Chapter 2
Turnpike Properties of Optimal Control Problems

2.1 Preliminaries

Denote by $|\cdot|$ the Euclidean norm in the k-dimensional Euclidean space R^k. Let m, n be natural numbers.

In this chapter we study a control system described by a differential equation

$$x'(t) = G(t, x(t), u(t)) \text{ a. e. } t \in \mathcal{I}, \tag{2.1}$$

where $\mathcal{I}$ is either R^1 or $[T_1, \infty)$ or $[T_1, T_2]$ $(-\infty < T_1 < T_2 < \infty)$, and $x : \mathcal{I} \to R^n$ is an absolutely continuous (a. c.) function which satisfies

$$(t, x(t)) \in A \text{ for all } t \in \mathcal{I}, \tag{2.2}$$

where A is a subset of R^{n+1}. The control function $u : \mathcal{I} \to R^m$ is Lebesgue measurable and satisfies the feedback control constraints

$$u(t) \in U(t, x(t)) \text{ a. e. } t \in \mathcal{I}, \tag{2.3}$$

where $U : A \to 2^{R^m}$ is a point to set mapping with a graph

$$M = \{(t, x, u) : \ (t, x) \in A, \ u \in U(t, x)\}. \tag{2.4}$$

We suppose that M is a Borel measurable subset of R^{n+m+1} and that the function $G : M \to R^n$ is borelian.

For any $t \in R^1$ set

$$A(t) = \{x \in R^n : \ (t, x) \in A\}. \tag{2.5}$$

We assume that the set $A(t) \neq \emptyset$ for any $t \in R^1$.

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The performance of the above control system is described by an integral functional

$$I^f(T_1, T_2, x, u) = \int_{T_1}^{T_2} f(t, x(t), u(t))dt, \tag{2.6}$$

where a borelian function $f : M \to R^1$ belongs to a complete metric space of functions $\mathfrak{M}$ defined below.

An a. c. function $x : \mathcal{I} \to R^n$, where $\mathcal{I}$ is either R^1 or $[T_1, \infty)$ or $[T_1, T_2]$ $(-\infty < T_1 < T_2 < \infty)$, will be called a *trajectory* if there exists a Lebesgue measurable function (referred to as a *control*) $u : \mathcal{I} \to R^m$ such that the pair (x, u) satisfies (2.1)–(2.3) and the function $t \to f(t, x(t), u(t))$ is locally Lebesgue integrable on $\mathcal{I}$.

For any $s \in R^1$ set $s_+ = \max\{s, 0\}$.

Let a_0 be a positive constant and let $\psi : [0, \infty) \to [0, \infty)$ be an increasing function such that

$$\psi(t) \to \infty \text{ as } t \to \infty.$$

Denote by $\mathfrak{M}$ the set of all borelian functions $f : M \to R^1$ which satisfy the following growth assumption:

(A)

$$f(t, x, u) \geq \max\{\psi(|x|),\ \psi(|u|),$$
$$\psi([|G(t, x, u)| - a_0|x|]_+)[|G(t, x, u)| - a_0|x|]_+\} - a_0$$

for each $(t, x, u) \in M$.

We equip the set $\mathfrak{M}$ with the uniformity which is determined by the following base:

$$\begin{aligned} E(N, \epsilon, \lambda) = \{&(f, g) \in \mathfrak{M} \times \mathfrak{M} :\ |f(t, x, u) - g(t, x, u)| \leq \epsilon \\ &\text{for each } (t, x, u) \in M \text{ satisfying } |x|, |u| \leq N\} \\ \cap\{&(f, g) \in \mathfrak{M} \times \mathfrak{M} :\ (|f(t, x, u)| + 1)(|g(t, x, u)| + 1)^{-1} \in [\lambda^{-1}, \lambda] \\ &\text{for each } (t, x, u) \in M \text{ satisfying } |x| \leq N\}, \end{aligned} \tag{2.7}$$

where $N > 0$, $\epsilon > 0$ and $\lambda > 1$.

Clearly, the uniform space $\mathfrak{M}$ is Hausdorff and has a countable base. Therefore $\mathfrak{M}$ is metrizable. It is not difficult to show that the uniform space $\mathfrak{M}$ is complete.

We consider functionals of the form $I^f(T_1, T_2, x, u)$ [see (2.6)], where $f \in \mathfrak{M}$, $-\infty < T_1 < T_2 < \infty$ and $x : [T_1, T_2] \to R^n$, $u : [T_1, T_2] \to R^m$ is a trajectory-control pair.

For $f \in \mathfrak{M}$, a pair of numbers $T_1 \in R^1$, $T_2 > T_1$ and (T_1, y), $(T_2, z) \in A$ set

$$U^f(T_1, T_2, y, z) = \inf\{I^f(T_1, T_2, x, u) : \ x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$
$$\text{is a trajectory-control pair satisfying } x(T_1) = y, \ x(T_2) = z\}, \qquad (2.8)$$
$$\sigma^f(T_1, T_2, y) = \inf\{U^f(T_1, T_2, y, h) : \ (T_2, h) \in A\}. \qquad (2.9)$$

Here we assume that the infimum over empty set is ∞.

2.2 The Main Results

Denote by $\mathfrak{M}_{reg}$ the set of all functions $f \in \mathfrak{M}$ which satisfy the following assumption:

(B) There exist a trajectory-control pair

$$x_f : R^1 \to R^n, \ u_f : R^1 \to R^m$$

and a number $b_f > 0$ such that:

(i)

$$U^f(T_1, T_2, x_f(T_1), x_f(T_2)) = I^f(T_1, T_2, x_f, u_f)$$

for each $T_1 \in R^1$ and each $T_2 > T_1$;

(ii)

$$\sup\{I^f(j, j+1, x_f, u_f) : j = 0, \pm 1, \pm 2, \dots\} < \infty;$$

(iii) For each $S_1 > 0$ there exist $S_2 > 0$ and an integer $c > 0$ such that

$$I^f(T_1, T_2, x_f, u_f) \le I^f(T_1, T_2, x, u) + S_2$$

for each $T_1 \in R^1$, each $T_2 \ge T_1 + c$, and each trajectory-control pair $x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$ which satisfies $|x(T_1)|, \ |x(T_2)| \le S_1$;

(iv) For each $\epsilon > 0$ there exists $\delta > 0$ such that for each $(T, z) \in A$ which satisfies

$$|z - x_f(T)| \le \delta$$

there are

$$\tau_1 \in (T, T + b_f] \text{ and } \tau_2 \in [T - b_f, T),$$

and trajectory-control pairs

$$x_1 : [T, \tau_1] \to R^n,\ u_1 : [T, \tau_1] \to R^m,$$
$$x_2 : [\tau_2, T] \to R^n,\ u_2 : [\tau_2, T] \to R^m$$

which satisfy

$$x_1(T) = x_2(T) = z,$$
$$x_i(\tau_i) = x_f(\tau_i),\ i = 1, 2,$$
$$|x_1(t) - x_f(t)| \le \epsilon \text{ for all } t \in [T, \tau_1],$$
$$|x_2(t) - x_f(t)| \le \epsilon \text{ for all } t \in [\tau_2, T],$$
$$I^f(T, \tau_1, x_1, u_1) \le I^f(T, \tau_1, x_f, u_f) + \epsilon,$$
$$I^f(\tau_2, T, x_2, u_2) \le I^f(\tau_2, T, x_f, u_f) + \epsilon.$$

Note that assumption (B) means that the trajectory-control pair

$$x_f : R^1 \to R^n,\ u_f : R^1 \to R^m$$

is a solution of the corresponding infinite horizon optimal control problem associated with the integrand f, and that certain controllability properties hold near this trajectory-control pair.

In this chapter we will establish the following result.

Theorem 2.1.

1. *Let $f \in \mathfrak{M}_{reg}$ and $S_0 > 0$. Then there exists $S > 0$ such that for each pair of real numbers $T_1 < T_2$ and each trajectory-control pair*

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

which satisfies $|x(T_1)| \le S_0$ the following inequality holds:

$$I^f(T_1, T_2, x_f, u_f) \le I^f(T_1, T_2, x, u) + S.$$

2. *Let $f \in \mathfrak{M}_{reg}$. Then for each $s \in R^1$ and each trajectory-control pair*

$$x : [s, \infty) \to R^n,\ u : [s, \infty) \to R^m$$

one of the following relations holds:

(a)

$$I^f(s, t, x, u) - I^f(s, t, x_f, u_f) \to \infty \text{ as } t \to \infty;$$

(b)

$$sup\{|I^f(s, t, x_f, u_f) - I^f(s, t, x, u)| :\ t \in (s, \infty)\} < \infty.$$

Moreover, if the relation (b) holds, then

$$\sup\{|x(t)| : t \in [s, \infty)\} < \infty.$$

For each $f \in \mathfrak{M}_{reg}$ and each $r > 0$ we define a function $f_r \in \mathfrak{M}$ by

$$f_r(t, x, u) = f(t, x, u) + r \min\{|x - x_f(t)|, 1\} \text{ for all } (t, x, u) \in M.$$

It is easy to see that $f_r \in \mathfrak{M}_{reg}$ for each $f \in \mathfrak{M}_{reg}$ and each $r > 0$.

