Chapter 2
The Lorentz Transformation

Imagination is more important than knowledge.
—Albert Einstein

2.1 Introduction

Now that we have seen the main consequences of the postulates of Special Relativity, i.e., the relativity of simultaneity, time dilation, and length contraction it is clear that the Galilei transformation, with its absolute time, is incorrect. These important physical phenomena can be seen as direct consequences of the correct transformation relating inertial frames, the Lorentz transformation. This transformation is the key for the formulation of Special Relativity in an enlightening four-dimensional formalism, which we will see in the next chapter. Here we study the Lorentz transformation and its properties and derive length contraction and time dilation directly from it, in addition to the transformation property of velocities. We must emphasize that, although the Lorentz transformation was discovered by studying the Maxwell equations, its validity is more general. The Lorentz transformation relates inertial frames without reference to the kind of physics studied in them. Lorentz-invariance is a general requirement for *any* physical theory, not just for electromagnetism.

2.2 The Lorentz Transformation

The Galilei transformation is not valid for speeds which are not negligible in comparison with the speed of light. The correct transformation relating space and time coordinates in two inertial frames $\{t, x, y, z\}$ and $\left\{t', x', y', z'\right\}$ moving with relative velocity v in standard configuration was discovered by Fitzgerald in 1889 and by

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Lorentz in 1892 as the transformation which leaves the Maxwell equations invariant.[1] The *Lorentz transformation* or *Lorentz boost* is[2]

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{2.1}$$

$$y' = y, \tag{2.2}$$

$$z' = z, \tag{2.3}$$

$$t' = \frac{t - v\,\frac{x}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{2.4}$$

The most striking feature of this linear coordinate transformation (bear in mind that v and $\gamma \equiv \left(1 - v^2/c^2\right)^{-1/2}$ are constants) is that it mixes the space and time coordinates. As a consequence, time intervals and 3-dimensional lengths are not invariant under this transformation and, therefore, time intervals and 3-dimensional lengths are not absolute quantities. The equations of electromagnetism are invariant under this transformation (it is said that they are *Lorentz-invariant*) but the equations of Newtonian mechanics are not.

The *inverse Lorentz transformation* is obtained by the exchange $\mathbf{x} \longleftrightarrow \mathbf{x}'$ and $v \longleftrightarrow -v$ according to the Principle of Relativity[3]:

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{2.5}$$

$$y = y', \tag{2.6}$$

[1] Lorentz was also trying to explain the null result of the Michelson-Morley experiment by a physical contraction of the apparatus in the direction of motion. His interpretation, however, is rather misleading: the Lorentz transformation relates measurements performed in two different inertial systems.

[2] A priori, the constant c appearing in the Lorentz transformation is a fundamental velocity which needs not coincide with the speed of electromagnetic waves in vacuo, and is only later identified with it. This is not the historical route, in which the Lorentz transformation was derived from the Maxwell equations. There are many facets to the quantity c in various areas of physics (see Ref. [1] for a review).

[3] This application of the Principle of Relativity is sometimes called "principle of reciprocity" and is a consequence of the isotropy of space [2, 3].

$$z = z', \tag{2.7}$$

$$t = \frac{t' + v\frac{x'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{2.8}$$

Although the Lorentz transformation was obtained before the formulation of Special Relativity as the correct transformation between inertial frames which respects the Maxwell equations, its meaning was not grasped until Einstein's 1905 paper.

2.3 Derivation of the Lorentz Transformation

The Lorentz transformation can be derived on the basis of the two postulates of Special Relativity. First, due to the isotropy of space contained in the second postulate, we can orient the spatial axes of an inertial frame S' with those of another inertial frame S and limit ourselves to considering motion of the two frames in standard configuration. As a starting point for deducing the transformation relating two inertial frames $S = \{t, x, y, z\}$ and $S' = \left\{t', x', y', z'\right\}$ in relative motion with velocity v in standard configuration, it is reasonable to assume that the transformation is linear

$$x' = G(x - vt), \tag{2.9}$$

where G is a dimensionless constant that depends only on v/c. This assumption corresponds to the homogeneity of space and time since G does not depend on (x, t). Since the spatial part of the Galilei transformation $x' = x - vt$ must be recovered in the limit $v/c \to 0$, G must tend to unity in this limit.

The laws of physics must have the same form in S and S', and then one must obtain the inverse Lorentz transformation by the exchange $\left(t', \mathbf{x}'\right) \longleftrightarrow (t, \mathbf{x})$ and $v \longleftrightarrow -v$ (this is the Principle of Relativity again):

$$x = G(x' + vt'). \tag{2.10}$$

There is no relative motion in the y and z directions, hence these coordinates must not be affected by the transformation,

$$y' = y, \quad z' = z. \tag{2.11}$$

Consider now a spherical pulse of electromagnetic radiation emitted at the origin of S at $t = 0$. It is received at a point on the x-axis and, during its propagation,

$$x = ct. \tag{2.12}$$

The same law must be true in S' due to the constancy of the speed of light,

$$x' = ct', \tag{2.13}$$

or $ct' = G(ct - vt)$ from Eq. (2.9), which leads to

$$t' = G\left(t - \frac{v}{c}t\right), \tag{2.14}$$

while

$$ct = G(ct' + vt') = G(c + v)t'. \tag{2.15}$$

By substituting Eq. (2.14) into Eq. (2.10), one obtains

$$ct = G(c + v)G(t - \frac{v}{c}t) = \frac{G^2}{c}(c + v)(c - v)t,$$

and

$$G^2 = \frac{c^2}{(c + v)(c - v)}.$$

Therefore, we have

$$G = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \equiv \gamma,$$

the Lorentz factor, and

$$x' = \gamma(x - vt). \tag{2.16}$$

Use then $x = \gamma(x' + vt')$ and $x' = \gamma(x - vt)$ and substitute x' into x to obtain

$$x = \gamma[\underbrace{\gamma(x - vt)}_{x'} + vt'],$$

$$x = \gamma^2 x - \gamma^2 vt + \gamma vt',$$

from which we obtain

$$t' = \frac{x - \gamma^2 x + \gamma^2 vt}{\gamma v}$$

and

$$t' = \frac{x}{v\gamma}(1 - \gamma^2) + \gamma t.$$

Since

$$1-\gamma^2 = 1 - \frac{1}{1-\frac{v^2}{c^2}} = \frac{1-\frac{v^2}{c^2}-1}{1-\frac{v^2}{c^2}} = \frac{-(\frac{v^2}{c^2})}{1-\frac{v^2}{c^2}},$$

we have

$$\frac{1-\gamma^2}{\gamma} = \sqrt{1-\frac{v^2}{c^2}}\,\frac{-(\frac{v^2}{c^2})}{(1-\frac{v^2}{c^2})} = \frac{-\frac{v^2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \equiv -\gamma\,\frac{v^2}{c^2},$$

$$t' = \frac{1-\gamma^2}{\gamma}\,\frac{x}{v} + \gamma t = -\gamma\,\frac{v^2}{c^2}\frac{x}{v} + \gamma t,$$

and, finally,

$$t' = \gamma\left(t - \frac{v}{c^2}x\right), \tag{2.17}$$

which completes the derivation. Equations (2.16), (2.11), and (2.17) constitute the Lorentz transformation. The fact that the Lorentz transformation can be derived from the two postulates of Special Relativity is conceptually important: it means that these two postulates constitute the physical explanation of the mathematical transformation and that this transformation should not be assumed in place of the two postulates, as Poincaré seem to have intended. While Poincaré, Lorentz, and FitzGerald stopped at the transformation (which is an important ingredient of Special Relativity and unveils the mixing of space and time of the 4-dimensional world view), they tried to explain it with an ether and with length contraction. It was Einstein's genius which reduced the physical explanation of the transformation to two simple and general postulates and led to a re-examination of the concepts of space and time, developing the full theory which was missed by other researchers.

