_1') \\
&\quad + \sum_{t=1}^{\infty}[\alpha g_t(x_t,x_{t+1})+(1-\alpha)g_t(x_t',x_{t+1}')] \\
&= \alpha v(\varphi)+(1-\alpha)v(\varphi').
\end{aligned}$$

This proves the concavity of v. The conclusion of the theorem follows from Theorem 2.3 in Fleming and Rishel [32, p. 4]. □

2.3 Solving Optimal Control Problems

In this section we solve several (stationary and nonstationary) OCPs by the EE approach. DSPs, like those in Sects. 2.3.2, 2.3.3, and 2.3.5, are typically solved by dynamic programming. In Sects. 2.3.1 and 2.3.4 we solve nonstationary problems.

We use standard methods to solve the difference equation (2.14). See, for instance, Kelley and Peterson [45] or Elaydi [28]. In particular, given the difference equation with constant coefficients

$$x_{t+1}+ax_t+bx_{t-1}=0, \quad t=1,2,\ldots, \tag{2.20}$$

we consider its *characteristic equation* $\lambda^2+a\lambda+b=0$. If λ_1,λ_2 are distinct roots of the characteristic equation, then the general solution of (2.20) is of the form

$$x_t=c_1\lambda_1^t+c_2\lambda_2^t, \quad t=1,2,\ldots,$$

where c_1,c_2 are arbitrary constants. We use the TC (2.18) and the initial condition x_0 to determine the constants c_1,c_2.

Note that all the examples in this section satisfy Assumptions 2.1–2.3. Then, by Theorem 2.1, the optimal plan $\hat{\varphi}$ has to verify the EE (2.14) and the TC (2.15) with $\varphi=\hat{\varphi}$. Assumption 2.4 holds for examples in Sects. 2.3.1–2.3.4; thus, Theorem 2.2 can be used. However, for the example in Sect. 2.3.5 we need Theorem 2.3, because Assumption 2.4(c) is not satisfied.

2.3.1 The Brock and Mirman Model

By (2.6), in Example 2.1 we want to maximize

$$\sum_{t=0}^{\infty}\beta^t\log(A_tk_t^{\alpha}-k_{t+1}), \quad k_0 \text{ given.} \tag{2.21}$$

We assume that A_t $(t = 0, 1, \ldots)$ belongs to some compact interval of positive numbers. The corresponding EE is the nonlinear difference equation

$$\frac{-1}{A_{t-1}\hat{k}_{t-1}^{\alpha} - \hat{k}_t} + \beta \frac{\alpha A_t \hat{k}_t^{\alpha-1}}{A_t \hat{k}_t^{\alpha} - \hat{k}_{t+1}} = 0, \quad t = 1, 2, \ldots. \tag{2.22}$$

Let

$$z_t := \frac{\hat{k}_{t+1}}{A_t \hat{k}_t^{\alpha}}, \quad t = 0, 1, 2, \ldots. \tag{2.23}$$

Then (2.22) becomes

$$\frac{-z_{t-1}}{1 - z_{t-1}} + \frac{\alpha\beta}{1 - z_t} = 0, \quad t = 1, 2, \ldots. \tag{2.24}$$

Equation (2.24) is a particular case of a Riccati equation (see [45, Sect. 3.6]). By making $z_t = w_{t+1}/w_t$, (2.24) can be transformed into a linear equation

$$w_{t+1} - (1 + \alpha\beta) w_t + \alpha\beta w_{t-1} = 0, \quad t = 1, 2, \ldots. \tag{2.25}$$

The general solution of (2.25) is $w_t = a_1 + a_2(\alpha\beta)^t$. Thus

$$z_t = \frac{a_1 + a_2(\alpha\beta)^{t+1}}{a_1 + a_2(\alpha\beta)^t}, \quad t = 0, 1, 2, \ldots. \tag{2.26}$$

Recall that $g_t(x, y) = \beta^t \log(A_t x^{\alpha} - y)$. By (2.23) and (2.26),

$$\begin{aligned}
\hat{k}_t \frac{\partial g_t}{\partial x}(\hat{k}_t, \hat{k}_{t+1}) &= \beta^t \hat{k}_t \frac{\alpha A_t \hat{k}_t^{\alpha-1}}{A_t \hat{k}_t^{\alpha} - \hat{k}_{t+1}} \\
&= \beta^t \frac{\alpha}{1 - z_t} \\
&= \frac{\alpha}{a_2(1 - \alpha\beta)} \left[\frac{a_1}{\alpha^t} + a_2 \beta^t \right].
\end{aligned}$$

The TC (2.18) implies that $a_1 = 0$. From (2.23) and (2.26), we get

$$\hat{k}_{t+1} = \alpha\beta A_t \hat{k}_t^{\alpha}, \quad t = 0, 1, 2, \ldots \tag{2.27}$$

or equivalently,

$$\log(\hat{k}_{t+1}) = \alpha \log(\hat{k}_t) + \log(\alpha\beta A_t), \quad t = 0, 1, 2, \ldots.$$

We can solve this linear difference equation in $\log(\hat{k}_t)$ to obtain

$$\log(\hat{k}_t) = \alpha^t \log(k_0) + \sum_{j=0}^{t-1} \alpha^{t-j-1} \log(\alpha\beta A_j), \quad t = 1, 2, \ldots. \tag{2.28}$$

By (2.3) and (2.27), we have the optimal Markov policy for consumption

$$\hat{c}_t = A_t \hat{k}_t^{\alpha} - \hat{k}_{t+1} = (1-\alpha\beta) A_t \hat{k}_t^{\alpha}, \quad t = 0,1,2,\ldots.$$

The value function can be found by using (2.28). In particular, if $A_t = A$ for every t, we get

$$\begin{aligned} v(k_0) &= \sum_{t=0}^{\infty} \beta^t \log(\hat{c}_t) \\ &= \frac{\alpha}{1-\alpha\beta} \log(k_0) + \frac{1}{1-\beta} \left[\frac{\alpha\beta}{1-\alpha\beta} \log(\alpha\beta) + \log(1-\alpha\beta) \right]. \end{aligned}$$

Remark 2.3.