Let $\mathfrak{A}$ be a subset of $\mathfrak{M}_{reg}$ such that $f_r \in \mathfrak{A}$ for each $f \in \mathfrak{A}$ and each $r \in (0, 1)$. Denote by $\bar{\mathfrak{A}}$ the closure of $\mathfrak{A}$ in the uniform space $\mathfrak{M}$ and consider the topological subspace $\bar{\mathfrak{A}} \subset \mathfrak{M}$ with the relative topology.

In this chapter we will establish the existence of a set $\mathcal{F} \subset \bar{\mathfrak{A}}$ which is a countable intersection of open everywhere dense sets in $\bar{\mathfrak{A}}$ and such the following theorems hold.

Theorem 2.2. *For each $f \in \mathcal{F}$ and each $S > 0$ there exist a neighborhood $\mathcal{U}$ of f in $\mathfrak{M}$ and positive numbers δ, Q such that the following assertions hold:*

$$\inf\{U^g(T_1, T_2, y_1, y_2) : (T_i, y_i) \in A, \ i = 1, 2\} < \infty$$

for each $g \in \mathcal{U}$, each $T_1 \in R^1$ and each $T_2 > T_1$;

for each $g \in \mathcal{U}$, each $T_1 \in R^1$, each $T_2 \geq T_1 + 1$ and each trajectory-control pair

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$

which satisfies

$$I^g(T_1, T_2, x, u) \leq \inf\{U^g(T_1, T_2, y_1, y_2) : (T_i, y_i) \in A, \ i = 1, 2\} + S$$

and

$$I^g(T_1, T_2, x, u) \leq U^g(T_1, T_2, x(T_1), x(T_2)) + \delta$$

the following inequality holds:

$$|x(t)| \leq Q \text{ for all } t \in [T_1, T_2].$$

Theorem 2.2 establishes uniform boundedness of approximate solutions of optimal control problems.

The next theorem is our first turnpike result.

Theorem 2.3. *Let $f \in \mathcal{F}$. Then there exists a bounded continuous function $X_f : R^1 \to R^n$ such that the following property holds.*

For each $S, \epsilon > 0$ *there exist a neighborhood* $\mathcal{U}$ *of* f *in* $\mathfrak{M}$ *and real numbers* $\Delta > 0$, $\delta \in (0, \epsilon)$ *such that for each* $g \in \mathcal{U}$, *each* $T_1 \in R^1$, *each* $T_2 \geq T_1 + 2\Delta$ *and each trajectory-control pair*

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$

which satisfies

$$I^g(T_1, T_2, x, u) \leq \inf\{U^g(T_1, T_2, y_1, y_2) : \ (T_i, y_i) \in A, \ i = 1, 2\} + S$$

and

$$I^g(T_1, T_2, x, u) \leq U^g(T_1, T_2, x(T_1), x(T_2)) + \delta$$

the following inequality holds:

$$|x(t) - X_f(t)| \leq \epsilon \textit{ for all } t \in [T_1 + \Delta, T_2 - \Delta].$$

Moreover, if $|x(T_1) - X_f(T_1)| \leq \delta$, *then*

$$|x(t) - X_f(t)| \leq \epsilon \textit{ for all } t \in [T_1, T_2 - \Delta]$$

and if $|x(T_2) - X_f(T_2)| \leq \delta$, *then*

$$|x(t) - X_f(t)| \leq \epsilon \textit{ for all } t \in [T_1 + \Delta, T_2].$$

Corollary 2.4. *Assume that* $f \in \mathcal{F}$, S, Δ *are positive numbers and that*

$$x : R^1 \to R^n, \ u : R^1 \to R^m$$

is a trajectory-control pair such that

$$I^f(T_1, T_2, x, u) = U^f(T_1, T_2, x(T_1), x(T_2))$$

for each $T_1 \in R^1$ *and each* $T_1 > T_1$, *and*

$$I^f(T_1, T_2, x, u) \leq \inf\{U^f(T_1, T_2, y_1, y_2) : \ (T_i, y_i) \in A, \ i = 1, 2\} + S$$

for each $T_1 \in R^1$ *and each* $T_2 > T_1 + \Delta$. *Then*

$$x(t) = X_f(t) \textit{ for all } t \in R^1.$$

The next theorem is our second turnpike result.

Theorem 2.5. *Let $f \in \mathcal{F}$, let a bounded continuous function $X_f : R^1 \to R^n$ be as guaranteed by Theorem 2.3. and let ϵ, M be a pair of positive numbers. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathfrak{M}$, real numbers $l > 0$, $L > 0$ and a natural number p such that for each $g \in \mathcal{U}$, each $T_1 \in R^1$, each $T_2 \geq T_1 + L$, and each trajectory-control pair*

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

which satisfies

$$I^g(T_1, T_2, x, u) \leq \inf\{U^g(T_1, T_2, y_1, y_2) :\ (T_i, y_i) \in A,\ i = 1, 2\} + M$$

there exist finite sequences

$$\{a_i\}_{i=1}^q,\ \{b_i\}_{i=1}^q \subset [T_1, T_2],$$

where $q \leq p$ is a natural number, such that

$$a_i \leq b_i \leq a_i + l \text{ for all integers } i = 1, \dots, q$$

and

$$|x(t) - X_f(t)| \leq \epsilon \text{ for all } t \in [T_1, T_2] \setminus \cup_{i=1}^q [a_i, b_i].$$

This chapter is organized as follows. In Sect. 2.3 we study uniform boundedness of trajectory-control pairs. Section 2.4 contains auxiliary results. Theorems 2.1–2.3 and 2.5 are proved in Sect. 2.5.

2.3 Uniform Boundedness of Trajectory-Control Pairs

Proposition 2.6. *Let M_0, M_1, τ_0 be positive numbers. Then there exists $M_2 > M_1$ such that for each $f \in \mathfrak{M}$, each $T_1 \in R^1$, each $T_2 \in (T_1, T_1 + \tau_0]$, and each trajectory-control pair*

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

which satisfies

$$\inf\{|x(t)| :\ t \in [T_1, T_2]\} \leq M_1,$$
$$I^f(T_1, T_2, x, u) \leq M_0 \tag{2.10}$$

the following inequality holds:

$$|x(t)| \le M_2 \text{ for all } t \in [T_1, T_2]. \tag{2.11}$$

Proof. Fix

$$\delta \in (0,\ \min\{8^{-1}(1 + a_0^{-1}),\ 16^{-1}\tau_0\}) \tag{2.12}$$

[recall a_0 in assumption (A)]. By assumption (A) there exist $h_0 > M_1 + 1$ such that

$$f(t, x, u) \ge 4(M_0 + a_0\tau_0)\delta^{-1}$$
$$\text{for each } (t, x, u) \in M \text{ satisfying } |x| \ge h_0 \tag{2.13}$$

and $\gamma_0 > 0$ such that

$$f(t, x, u) \ge 8[|G(t, x, u)| - a_0|x|]_+ \text{ for each } (t, x, u) \in M$$
$$\text{satisfying } |G(t, x, u)| - a_0|x| \ge \gamma_0. \tag{2.14}$$

Choose a number

$$M_2 > 8(M_0 + \tau_0 a_0 + \gamma_0\delta + h_0) + 8M_1. \tag{2.15}$$

Let $f \in \mathfrak{M}$, $T_1 \in R^1$, $T_2 \in (T_1, T_1 + \tau_0]$ and

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

be a trajectory-control pair satisfying (2.10). We show that (2.11) holds.

Assume the contrary. Then there exists $t_0 \in [T_1, T_2]$ such that

$$|x(t_0)| > M_2. \tag{2.16}$$

By the choice of h_0, (2.10), (2.12), (2.13), and assumption (A), there exists $t_1 \in [T_1, T_2]$ such that

$$|x(t_1)| \le h_0 \text{ and } |t_1 - t_0| \le \delta. \tag{2.17}$$

There exists

$$t_2 \in [\min\{t_0, t_1\},\ \max\{t_0, t_1\}]$$

such that

$$|x(t_2)| \ge |x(t)| \text{ for all } t \in [\min\{t_0, t_1\},\ \max\{t_0, t_1\}]. \tag{2.18}$$

We define

$$\begin{aligned} E_1 = \{t \in [\min\{t_1,t_2\},\ \max\{t_1,t_2\}] :& \\ |G(t,x(t),u(t))| \geq a_0|x(t)| + \gamma_0\},& \\ E_2 = [\min\{t_1,t_2\},\ \max\{t_1,t_2\}] \setminus E_1.& \end{aligned} \tag{2.19}$$

It follows from (2.1), (2.19), (2.17), (2.18), (2.14), (2.10), and assumption (A) that

$$\begin{aligned} |x(t_2) - x(t_1)| = |\int_{t_1}^{t_2} G(t,x(t),u(t))dt| \\ \leq a_0|\int_{t_1}^{t_2} |x(t)|dt| + \int_{E_1} (|G(t,x(t),u(t))| - a_0|x(t)|)_+ dt \\ + \int_{E_2} (|G(t,x(t),u(t))| - a_0|x(t)|)_+ dt \\ \leq a_0|x(t_2)|\delta + \gamma_0\delta + 8^{-1}\int_{E_1} f(t,x(t),u(t))dt \\ \leq a_0|x(t_2)|\delta + \gamma_0\delta + 8^{-1}(M_0 + \tau_0 a_0). \end{aligned}$$

Together with (2.12), (2.18), (2.16), and (2.17) this implies that

$$(7/8)M_2 - h_0 \leq (7/8)|x(t_2)| - |x(t_1)| \leq \gamma_0\delta + 8^{-1}(M_0 + \tau_0 a_0).$$

This contradicts (2.15). The contradiction we have reached proves Proposition 2.6. □

Proposition 2.6 and assumption (A) imply the following result.