2.4 Mathematical Properties of the Lorentz Transformation

Let us examine the properties of the Lorentz transformation.

- Qualitatively, the Lorentz transformation mixes t and x, therefore 3-dimensional lengths and time intervals cannot be left invariant. Quantitatively, length contraction and time dilation can be derived from the Lorentz transformation as a consequence, which will be done in the following sections.

The quantity

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

("Minkowski line element") is invariant under Lorentz transformations.

This invariance is easy to see: by using the Lorentz transformation we have

$$
\begin{aligned}
(\mathrm{d}s')^2 &\equiv -c^2(\mathrm{d}t')^2 + (\mathrm{d}x')^2 + (\mathrm{d}y')^2 + (\mathrm{d}z')^2 \\
&= -c^2\gamma^2\left(\mathrm{d}t - \frac{v}{c^2}\,\mathrm{d}x\right)^2 + \gamma^2\,(\mathrm{d}x - v\mathrm{d}t)^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\
&= -c^2\gamma^2\left(1 - \frac{v^2}{c^2}\right)\mathrm{d}t^2 + 2c^2\gamma^2\,\frac{v}{c^2}\,\mathrm{d}t\mathrm{d}x + \gamma^2\mathrm{d}x^2\left(1 - \frac{v^2}{c^2}\right) \\
&\qquad -2\gamma^2\,v\,\mathrm{d}t\mathrm{d}x + \mathrm{d}y^2 + \mathrm{d}z^2 \\
&= -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \equiv \mathrm{d}s^2.
\end{aligned}
$$

We will discuss extensively this aspect of Special Relativity in the following chapters.

- The Lorentz transformation is symmetric under the exchange $x \longleftrightarrow ct$:

$$
\begin{aligned}
x' &= \gamma\,(x - vt)\,, \\
y' &= y, \\
z' &= z, \\
ct' &= \gamma\left(ct - \frac{vx}{c}\right),
\end{aligned}
$$

becomes

$$
\begin{aligned}
ct' &= \gamma\left(ct - \frac{vx}{c}\right), \\
y' &= y, \\
z' &= z, \\
x' &= \gamma\,(x - vt).
\end{aligned}
$$

In standard configuration, the Lorentz transformation is also symmetric under the exchange $y \longleftrightarrow z$.

- The Galilei transformation can somehow be recovered from the Lorentz transformation in the limit of small velocities $|v|/c \ll 1$, although the derivation is a bit finicky. First, expand the Lorentz factor γ to first order in v/c:

$$\gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 1 + \frac{v^2}{2c^2} + \ldots \approx 1 \tag{2.18}$$

and

$$x' \approx x - vt,$$
$$y' = y,$$
$$z' = z.$$

Strictly speaking, the transformation of the time coordinate gives, to first order in v/c,

$$t' = t - \frac{v}{c^2}\,x,$$

not $t' = t$ as in the Galilei transformation: the relativity of simultaneity persists to first order (it is a first order effect in v/c). If the Lorentz transformation reduced to the Galilei transformation to first order, then infinitesimal Lorentz transformations and infinitesimal Galilei transformations would coincide, which is not the case.[4] However, time dilation is computed by considering time differences, recording two spatial events at the same location. Since $\Delta t' = \Delta t - \frac{v}{c^2}\,\Delta x$, by setting $\Delta x = 0$, time dilation is eliminated to first order. In practice, when speeds are small, time intervals Δt are measured over spatial distances Δx such that $c\Delta t \gg \Delta x \gg \frac{v}{c}\,\Delta x$ and the Δx term can be dropped. Although the Lorentz transformation does not quite reduce to the Galilei transformation, which is recovered only in the limit $v/c \longrightarrow 0$, Newtonian mechanics and the Galilei transformation turn out to be adequate[5] in the limit $|v| \ll c$.

- Since the Lorentz transformation is linear and homogeneous, finite coordinate differences transform in the same way as infinitesimal coordinate differences:

$$\begin{aligned} \Delta x' &= \gamma\,(\Delta x - v\Delta t), \\ \Delta y' &= \Delta y, \\ \Delta z' &= \Delta z, \\ \Delta t' &= \gamma\left(\Delta t - \tfrac{v}{c^2}\Delta x\right), \end{aligned} \quad \text{and} \quad \begin{aligned} \mathrm{d}x' &= \gamma\,(\mathrm{d}x - v\mathrm{d}t), \\ \mathrm{d}y' &= \mathrm{d}y, \\ \mathrm{d}z' &= \mathrm{d}z, \\ \mathrm{d}t' &= \gamma\left(\mathrm{d}t - \tfrac{v}{c^2}\mathrm{d}x\right). \end{aligned}$$

[4] This point is made clearly in Refs. [4, 5].

[5] Our derivation of the Lorentz transformation from the postulates of Special Relativity requires only that the *spatial part* of the Lorentz transformation $x' = G(v)\,(x - vt)$ reduces to the spatial part of the Galilei transformation $x' = x - vt$ in the limit $|v|/c \ll 1$, from which we deduced that $G \to 1$. We did not assume that we recover $t' = t$ in this limit, hence the proof is correct.

2.5 Absolute Speed Limit and Causality

At this point, you are certainly aware that $\gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, the ratio between coordinate and proper times, diverges as $v \to c$. The inequality $v > c$ leads to a purely imaginary γ, therefore,

the relative velocity of two inertial frames must be strictly smaller than c.

Since an inertial frame can be associated with any non-accelerated particle or object moving with subluminal (i.e., $|v| < c$) speed, this statement translates into the requirement that the speed of particles and of all physical signals be limited by c (remember that c is the sped of light in vacuo: the speed of particles traveling in a medium can be larger than the speed of light *in that medium*).[6] Never mind the fact that the Lorentz factor becomes imaginary: we can agree to define γ as the modulus $\left|\left(1-v^2/c^2\right)^{-1/2}\right|$ if $|v| > c$. What is truly important[7] is that

the restriction $|v| \leq c$ preserves the notion of cause and effect.

In fact, consider a process in which an event P causes, or affects, an event Q by sending a signal containing some information from P to Q. If a signal were sent from P to Q at superluminal speed $u > c$ in some inertial frame S, we could orient the axes of S so that both events P and Q occur on the x-axis and their time and spatial separations satisfy $\Delta t > 0$ and $\Delta x > 0$ in this frame. Then, in an inertial frame S' moving with respect to S with speed v in standard configuration, we would have

$$\Delta t' = \gamma \left(\Delta t - v\,\frac{\Delta x}{c^2}\right) = \gamma\,\Delta t\left(1-\frac{uv}{c^2}\right), \tag{2.19}$$

where we used $\Delta x = u\Delta t$. Now, because $u > c$ it is also $-u < -c$ which, together with $0 < v < c$, implies that $-uv < -c^2$ or $-uv/c^2 < -1$, so

$$\Delta t' = \gamma\,\Delta t\left(1-\frac{uv}{c^2}\right) < 0. \tag{2.20}$$

According to this result, in the frame S' the event Q precedes P: cause and effect are reversed or, the signal goes backward in time. The signal reaches Q before being emitted by P, which creates a logical problem. The fact that there is an absolute speed limit c comes to the rescue and enforces causality: if both $|u|, |v| < c$, then $\Delta t'$ in Eq. (2.19) has the same sign as Δt. The possibility of reversing cause and effect and travelling in time would lead to logical paradoxes, which have been discussed at length in the literature (see, e.g., [9] and the references therein).