(a) The change of variable (2.23) is suggested by Stokey and Lucas [70, Exercise 2.2, p. 12] for the finite horizon (stationary) case. They solve recursively a finite horizon version of (2.24), with the boundary condition $z_{T+1} = 0$, and then they take the limit as $T \to \infty$. In contrast, we solve (2.24) as a Riccati equation, and replace the boundary condition with (2.18).

(b) For the (stationary) Brock and Mirman model with $A_t = 1$ ($t = 0,1,2,\ldots$), Acemoglu [3, Example 6.4] uses (2.22) and a guess-and-verify method. He conjectures that $\hat{k}_{t+1} = a k_t^{\alpha}$ ($t = 0,1,2,\ldots$). Substituting this in (2.22), he obtains $a = \alpha\beta$, as in (2.27).

(c) Cruz–Suárez and Montes-de-Oca [19, 20] study OCPs by combining the value iteration algorithm (briefly described in Sect. 1.2.1) with the EE or *envelope results*. As an example they solve, in both references, the (stationary) Brock and Mirman model. ♢

2.3.2 An Optimal Growth Model with Linear Production

This is an optimal growth model known as the *Ak* model; it can be found in LeVan and Dana [51, Example 5.5.2, pp. 118–119], where it is solved by the dynamic programming technique.

Assume that capital k_t and consumption c_t move according to the law

$$k_{t+1} = A k_t + (1-\delta) k_t - c_t, \quad t = 0,1,\ldots,$$

where A, δ are positive real numbers and k_0 is given. Let $0 < \beta < 1$ be the discount factor and let $a := A + 1 - \delta$. Assume

$$a > 1, \quad 1 - \delta < a\beta < 1. \tag{2.29}$$

Let $\theta < 0$. We want to maximize

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{\theta}}{\theta}.$$

Hence, with the notation of Sect. 2.2, $g_t(x,y) = \beta^t \theta^{-1}(ax-y)^\theta$. The EE

$$-(a\hat{k}_{t-1} - \hat{k}_t)^{\theta-1} + \beta a(a\hat{k}_t - \hat{k}_{t+1})^{\theta-1} = 0, \quad t = 1,2,\ldots,$$

is equivalent to

$$b\hat{k}_{t+1} - (1+ab)\hat{k}_t + a\hat{k}_{t-1} = 0,$$

with $b := (a\beta)^{\frac{1}{\theta-1}}$. Its general solution is

$$\hat{k}_t = a_1 b^{-t} + a_2 a^t, \quad t = 0,1,\ldots,$$

for some constants a_1, a_2. Notice that

$$\hat{k}_t \frac{\partial g_t}{\partial x}(\hat{k}_t, \hat{k}_{t+1}) = q\left[a_1(\beta b^{-\theta})^t + a_2(a\beta b^{1-\theta})^t\right], \tag{2.30}$$

where $q := a[a_1(a-b^{-1})]^{\theta-1}$.

We now claim that

$$a\beta b^{1-\theta} = 1, \tag{2.31}$$

$$0 < \beta b^{-\theta} < 1. \tag{2.32}$$

Indeed, the equality (2.31) follows because $b = (a\beta)^{\frac{1}{\theta-1}}$. Since $\theta < 0$, from (2.29) we see that

$$\frac{1}{\theta-1} < 0, \quad 0 < a\beta < 1.$$

Thus, $1 < (a\beta)^{\frac{1}{\theta-1}}$. Moreover, $1 < a(a\beta)^{\frac{1}{\theta-1}}$, and $a(a\beta)^{\frac{1}{\theta-1}} = \beta^{-1}b^\theta$. This proves (2.32).

From (2.30) to (2.32) and the TC (2.18), we observe that $a_2 = 0$. The initial condition implies $a_1 = k_0$. Therefore we obtain

$$\hat{k}_t = k_0 b^{-t}, \quad \hat{c}_t = (a-b^{-1})\hat{k}_t, \quad t = 0,1,2,\ldots,$$

and the value function

$$v(k_0) = \frac{(a-b^{-1})^\theta}{\theta(1-\beta b^{-\theta})} k_0^\theta.$$

2.3.3 A Consumption–Investment Problem

Let $\gamma, \beta \in (0,1)$ and $R > 0$ be such that $\beta R^{1-\gamma} < 1$. Assume that x_t is the wealth of a certain investor at date $t = 0,1,2,\ldots$. For each t, the investor consumes a fraction $s_t \in (0,1)$ of the assets x_t. Suppose that the investor wishes to maximize

$$\sum_{t=0}^{\infty} \beta^t (s_t x_t)^{1-\gamma},$$

subject to the dynamics of the assets

$$x_{t+1} = R(1-s_t)x_t, \quad t = 0,1,2,\ldots,$$

where $x_0 > 0$ is given.

In this case the functions g_t $(t = 0,1,\ldots)$ are of the form $g_t(x,y) = \beta^t(x - y/R)^{1-\gamma}$. Hence, the EE (2.14) becomes the nonlinear equation

$$-\frac{1-\gamma}{R}\left[\hat{x}_{t-1} - \frac{\hat{x}_t}{R}\right]^{-\gamma} + \beta(1-\gamma)\left[\hat{x}_t - \frac{\hat{x}_{t+1}}{R}\right]^{-\gamma} = 0, \quad t = 1,2,\ldots,$$

but it can be rewritten as a linear one:

$$\hat{x}_{t+1} - [R + (R\beta)^{1/\gamma}]\hat{x}_t + R(R\beta)^{1/\gamma}\hat{x}_{t-1} = 0, \quad t = 1,2,\ldots. \tag{2.33}$$