Proposition 2.7. *Let $M_1 > 0$ and $0 < \tau_0 < \tau_1$. Then there exists $M_2 > 0$ such that for each $f \in \mathfrak{M}$, each $T_1 \in R^1$, each $T_2 \in [T_1 + \tau_0, T_1 + \tau_1]$ and each trajectory-control pair*

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

which satisfies

$$I^f(T_1, T_2, x, u) \leq M_1 \tag{2.20}$$

relation (2.11) holds.

Proposition 2.8. *Let $M_1, \epsilon > 0$ and $0 < \tau_0 < \tau_1$. Then there exists $\delta > 0$ such that for each $f \in \mathfrak{M}$, each $T_1 \in R^1$, each $T_2 \in [T_1 + \tau_0, T_1 + \tau_1]$, and each trajectory-control pair*

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

which satisfies (2.20) and each $t_1, t_2 \in [T_1, T_2]$ satisfying $|t_1 - t_2| \le \delta$ the inequality

$$|x(t_1) - x(t_2)| \le \epsilon$$

holds.

Proof. Let a number $M_2 > 0$ be as guaranteed in Proposition 2.7. By assumption (A) there exists $h_0 > 0$ such that

$$f(t, x, u) \ge 4\epsilon^{-1}(M_1 + a_0\tau_1 + 8)(|G(t, x, u)| - a_0|x|)_+$$
$$\text{for each } (t, x, u) \in M \text{ satisfying } |G(t, x, u)| - a_0|x| \ge h_0. \tag{2.21}$$

Fix a number

$$\delta \in (0, \epsilon(4a_0M_2 + 4h_0 + 4)^{-1}). \tag{2.22}$$

Let $f \in \mathfrak{M}$, $T_1 \in R^1$, $T_2 \in [T_1 + \tau_0, T_1 + \tau_1]$,

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$

be a trajectory-control pair satisfying (2.20) and let

$$t_1, \ t_2 \in [T_1, T_2], \ 0 < t_2 - t_1 \le \delta. \tag{2.23}$$

Set

$$E_1 = \{t \in [t_1, \ t_2] : \ |G(t, x(t), u(t))| - a_0|x(t)| \ge h_0\},$$
$$E_2 = [t_1, \ t_2] \setminus E_1. \tag{2.24}$$

It follows from the choice of M_2, (2.20), (2.11), (2.23), (2.24), (2.21), (2.22), and (2.1) that

$$\begin{aligned}|x(t_2) - x(t_1)| &\le a_0 \int_{t_1}^{t_2} |x(t)|dt + \int_{t_1}^{t_2} (|G(t, x(t), u(t))| - a_0|x(t)|)_+ dt \\ &\le a_0\delta M_2 + \delta h_0 + \int_{E_1} (|G(t, x(t), u(t))| - a_0|x(t)|)dt \\ &\le a_0\delta M_2 + \delta h_0 + \epsilon(4(M_1 + a_0\tau_1 + 8))^{-1} \int_{E_1} f(t, x(t), u(t))dt \\ &\le a_0\delta M_2 + \delta h_0 + 4^{-1}\epsilon \le \epsilon.\end{aligned}$$

Proposition 2.8 is proved. □

Proposition 2.9. *Let $f \in \mathfrak{M}$, $0 < c_1 < c_2$, and $D, \epsilon > 0$. Then there exists a neighborhood V of f in $\mathfrak{M}$ such that for each $g \in V$, each $T_1 \in R^1$, each $T_2 \in [T_1 + c_1, T_1 + c_2]$, and each trajectory-control pair*

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

which satisfies

$$\min\{I^f(T_1, T_2, x, u),\ I^g(T_1, T_2, x, u)\} \le D$$

the inequality

$$|I^f(T_1, T_2, x, u) - I^g(T_1, T_2, x, u)| \le \epsilon$$

holds.

Proof. By Proposition 2.7 there exists $S > 0$ such that for each $g \in \mathfrak{M}$, each $T_1 \in R^1$, each $T_2 \in [T_1 + c_1, T_1 + c_2]$, and each trajectory-control pair

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

which satisfies

$$I^g(T_1, T_2, x, u) \le D + 1$$

the following inequality holds:

$$|x(t)| \le S \text{ for all } t \in [T_1, T_2]. \tag{2.25}$$

There exist $\delta \in (0, 1)$, $N > S$ and $\Gamma > 1$ such that

$$\begin{aligned} &\psi(N) \ge 4a_0 + 4,\ \delta c_2 \le 8^{-1}\epsilon, \\ &(\Gamma - 1)(D + a_0 c_2 + c_2) \le 8^{-1}\epsilon. \end{aligned} \tag{2.26}$$

Define

$$V = \{g \in \mathfrak{M} :\ (f, g) \in E(N, \delta, \Gamma)\}. \tag{2.27}$$

Assume that $g \in V$, each $T_1 \in R^1$, each $T_2 \in [T_1 + c_1, T_1 + c_2]$ and

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

is a trajectory-control pair which satisfies

$$\min\{I^f(T_1, T_2, x, u),\ I^g(T_1, T_2, x, u)\} \le D. \tag{2.28}$$

Set

$$E_1 = \{t \in [T_1, T_2] : \ |u(t)| \le N\}, \ E_2 = [T_1, T_2] \setminus E_1. \tag{2.29}$$

It follows from (2.28), (2.29), (2.27), and the definition of S that (2.25) holds and that

$$|f(t, x(t), u(t)) - g(t, x(t), u(t))| \le \delta \text{ for all } t \in E_1. \tag{2.30}$$

Define

$$h(t) = \min\{f(t, x(t), u(t)), \ g(t, x(t), u(t))\}$$

for all $t \in [T_1, T_2]$. By (2.25), (2.27), (2.29), (2.26), and assumption (A), for each $t \in E_2$,

$$(f(t, x(t), u(t)) + 1)(g(t, x(t), u(t)) + 1)^{-1} \in [\Gamma^{-1}, \Gamma]$$

and

$$|f(t, x(t), u(t)) - g(t, x(t), u(t))| \le (\Gamma - 1)(h(t) + 1).$$

It follows from (2.29), (2.30), assumption (A), (2.28), (2.26) and the inequality above that

$$\begin{aligned} |I^f(T_1, T_2, x, u) - I^g(T_1, T_2, x, u)| &\le \delta c_2 + (\Gamma - 1)\int_{E_2} h(t)dt + (\Gamma - 1)c_2 \\ &\le \delta c_2 + (\Gamma - 1)c_2 + (\Gamma - 1)(D + a_0 c_2) \le \epsilon. \end{aligned}$$

Proposition 2.9 is proved. □

2.4 Auxiliary Results

Consider the control system described by (2.1)–(2.5). Suppose that $f \in \mathfrak{M}$ and that there exists a trajectory-control pair

$$x_* : R^1 \to R^n, \ u_* : R^1 \to R^m$$

such that:

(i)

$$U^f(T_1, T_2, x_*(T_1), x_*(T_2)) = I^f(T_1, T_2, x_*, u_*)$$

for each $T_1 \in R^1$ and each $T_2 > T_1$;

(ii)

$$\sup\{I^f(j, j+1, x_*, u_*) : j = 0, \pm 1, \pm 2, \dots\} < \infty;$$

(iii) For each $S_1 > 0$ there exist $S_2 > 0$ and an integer $c > 0$ such that

$$I^f(T_1, T_2, x_*, u_*) \le I^f(T_1, T_2, x, u) + S_2$$

for each $T_1 \in R^1$, each $T_2 \ge T_1 + c$ and each trajectory-control pair

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

which satisfies $|x(T_1)|,\ |x(T_2)| \le S_1$;

(iv) There exists $b_* > 0$ such that for each $\epsilon > 0$ there exists $\delta > 0$ such that for each $(T, z) \in A$ which satisfies

$$|z - x_*(T)| \le \delta$$

there are

$$\tau_1 \in (T, T + b_*] \text{ and } \tau_2 \in [T - b_*, T),$$

and trajectory-control pairs

$$x_1 : [T, \tau_1] \to R^n,\ u_1 : [T, \tau_1] \to R^m,$$
$$x_2 : [\tau_2, T] \to R^n,\ u_2 : [\tau_2, T] \to R^m$$

which satisfy

$$x_1(T) = x_2(T) = z,$$
$$x_i(\tau_i) = x_*(\tau_i),\ i = 1, 2,$$
$$|x_1(t) - x_*(t)| \le \epsilon \text{ for all } t \in [T, \tau_1],$$
$$|x_2(t) - x_*(t)| \le \epsilon \text{ for all } t \in [\tau_2, T],$$
$$I^f(T, \tau_1, x_1, u_1) \le I^f(T, \tau_1, x_*, u_*) + \epsilon,$$
$$I^f(\tau_2, T, x_2, u_2) \le I^f(\tau_2, T, x_*, u_*) + \epsilon.$$

It follows from property (ii) and Proposition 2.7 that

$$\sup\{|x_*(t)| :\ t \in R^1\} < \infty. \tag{2.31}$$

Note that properties (i)–(v) mean that the trajectory-control pair

$$x_* : R^1 \to R^n, \ u_* : R^1 \to R^m$$

is a solution of the corresponding infinite horizon optimal control problem associated with the integrand f and that certain controllability properties hold near this trajectory-control pair.