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[6] If the particle traveling faster than light in that medium is charged, Cerenkov radiation is emitted.

[7] In the past, "tachyons" traveling at speed larger than c and incapable of slowing down to speeds less than c were considered theoretically and searched for experimentally (see, e.g., [6–8]) but no tachyon has been convincingly detected.

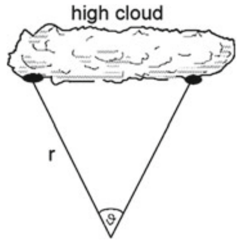

Fig. 2.1 If r becomes sufficiently large, a searchlight spot on a high cloud can attain an arbitrarily large apparent linear velocity

Another argument against superluminal travel is the following.[8] Suppose that an observer O, at rest in an inertial frame, emits a flash of laser light in a certain direction (called the "forward" direction). A second observer O' at rest in the same inertial frame sees the light, measuring that it moves with speed c. A third inertial observer O'' is moving with respect to O and O' in the "backward" direction with constant speed v. According to the second postulate, he measures light moving with speed c in the "forward" direction. He also sees O and O' moving in the "forward" direction toward him with speed v. Now, it is impossible for this speed v of the observers O and O' measured by O'' to be larger than c. For, were this possible, the flash of laser light, when emitted by O, will remain behind O. According to O'', the laser light would always remain behind O and the observer O' (who is "forward" of O) would never see this light. But then whether O' sees the light or not depends on the inertial frame, which contradicts the Principle of Relativity stating that all inertial frames are physically equivalent. Therefore, the relative speed of two inertial frames cannot be larger than c. Formally, the argument works also if the relative speed of inertial frames equals c for, in that case, light will never reach O' according to the observer O''.

If there is an absolute speed limit c then, according to the Principle of Relativity, this limit must be the same in all inertial frames, consistent with the postulate of the constancy of the speed of light.[9]

The absolute speed limit refers to *physical, propagating signals*. Apparent motions which carry no information can have arbitrarily large speeds. In these cases, certain motions *appear* to be faster than light, but they are illusions and not real motions. For example, consider a searchlight spot on high clouds (Fig. 2.1).

Let $\Delta\theta$ be the angle spanned by the light spot across a cloud in the time Δt. The linear velocity of the spot perceived by an observer on the ground is $r\Delta\theta/\Delta t$, where r is the distance to the cloud. If $\Delta\theta/\Delta t = 10\,\text{rad/s}$, the apparent linear velocity of the spot v_{spot} is larger than c if $r > 3 \cdot 10^7$ m. This distance is too large for a searchlight in the atmosphere but it illustrates the principle and it is not too large for astrophysical phenomena (apparent superluminal motions do occur in astrophysics and they constituted a puzzle when they were first discovered [11, 12]). In any case,

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[8] This argument is due to E. F. Taylor [10].

[9] When introducing a set of axioms for a theory, one should always worry about the mutual consistency of these axioms. If two axioms are not consistent with each other, one is building an empty theory.

everyday experience shows the apparent amplification of velocity when a searchlight spot moves on clouds.[10] The point is, the perceived velocity of the spot is not related to the velocity of propagation of the light beam.

Unphysical velocities occur also when waves propagate in a dispersive medium. In general, a wave signal[11] is composed of many, or infinite, monochromatic waves of angular frequencies ω and wave vectors $\mathbf{k}$, with $k \equiv |\mathbf{k}|$. The properties of the medium are described by a *dispersion relation* $\omega = \omega(k)$, a functional relation between ω and k. The individual monochromatic waves composing the complex wave propagate at the *phase velocity*[12]

$$v_p \equiv \frac{\omega}{k}, \tag{2.21}$$

while the "envelope" composed of the individual monochromatic waves travels at the *group velocity*

$$v_g \equiv \frac{d\omega}{dk}. \tag{2.22}$$

In Eqs. (2.21) and (2.22) the right hand sides must be evaluated at a central value[13] of k.

If the dispersion relation $\omega(k)$ is linear, the medium is non-dispersive; if this relation is non-linear, it is dispersive. Then the individual component waves have phase velocities which depend on their wave vectors (or wavelengths $\lambda = \frac{2\pi}{k}$) and the waveform is altered as it propagates through the medium. Phase velocity and group velocity then differ, and the physical velocity of the wave (the velocity at which energy and information propagate) is the group velocity. It is possible that, formally, phase velocities be larger than c. This fact does not violate Special Relativity because v_p is not the true velocity of propagation of the signal. When wavepackets are not too spread out and group velocities are well defined and physically meaningful,[14] they have values which are no larger than c.

A consequence of the existence of an absolute speed limit is that the idealizations of *rigid body* and *incompressible fluid* used in Newtonian mechanics, which imply infinite sound speed, are impossible in Special Relativity. By definition, such systems

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[10] This effect is the same phenomenon used advantageously in the optical lever of the torsion balance.

[11] A general wave, not necessarily electromagnetic, is discussed here. Also, quantum vacuum can behave as a medium and give rise to apparent velocities larger than c. This is not, however, the propagation velocity of a physical signal and does not threaten causality [13].

[12] The phase velocity is usually identified with the velocity of a point of constant phase, for example a point where the amplitude of the wavepacket envelope vanishes.

[13] For wavepackets which are too spread out, a situation that occurs with high absorption or near resonances, the concepts of group and phase velocity may become largely unphysical and a more detailed discussion is necessary. No less than eight "wave velocities" can be defined in this case [14, 15].

[14] Even group velocities can be larger than c if the wavepacket is too spread out in a medium with high absorption [16]. Again, we are not talking about physical velocities here.

would transmit information instantaneously by means of sound waves propagating with infinite speed.

Example 2.1 The fastest spinning pulsar known to date, PSR J1746-2446ad, spins with a frequency of 716 Hz [17]. What limit is imposed by fundamental physics on its radius R?

The answer is that the equatorial speed, which is the largest rotational speed of a particle on the pulsar, must be less than c, yielding

$$R < \frac{c}{\omega} = \frac{3 \cdot 10^8 \text{ m/s}}{2\pi \cdot 716 \text{ s}^{-1}} \simeq 67 \text{ km}.$$

Neutron stars are believed to have sizes $\sim 10 - 20$ km, which brings them fairly close to achieving the largest rotational speeds that are possible in nature.

2.6 Length Contraction from the Lorentz Transformation

Length contraction and time delay can be derived directly from the Lorentz transformation. Consider a rod at rest in the inertial frame $S' = \left\{t', x', y', z'\right\}$ and moving with speed v with respect to another inertial frame $S = \{t, x, y, z\}$ as in Fig. 2.2. The endpoints of the rod are x'_A and x'_B with $l_0 \equiv x'_B - x'_A$ the rest length of the rod. According to the Lorentz transformation, it is

$$x'_A = \frac{x_A - vt_A}{\sqrt{1 - \frac{v^2}{c^2}}}, \qquad x'_B = \frac{x_B - vt_B}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{2.23}$$

In order to measure the rod, we must find the coordinates of the endpoints *at the same time* $t_A = t_B \equiv t$ (the two endpoints are observed simultaneously). We have

$$l_0 = x'_B - x'_A = \frac{(x_B - vt) - (x_A - vt)}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{x_B - x_A}{\sqrt{1 - \frac{v^2}{c^2}}}$$

or

$$l_0 = \frac{l}{\sqrt{1 - \frac{v^2}{c^2}}},$$

and

$$l = l_0\sqrt{1 - \frac{v^2}{c^2}}, \tag{2.24}$$

the Lorentz-FitzGerald formula for length contraction.