The general solution to (2.33) is $\hat{x}_t = a_1 R^t + a_2 (R\beta)^{t/\gamma}$ for some constants a_1, a_2. Observe that

$$\hat{x}_t \frac{\partial g_t}{\partial x}(\hat{x}_t, \hat{x}_{t+1}) = d\left[a_1 + a_2\rho^t\right] \tag{2.34}$$

where $d := (1-\gamma)[a_2(1-\rho)]^{-\gamma}$ and $\rho := R^{-1}(R\beta)^{1/\gamma}$. The assumption $0 < \beta R^{1-\gamma} < 1$ yields that $0 < \rho < 1$. Thus, by (2.34) and the TC (2.18), we have $a_1 = 0$. In addition, the initial condition gives $a_2 = x_0$. Therefore,

$$\hat{x}_t = x_0(R\beta)^{t/\gamma}, \quad \hat{s}_t = 1 - \rho, \quad t = 0,1,2,\ldots.$$

Finally we have the value function

$$v(x_0) = (1-\rho)^{-\gamma} x_0^{1-\gamma}.$$

Remark 2.4. Sydsæter et al. [71, Example 12.3.1, pp. 437–438] solve this consumption-investment problem using dynamic programming. They guess that the value function takes the form $v(x) = kx^{1-\gamma}$ for some constant k, and then they find the value of k by means of the Bellman equation. ♢

2.3.4 The Great Fish War of Levhari and Mirman

This problem concerns fisheries. Let x_t $(t = 0,1,2,\ldots)$ be the stock of fish at time t, in a specific fishing zone. Assume there are k countries deriving utility from fish consumption. More precisely, country i wants to maximize

$$\sum_{t=0}^{\infty} \beta_i^t \log(c_t^i), \quad i = 1,\ldots,k,$$

where β_i is a discount factor and c_t^i is the consumption corresponding to country i. The fish population follows the dynamics

$$x_{t+1} = (x_t - c_t^1 - \cdots - c_t^k)^\alpha, \quad t = 0, 1, 2, \ldots, \tag{2.35}$$

where x_0 is given and $0 < \alpha < 1$.

In this example, we want to find a *Pareto (or cooperative) solution* to this problem, that is, we want to maximize the convex combination

$$\sum_{t=0}^{\infty} [\lambda_1 \beta_1^t \log(c_t^1) + \cdots + \lambda_k \beta_k^t \log(c_t^k)],$$

subject to (2.35), where $\lambda_1 + \cdots + \lambda_k = 1$ and each $\lambda_i > 0$. Define

$$g_t(x_t, x_{t+1}) := \max\{\lambda_1 \beta_1^t \log c_t^1 + \cdots + \lambda_k \beta_k^t \log c_t^k \mid c_t^1 + \cdots + c_t^k = x_t - x_{t+1}^{1/\alpha}\}.$$

The maximization problem of the right-hand side can be solved by the Lagrange multipliers method. We find

$$c_t^i = \frac{\lambda_i \beta_i^t}{\beta_\lambda(t)} (x_t - x_{t+1}^{1/\alpha}), \quad i = 1, \ldots, k, \tag{2.36}$$

where $\beta_\lambda(t) := \lambda_1 \beta_1^t + \cdots + \lambda_k \beta_k^t$. Thus,

$$g_t(x_t, x_{t+1}) = \beta_\lambda(t) \log(x_t - x_{t+1}^{1/\alpha}) + \sum_{i=1}^{k} \lambda_i \beta_i^t [\log(\lambda_i \beta_i^t) - \log \beta_\lambda(t)].$$

The EE (2.14) for this OCP is

$$-\beta_\lambda(t-1) \frac{1}{z_{t-1} - 1} + \alpha \beta_\lambda(t) \frac{z_t}{z_t - 1} = 0, \tag{2.37}$$

where $z_t := \hat{x}_t / \hat{x}_{t+1}^{1/\alpha}$ $(t = 0, 1, \ldots)$. Making $z_t := w_t / w_{t+1}$, (2.37) can be written as

$$\frac{(w_{t+1} - w_t)/\beta_\lambda(t)}{(w_t - w_{t-1})/\beta_\lambda(t-1)} = \alpha.$$

Then $(w_{t+1} - w_t)/\beta_\lambda(t) = a_1 \alpha^t$, for some constant a_1. Equivalently,

$$w_{t+1} - w_t = a_1 \alpha^t \beta_\lambda(t). \tag{2.38}$$

Equation (2.38) is a nonhomogeneous linear equation. Recall that $\beta_\lambda(t) = \lambda_1 \beta_1^t + \cdots + \lambda_k \beta_k^t$. We propose a particular solution w_t^p to (2.38) of the form

$$w_t^p = (b_1 \lambda_1 \beta_1^t + \cdots + b_k \lambda_k \beta_k^t) \alpha^t,$$

for some undetermined coefficients $b_1,\dots,b_k$. Substituting w_t^p in (2.38) we obtain $b_i = (1-\alpha\beta_i)^{-1}a_1$ $(i=1,\dots,k)$. The general solution to (2.38) is of the form

$$w_t = a_2 + w_t^p = a_2 + a_1 \sum_{i=1}^{k} \frac{\lambda_i(\alpha\beta_i)^t}{1-\alpha\beta_i}, \quad t=0,1,\dots, \tag{2.39}$$

for some constants a_1, a_2. Since $z_t := w_t/w_{t+1}$, observe that

$$\begin{aligned}
\hat{x}_t \frac{\partial g_t}{\partial x}(\hat{x}_t,\hat{x}_{t+1}) &= \frac{\beta_\lambda(t) z_t}{z_t - 1} \\
&= \frac{\beta_\lambda(t)(a_2 + w_t^p)}{w_t^p - w_{t+1}^p} \\
&= \frac{(a_2 + w_t^p)}{a_1 \alpha^t} \\
&= \frac{a_2}{a_1\alpha^t} + a_1^{-1} \sum_{i=1}^{k} \frac{\lambda_i \beta_i^t}{1-\alpha\beta_i}.
\end{aligned}$$

The TC (2.18) implies $a_2 = 0$. Define

$$\beta_{\alpha\lambda}(t) := \sum_{i=1}^{k} \frac{\lambda_i\beta_i^t}{1-\alpha\beta_i}, \quad t = 0,1,\dots.$$