Lemma 2.10. *Let $S_0 > 0$. Then there exists $S > 0$ and an integer $c \geq 1$ such that for each $T_1 \in R^1$, each $T_2 \geq T_1 + c$ and each trajectory-control pair*

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m,$$

satisfying $|x(T_1)| \leq S_0$ the inequality

$$I^f(T_1, T_2, x_*, u_*) \leq I^f(T_1, T_2, x, u) + S \tag{2.32}$$

holds.

Proof. Fix a number S_2 which satisfies

$$S_2 > S_0,$$
$$\psi(S_2) > a_0 + \sup\{|I^f(j, j+1, x_*, u_*)| : \ j = 0, \pm 1, \pm 2, \dots\}. \tag{2.33}$$

By property (iii) there exists a number $S_1 > 0$ and an integer $c > 0$ such that for each $T_1 \in R^1$, each $T_2 \geq T_1 + c$, and each trajectory-control pair

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$

which satisfies $|x(T_1)|, \ |x(T_2)| \leq S_2$ the inequality

$$I^f(T_1, T_2, x_*, u_*) \leq I^f(T_1, T_2, x, u) + S_1$$

holds.

Fix a number

$$S > S_1 + 2 + 2a_0(2 + c)$$
$$+(c+4)\sup\{|I^f(j, j+1, x_*, u_*)| : \ j = 0, \pm 1, \pm 2, \dots\}. \tag{2.34}$$

Let $T_1 \in R^1$, $T_2 \geq T_1 + c$ and

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$

be a trajectory-control pair satisfying $|x(T_1)| \leq S_0$. We show that (2.32) holds. By the choice of S, S_1, and c, we may assume that

$$|x(T_2)| > S_2. \tag{2.35}$$

Set

$$T_3 = \sup\{t \in [T_1, T_2] : \; |x(t)| \le S_2\}. \tag{2.36}$$

It follows from the choice of S_1, (2.36), (2.33), and assumption (A) that

$$\begin{aligned} & I^f(T_1, T_3, x_*, u_*) - I^f(T_1, T_3, x, u) \\ & \le S_1 + 2a_0(1+c) + (c+2)\sup\{|I^f(j, j+1, x_*, u_*)| : \; j = 0, \pm 1, \pm 2, \dots\}. \end{aligned} \tag{2.37}$$

By assumption (A), (2.33) and (2.36),

$$\begin{aligned} & I^f(T_3, T_2, x_*, u_*) - I^f(T_3, T_2, x, u) \\ & \le (T_2 - T_3 + 2)\sup\{|I^f(j, j+1, x_*, u_*)| : \; j = 0, \pm 1, \pm 2, \dots\} \\ & \quad +2a_0 - (\psi(S_2) - a_0)(T_2 - T_3) \\ & \le 2\sup\{|I^f(j, j+1, x_*, u_*)| : j = 0, \pm 1, \pm 2, \dots\}. \end{aligned}$$

Together with (2.34) and (2.37) this implies (2.32). Lemma 2.10 is proved. □

Fix a number $d_0 > 0$ and define a continuous function $\phi : A \to R^1$ by

$$\phi(t, x) = \min\{|x - x_*(t)|, \; d_0\} \text{ for each } (t, x) \in A. \tag{2.38}$$

For any $r > 0$ we define $f_r \in \mathfrak{M}$ by

$$f_r(t, x, u) = f(t, x, u) + r\phi(t, x) \text{ for each } (t, x, u) \in M. \tag{2.39}$$

We have the following simple auxiliary result.

Lemma 2.11. *Let V be a neighborhood of f in $\mathfrak{M}$. Then there exists $r_0 > 0$ such that $f_r \in V$ for every $r \in (0, r_0)$.*

Fix $r \in (0, 1]$ and set

$$\begin{aligned} \Lambda_0 = \sup\{|I^f(j, j+1, x_*, u_*)| : \; j = 0, \pm 1, \pm 2, \dots\}, \\ \Lambda_1 = \sup\{|x_*(t)| : \; t \in R^1\}. \end{aligned} \tag{2.40}$$

Lemma 2.12. *Let $b_1 \ge 1$ be an integer and let $b_2, Q, S > 0$. Then there exists $S_1 > S$ such that*

$$|x(t)| \le S_1 \textit{ for all } t \in [T_1, T_2] \tag{2.41}$$

for each $T_1 \in R^1$, each $T_2 \geq T_1 + 2b_1$, and each trajectory-control pair

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

which satisfies the following conditions:

(a)

$$|x(T_1)|,\ |x(T_2)| \leq S$$

and

$$I^{f_r}(T_1, T_2, x, u) \leq U^{f_r}(T_1, T_2, x(T_1), x(T_2)) + Q;$$

(b) There exist

$$\tau_1 \in (T_1, T_1 + b_1] \text{ and } \tau_2 \in [T_2 - b_1, T_2)$$

such that

$$U^{f_r}(T_1, \tau_1, x(T_1), x_*(\tau_1)) \leq b_2,$$
$$U^{f_r}(\tau_2, T_2, x_*(\tau_2), x(T_2)) \leq b_2. \tag{2.42}$$

Proof. Choose a number S_0 such that

$$S_0 > S + 1,$$
$$\psi(S_0) > a_0 + \sup\{|I^f(j, j+1, x_*, u_*)| :\ j = 0, \pm 1, \pm 2, \dots\} + 4. \tag{2.43}$$

By property (iii), there exist $Q_1 > 0$ and an integer $c_1 \geq 1$ such that for each $T_1 \in R^1$, each $T_2 \geq T_1 + c_1$, and each trajectory-control pair

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

which satisfies

$$|x(T_i)| \leq S_0 + 1,\ i = 1, 2$$

the following inequality holds:

$$I^f(T_1, T_2, x_*, u_*) \leq I^f(T_1, T_2, x, u) + Q_1. \tag{2.44}$$

Fix a number

$$\tilde{S} > 4 + Q + 2Q_1 + 2a_0(b_1 + c_1 + 3) + 2b_2 + 2$$

$$+ \Lambda_0[2(c_1 + 3) + 2\Lambda_0(c_1 + 3) + 2a_0(b_1 + c_1 + 3) + 2b_2 + 2 + Q + 2Q_1]. \tag{2.45}$$

By Proposition 2.6 there exists $S_1 > S_0+2$ such that for each $g \in \mathfrak{M}$, each $T_1 \in R^1$, each

$$T_2 \in (T_1, T_1 + Q + 2b_2 + 2 + 2a_0(b_1 + c_1 + 3) + 2Q_1 + 2(c_1 + 3)\Lambda_0]$$

and each trajectory-control pair

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$

which satisfies

$$\inf\{|x(t)| : \ t \in [T_1, T_2]\} \le S_0,$$
$$I^g(T_1, T_2, x, u) \le \tilde{S} \tag{2.46}$$

the following inequality holds:

$$|x(t)| \le S_1 \text{ for all } t \in [T_1, T_2]. \tag{2.47}$$

Assume that $T_1 \in R^1$, $T_2 \ge T_1 + 2b_1$ and that a trajectory-control pair

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$

satisfies conditions (a) and (b). Let τ_1 and τ_2 be as guaranteed in condition (b). We show that (2.41) holds.

Assume the contrary. Then there exists $T_0 \in [T_1, T_2]$ such that

$$|x(T_0)| > S_1. \tag{2.48}$$

By condition (b) there exists a trajectory-control pair

$$x_1 : [T_1, T_2] \to R^n, \ u_1 : [T_1, T_2] \to R^m$$

such that

$$x_1(T_j) = x(T_j), \ x_1(\tau_j) = x_*(\tau_j), \ j = 1, 2,$$
$$x_1(t) = x_*(t) \text{ and } u_1(t) = u_*(t) \text{ for all } t \in [\tau_1, \tau_2],$$
$$I^{f_r}(T_1, \tau_1, x_1, u_1) \le b_2 + 1,$$
$$I^{f_r}(\tau_2, T_2, x_1, u_1) \le b_2 + 1. \tag{2.49}$$

It follows from (2.49) conditions (a) and (b), and assumption (A) that

$$I^{f_r}(T_1, T_2, x, u) \le I^{f_r}(T_1, T_2, x_1, u_1) + Q$$
$$\le Q + I^{f_r}(T_1, T_2, x_*, u_*) + 2a_0b_1 + 2b_2 + 2. \tag{2.50}$$

Set

$$t_1 = \sup\{t \in [T_1, T_0] : \ |x(t)| \le S_0\},$$
$$t_2 = \inf\{t \in [T_0, T_2] : \ |x(t)| \le S_0\}. \tag{2.51}$$