There is no contraction in the directions transversal to the motion. As a result of length contraction, a moving rod can be fit momentarily in a space in which it would not fit at rest (which originates the car-in-the-garage "paradox"). However, nothing

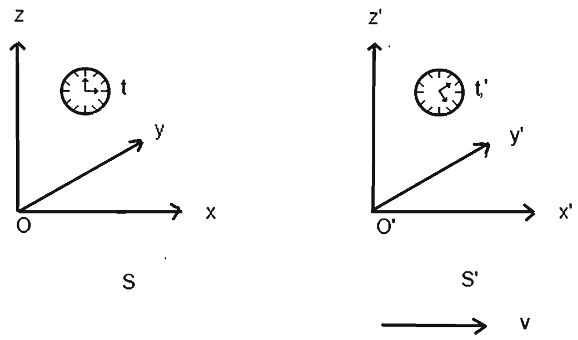


Fig. 2.2 The inertial frames S and S'

has happened to the rod. If you measure it again in its rest frame, it still has the same rest length l_0.

2.7 Time Dilation from the Lorentz Transformation

Let us now derive time dilation directly from the Lorentz transformation. Let a clock at rest at x' in the inertial frame S' record two events happening *at the same location* x' and separated by the proper time interval $\Delta\tau$. The two events have coordinates $\left(t_1', x', 0, 0\right)$ and $\left(t_2', x', 0, 0\right) \equiv \left(t_1' + \Delta\tau, x', 0, 0\right)$. What is the time interval measured by a clock in the inertial frame S which is moving with constant speed v with respect to S'? The inverse Lorentz transformation (2.5)–(2.8) gives, for these two events,

$$t_1 = \frac{t_1' + \frac{v}{c^2}\, x'}{\sqrt{1 - \frac{v^2}{c^2}}}, \qquad t_2 = \frac{t_2' + \frac{v}{c^2}\, x'}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.25}$$

and the time interval in S is

$$\Delta t \equiv t_2 - t_1 = \frac{\left(t_2' + \frac{v}{c^2}\, x'\right) - \left(t_1' + \frac{v}{c^2}\, x'\right)}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{t_2' - t_1'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

or

$$\Delta t = \frac{\Delta\tau}{\sqrt{1 - \frac{v^2}{c^2}}} \equiv \gamma\, \Delta\tau. \tag{2.26}$$

A moving clock "runs slower" than a static one by the Lorentz factor γ.

An *ideal clock* is defined as one that is not affected by acceleration. The finite interval of proper time recorded by an (ideal) clock between proper instants t_0 and t is

$$\Delta\tau = \int_{t_0}^{t} \mathrm{d}t \sqrt{1 - \frac{v^2}{c^2}}. \tag{2.27}$$

Example 2.2 Because they are light, electrons can be easily accelerated to become very relativistic. Consider an electron traveling at speed $v = 0.99c$ in an accelerator. The time interval $\Delta t = 1$ s in the laboratory frame corresponds, in the rest frame of this electron, to the interval $\Delta\tau = \Delta t/\gamma = \sqrt{1 - 0.99^2}\,(1\text{ s}) = 0.14\text{ s}$.

2.8 Transformation of Velocities and Accelerations in Special Relativity

Contrary to Newtonian mechanics, velocities do not simply "add up" in Special Relativity, otherwise an observer moving toward a light source would measure the speed of light to be larger than c, which contradicts the second postulate. In order to derive the correct formula for the composition of relativistic velocities,[15] suppose that a particle has velocity $u^{x'} \equiv dx'/dt'$ relative to an inertial frame $S' = \{t', x', y', z'\}$; we want to find its velocity u^x with respect to another inertial frame $S = \{t, x, y, z\}$, with respect to which $\{t', x', y', z'\}$ is moving with constant velocity v (Fig. 2.3). Remember the convention that the velocity v is positive if the inertial frame S' is moving *away* from S. Differentiate the Lorentz transformation (2.1)–(2.4) to obtain

$$\mathrm{d}x' = \gamma\,(\mathrm{d}x - v\mathrm{d}t),$$

$$\mathrm{d}y' = \mathrm{d}y,$$

$$\mathrm{d}z' = \mathrm{d}z,$$

$$\mathrm{d}t' = \gamma\left(\mathrm{d}t - \frac{v}{c^2}\mathrm{d}x\right),$$

and

$$u^{x'} \equiv \frac{dx'}{dt'} = \frac{dx - vdt}{dt - \frac{v}{c^2}dx} = \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\frac{dx}{dt}} = \frac{u^x - v}{1 - \frac{vu^x}{c^2}}.$$

Analogously, $\mathrm{d}y' = \mathrm{d}y,\ \mathrm{d}z' = \mathrm{d}z$, and

[15] It is possible to derive the relativistic law of transformation of velocities without using the Lorentz transformation and relying only on the two postulates (e.g., [18–20]). Here we present only the "standard" derivation from the Lorentz transformation.

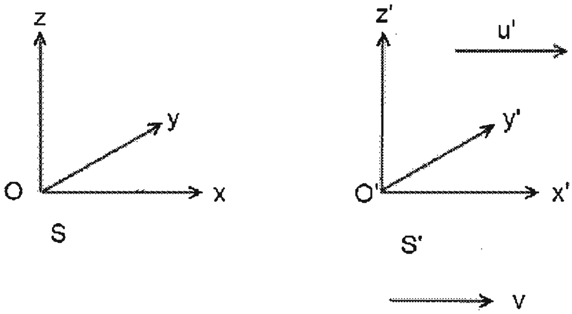


Fig. 2.3 The composition of velocities v and u' in Special Relativity

$$u^{y'} \equiv \frac{dy'}{dt'} = \frac{dy}{\gamma\left(dt - \frac{v}{c^2}dx\right)} = \frac{\frac{dy}{dt}}{\gamma\left(1 - \frac{vu^x}{c^2}\right)} = \frac{u^y}{\gamma\left(1 - \frac{vu^x}{c^2}\right)},$$

$$u^{z'} \equiv \frac{dz'}{dt'} = \frac{dz}{\gamma\left(dt - \frac{v}{c^2}dx\right)} = \frac{\frac{dz}{dt}}{\gamma\left(1 - \frac{vu^x}{c^2}\right)} = \frac{u^z}{\gamma\left(1 - \frac{vu^x}{c^2}\right)}.$$

The *relativistic velocity addition formulae* are, therefore,

$$u^{x'} = \frac{u^x - v}{1 - \frac{vu^x}{c^2}}, \tag{2.28}$$

$$u^{y'} = \frac{u^y}{\gamma\left(1 - \frac{vu^x}{c^2}\right)}, \tag{2.29}$$

$$u^{z'} = \frac{u^z}{\gamma\left(1 - \frac{vu^x}{c^2}\right)}. \tag{2.30}$$

Note that we did not assume that the particle has uniform velocity $\mathbf{u}'$ in S; the derivation is valid for *instantaneous* velocities. In addition, while $|v|$ is restricted to be less than c, u^x, u^y, and u^z can be the coordinate velocity components of a light ray.[16]

Let us consider two limiting cases. In the *Newtonian limit* $|u^x|,\ |v| \ll c$ we have, to first order,

[16] This possibility will be applied to the derivation of the laws describing the aberration of light in Chap. 7

$$u^{x'} \approx u^x - v,$$

$$u^{y'} \approx u^y,$$

$$u^{z'} \approx u^z.$$

In the *ultrarelativistic limit* $u^x \to c$ we have $u^{x'} \to \frac{c-v}{1-\frac{v}{c}} = c$, in agreement with the postulate that the speed of light is c in every inertial frame. This conclusion is not surprising since it is built into the Lorentz transformation used here to derive the addition law of velocities.