Thus, $z_t = w_t^p / w_{t+1}^p = \alpha^{-1}\beta_{\alpha\lambda}(t)/\beta_{\alpha\lambda}(t+1)$. Finally we get a (nonlinear) first-order difference equation for $\{\hat{x}_t\}$:

$$\hat{x}_{t+1}^{1/\alpha} = \frac{\alpha\beta_{\alpha\lambda}(t+1)}{\beta_{\alpha\lambda}(t)}\hat{x}_t, \quad t=0,1,\dots.$$

Therefore, by (2.36), for each $i=1,\dots,k$ the nonstationary Markov strategy for consumption is

$$\hat{c}_t^i = \frac{\lambda_i\beta_i^t}{\beta_{\alpha\lambda}(t)}\hat{x}_t, \quad t=0,1,\dots.$$

The great fish war problem was first studied, for $k=2$, by Levhari and Mirman [52]. Okuguchi [62] considers the model with k countries; nonetheless, he assumes $\beta_i = \beta$ $(i=1,\dots,k)$ for the Pareto solution.

2.3.5 The Discounted LQ Problem

An OCP with a linear system equation and a quadratic cost function is known as a *LQ problem* (also called a *linear regulator problem*). *LQ* problems have been widely studied. See, for instance, Ljungqvist and Sargent [53, Chap. 5] for discrete-time; or Engwerda [29] for continuous-time.

We consider the deterministic scalar case. The state of the system evolves according to

$$x_{t+1} = \alpha x_t + \gamma u_t, \quad t = 0, 1, \ldots, \tag{2.40}$$

with $\alpha\gamma \neq 0$. The performance index is

$$\sum_{t=0}^{\infty} \beta^t [q x_t^2 + r u_t^2], \tag{2.41}$$

where $q, r > 0$, and $0 < \beta < 1$. Given x_0, we want to minimize (2.41) subject to (2.40).

Remark 2.5. Note that the control variable u_t is unconstrained, in the sense that the control set $U_t(x) \equiv \mathbb{R}$ for all $t = 0, 1, \ldots$ and $x \in \mathbb{R}$. Therefore, the state space X is $\mathbb{R}$, and we cannot apply Theorem 2.2. Nonetheless, Theorems 2.1 and 2.3 can be used. $\Diamond$

For this problem, $g_t(x,y) = \beta^t [q x^2 + r\gamma^{-2}(y - \alpha x)^2]$ for $t = 0, 1, \ldots$. The EE (2.14) becomes the linear equation

$$\alpha\beta r \hat{x}_{t+1} - (r + Q)\hat{x}_t + \alpha r \hat{x}_{t-1} = 0, \tag{2.42}$$

where $Q := (\alpha^2 r + \gamma^2 q)\beta$. The general solution to (2.42) is $\hat{x}_t = k_1 \lambda_1^t + k_2 \lambda_2^t$ for some constants k_1, k_2, and

$$\lambda_1 := \frac{r + Q + \sqrt{(r+Q)^2 - 4\alpha^2 \beta r^2}}{2\alpha\beta r},$$

$$\lambda_2 := \frac{r + Q - \sqrt{(r+Q)^2 - 4\alpha^2 \beta r^2}}{2\alpha\beta r}.$$

Remark 2.6. Observe that $(r+Q)^2 - 4\alpha^2\beta r^2 = (r-Q)^2 + 4\beta\gamma q r$. Because λ_1, λ_2 are the roots of the equation $\alpha\beta r \lambda^2 - (r+Q)\lambda + \alpha r = 0$, we have the following:

(a) λ_1, λ_2 are real and distinct,
(b) $\lambda_1 + \lambda_2 = (\alpha\beta r)^{-1}(r+Q)$, $\lambda_1 \lambda_2 = \beta^{-1}$,
(c) $\lambda_1 > 0$, $\lambda_2 > 0$,
(d) $0 < \lambda_1^{-1}\lambda_2 < 1$, and $\beta\lambda_1^2 = \lambda_1 \lambda_2^{-1} > 1$. $\Diamond$

The sequence $\hat{x}_t = k_1 \lambda_1^t + k_2 \lambda_2^t$ $(t = 0, 1, \ldots)$ satisfies

$$
\begin{aligned}
\hat{x}_t \frac{\partial g_{t-1}}{\partial y}(\hat{x}_{t-1}, \hat{x}_t) &= 2\gamma^{-2} r \beta^{t-1} \hat{x}_t (\hat{x}_t - \alpha \hat{x}_{t-1}) \\
&= \frac{2r\beta^{t-1}}{\gamma^2} [k_1 \lambda_1^t + k_2 \lambda_2^t][k_1 \lambda_1^{t-1}(\lambda_1 - \alpha) + k_2 \lambda_2^{t-1}(\lambda_2 - \alpha)] \\
&= \frac{2r(\beta\lambda_1^2)^{t-1}}{\gamma^2} \left[k_1 \lambda_1 + k_2 \lambda_2 \left(\frac{\lambda_2}{\lambda_1} \right)^{t-1} \right] \\
&\quad \times \left[k_1(\lambda_1 - \alpha) + k_2(\lambda_2 - \alpha) \left(\frac{\lambda_2}{\lambda_1} \right)^{t-1} \right].
\end{aligned}
$$

Then, the TC (2.15), Remark 2.6(d), and the initial condition imply $k_1 = 0, k_2 = x_0$. Therefore, $\hat{x}_t = x_0 \lambda_2^t$, and

$$
\begin{aligned}
\hat{u}_t &= \gamma^{-1}(\hat{x}_{t+1} - \alpha \hat{x}_t) \\
&= \gamma^{-1}(\lambda_2 - \alpha)\hat{x}_t
\end{aligned}
$$

for each $t = 0, 1, \ldots$. By substituting the plan $\hat{\pi} = \{\hat{x}_t\}$ in (2.41), we obtain

$$
v(\hat{\pi}) = \frac{q\gamma^2 + r(\lambda_2 - \alpha)^2}{\gamma^2(1 - \beta\lambda_2^2)} x_0^2. \tag{2.43}
$$