It follows from the choice of Q_1 and c_1, (2.44), (2.51), (2.40), condition (a), and assumption (A) that

$$I^{f_r}(T_1, t_1, x_*, u_*) - I^{f_r}(T_1, t_1, x, u)$$
$$\le Q_1 + a_0(c_1 + 2) + (c_1 + 2)\Lambda_0,$$
$$I^{f_r}(t_2, T_2, x_*, u_*) - I^{f_r}(t_2, T_2, x, u)$$
$$\le Q_1 + a_0(c_1 + 2) + (c_1 + 2)\Lambda_0. \tag{2.52}$$

By (2.39), (2.40), (2.51), (2.43), and assumption (A),

$$I^{f_r}(t_1, t_2, x_*, u_*) - I^{f_r}(t_1, t_2, x, u)$$
$$\le (t_2 - t_1 + 2)\Lambda_0 + 2a_0 - (t_2 - t_1)(\psi(S_0) - a_0) \le 2\Lambda_0 + 2a_0 - 4(t_2 - t_1).$$

Together with (2.50) and (2.52) this implies that

$$t_2 - t_1$$
$$\le 2\Lambda_0 + 2a_0 + Q + 2a_0 b_1 + 2b_2 + 2 + 2Q_1 + 2a_0(c_1 + 2) + 2(c_1 + 2)\Lambda_0. \tag{2.53}$$

It follows from the choice of S_1, (2.53), (2.46), (2.47), (2.48), and (2.51) that

$$I^{f_r}(t_1, t_2, x, u) > \tilde{S}.$$

Together with (2.39), (2.50), (2.52), (2.40), and assumption (A) this implies that

$$-Q - 2a_0 b_1 - 2b_2 - 2$$
$$\le I^{f_r}(T_1, T_2, x_*, u_*) - I^{f_r}(T_1, T_2, x, u)$$
$$\le 2Q_1 + 2a_0(c_1 + 2) + 2(c_1 + 2)\Lambda_0$$
$$+ I^{f_r}(t_1, t_2, x_*, u_*) - I^{f_r}(t_1, t_2, x, u)$$
$$\le 2Q_1 + 2a_0(c_1 + 2) + 2(c_1 + 2)\Lambda_0 + \Lambda_0(t_2 - t_1 + 2) + 2a_0 - \tilde{S}.$$

By this relation and (2.53),

$$\tilde{S} \le Q + 2Q_1 + 2a_0(b_1 + c_1 + 3) + 2(b_2 + 1)$$
$$+\Lambda_0[2(c_1 + 3) + 2\Lambda_0(c_1 + 3) + 2a_0(b_1 + c_1 + 3) + 2b_2 + 2 + Q + 2Q_1].$$

This contradicts (2.45). The contradiction we have reached proves Lemma 2.12. □

Lemma 2.13. *Let $b_1 \ge 1$ be an integer and let $b_2, Q, S > 0$. Then there exists $S_0 > 0$ such that for each $T_1 \in R^1$, each $T_2 \ge T_1 + 2b_1$, and each trajectory-control pair*

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$

which satisfies conditions (a) and (b) of Lemma 2.12 the following inequality holds:

$$I^{f_r}(T, T + 1, x, u) \le S_0 \text{ for any } T \in [T_1, T_2 - 1]. \tag{2.54}$$

Proof. Let $S_1 > S$ be as guaranteed in Lemma 2.12. By property (iii), there exist $Q_1 > 0$ and an integer $c_1 \ge 1$ such that for each $T_1 \in R^1$, each $T_2 \ge T_1 + c_1$, and each trajectory-control pair

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$

which satisfies

$$|x(T_i)| \le S_1 + 1, \ i = 1, 2$$

the following inequality holds:

$$I^f(T_1, T_2, x_*, u_*) \le I^f(T_1, T_2, x, u) + Q_1. \tag{2.55}$$

Fix a number

$$S_0 > 4 + Q + 2Q_1 + 2(b_2 + 1) + 2a_0(b_1 + c_1 + 4) + 2(c_1 + 4)\Lambda_0 \tag{2.56}$$

[recall a_0 in assumption (A)].

Assume that $T_1 \in R^1$, $T_2 \ge T_1 + 2b_1$ and that a trajectory-control pair

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$

satisfies conditions (a) and (b) of Lemma 2.12.

Let τ_1 and τ_2 be as guaranteed in conditions (a) and (b) of Lemma 2.12. We show that (2.54) holds for any $T \in [T_1, T_2 - 1]$.

Assume the contrary. Then there exists $T \in [T_1, T_2 - 1]$ such that

$$I^{f_r}(T, T + 1, x, u) > S_0. \tag{2.57}$$

By condition (b) of Lemma 2.12, there exists a trajectory-control pair

$$x_1 : [T_1, T_2] \to R^n, \; u_1 : [T_1, T_2] \to R^m$$

which satisfies (2.49). By (2.49), conditions (a) and (b) of Lemma 2.12 and assumption (A), relation (2.50) is true. It follows from the choice of S_1 and Lemma 2.12 that

$$|x(t)| \le S_1 \text{ for all } t \in [T_1, T_2] \tag{2.58}$$

In view of the choice of Q_1, c_1, (2.58), (2.40), and (2.39),

$$\begin{aligned} I^{f_r}(T_1, T, x_*, u_*) - I^{f_r}(T_1, T, x, u) &\le Q_1 + a_0(c_1 + 2) + (c_1 + 2)\Lambda_0, \\ I^{f_r}(T + 1, T_2, x_*, u_*) - I^{f_r}(T + 1, T_2, x, u) &\le Q_1 + a_0(c_1 + 2) + (c_1 + 2)\Lambda_0. \end{aligned} \tag{2.59}$$

It follows from (2.50), (2.59), (2.57), (2.40), and assumption (A) that

$$\begin{aligned} &-Q - 2a_0 b_1 - 2b_2 - 2 \\ &\le I^{f_r}(T_1, T_2, x_*, u_*) - I^{f_r}(T_1, T_2, x, u) \\ &\le 2Q_1 + 2a_0(c_1 + 2) + 2(c_1 + 2)\Lambda_0 \\ &+ I^{f_r}(T, T + 1, x_*, u_*) - I^{f_r}(T, T + 1, x, u) \\ &\le 2Q_1 + 2a_0(c_1 + 4) + 2(c_1 + 4)\Lambda_0 - S_0. \end{aligned}$$

This contradicts (2.56). The contradiction we have reached proves Lemma 2.13. □

Lemma 2.14. *Let $\Delta > 0$. Then there exists $\delta \in (0, \Delta)$ such that for each $T_1 \in R^1$, each $T_2 > T_1$, and each trajectory-control pair*

$$x : [T_1, T_2] \to R^n, \; u : [T_1, T_2] \to R^m$$

which satisfies

$$|x(T_i) - x_*(T_i)| \le \delta, \; i = 1, 2 \tag{2.60}$$

the following inequality holds:

$$I^f(T_1, T_2, x, u) \ge I^f(T_1, T_2, x_*, u_*) - \Delta. \tag{2.61}$$

Proof. There exists $\delta \in (0, \Delta)$ such that property (iv) holds with $\epsilon = 3^{-1}\Delta$ (see the definition of f).

Assume that $T_1 \in R^1$, $T_2 > T_1$ and that a trajectory-control pair

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

satisfies (2.60). By (2.60), the choice of δ and property (iv), there exist

$$\tau_1 \in [T_1 - b_*, T_1) \text{ and } \tau_2 \in (T_2, T_2 + b_*]$$

and trajectory-control pairs

$$x_1 : [\tau_1, T_1] \to R^n,\ u_1 : [\tau_1, T_1] \to R^m$$

and

$$x_2 : [T_2, \tau_2] \to R^n,\ u_2 : [T_2, \tau_2] \to R^m$$

such that

$$\begin{aligned}
x_i(T_i) &= x(T_i),\ i = 1, 2,\\
x_i(\tau_i) &= x_*(\tau_i),\ i = 1, 2,\\
I^f(\tau_1, T_1, x_1, u_1) &\le I^f(\tau_1, T_1, x_*, u_*) + 3^{-1}\Delta,\\
I^f(T_2, \tau_2, x_2, u_2) &\le I^f(T_2, \tau_2, x_*, u_*) + 3^{-1}\Delta. \qquad (2.62)
\end{aligned}$$

It follows from the choice of (x_1, u_1), (x_2, u_2) and property (i) that

$$\begin{aligned}
I^f(\tau_1, T_1, x_1, u_1) + I^f(T_1, T_2, x, u) + I^f(T_2, \tau_2, , x_2, u_2)\\
\ge I^f(\tau_1, \tau_2, x_*, u_*).
\end{aligned}$$

Together with (2.62) this implies (2.61). Lemma 2.14 is proved. □

Lemma 2.15. *Let $\epsilon \in (0, d_0)$. Then there exist $\delta \in (0, \epsilon)$ and $\Delta > 0$ such that for each $T_1 \in R^1$, each $T_2 \ge T_1 + \Delta$, and each trajectory-control pair*

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

which satisfies

$$\begin{aligned}
|x(T_i) - x_*(T_i)| &\le \delta,\ i = 1, 2,\\
I^{f_r}(T_1, T_2, x, u) &\le U^{f_r}(T_1, T_2, x(T_1), x(T_2)) + \delta \qquad (2.63)
\end{aligned}$$

the following inequality holds:

$$|x(t) - x_*(t)| \le \epsilon \text{ for all } t \in [T_1, T_2]. \qquad (2.64)$$

Proof. There exists $\delta_0 \in (0, 8^{-1})$ such that property (iv) holds with $\epsilon = 1$ and $\delta = \delta_0$. There exists an integer b_1 such that

$$b_* + 1 < b_1 \leq b_* + 2 \tag{2.65}$$

[recall b_* in the definition of f, property (iv)].