If the two inertial frames are in standard configuration, planar motions remain planar under the change of frame. In fact if, for example, the motion of a particle occurs in the (x, y) plane according to S, then $u^z = 0$ and, according to Eq. (2.30), also $u^{z'} = 0$. A rectilinear motion along the x-axis of O (with $u^y = u^z = 0$) appears as a rectilinear motion along the x'-axis of O' (with $u^{y'} = u^{z'} = 0$). The rectilinear motion of a particle along the y-axis of O is, of course, distorted according to O' (since $u^{x'} \neq 0$, $u^{y'} \neq 0$, and $u^{z'} = 0$), as is rectilinear motion along the z-axis (since $u^{x'} \neq 0$, $u^{y'} = 0$, and $u^{z'} \neq 0$).

According to the Principle of Relativity, the inverse velocity transformation is obtained with the exchange $\left(u^i, v\right) \longleftrightarrow \left(u^{i'}, -v\right)$ yielding

$$u^x = \frac{u^{x'} + v}{1 + \frac{vu^{x'}}{c^2}}, \tag{2.31}$$

$$u^y = \frac{u^{y'}}{\gamma\left(1 + \frac{vu^{x'}}{c^2}\right)}, \tag{2.32}$$

$$u^z = \frac{u^{z'}}{\gamma\left(1 + \frac{vu^{x'}}{c^2}\right)}. \tag{2.33}$$

2.8.1 Relative Velocity of Two Particles

Consider two particles moving instantaneously along the same line with *speeds* v_1 and v_2 in an inertial frame $S = \{t, x, y, z\}$. Their relative velocity is computed by using an inertial frame $S' = \left\{t', \mathbf{x}'\right\}$ in which particle 1 is at rest. In this frame, which has a velocity (and speed) v_1 with respect to S, particle 2 has the velocity

$$u^{x'}_{(2)} = \frac{u^x - v_1}{1 - \frac{u^x v_1}{c^2}} \tag{2.34}$$

where $u^x = -v_2$ is the velocity of particle 2 in the frame $\{t', x', y', z'\}$. This is the *relativistic law of composition of velocities*. Then, it is

$$u^{x'}_{(2)} = -\frac{(v_1 + v_2)}{1 + \frac{v_1 v_2}{c^2}}. \tag{2.35}$$

The *relative speed* of the two particles is given by

$$\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}. \tag{2.36}$$

Example 2.3 In a science fiction movie two spaceships are moving head-on toward each other with speeds $0.65c$ and $0.90c$ with respect to an observer on earth. What is the relative speed measured by the astronauts on each ship?
The relative speed is

$$\frac{0.65c - (-0.90c)}{1 - \frac{0.65c(-0.90c)}{c^2}} = 0.98c,$$

which is obviously larger than the speed of each spaceship with respect to earth but still less than c.

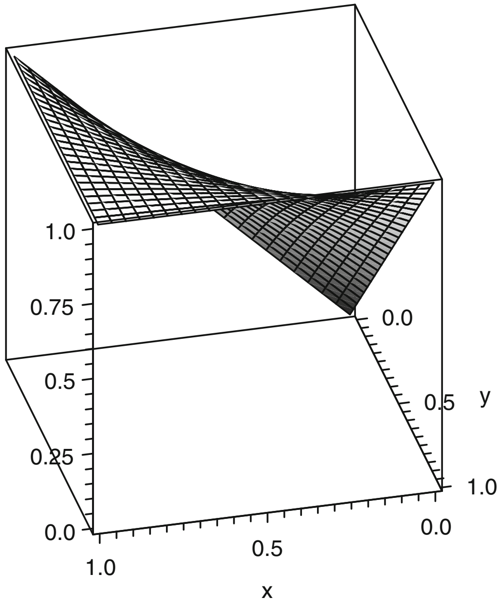
* * *

Let us study now the function of two variables

$$f(x, y) \equiv \frac{x + y}{1 + xy} \tag{2.37}$$

appearing in the law of composition of relativistic speeds, in the relevant range $(x, y) \in [0, 1] \times [0, 1]$. Here $x \equiv v_1/c$ and $y \equiv v_2/c$. Physics tells us that the value of this function should never exceed unity, which is confirmed by the following mathematical considerations. Note that $f(0, 0) = 0$, f is continuous with all its derivatives of any order in $[0, 1] \times [0, 1]$, $f(y, x) = f(x, y)$,

Fig. 2.4 The function $f(x, y) = \dfrac{x+y}{1+xy}$ of $x = v_1/c$ and $y = v_2/c$



$$\frac{\partial f}{\partial x} = \frac{1-y^2}{(1+xy)^2} \geq 0 \quad \text{if} \quad y \leq 1,$$
$$\frac{\partial f}{\partial y} = \frac{1-x^2}{(1+xy)^2} \geq 0 \quad \text{if} \quad x \leq 1,$$

and $\nabla f = (0, 0)$ at $(x, y) = (1, 1)$. The differential of f is

$$\mathrm{d}f = \nabla f \cdot \mathrm{d}\mathbf{x} = \frac{\left(1-y^2\right)\mathrm{d}x + \left(1-x^2\right)\mathrm{d}y}{(1+xy)^2}.$$

The maximum of the function f is attained at $(x, y) = (1, 1)$ and

$$f(1, 1) = 1$$

hence $0 \leq f(x, y) < 1 \quad \forall \quad (x, y) \in [0, 1) \times [0, 1)$. The function $f(x, y)$ is plotted in Fig. 2.4.

2.8.2 Relativistic Transformation Law of Accelerations

In a way similar to how the relativistic transformation law of velocities is derived, one can obtain the relativistic law of transformation of accelerations under a change of inertial frames (found by Tolman in 1912 [21])

$$a^{x'} = \frac{a^x}{\gamma^3\left(1-\frac{vu^x}{c^2}\right)^3}, \tag{2.38}$$

$$a^{y'} = \frac{1}{\gamma^2\left(1-\frac{vu^x}{c^2}\right)^3}\left[\left(1-\frac{vu^x}{c^2}\right)a^y + \frac{v}{c^2}u^y a^x\right], \tag{2.39}$$

$$a^{z'} = \frac{1}{\gamma^2\left(1-\frac{vu^x}{c^2}\right)^3}\left[\left(1-\frac{vu^x}{c^2}\right)a^z + \frac{v}{c^2}u^z a^x\right], \tag{2.40}$$

where $\gamma = \gamma(v)$ (the detailed derivation is left as an exercise). From these transformation properties it follows that all inertial observers agree on whether a particle is accelerated or not. Moreover, if a particle has 3-dimensional acceleration $\mathbf{a}$ =constant in one inertial frame, its acceleration is necessarily non-constant in another inertial frame. Finally, we note that in the Newtonian limit $v/c \longrightarrow 0$ the acceleration is Galilei-invariant, $\mathbf{a'} = \mathbf{a}$, and Newton's second law is invariant under Galilei transformations, as already discussed.