In this LQ problem, the TC (2.15) does not hold for all directions $\pi \in \Lambda$. However, we claim the following:

Remark 2.7. Let $\pi_0 \in \Lambda$ be the plan with entries that are all zero. Thus,

$$
v(\pi_0) = [q + r(\alpha/\gamma)^2] x_0^2. \tag{2.44}
$$

Define the subset Π_0 of Λ by

$$
\Pi_0 := \{\pi \in \Lambda \mid v(\pi) \le v(\pi_0)\}.
$$

Then

(a)

$$
\lim_{t \to \infty} x_t \frac{\partial g_{t-1}}{\partial y}(\hat{x}_{t-1}, \hat{x}_t) = 0 \quad \forall \{x_t\} \in \Pi_0, \tag{2.45}
$$

that is, the TC (2.15) holds for each $\pi \in \Pi_0$.

(b) $\hat{\pi}$ is in Π_0.

Proof of (a). If $\pi = \{x_t\}$ belongs to Π_0, then

$$
\begin{aligned}
v(\pi_0) &\ge v(\pi) \\
&= \sum_{t=0}^{\infty} \beta^t [q x_t^2 + r\gamma^{-2}(x_{t+1} - \alpha x_t)^2] \\
&\ge \sum_{t=0}^{\infty} \beta^t q x_t^2.
\end{aligned}
$$

This inequality implies that each term of the series is less than or equal to $v(\pi_0)$. Therefore,

$$|x_t| \leq \beta^{-t/2}\sqrt{q^{-1}v(\pi_0)}, \quad t = 1,2,\ldots. \tag{2.46}$$

On the other hand

$$\frac{\partial g_{t-1}}{\partial y}(\hat{x}_{t-1},\hat{x}_t) = \frac{2r(\lambda_2-\alpha)x_0}{\beta\lambda_2\gamma^2}(\beta\lambda_2)^t, \quad t = 1,2,\ldots. \tag{2.47}$$

By (2.46) and (2.47), there is a constant M such that

$$\left|x_t\frac{\partial g_{t-1}}{\partial y}(\hat{x}_{t-1},\hat{x}_t)\right| \leq M(\beta\lambda_2^2)^{t/2}, \quad t = 1,2,\ldots. \tag{2.48}$$

Note that Remark 2.6(b) and (d) yield that $\beta\lambda_2^2 < 1$. If we let $t \to \infty$, (2.45) follows. This proves Remark 2.7(a).
Proof of (b). By (2.43) and (2.44), we need to show that

$$\frac{q\gamma^2 + r(\lambda_2-\alpha)^2}{\gamma^2(1-\beta\lambda_2^2)} \leq q + \frac{r\alpha^2}{\gamma^2}. \tag{2.49}$$

The inequality (2.49) is equivalent to

$$\lambda_2 \leq \frac{2r\alpha}{r+Q},$$

which follows from the definition of λ_2. $\Diamond$

Remark 2.7 implies that $\hat{\pi}$ is an optimal plan in the set Π_0 by virtue of Theorem 2.3. Moreover, the definition of Π_0 implies $v(\pi) > v(\pi_0)$ for each $\pi \notin \Pi_0$. Therefore, $\hat{\pi}$ is a global optimal plan.

2.3.6 On the Assumptions

Assumptions 2.1 and 2.4 are standard in infinite-horizon problems. For instance, Stokey and Lucas [70, Sect. 4.4] verify that the discounted stationary Brock and Mirman model satisfies Assumption 2.1(b); the same procedure is valid for the non-stationary model in Example 2.1, since by our hypotheses in Sect. 2.3.1, $\{A_t\}$ is a sequence in a compact interval of positive numbers. Assumption 2.4(b) in the Brock and Mirman model requires, for each λ in $[0,1]$ and $t = 0,1,\ldots$, that

$$\lambda k_{t+1}^1 + (1-\lambda)k_{t+1}^2 \in \Gamma_t(\lambda k_t^1 + (1-\lambda)k_t^2) \tag{2.50}$$

whenever $\{k_t^1\}$ and $\{k_t^2\}$ belong to $\Pi(k_0)$. Condition (2.50) is a direct consequence of the concavity of the function $h(k) = k^\alpha$.

In contrast, Assumptions 2.2 and 2.3, in general, cannot be verified a priori for a given OCP because the optimal plans are unknown. In this case, we follow the standard procedure. Namely, first, we propose "potentially" optimal plans, and then we go back and check that they indeed verify those assumptions. As an example, we will next show that Assumptions 2.2 and 2.3 hold for the Brock and Mirman model and the plan (2.27).

For Assumption 2.2 we need two plans $\hat{\pi} = \{\hat{x}_t\}$ and $\pi = \{x_t\}$; however, as we can see in Sects. 2.3.1–2.3.4, it is common to choose $\pi = \hat{\pi}$. Thus, we need to verify that $\hat{\pi}$ is an internal plan in the direction $\hat{\pi}$ and that $\sum h_t'(\varepsilon)$ converges uniformly on some interval $(-\varepsilon_0, \varepsilon_0)$, where

$$h_t'(\varepsilon) = \frac{\partial g_t}{\partial x}((1+\varepsilon)\hat{x}_t, (1+\varepsilon)\hat{x}_{t+1}) \cdot \hat{x}_t + \frac{\partial g_t}{\partial y}((1+\varepsilon)\hat{x}_t, (1+\varepsilon)\hat{x}_{t+1}) \cdot \hat{x}_{t+1},$$

for $t = 1, 2, \ldots$. In the Brock and Mirman model, it can be checked that $\{\hat{k}_t\}$ is an internal plan in the direction $\{\hat{k}_t\}$, that is,

$$(1+\varepsilon)\hat{k}_{t+1} \in \Gamma_t((1+\varepsilon)\hat{k}_t) \quad \forall \varepsilon \in (-\varepsilon_0, \varepsilon_0),\ t = 0, 1, \ldots,$$

for some $\varepsilon_0 > 0$. Further, for each $t = 1, 2, \ldots$,

$$\begin{aligned} h_t'(\varepsilon) &= \beta^t \frac{\alpha A_t (1+\varepsilon)^{\alpha-1} \hat{k}_t^{\alpha} - \hat{k}_{t+1}}{A_t (1+\varepsilon)^{\alpha} \hat{k}_t^{\alpha} - (1+\varepsilon)\hat{k}_{t+1}} \\ &= \beta^t \frac{\alpha (1+\varepsilon)^{\alpha-1} - \alpha\beta}{(1+\varepsilon)^{\alpha} - (1+\varepsilon)\alpha\beta}. \end{aligned}$$