Set

$$\begin{aligned} Q &= 1, \\ S &= \sup\{|x_*(t)| : \ t \in R^1\} + 4, \\ b_2 &= 2 + b_* + 2a_0 + (b_* + 3)\Lambda_0. \end{aligned} \tag{2.66}$$

Let S_0 be as guaranteed in Lemma 2.13. By Proposition 2.8 there exists $\delta_1 \in (0, 8^{-1})$ such that for each $g \in \mathfrak{M}$, each $T \in R^1$, each trajectory-control pair

$$x : [T, T+1] \to R^n, \ u : [T, T+1] \to R^m$$

which satisfies

$$I^g(T, T+1, x, u) \leq 1 + S_0 + 2\Lambda_0 + 2a_0 \tag{2.67}$$

and each $t_1, t_2 \in [T, T+1]$ satisfying $|t_1 - t_2| \leq \delta_1$ the inequality

$$|x(t_1) - x(t_2)| \leq 16^{-1}\epsilon$$

holds.

Choose a number

$$\epsilon_1 \in (0, \min\{8^{-1}, 8^{-1}\epsilon, 12^{-1}\epsilon r \delta_1 (4 + 4b_*)^{-1}\}). \tag{2.68}$$

There exists

$$\delta_2 \in (0, 4^{-1}\epsilon_1) \tag{2.69}$$

such that property (iv) (see the definition of f) holds with $\epsilon = \epsilon_1$, $\delta = \delta_2$. There exists

$$\delta \in (0, \delta_2) \tag{2.70}$$

such that Lemma 2.14 holds with $\Delta = 16^{-1}\epsilon\delta_1 r$. Fix a number

$$\Delta > 2b_* + 2b_1 + 4. \tag{2.71}$$

Assume that $T_1 \in R^1$, $T_2 \geq T_1 + \Delta$ and that a trajectory-control pair

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

satisfies (2.63). We show that (2.64) holds.

Assume the contrary. Then there exists a number t_1 such that

$$t_1 \in [T_1, T_2],$$
$$|x(t_1) - x_*(t_1)| > \epsilon. \tag{2.72}$$

By (2.63), property (iv), and the choice of δ_2 there exist

$$\tau_1 \in (T_1, T_1 + b_*] \text{ and } \tau_2 \in [T_2 - b_*, T_2) \tag{2.73}$$

and trajectory-control pairs

$$x_1 : [T_1, \tau_1] \to R^n,\ u_1 : [T_1, \tau_1] \to R^m,$$
$$x_2 : [\tau_2, T_2] \to R^n,\ u_2 : [\tau_2, T_2] \to R^m$$

which satisfy

$$x_i(T_i) = x(T_i),\ i = 1, 2,$$
$$x_i(\tau_i) = x_*(\tau_i),\ i = 1, 2,$$
$$|x_1(t) - x_*(t)| \leq \epsilon_1 \text{ for all } t \in [T_1, \tau_1],$$
$$|x_2(t) - x_*(t)| \leq \epsilon_1 \text{ for all } t \in [\tau_2, T_2],$$
$$I^f(T_1, \tau_1, x_1, u_1) \leq I^f(T_1, \tau_1, x_*, u_*) + \epsilon_1,$$
$$I^f(\tau_2, T_2, x_2, u_2) \leq I^f(\tau_2, T_2, x_*, u_*) + \epsilon_1. \tag{2.74}$$

Define a trajectory-control pair

$$x_3 : [T_1, T_2] \to R^n,\ u_3 : [T_1, T_2] \to R^m$$

by

$$x_3(t) = x_1(t),\ u_3(t) = u_1(t) \text{ for all } t \in [T_1, \tau_1],$$
$$x_3(t) = x_*(t),\ u_3(t) = u_*(t) \text{ for all } t \in (\tau_1, \tau_2],$$
$$x_3(t) = x_2(t),\ u_3(t) = u_2(t) \text{ for all } t \in (\tau_2, T_2].$$

The relations above, (2.63) and (2.74) imply that

$$I^{f_r}(T_1, T_2, x, u) \leq I^{f_r}(T_1, T_2, x_3, u_3) + \delta. \tag{2.75}$$

It follows from (2.75), (2.39), (2.74), (2.73), (2.65), and the definition of x_3 that

$$\begin{aligned}
&I^{f_r}(T_1, T_2, x_3, u_3) - I^{f_r}(T_1, T_2, x_*, u_*) \\
&\leq \int_{T_1}^{\tau_1} \phi(t, x_1(t))dt + I^f(T_1, \tau_1, x_1, u_1) \\
&-I^f(T_1, \tau_1, x_*, u_*) + \int_{\tau_2}^{T_2} \phi(t, x_2(t))dt \\
&+I^f(\tau_2, T_2, x_2, u_2) - I^f(\tau_2, T_2, x_*, u_*) \\
&\leq 2\epsilon_1 + 2\epsilon_1 b_1.
\end{aligned}$$

Together with (2.75) and (2.39) this implies that

$$\begin{aligned}
I^f(T_1, T_2, x_*, u_*) + 2\epsilon_1 + 2\epsilon_1 b_1 + \delta &\geq I^{f_r}(T_1, T_2, x, u) \\
&\geq I^f(T_1, T_2, x, u) + r\int_{T_1}^{T_2} \phi(t, x(t))dt. \qquad (2.76)
\end{aligned}$$

By (2.74), (2.73), (2.39), and (2.68),

$$\begin{aligned}
&U^{f_r}(T_1, \tau_1, x(T_1), x_*(\tau_1)) \leq I^f(T_1, \tau_1, x_*, u_*) + 1 + b_*, \\
&U^{f_r}(\tau_2, T_2, x_*(\tau_2), x(T_2)) \leq I^f(\tau_2, T_2, x_*, u_*) + 1 + b_*. \qquad (2.77)
\end{aligned}$$

There exists an interval

$$[d_1, d_2] \subset [T_1, T_2]$$

such that

$$\begin{aligned}
&d_2 - d_1 = 1, \\
&t_1 \in [d_1, d_2]. \qquad (2.78)
\end{aligned}$$

It follows from (2.63), (2.68)–(2.70), (2.73), and (2.77) that the trajectory-control pair

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$

satisfies conditions (a) and (b) of Lemma 2.12 with b_1, b_2, Q, S defined by (2.65) and (2.66). Therefore by the choice of S_0 and Lemma 2.13,

$$I^{f_r}(d_1, d_2, x, u) \leq S_0. \tag{2.79}$$

It is easy to see that

$$I^{f_r}(d_1, d_2, x_*, u_*) \le 2\Lambda_0 + 2a_0. \tag{2.80}$$

It follows from the choice of δ_1, (2.67), (2.78)–(2.80), and (2.72) that for every

$$t \in [d_1, d_2] \cap [t_1 - \delta_1, t_1 + \delta_1]$$

we have

$$|x(t) - x(t_1)| \le 16^{-1}\epsilon,$$
$$|x_*(t) - x_*(t_1)| \le 16^{-1}\epsilon$$

and

$$|x(t) - x_*(t)| \ge (3/4)\epsilon.$$

Therefore

$$r\int_{T_1}^{T_2} \phi(t, x(t))dt \ge (3/4)\epsilon r\delta_1$$

and in view of (2.76) and (2.68)–(2.70),

$$I^f(T_1, T_2, x, u) \le I^f(T_1, T_2, x_*, u_*) - 4^{-1}\epsilon r\delta_1.$$

On the other hand in view of the choice of δ [see (2.70)], (2.63) and Lemma 2.14,

$$I^f(T_1, T_2, x, u) \ge I^f(T_1, T_2, x_*, u_*) - 16^{-1}\epsilon r\delta_1.$$

The contradiction we have reached proves Lemma 2.15. □

Lemma 2.16. *Let $\epsilon \in (0, \min\{1, d_0\})$. Then there exist $\delta \in (0, \epsilon)$ and $\Delta_1 > 0$ such that for each $\Delta_2 > \Delta_1$ there exists a neighborhood $\mathcal{U}$ of f_r in $\mathfrak{M}$ for which the following property holds:*

For each $g \in \mathcal{U}$, each $T_1 \in R^1$, each $T_2 \in [T_1 + \Delta_1, T_1 + \Delta_2]$, and each trajectory-control pair

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

which satisfies

$$|x(T_i) - x_*(T_i)| \le \delta,\ i = 1, 2,$$
$$I^g(T_1, T_2, x, u) \le U^g(T_1, T_2, x(T_1), x(T_2)) + \delta \tag{2.81}$$

the following inequality holds:

$$|x(t) - x_*(t)| \le \epsilon \text{ for all } t \in [T_1, T_2]. \tag{2.82}$$

Proof. There exists $\delta_0 \in (0, 8^{-1})$ such that property (iv) (see the definition of f) holds with $\epsilon = 1$ and $\delta = \delta_0$. By Lemma 2.15 there exist

$$\delta_1 \in (0, \min\{4^{-1}\epsilon, 4^{-1}\delta_0\}),$$

$$\Delta_1 \ge 2b_* + 2 \tag{2.83}$$

such that for each $T_1 \in R^1$, each $T_2 \ge T_1 + \Delta_1$, and each trajectory-control pair