2.9 Matrix Representation of the Lorentz Transformation

The Lorentz transformation

$$\hat{L}_v : \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

is a linear homogeneous coordinate transformation in the space (ct, x, y, z) and can be represented by a symmetric 4×4 matrix $\hat{L}_v$ with components given by

$$\hat{L}_v = \left(L_{(v)}{}^{\alpha}{}_{\beta}\right) \equiv \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.41}$$

where $\beta \equiv v/c$ and $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$. This is a real symmetric 4×4 matrix parametrized by the parameter v. To check that this representation is correct, take the product

$$\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma ct - \gamma\beta x \\ -\gamma\beta ct + \gamma x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma \left(ct - \frac{v}{c} x \right) \\ \gamma \left(x - vt \right) \\ y \\ z \end{pmatrix},$$

therefore,

$$ct' = \gamma \left(ct - \frac{v}{c} x \right),$$

$$x' = \gamma \left(x - vt \right),$$

$$y' = y,$$

$$z' = z,$$

which is the Lorentz transformation.

The matrix $\hat{L}_v$ of the Lorentz transformation has unit determinant:

$$\begin{aligned} \text{Det}\left(\hat{L}_v\right) &= \text{Det} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \gamma \begin{vmatrix} \gamma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - (-\gamma\beta) \begin{vmatrix} -\gamma\beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= \gamma \cdot \gamma + \gamma\beta \left(-\gamma\beta \right) = \gamma^2 \left(1 - \beta^2 \right) = \left(\frac{1}{\sqrt{1-\beta^2}} \right)^2 \left(1 - \beta^2 \right) \\ &= 1. \end{aligned}$$

The inverse of the matrix $\hat{L}_v$ is the matrix $\hat{L}_{(-v)}$ corresponding to the inverse Lorentz transformation,

$$\hat{L}_{(-v)} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.42}$$

In fact, we have

$$\hat{L}_v \hat{L}_{(-v)} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\hat{L}_{(-v)} \hat{L}_v = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The matrix corresponding to $v = 0$ and $\gamma = 1$ is obviously the identity matrix.

2.10 *The Lorentz Group

As Poincaré realized, the Lorentz transformations $\hat{L}_v$ form a group with respect to the composition of transformations $\circ$. In fact,

- if $\hat{L}_u$, $\hat{L}_v$ are Lorentz transformations, then $\hat{L}_u \circ \hat{L}_v$ is a Lorentz transformation $\hat{L}_w$ with parameter

$$w = -\frac{(u+v)}{1+\frac{uv}{c^2}} \tag{2.43}$$

  given by the relativistic law of composition of velocities. Of course, one can obtain this result directly using the matrix representation of $\hat{L}_u$ and $\hat{L}_v$, which has the advantage of providing an alternative derivation of the law of composition of velocities. Let $\hat{L}_v$ and $\hat{L}_u$ be Lorentz transformations represented by

$$\begin{pmatrix} \gamma_v & -\gamma_v \frac{v}{c} & 0 & 0 \\ -\gamma_v \frac{v}{c} & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} \gamma_u & -\gamma_u \frac{u}{c} & 0 & 0 \\ -\gamma_u \frac{u}{c} & \gamma_u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

respectively. Then

$$\hat{L}_v \, \hat{L}_u = \begin{pmatrix} \gamma_v \gamma_u \left(1 + \frac{uv}{c^2}\right) & -\gamma_v \gamma_u \frac{(v+u)}{c} & 0 & 0 \\ -\gamma_v \gamma_u \frac{(v+u)}{c} & \gamma_v \gamma_u \left(\frac{uv}{c^2} + 1\right) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and letting $\gamma_u \gamma_v \left(1 + \dfrac{uv}{c^2}\right) \equiv \gamma_w$, we have

$$\gamma_w \equiv \frac{1}{\sqrt{1 - \frac{w^2}{c^2}}} = \frac{1 + \frac{uv}{c^2}}{\sqrt{1 - \frac{u^2}{c^2}}\sqrt{1 - \frac{v^2}{c^2}}},$$

$$1 - \frac{w^2}{c^2} = \frac{\left(1 - \frac{u^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)}{\left(1 + \frac{uv}{c^2}\right)^2},$$

$$1 - \frac{w^2}{c^2} = \frac{1 - \frac{u^2}{c^2} - \frac{v^2}{c^2} + \frac{u^2 v^2}{c^4}}{\left(1 + \frac{uv}{c^2}\right)^2}.$$

Then

$$\left(1 + \frac{uv}{c^2}\right)^2 - \frac{w^2}{c^2}\left(1 + \frac{uv}{c^2}\right)^2 = 1 - \frac{u^2}{c^2} - \frac{v^2}{c^2} + \frac{u^2 v^2}{c^4},$$

$$1 + \frac{u^2 v^2}{c^4} + 2\,\frac{uv}{c^2} - \frac{w^2}{c^2}\left(1 + \frac{uv}{c^2}\right)^2 = 1 - \frac{u^2}{c^2} - \frac{v^2}{c^2} + \frac{u^2 v^2}{c^4},$$

$$\frac{w^2}{c^2}\left(1 + \frac{uv}{c^2}\right)^2 = \left(\frac{u + v}{c}\right)^2,$$

and finally

$$w = -\frac{(u + v)}{1 + \frac{uv}{c^2}}, \tag{2.44}$$

which is the law of composition of velocities. Two consecutive Lorentz transformations *with parallel velocity vectors* $\mathbf{v}_1$ and $\mathbf{v}_2$ commute, i.e., the result of the combined transformations does not depend on the order in which they are performed. This is no longer true if $\mathbf{v}_1$ and $\mathbf{v}_2$ are not parallel, contrary to the case of Galilei transformations. To conclude, the composition of two Lorentz transformations is a Lorentz transformation.

- The operation $\circ$ is *associative*.
- The transformation $\hat{I}_d = (\delta^\alpha_\beta) = \hat{L}_0$ corresponding to $v = 0$ is the *neutral element* of the group.
- For any Lorentz transformation $\hat{L}_v$ there is an *inverse* Lorentz transformation

$$\left(\hat{L}_v\right)^{-1} = \hat{L}_{(-v)}. \tag{2.45}$$

The fact that $\hat{L}_v$ is invertible follows from the fact that its determinant is unity.

To conclude, the Lorentz transformations $\left\{\hat{L}_v\right\}$ form a group. Since they depend on a continuous parameter v, this is called a continuous *1-parameter group*.

Pure Lorentz transformations in standard configuration form a group but other linear coordinate transformations which leave the interval $\mathrm{d}s^2$ invariant can be added, including continuous transformations such as purely spatial rotations (which themselves form a 3-parameter group called *special orthogonal group* SO(3)), spatial translations $\mathbf{x} \longrightarrow \mathbf{x} + \mathbf{x}_{(0)}$, and time translations $t \longrightarrow t + t_{(0)}$; and discrete transformations such as reflections of the spatial axes and time reflection. A *proper* transformation is one with determinant equal to unity and an *orthochronous* transformation is one which preserves the time orientation, i.e., $L^0{}_0 \geq 0$. The *proper orthochronous Lorentz group* is a 6-parameter continuous group consisting of one Lorentz boost in standard configuration (parametrized by one continuous parameter v), two spatial rotations needed to align the x-axis of the inertial frame S with the velocity v of the inertial observer O' (which needs two angles as continuous parameters), and three spatial rotations to rotate the frame S of the inertial observer O in the same orientation of the frame S' of the inertial observer O', accounting for the remaining three continuous parameters, which are rotation angles about the three spatial axes).