Therefore, $\sum h_t'(\varepsilon)$ converges uniformly on some interval $(-\varepsilon_0, \varepsilon_0)$. This shows that the OCP in Sect. 2.3.1 verifies Assumption 2.2 for $\pi = \hat{\pi}$.

Notice that the EE (2.14) is a consequence of Assumption 2.3 (and Assumption 2.1). In fact, Assumption 2.3 implies the existence of a plan $\pi_\tau(x)$ (see the proof of Theorem 2.1, above) such that $\hat{\pi}$ and $\pi_\tau(x)$ satisfy Assumption 2.2. For the Brock and Mirman model, Assumption 2.3(a) is clearly valid, whereas Assumption 2.3(b) holds with $\varepsilon_t = (1 - (\alpha\beta)^{1/\alpha})\hat{k}_t$.

Since the LQ problem in Sect. 2.3.5 is unconstrained, Assumption 2.3 is automatically satisfied. Thus, Theorem 2.1(a) can be used to obtain the EE (2.14). Instead of using Theorem 2.1(b), which requires verifying Assumption 2.2, we applied Theorem 2.3 and directly verified the TC (2.15); see Remark 2.7.

As we mentioned in Sect. 2.1, Bar-Ness [6, Theorem 3.2] derives the EE for an optimal plan converging to zero. For our results we do not require this assumption. Actually, we allow an optimal plan to be unbounded. For instance, let $\gamma \neq 0$, $q > 0$, and $\beta = 2^{-4}$ in the LQ model; define $r := \gamma^2 q\beta$ and α such that $\alpha^2\beta = 1$. Then $\lambda_2 > 1$, and hence the optimal plan $\hat{x}_t = x_0\lambda_2^t$ is unbounded, whenever $x_0 \neq 0$.

2.4 Extension to Stochastic Control Problems

In this section we generalize Theorem 2.1 to the stochastic case. This generalization is straightforward with the appropriate changes. We now deal with the objective function

$$\mathbb{E}\sum_{t=0}^{\infty} g_t(x_t, x_{t+1}, \xi_t), \tag{2.51}$$

where $\{\xi_t\}$ is a sequence of independent random variables. We suppose that each random variable ξ_t takes values in a Borel space S_t ($t = 0, 1, \ldots$), that is, a Borel subset of a complete and separable metric space. We also suppose that the initial state $x_0 \in X_0$ and the initial value $\xi_0 = s_0$ are given. In this problem each x_{t+1} has to be chosen at time t after the value of ξ_t has been observed. The family of feasible sets takes the form

$$\{\Gamma_t(x,s) \subseteq X_{t+1} \mid (x,s) \in X_t \times S_t\}.$$

Let $\varphi = (\mu_0, \mu_1, \ldots)$ be a sequence of measurable functions $\mu_t : X_t \times S_t \to X_{t+1}$ ($t = 0, 1, \ldots$). For each $(x_0, s_0) \in X_0 \times S_0$, the sequence φ determines a Markov process $\{x_t^\varphi \mid t = 1, 2, \ldots\}$ given by

$$\begin{aligned} x_1^\varphi &:= \mu_0(x_0, s_0), \\ x_{t+1}^\varphi &:= \mu_t(x_t^\varphi, \xi_t), \quad t = 1, 2, \ldots. \end{aligned}$$

The sequence $\varphi = (\mu_0, \mu_1, \ldots)$ is said to be a *feasible plan* from (x_0, s_0) if $x_1^\varphi \in \Gamma_0(x_0, s_0)$ and

$$x_{t+1}^\varphi \in \Gamma_t(x,s) \quad \forall (x,s) \in X_t \times S_t,$$

for $t = 1, 2, \ldots$. The set of all feasible plans from (x_0, s_0) is denoted by $\Phi(x_0, s_0)$.

The following assumption is analogous to Assumption 2.1 and it will be supposed to hold throughout the remainder of the section.

Assumption 2.5 The stochastic control model

$$(\{X_t\}, \{\xi_t\}, \{g_t\}, \Phi(x_0, s_0)) \tag{2.52}$$

satisfies the following for each $(x_0, s_0) \in X_0 \times S_0$:

(a) The set $\Phi(x_0, s_0)$ is nonempty;
(b) There is a sequence $\{m_t(x_0, s_0)\}$ of nonnegative real numbers such that

$$\mathbb{E}[g_t(x_t^\varphi, x_{t+1}^\varphi, \xi_t)] \le m_t(x_0, s_0),$$

for each $t = 0, 1, \ldots$ and $\varphi \in \Phi(x_0, s_0)$, and, moreover, $\sum_{t=0}^{\infty} m_t(x_0, s_0) < \infty$;
(c) For each $\varphi \in \Phi(x_0, s_0)$, the limit $\lim_{T\to\infty} \mathbb{E}\sum_{t=0}^{T} g_t(x_t^\varphi, x_{t+1}^\varphi, \xi_t)$ exists (it may be $-\infty$);
(d) There exists $\varphi \in \Phi(x_0, s_0)$ such that $\mathbb{E}\sum_{t=0}^{\infty} g_t(x_t^\varphi, x_{t+1}^\varphi, \xi_t) > -\infty$.
(e) For each $t = 0, 1, \ldots$ and each possible value $s_t \in S_t$ of ξ_t, the function $g_t(\cdot, \cdot, s_t)$ is differentiable in the interior of $X_t \times X_{t+1}$.