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$

which satisfies

$$|x(T_i) - x_*(T_i)| \le \delta_1, \ i = 1, 2,$$

$$I^{f_r}(T_1, T_2, x, u) \le U^{f_r}(T_1, T_2, x(T_1), x(T_2)) + \delta_1 \tag{2.84}$$

relation (2.82) holds for all $t \in [T_1, T_2]$. Fix

$$\delta \in (0, 16^{-1}\delta_1). \tag{2.85}$$

Let $\Delta_2 > \Delta_1$. By Proposition 2.9 there exists a neighborhood $\mathcal{U}$ of f_r in $\mathfrak{M}$ such that for each $g \in \mathcal{U}$, each $T_1 \in R^1$, each $T_2 \in [T_1 + \Delta_1, T_1 + \Delta_2]$, and each trajectory-control pair

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$

which satisfies

$$\min\{I^{f_r}(T_1, T_2, x, u), \ I^g(T_1, T_2, x, u)\} \le 4 + 2a_0 + \Delta_2 + \Lambda_0(\Delta_2 + 2) \tag{2.86}$$

the inequality

$$|I^{f_r}(T_1, T_2, x, u) - I^g(T_1, T_2, x, u)| \le \delta \tag{2.87}$$

holds.

Assume that

$$g \in \mathcal{U}, \ T_1 \in R^1, \ T_2 \in [T_1 + \Delta_1, T_1 + \Delta_2]$$

and a trajectory-control pair

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

satisfies (2.81). We show that (2.82) holds.

By (2.81), (2.83), property (iv), and the choice of δ_0 there exist

$$\tau_1 \in (T_1, T_1 + b_*] \text{ and } \tau_2 \in [T_2 - b_*, T_2) \tag{2.88}$$

and trajectory-control pairs

$$x_1 : [T_1, \tau_1] \to R^n,\ u_1 : [T_1, \tau_1] \to R^m,$$
$$x_2 : [\tau_2, T_2] \to R^n,\ u_2 : [\tau_2, T_2] \to R^m$$

such that

$$\begin{aligned} &x_i(T_i) = x(T_i),\ i = 1, 2,\\ &x_i(\tau_i) = x_*(\tau_i),\ i = 1, 2,\\ &|x_1(t) - x_*(t)| \le 1 \text{ for all } t \in [T_1, \tau_1],\\ &|x_2(t) - x_*(t)| \le 1 \text{ for all } t \in [\tau_2, T_2],\\ &I^f(T_1, \tau_1, x_1, u_1) \le I^f(T_1, \tau_1, x_*, u_*) + 1,\\ &I^f(\tau_2, T_2, x_2, u_2) \le I^f(\tau_2, T_2, x_*, u_*) + 1. \end{aligned} \tag{2.89}$$

Define a trajectory-control pair

$$x_3 : [T_1, T_2] \to R^n,\ u_3 : [T_1, T_2] \to R^m$$

by

$$\begin{aligned} &x_3(t) = x_1(t),\ u_3(t) = u_1(t) \text{ for all } t \in [T_1, \tau_1],\\ &x_3(t) = x_*(t),\ u_3(t) = u_*(t) \text{ for all } t \in (\tau_1, \tau_2],\\ &x_3(t) = x_2(t),\ u_3(t) = u_2(t) \text{ for all } t \in (\tau_2, T_2]. \end{aligned} \tag{2.90}$$

Relations (2.88), (2.89), (2.90), (2.39), and (2.40) imply that

$$\begin{aligned} &U^{f_r}(T_1, T_2, x(T_1), x(T_2)) \le I^{f_r}(T_1, T_2, x_3, u_3)\\ &\le r \int_{T_1}^{T_2} \phi(t, x_3(t))dt + I^f(T_1, T_2, x_3, u_3)\\ &\le 2b_* + 2 + I^f(T_1, T_2, x_*, u_*) \le 2b_* + 2 + \Lambda_0(\Delta_2 + 2) + 2a_0. \end{aligned} \tag{2.91}$$

It follows from (2.91), (2.83), (2.81), and the choice of $\mathcal{U}$ [see (2.86), (2.87)] that

$$|U^{f_r}(T_1, T_2, x(T_1), x(T_2)) - U^g(T_1, T_2, x(T_1), x(T_2))| \le \delta$$

and

$$|I^{f_r}(T_1, T_2, x, u) - I^g(T_1, T_2, x, u)| \le \delta.$$

Together with (2.81) and (2.85) this implies that

$$\begin{aligned} I^{f_r}(T_1, T_2, x, u) &\le U^{f_r}(T_1, T_2, x(T_1), x(T_2)) + 3\delta \\ &\le U^{f_r}(T_1, T_2, x(T_1), x(T_2)) + \delta_1. \end{aligned}$$

Now (2.82) follows from the relation above, (2.81), (2.85) and the choice of δ_1, Δ_1 [see (2.83), (2.84)]. This completes the proof of Lemma 2.16. □

Lemma 2.17. *Let $S > 0$ and $\delta \in (0, \min\{1, d_0\})$. Then there exist $\Delta \ge 1$ and a neighborhood $\mathcal{U}$ of f_r in $\mathfrak{M}$ such that for each $g \in \mathcal{U}$, each $T_1 \in R^1$, each $T_2 \ge T_1 + \Delta$ and each trajectory-control pair*

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$

which satisfies

$$|x(t) - x_*(t)| \ge \delta \text{ for all } t \in [T_1, T_2] \tag{2.92}$$

the following inequality holds:

$$I^g(T_1, T_2, x, u) > I^g(T_1, T_2, x_*, u_*) + S. \tag{2.93}$$

Proof. Fix a number

$$D_1 > 16(\Lambda_0 + a_0 + b_* + S + 22). \tag{2.94}$$

By Proposition 2.7 there exists $D_2 > D_1$ such that for each $g \in \mathfrak{M}$, each $T_1 \in R^1$, each $T_2 \in [T_1 + 8^{-1}, T_1 + 8]$, and each trajectory-control pair

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$

which satisfies

$$I^g(T_1, T_2, x, u) \le 2D_1 + 4 \tag{2.95}$$

we have

$$|x(t)| \le D_2 \text{ for all } t \in [T_1, T_2].$$

By Lemma 2.10 there exist $D_3 > D_2$ and an integer $c_1 \geq 1$ such that for each $T_1 \in R^1$, each $T_2 \geq T_1 + c_1$, and each trajectory-control pair

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

satisfying $|x(T_1)| \leq 2D_2 + 1$ the inequality

$$I^f(T_1, T_2, x_*, u_*) \leq I^f(T_1, T_2, x, u) + D_3 \tag{2.96}$$

holds.

Choose numbers

$$\begin{aligned} S_1 > 8(D_3 + 20 + c_1 + 4(b_* + 4 + c_1)(\Lambda_0 + a_0) + S), \\ D_4 > 8S_1(2 + 4a_0 + 4\Lambda_0). \end{aligned} \tag{2.97}$$

By Proposition 2.9 there exists a neighborhood $\mathcal{U}$ of f_r in $\mathfrak{M}$ such that for each $g \in \mathcal{U}$, each $T_1 \in R^1$, each $T_2 \in [T_1 + 4^{-1}, T_1 + 8]$, and each trajectory-control pair

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

which satisfies

$$\min\{I^{f_r}(T_1, T_2, x, u),\ I^g(T_1, T_2, x, u)\} \leq 2D_4 + 4 \tag{2.98}$$

the inequality

$$|I^g(T_1, T_2, x, u) - I^{f_r}(T_1, T_2, x, u)| \leq 64^{-1} r\delta \tag{2.99}$$

holds.

Fix a number

$$\Delta > 64(r\delta)^{-1}[2S_1 + (c_1 + 2)(a_0 + \Lambda_0)]. \tag{2.100}$$

Assume that $g \in \mathcal{U}$, $T_1 \in R^1$, $T_2 \geq T_1 + \Delta$, and that a trajectory-control pair

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

satisfies (2.92). By assumption (A), (2.40), (2.97), and the choice of $\mathcal{U}$ [see (2.98), (2.99)] for each $t_1, t_2 \in [T_1, T_2]$ satisfying $t_2 - t_1 \in [2^{-1}, 2]$

$$I^g(t_1, t_2, x_*, u_*) \leq I^{f_r}(t_1, t_2, x_*, u_*) + 64^{-1}\delta \leq 3\Lambda_0 + 2a_0 + 1. \tag{2.101}$$

We show that (2.93) holds. There exists an integer p such that

$$p - 1 < T_2 - T_1 \leq p. \tag{2.102}$$

Set

$$b_0 = T_1,\ b_j = T_1 + j,\ j = 1, \dots, p-2,\ b_{p-1} = T_2,\ \tau_0 = T_1. \tag{2.103}$$

By induction we define a sequence $\{\tau_j\} \subset \{b_i\}_{i=0}^{p-1}$.