By adding the translations in space and time $x^\mu \longrightarrow x^{\mu'} = x^\mu + x^\mu_{(0)}$, where the $x^\mu_{(0)}$ are constants, one obtains the ten-parameter *Poincaré group* consisting of linear inhomogeneous transformations $x^\mu \longrightarrow x^{\mu'} = L^\mu{}_\alpha x^\alpha + x^\mu_{(0)}$ which leave the interval $\mathrm{d}s^2$ invariant.

2.11 The Lorentz Transformation as a Rotation by an Imaginary Angle with Imaginary Time

An interesting mathematical representation of the Lorentz transformation is the following. Considering again two inertial frames in relative motion with speed v in standard configuration, it is straightforward to check that the quantity $-c^2t^2 + x^2$ is invariant under Lorentz transformations. Define the imaginary "times" $T \equiv \mathrm{i}ct$ and $T' \equiv \mathrm{i}ct'$ in the two inertial frames. Then $T^2 + x^2$ is a Lorentz invariant, i.e.,

$$\underbrace{T^2 + x^2}_{\substack{\text{distance from}\\ \text{the origin in}\\ \text{the (x, T) plane}}} = \underbrace{\left(T'\right)^2 + \left(x'\right)^2}_{\substack{\text{distance from}\\ \text{the origin in}\\ \text{the } \left(x', T'\right) \text{ plane}}} .$$

The distance from the origin is invariant under a rotation in the (x, T) plane described by

$$x' = x\cos\theta + T\sin\theta,$$
$$T' = -x\sin\theta + T\cos\theta, \tag{2.46}$$

or, in matrix form,

$$\begin{pmatrix} x' \\ T' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ T \end{pmatrix}. \tag{2.47}$$

We see that the T'-axis has equation $x' = 0$ equivalent to $x = vt = v\,\dfrac{T}{\mathrm{i}c}$ and that a rotation producing this axis satisfies

$$\overbrace{0}^{x'} = \overbrace{\frac{vT}{\mathrm{i}c}\cos\theta}^{x\cos\theta} + T\sin\theta.$$

Therefore,

$$\tan\theta = -\frac{v}{\mathrm{i}c} = \frac{\mathrm{i}v}{c} \tag{2.48}$$

corresponds to an imaginary rotation angle θ. Since

$$\underbrace{1 + \tan^2\theta}_{1-\frac{v^2}{c^2}} = \frac{1}{\cos^2\theta},$$

we have $\cos\theta = \left(1 - \dfrac{v^2}{c^2}\right)^{-1/2}$ and

$$x' = x\cos\theta + T\sin\theta = (x + T\tan\theta)\cos\theta$$

$$= \gamma\left(x + \tfrac{\mathrm{i}v}{c}\,T\right) = \gamma\,(x - vt),$$

while

$$T' = -x\sin\theta + T\cos\theta = (-x\tan\theta + T)\cos\theta$$

$$= \gamma\left(-x\,\tfrac{\mathrm{i}v}{c} + \mathrm{i}ct\right),$$

or

$$t' = \gamma\left(t - \frac{v}{c^2}\,x\right)$$

so that

$$ct' = \gamma\left(ct - \tfrac{v}{c}\,x\right),$$

$$x' = \gamma\,(x - vt).$$

Then $\tan\theta = \mathrm{i}\,v/c$, $\cos\theta = \gamma$, and $\sin\theta = \mathrm{i}\gamma\,v/c$. It is customary to define the *rapidity* ϕ by[17]

$$\tanh\phi \equiv \beta \quad \text{or} \quad \phi \equiv \tanh^{-1}\left(\frac{\mathrm{v}}{\mathrm{c}}\right); \tag{2.49}$$

then the relation $\tan\theta = \mathrm{i}\,\dfrac{v}{c}$ gives $\tanh\phi \equiv \dfrac{v}{c} = -\mathrm{i}\,\tan\theta$ and, using the identity $\tanh(\mathrm{i}\theta) = \mathrm{i}\tan\theta$, one obtains $-\tanh\phi = \tanh(-\phi) = \mathrm{i}\tan\theta = \tanh(\mathrm{i}\theta)$, or

$$\phi = -\mathrm{i}\theta \quad \in \mathbb{R}$$

and

$$\theta = \mathrm{i}\phi. \tag{2.50}$$

We can revisit the fact that Lorentz transformations form a group by viewing Lorentz boosts as rotations by imaginary angles in a space with imaginary time.

The composition of $\hat{L}_{v_1}$ and $\hat{L}_{v_2}$ is a Lorentz boost $\hat{L}_w$ with speed $w = -\frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}$.

To prove this statement, note that

[17] The name "rapidity" arises from the one-to-one correspondence of ϕ with the velocity v and the fact that $\phi \approx \beta$ for $|v| \ll c$.

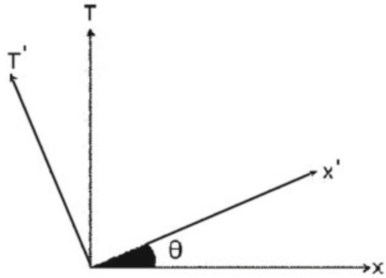


Fig. 2.5 A rotation by an imaginary angle θ in the $(x, T = \mathrm{i}ct)$ plane

$$\tan\theta = \tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}, \tag{2.51}$$

or

$$\theta = \tan^{-1}\left(\mathrm{i}\,\frac{w}{c}\right) = -\tan^{-1}\left(\frac{\mathrm{i}\,\frac{v_1}{c} + \mathrm{i}\,\frac{v_2}{c}}{1 - \mathrm{i}^2\frac{v_1}{c}\frac{v_2}{c}}\right) = -\tan^{-1}\left[\frac{\mathrm{i}\,(v_1 + v_2)}{c + \frac{v_1 v_2}{c}}\right] = \theta_1 + \theta_2, \tag{2.52}$$

in other words, the rapidity $\phi = -\mathrm{i}\theta$ is additive, like all angles (Fig 2.5).

The trivial transformation $\hat{L}_0 = \hat{I}_d$ is a rotation by an angle $\theta = 0$. Moreover, $\left(\hat{L}_v\right)^{-1} = \hat{L}_{-v}$ because $\hat{L}_v$ corresponds to $\tan\theta = \mathrm{i}\,v/c$ and $\hat{L}_{-v}$ corresponds to $-\tan\theta = -\mathrm{i}\,v/c$.

2.12 *The GPS System

The Global Positioning System (GPS) nowadays used for navigating aircrafts, shipping, in private and commercial vehicles, and for urban navigation and wilderness recreation, originated in the 1970s for military navigation purposes following a few predecessors and early ideas dating back to the 1940s [22, 23]. More exotic applications include the monitoring of shifts in plate tectonics and more mundane applications include the precise time-stamping of financial transactions. GPS receivers available in outdoor stores have a typical position accuracy of 15 m, while differential techniques using multiple receivers next to each other can potentially achieve an accuracy of centimeters ("survey grade GPS").

The GPS system consists of a constellation of twenty-four satellites (plus spares) in six orbital planes, in each of which reside four satellites, in high (~20000 km radius) orbits all with a period of twelve hours. Each satellite carries a stable atomic clock which keeps track of time with fractional stability better than one part in 10^{13}. This network of satellites is designed so that, from any point on the earth with unobstructed line of sight, at least four satellites are visible above the horizon at any time. GPS receivers on the ground, in flight, or on a ship detect signals emitted by these satellites and determine their own position by means of triangulation.