For each $(x_0, s_0) \in X_0 \times S_0$, we now define $v : \Phi(x_0, s_0) \to \mathbb{R} \cup \{-\infty\}$ by

$$v(\varphi) := \mathbb{E} \sum_{t=0}^{\infty} g_t(x_t^{\varphi}, x_{t+1}^{\varphi}, \xi_t). \tag{2.53}$$

The stochastic OCP is to find a feasible plan $\hat{\varphi} \in \Phi(x_0, s_0)$ such that

$$\sup\{v(\varphi) \mid \varphi \in \Phi(x_0, s_0)\} = v(\hat{\varphi}). \tag{2.54}$$

Since a feasible plan φ is, by definition, a sequence of measurable functions, the space Λ introduced in Sect. 2.2 is now replaced by the vector space

$$\tilde{\Lambda} := \{(\mu_0, \mu_1, \ldots) \mid \mu_t : X_t \times S_t \to \mathbb{R}^n,\ t = 0, 1, \ldots\}.$$

Assumption 2.6 Let $\hat{\varphi} \in \Phi(x_0, s_0)$ be some internal plan in the direction $\varphi \in \tilde{\Lambda}$. Denote by $\{\hat{x}_t^{\varphi}\}$ and $\{x_t^{\varphi}\}$ the corresponding state (Markov) processes induced by $\hat{\varphi}$ and φ, respectively. Define

$$h_0(\varepsilon) := g_0(x_0, \hat{x}_1^{\varphi} + \varepsilon x_1^{\varphi}, s_0), \quad h_t(\varepsilon) := g_t(\hat{x}_t^{\varphi} + \varepsilon x_t^{\varphi}, \hat{x}_{t+1}^{\varphi} + \varepsilon x_{t+1}^{\varphi}, \xi_t),$$

for $t = 1, 2, \ldots$. There exists $\varepsilon_0 > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \mathbb{E} \sum_{t=0}^{\infty} h_t(\varepsilon) = \mathbb{E} \sum_{t=0}^{\infty} \frac{\mathrm{d}h_t}{\mathrm{d}\varepsilon}(\varepsilon) \quad \forall \varepsilon \in (-\varepsilon_0, \varepsilon_0).$$

If $\hat{\varphi}$ and φ verify Assumption 2.6, then the Gâteaux differential of the function v in (2.53) is

$$\begin{aligned} \delta v(\hat{\varphi}; \varphi) = \mathbb{E}\Big[&\frac{\partial g_0}{\partial y}(x_0, \hat{x}_1^{\varphi}, s_0) \cdot x_1^{\varphi} + \\ &\sum_{t=1}^{\infty} \frac{\partial g_t}{\partial x}(\hat{x}_t^{\varphi}, \hat{x}_{t+1}^{\varphi}, \xi_t) \cdot x_t^{\varphi} + \frac{\partial g_t}{\partial y}(\hat{x}_t^{\varphi}, \hat{x}_{t+1}^{\varphi}, \xi_t) \cdot x_{t+1}^{\varphi} \Big]. \end{aligned} \tag{2.55}$$

Assumption 2.7 Let $\hat{\varphi} \in \Phi(x_0, s_0)$ be an optimal plan for the OCP (2.52)–(2.54). For each sequence $\{\xi_t = s_t\}$ of observed values,

(a) $\hat{x}_{t+1}^{\varphi}$ is an interior point of the set $\Gamma_t(\hat{x}_t{}^{\varphi}, s_t)$;
(b) there exists $\varepsilon_t > 0$ such that $\|x - \hat{x}_t^{\varphi}\| < \varepsilon_t$ implies $\hat{x}_{t+1} \in \Gamma_t(x, s_t)$.

Repeating the proof of Theorem 2.1, with the appropriate changes (e.g., replace the derivative (2.13) by (2.55)), we obtain the following.

Theorem 2.5. *Let $\hat{\varphi} \in \Phi(x_0, s_0)$ be an optimal plan for the OCP (2.52)–(2.54). Suppose that Assumption 2.7 holds. Then:*

(a) $\hat{\varphi}$ satisfies the stochastic Euler equation (SEE)

$$\mathbb{E}\left[\frac{\partial g_{t-1}}{\partial y}(\hat{x}_{t-1}^{\varphi}, \hat{x}_t^{\varphi}, \xi_{t-1}) + \frac{\partial g_t}{\partial x}(\hat{x}_t^{\varphi}, \hat{x}_{t+1}^{\varphi}, \xi_t) \right] = 0, \quad t = 1, 2, \ldots. \tag{2.56}$$

(b) Suppose that, in addition, $\hat{\varphi}$ and $\varphi \in \tilde{\Lambda}$ satisfy Assumption 2.7. Then $\hat{\varphi}$ and φ satisfy the TC

$$\lim_{t\to\infty} \mathbb{E}\left[\frac{\partial g_{t-1}}{\partial y}(\hat{x}^{\varphi}_{t-1}, \hat{x}^{\varphi}_{t}, \xi_{t-1}) \cdot x^{\varphi}_{t}\right] = 0. \tag{2.57}$$

Remark 2.8. In this stochastic model it is assumed that x_t is chosen after the value $\xi_{t-1} = s_{t-1}$ has been observed, for $t = 1, 2, \ldots$. Taking into account this assumption, the SEE (2.56) is also written as

$$\frac{\partial g_{t-1}}{\partial y}(\hat{x}^{\varphi}_{t-1}, \hat{x}^{\varphi}_{t}, s_{t-1}) + \mathbb{E}\left[\frac{\partial g_t}{\partial x}(\hat{x}^{\varphi}_{t}, \hat{x}^{\varphi}_{t+1}, \xi_t)\right] = 0, \quad t = 1, 2, \ldots. \tag{2.58}$$

See, for instance, Stokey and Lucas [70, Sect. 9.5] (for discounted stationary models) or Acemoglu [3, Sect. 16.3]. ♢

Sufficient conditions for optimality can be given, as in Theorem 2.3, if $\hat{\varphi}$ satisfies the SEE (2.56) and the TC (2.57) for each $\varphi \in \Phi(x_0, s_0)$. We also have to require convexity of the set $\Phi(x_0, s_0)$ and concavity of the function $g_t(\cdot, \cdot, s_t)$ for each $s_t \in S_t$ $(t = 0, 1, \ldots)$.

