

Preface

Optimal control problems and dynamic games are mathematical models to analyze decision-making processes that arise in economics, engineering, and resource management, among many other fields. Dynamic games concern several, say $N > 1$, decision-makers called players or agents or controllers, depending on the context. In contrast, an optimal control problem involves a single ($N = 1$) decision-maker. In this book, we are interested in so-called noncooperative dynamic games in which we wish to find (if they exist) *noncooperative equilibria* also known as *Nash equilibria*. To find these equilibria usually requires studying N particular optimal control problems (one for each of the N players). There is, however, a class of noncooperative dynamic games, called *potential games*, which reduce to study a *single* optimal control problem. Hence, it is important to characterize potential games because they are easier to study than general noncooperative games. In addition, potential games have applications in different areas; see, for instance, [8, 38, 57, 65, 69], and the references therein.

There are several techniques to study noncooperative dynamic games, such as dynamic programming and the maximum principle (also called the Lagrange method). It turns out, however, that one way to characterize dynamic potential games requires analyzing *inverse* optimal control problems, and it is here where the Euler equation approach comes in because it is particularly well-suited to solve inverse problems.

Despite the importance of dynamic potential games, there is no systematic study about them. There is just a handful of contributions about this topic, and mostly about particular cases, such as the stochastic lake game by Dechert and O'Donnell [23]. Our book is, to the best of our knowledge, the first attempt to provide a systematic, self-contained presentation of stochastic dynamic potential games. We characterize particular classes of (deterministic and stochastic) dynamic potential games. Actually, we identify two such classes. The first one is characterized by a generalization to the stochastic case of an inverse optimal control problem studied by Dechert [21]. Similarly, the second class is obtained by extending the work of Slade [69] for deterministic games. Moreover, we show that for certain stochastic games the results by Dechert and Slade are in fact equivalent. To reach these aims we start by extending the Euler equation approach to a general class of (direct) stochastic

control problems. We then continue our study with stochastic inverse problems and dynamic games.

The literature on optimal control and dynamic games includes several classes of models. In particular, discrete-time *discounted stationary* control problems have been widely studied, for instance, in the books [18, 41, 51], and [70]. The discrete-time *nonstationary* case, which generalizes the discounted stationary one, can be found only in just a few works such as [39] or [3, Theorem 6.11]. Discrete-time models are usually studied in the mentioned references by means of dynamic programming and the Lagrange method. Continuous-time models are studied, for instance, in [7, 17, 24, 32, 33], among many others. Here we are interested in discrete-time nonstationary stochastic models with infinite horizon.

For *static* potential games there is an extensive literature—see, for instance, [12, 14, 25, 48, 61, 67, 72], etc. In contrast, for either deterministic or stochastic *dynamic* potential games there are just a few papers: [8, 22, 23, 69].

This book consists of five chapters organized as follows. Chapter 1 is an introduction to discrete-time stochastic control and dynamic games. We introduce some concepts that are illustrated by means of examples. Dynamic programming and the Lagrange method for solving optimal control problems are briefly reviewed. Some important results about static potential games are also presented. We close with a detailed description of the remaining chapters.

The optimal control model we are interested in and the Euler equation approach are described in Chap. 2. We derive the Euler equation and a *transversality condition* as necessary conditions for optimality. Both conditions are derived by using Gâteaux differentials and requiring assumptions milder than in previous works. Under suitable convexity hypotheses, we show that the Euler equation and the transversality condition are also sufficient for optimality. Our results are used to find explicit solutions to several optimal control problems.

Chapter 3 is devoted to an inverse optimal control problem in stochastic control. We generalize in several ways the results of Dechert [21], in particular to the stochastic case. This inverse problem is used to determine a class of stochastic dynamic potential games.

The purpose of Chap. 4 is twofold. First, to determine Nash equilibria in dynamic games using the Euler equation along with the transversality condition. Sufficient conditions are provided to determine Markov–Nash equilibria in stochastic games and open-loop equilibria in deterministic games. Second, to characterize dynamic potential games. Two classes of dynamic potential games and a particular subclass are identified. A feature of such a subclass is that a Pareto equilibrium (or cooperative solution) is also a (noncooperative) Nash equilibrium.

Finally, in Chap. 5 we conclude with some remarks about our contributions and suggest possible future research.

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Discrete-Time Stochastic Control and Dynamic Potential
Games

The Euler-Equation Approach

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2013, XIV, 69 p., Softcover

ISBN: 978-3-319-01058-8