When a signal is emitted from a satellite, it carries encoded with it information about the precise time and position at which it was emitted. The distance between the satellite and a GPS receiver detecting it is the time difference between emission and detection of this signal multiplied by c. The receiver communicates with several satellites and it is a straightforward triangulation problem to compute the receiver's location using four or more satellites. However, for this task to be performed, the time must be kept to high accuracy, and this is where relativistic effects come into play. In order to achieve an accuracy of 15 m, times must be known with an error not less than $15\,\mathrm{m/c} = 5 \cdot 10^{-8}\,\mathrm{s}$ (50 ns). Since the satellites are moving with respect to an observer on the ground, a ship, or an aircraft, the relativistic time dilation effect is present. The linear speed of a satellite in a circular orbit of radius r and angular velocity ω with respect to the ground is

$$v = \omega r \simeq \frac{2\pi}{12 \cdot (3600\,\mathrm{s})} \cdot (2 \cdot 10^7\,\mathrm{m}) \simeq 3 \cdot 10^3\,\frac{\mathrm{m}}{\mathrm{s}} \simeq 10^{-5}\mathrm{c}.$$

The proper time τ of the satellite and the time t of the observer on the ground are related by $\Delta t = \gamma \Delta\tau$; the ratio $\Delta\tau/\Delta t = \sqrt{1 - v^2/c^2} \simeq 1 - \frac{v^2}{2c^2}$ is the percent frequency shift of a signal $\delta\nu/\nu$ and $\dfrac{v^2}{2c^2} \simeq 5 \cdot 10^{-11}$. Over 24 h, the time error is $5 \cdot 10^{-11} \cdot (24 \cdot 3600\,\mathrm{s}) = 4.3 \cdot 10^{-6}\,\mathrm{s}$. A more precise calculation takes into account the fact that the earth rotates and the position is referred to a rotating reference frame and not an inertial frame. Moreover, the satellites' orbits are not exactly circular but elliptical and are perturbed by the moon and the sun, while the earth is not perfectly spherical and has local overdensities and underdensities affecting these orbits. When all these effects are taken into account, the error arising from neglecting Special Relativity would amount to 7 ms per day. Even more important is a general-relativistic effect which consists in the slowing down of clocks in a gravitational potential well with respect to clocks far away. If Φ is the Newtonian gravitational potential and $\Delta\Phi$ is the difference in the values of this quantity at the emission and detection points, the frequency shift of a signal is $\Delta\Phi/c^2$. This second effect, if not accounted for, would be responsible for an error of 45 µs per day. The two effects subtract from each other, since a moving click ticks slower than a stationary one while a clock far away from a mass ticks faster than one closer to it. As a result, there would be a net error of 38 µs per day in neglecting these effects. Since errors larger than $5 \cdot 10^{-8}$ s ruin the required 15 m precision, at the rate of $3.8 \cdot 10^{-5}\,\mathrm{s/day}$ it would take $114\,\mathrm{s} \simeq 2$ minutes to build up the necessary error for the GPS to fail. In a full day, the accumulated time error of $3.8 \cdot 10^{-5}$ s would correspond to a position error of $(3.8 \cdot 10^{-5}\,\mathrm{s})\mathrm{c} = 11.4\,\mathrm{km}$, certainly not what you want when landing a commercial aircraft in poor visibility, piloting a large ship in narrow straits, or trying to find a precise spot on an arctic expedition or crossing a desert. The GPS system automatically corrects for the general- and special- relativistic effects. Without these corrections the GPS system would become completely useless in a matter of minutes or hours. This example shows how the seemingly abstract theory of Special Relativity and the seemingly even more abstract theory of General Relativity (whose effects are actually about six times more pronounced than Special Relativity in the GPS system), have become essential to the functioning of modern life.

2.13 Conclusion

The theoretical discovery of the Lorentz transformation was an important step of the learning process leading to Special Relativity, but its deep meaning was not understood before Einstein. In our presentation we have made it clear that the Lorentz transformation can be derived from the two postulates of Special Relativity, which are physically more transparent than what, at first sight, appears "only" as a mathematical transformation. From the physical point of view it is more satisfactory to construct the theory beginning from two very clear ideas rather than from a telling, but less transparent, mathematical symmetry. However, the Lorentz group constitutes the symmetry group of Special Relativity and suggests a unified view of space and time, a new way of looking at nature which we present in the next chapter.

Problems

2.1. Write the mathematical expression of a Lorentz boost with the $\{ct', x', y', z'\}$ inertial frame sliding along the z- (or z'-) axis, and with the x'- and the y'- axes parallel to the x-axis and the y-axis, respectively.
2.2. Find eigenvalues and eigenspaces of the matrix describing the Lorentz transformation and interpret them physically.
2.3. In an inertial frame S, two laser pulses are emitted by points on the x-axis 10 km apart and separated by 3 μs. They reach an inertial observer O' travelling in standard configuration with velocity v away from S. O' receives the two laser pulses simultaneously. Find v.
2.4. Show that the rapidity ϕ satisfies the relations

$$\begin{aligned}
e^{\phi} &= \gamma\,(1+\beta)\,, \\
e^{-\phi} &= \gamma\,(1-\beta)\,, \\
\gamma &= \cosh\phi, \\
\beta\gamma &= \sinh\phi,
\end{aligned}$$

so that the Lorentz transformation can be written as

$$ct' = ct \cosh\phi - x \sinh\phi,$$
$$x' = -ct \sinh\phi + x \cosh\phi.$$

2.5. Show that, given the two events $x^{\mu}_{(1,2)} = \left(ct_{(1,2)}, \mathbf{x}_{(1,2)}\right)$, the quantity

$$\mathcal{I}\left(x^{\mu}_{(1)}, x^{\mu}_{(2)}\right) \equiv \frac{(x_1 - ct_1)(x_2 + ct_2)}{(x_1 + ct_1)(x_2 - ct_2)}$$

is an invariant of the Lorentz transformation (2.1)–(2.4) [24, 25].

2.6. Derive the inverse law of addition of velocities (2.31)–(2.33) without invoking the Principle of Relativity, i.e., without the exchange $\left(u^i, v\right) \longleftrightarrow \left(u^{i'}, -v\right)$.

2.7. Derive the relativistic law of transformation of accelerations (2.38)–(2.40) under a change of inertial frames, and its inverse. Argue that all inertial observers agree on whether a particle is accelerated or not, however, if a particle has uniform acceleration in one inertial frame, its acceleration is necessarily non-uniform in another inertial frame.

2.8. A laser beam is shone from the surface of the earth onto the moon and the laser spot sweeps the surface of the full moon in the time $\Delta t = 0.010$ s. The radius of the moon is $R_m = 1.737 \cdot 10^6$ m and the earth-moon distance is $d = 3.844 \cdot 10^8$ m. What is the linear velocity of the laser spot? Comment.

2.9. The dispersion relation of electromagnetic waves propagating in a dilute plasma is

$$\omega(k) = \sqrt{c^2 k^2 + \omega_p^2},$$

where the constant ω_p (*plasma frequency*) is given by $\sqrt{\frac{4\pi e^2 n_e}{m_e}}$ for non-relativistic electrons and by $\frac{1}{\gamma}\sqrt{\frac{4\pi e^2 n_e}{m_e}}$ for relativistic electrons, where n_e is the number density of electrons (with charge e and mass m_e). Compute the phase velocity and group velocity as functions of k and sketch their graphs. Discuss their magnitudes with respect to c and compute their geometric average $\sqrt{v_p\, v_g}$. Discuss the propagation of a plane monochromatic electromagnetic wave with electric field $\mathbf{E} = \mathbf{E}_0\, \mathrm{e}^{\mathrm{i}(\mathrm{kx} - \omega\, \mathrm{t})}$ as the ratio ω/ω_p varies (here $\mathbf{E}_0$ is a constant amplitude).

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