Example 2.2 (A stochastic LQ problem). In this example we consider a stochastic version of the OCP studied in Sect. 2.3.5. We assume the dynamics

$$x_{t+1} = \alpha x_t + \gamma u_t + \xi_t, \quad t = 0, 1, \ldots, \tag{2.59}$$

where $\alpha\gamma \neq 0$ and $\{\xi_t\}$ is a sequence of i.i.d. random variables with zero mean and variance σ^2. Let $q, r > 0$, and $0 < \beta < 1$. Given an initial state x_0, we want to minimize

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t [q x_t^2 + r u_t^2], \tag{2.60}$$

subject to (2.59).

Note that $g_t(x, y, s) = \beta^t [q x^2 + r\gamma^{-2}(y - \alpha x - s)^2]$. Then the SEE for the optimal plan $\hat{x}^{\pi}_t$ is

$$\mathbb{E}[\alpha\beta r \hat{x}^{\pi}_{t+1} - (r + Q)\hat{x}^{\pi}_t + \alpha r \hat{x}^{\pi}_{t-1} + r\xi_{t-1} - r\alpha\beta\xi_t] = 0, \tag{2.61}$$

where $Q := (\alpha^2 r + \gamma^2 q)\beta$. Let $\bar{x}_t$ denote the expected value of $\hat{x}^{\pi}_t$ for $t = 0, 1, \ldots$. Thus, (2.61) becomes a deterministic difference equation

$$\alpha\beta r \bar{x}_{t+1} - (r + Q)\bar{x}_t + \alpha r \bar{x}_{t-1} = 0, \quad t = 1, 2, \ldots. \tag{2.62}$$

Using the same notation of Sect. 2.3.5, the general solution to (2.62) is given by $\bar{x}_t = k_1 \lambda_1^t + k_2 \lambda_2^t$, where k_1, k_2 are constants to be determined.

On the other hand

$$\begin{aligned}
\mathbb{E}\left[\bar{x}_t \frac{\partial g_{t-1}}{\partial y}(\hat{x}^{\pi}_{t-1}, \hat{x}^{\pi}_{t}, \xi_{t-1})\right] &= 2r\gamma^{-2}\beta^{t-1}\bar{x}_t \mathbb{E}(\hat{x}^{\pi}_t - \alpha\hat{x}^{\pi}_{t-1} - \xi_{t-1}) \\
&= 2r\gamma^{-2}\beta^{t-1}\bar{x}_t(\bar{x}_t - \alpha\bar{x}_{t-1}).
\end{aligned}$$

Therefore, we can use the TC (2.57) with $x_t^\pi = \bar{x}_t$ to solve (2.62) as in Sect. 2.3.5. The expected values are $\bar{x}_t = \lambda_2^t x_0$ for $t = 1, 2, \ldots$.

By virtue of Remark 2.8, we have

$$\begin{aligned} 0 &= r(\hat{x}_t^\pi - \alpha \hat{x}_{t-1}^\pi - s_{t-1}) + \beta \mathbb{E}[q\gamma^2 \hat{x}_t^\pi - \alpha r(\hat{x}_{t+1}^\pi - \alpha \hat{x}_t^\pi - \xi_t)] \\ &= r(\hat{x}_t^\pi - \alpha \hat{x}_{t-1}^\pi - s_{t-1}) + \beta[(q\gamma^2 + r\alpha^2)\bar{x}_t - \alpha r \bar{x}_{t+1}] \\ &= r(\hat{x}_t^\pi - \alpha \hat{x}_{t-1}^\pi - s_{t-1}) + \beta(q\gamma^2 + r\alpha^2 - \alpha r \lambda_2)\lambda_2^t x_0. \end{aligned}$$

Hence $\hat{x}_t^\pi = \mu_{t-1}(\hat{x}_{t-1}^\pi, s_{t-1})$, where

$$\mu_{t-1}(x,s) := -\beta r^{-1}(q\gamma^2 + r\alpha^2 - \alpha r \lambda_2)\lambda_2^t x_0 + \alpha x + s, \quad t = 1, 2, \ldots.$$

This is the optimal plan for the stochastic LQ problem (2.59)–(2.60). ◊

Example 2.3. We now consider random shocks in the Brock and Mirman model of Example 1.2 Specifically, the state follows the dynamics

$$k_{t+1} = \xi_t k_t^\alpha - c_t,$$

where $\{\xi_t\}$ is a sequence of independent random variables taking positive values. It should be noted that each parameter A_t in the OCP of Sect. 2.3.1 is replaced by a random shock ξ_t. For this OCP the objective function takes the form

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t \log(\xi_t k_t^\alpha - k_{t+1}).$$

The associated SEE is

$$\mathbb{E}\left[\frac{-1}{\xi_{t-1}\hat{k}_{t-1}^\alpha - \hat{k}_t} + \beta \frac{\alpha \xi_t \hat{k}_t^{\alpha-1}}{\xi_t \hat{k}_t^\alpha - \hat{k}_{t+1}}\right] = 0, \quad t = 1, 2, \ldots. \tag{2.63}$$

In analogy with the deterministic OCP of Sect. 2.3.1 [see (2.27)], we propose an optimal plan $\hat{k}_{t+1} = \mu_t(\hat{k}_t, \xi_t)$, where

$$\mu_t(k,s) = \alpha\beta s k^\alpha.$$

It can be verified that this is indeed an optimal plan. ◊

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