Assume that an integer $q \geq 0$, the sequence $\{\tau_i\}_{i=0}^{q}$ has been defined and that $\tau_q = b_{j(q)}$, where $0 \leq j(q) < p - 1$. If

$$I^g(b_{j(q)}, b_{j(q)+1}, x, u) \geq D_1, \tag{2.104}$$

then we set $\tau_{q+1} = b_{j(q)+1}$. If (2.104) does not hold and there exists an integer $k \in \{j(q)+1, \dots, p-2\}$ such that

$$\begin{aligned} I^g(b_i, b_{i+1}, x, u) < D_4,\ i = j(q), \dots, k-1, \\ I^g(b_k, b_{k+1}, x, u) \geq D_4, \end{aligned} \tag{2.105}$$

then $\tau_{q+1} = b_{k+1}$. Otherwise $\tau_{q+1} = b_{p-1}$.

Evidently, the construction of the sequence $\{\tau_i\}$ is completed in a finite number of steps. Let τ_Q be the last element of the sequence and let

$$q \in \{0, \dots, Q-1\},\ j(q) \in \{0, \dots, p-2\},\ \tau_q = b_{j(q)}. \tag{2.106}$$

We estimate

$$I^g(\tau_q, \tau_{q+1}, x, u) - I^g(\tau_q, \tau_{q+1}, x_*, u_*).$$

If (2.204) holds, then by (2.101) which holds with $t_1 = \tau_q$, $t_2 = \tau_{q+1}$ and (2.94),

$$I^g(\tau_q, \tau_{q+1}, x, u) - I^g(\tau_q, \tau_{q+1}, x_*, u_*) \geq (3/4)D_1. \tag{2.107}$$

Assume that (2.104) does not hold and there exists $k \in \{j(q)+1, \dots, p-2\}$ which satisfies (2.105). We show that

$$I^g(\tau_q, \tau_{q+1}, x, u) - I^g(\tau_q, \tau_{q+1}, x_*, u_*) \geq (3/4)D_4. \tag{2.108}$$

It follows from (2.105), the choice of τ_{q+1}, (2.101), and (2.97) that

$$\begin{aligned} &I^g(\tau_q, \tau_{q+1}, x, u) - I^g(\tau_q, \tau_{q+1}, x_*, u_*) \\ &\geq I^g(b_{j(q)}, b_k, x, u) - I^g(b_{j(q)}, b_k, x_*, u_*) + (7/8)D_4. \end{aligned} \tag{2.109}$$

By the choice of $\mathcal{U}$ [see (2.98), (2.99)], (2.105), (2.101), and (2.97) for $i = j(q), \dots, k-1$,

$$|I^g(b_i, b_{i+1}, y, v) - I^{f_r}(b_i, b_{i+1}, y, v)| \le 64^{-1}\delta r,$$
$$(y, v) \in \{(x, u),\ (x_*, u_*)\}. \tag{2.110}$$

It follows from (2.38), (2.39), and (2.92) that

$$I^{f_r}(b_{j(q)}, b_k, x, u) \ge I^f(b_{j(q)}, b_k, x, u) + r\delta(k - j(q)). \tag{2.111}$$

Since (2.104) does not hold it follows from the definition of D_2 [see (2.95)] that

$$|x(t)| \le D_2 \text{ for all } t \in [b_{j(q)}, b_{j(q)+1}]. \tag{2.112}$$

There are two cases: (i) $k - j(q) \ge S_1$; (ii) $k - j(q) < S_1$.

Consider the case (i). By (2.112), the choice of D_3, c_1 [see (2.96)] and the inequality $k - j(q) \ge S_1$,

$$I^f(b_{j(q)}, b_k, x, u) \ge I^f(b_{j(q)}, b_k, x_*, u_*) - D_3. \tag{2.113}$$

Combining (2.109), (2.110), (2.111), (2.113), (2.97), and the inequality

$$k - j(q) \ge S_1$$

we obtain (2.108). If $k - j(q) < S_1$, then (2.108) follows from (2.109), (2.101), (2.97), and assumption (A).

Assume that (2.104) does not hold and there is no $k \in \{j(q)+1, \dots, p-2\}$ satisfying (2.105). Then

$$\tau_{q+1} = b_{p-1},\ q = Q - 1,$$
$$I^g(b_i, b_{i+1}, x, u) < D_4,\ i = j(q), \dots, p-2. \tag{2.114}$$

By the choice of $\mathcal{U}$ [see (2.98), (2.99)], (2.114), (2.101), and (2.97), relation (2.110) holds for $i = j(q), \dots, p-2$. Relation (2.110), which holds for $i = j(q), \dots, p-2$, (2.92) and (2.114) imply that

$$I^g(\tau_q, \tau_{q+1}, x, u) - I^g(\tau_q, \tau_{q+1}, x_*, u_*)$$
$$\ge I^g(\tau_q, \tau_{q+1}, x, u) - I^g(\tau_q, \tau_{q+1}, x_*, u_*) + r(\delta - 32^{-1}\delta)(T_2 - \tau_q). \tag{2.115}$$

Since (2.104) does not hold it follows from the choice of D_2 [see (2.95)] that

$$|x(t)| \le D_2 \text{ for all } t \in [\tau_q, \tau_q + 1].$$

By this relation, the choice of D_3, c_1 [see (2.96)] and assumption (A),

$$I^f(\tau_q, T_2, x, u) - I^f(\tau_q, T_2, x_*, u_*)$$
$$\geq -D_3 - a_0 c_1 - \Lambda_0(c_1 + 2) - 2a_0.$$

Together with (2.115) this implies that

$$I^g(\tau_q, T_2, x, u) - I^g(\tau_q, T_2, x_*, u_*)$$
$$\geq 2^{-1} r\delta(T_2 - \tau_q) - D_3 - (a_0 + \Lambda_0)(c_1 + 2) \tag{2.116}$$

(here $q = Q - 1$ [see (2.114)].

We showed that if (2.104) holds, then (2.107) is valid, if (2.104) does not hold and there is $k \in \{j(q) + 1, \ldots, p - 2\}$ satisfying (2.105), then (2.108) holds; otherwise $q = Q - 1$ and (2.116) holds.

We show that (2.93) holds. For any integer $q \in \{0, \ldots, Q\}$ there is an integer $j(q) \in \{0, \ldots, p - 1\}$ such that $\tau_q = b_{j(q)}$. We may assume that for $q = Q - 1$ (2.104) does not hold and there is not an integer

$$k \in \{j(Q - 1) + 1, \ldots, p - 2\}$$

satisfying (2.105) with $q = Q - 1$. If $T_2 - \tau_{Q-1} \geq 8^{-1}\Delta$, then (2.93) follows from (2.100), (2.116), which holds with $q = Q - 1$, (2.107), (2.108), and (2.94). Therefore we may assume that

$$T_2 - \tau_{Q-1} < 8^{-1}\Delta.$$

This implies that

$$(7/8)\Delta \leq \tau_{Q-1} - T_1. \tag{2.117}$$

There are two cases:

(i) Relation (2.104) holds for each integer $q \in \{0, \ldots, Q - 2\}$;
(ii) There is an integer $q \in \{0, \ldots, Q - 2\}$ which does not satisfy (2.104).

Consider the case (i). By the choice of $\{\tau_i\}_{i=0}^{Q}$, (2.116), which holds with $q = Q - 1$, (2.117), (2.107), (2.100), and (2.94),

$$Q - 1 = \tau_Q - T_1,$$
$$I^g(T_1, T_2, x, u) - I^g(T_1, T_2, x_*, u_*)$$
$$\geq (3/4)(Q - 1)D_1 - D_3 - (c_1 + 2)(\Lambda_0 + a_0)$$
$$\geq 2^{-1}\Delta D_1 - D_3 - (c_1 + 2)(\Lambda_0 + a_0) \geq 2S$$

and (2.93) holds.

Consider the case (ii). It is easy to see that (2.108) holds. Relation (2.93) follows from (2.116), which holds with $q = Q - 1$, (2.108), (2.107), and (2.97). This completes the proof of Lemma 2.17. □

Lemma 2.18. *There exists $\gamma > 0$ such that for each $\delta \in (0, \gamma)$ and each $S > 0$ there exist $\Delta \geq 1$ and a neighborhood $\mathcal{U}$ of f_r in $\mathfrak{M}$ such that for each $g \in \mathcal{U}$, each $T_1 \in R^1$, each $T_2 \geq T_1 + \Delta$, and each trajectory-control pair*

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$

which satisfies

$$|x(T_i) - x_*(T_i)| \leq \gamma, \ i = 1, 2,$$
$$|x(t) - x_*(t)| > \delta \text{ for all } t \in [T_1, T_2] \tag{2.118}$$

the inequality

$$I^g(T_1, T_2, x, u) > U^g(T_1, T_2, x(T_1), x(T_2)) + S$$

holds.

Proof. There exists

$$\gamma \in (0, \min\{8^{-1}, d_0\})$$

such that property (iv) (see the definition of f) holds with $\epsilon = 1$ and $\delta = \gamma$.

Let $\delta \in (0, \gamma)$ and $S > 0$. By Proposition 2.9 there exists a neighborhood $\mathcal{U}_1$ of f_r in $\mathfrak{M}$ such that for each $g \in \mathcal{U}_1$, each $T_1 \in R^1$, each $T_2 \in [T_1 + 8^{-1}, T_1 + 8(b_* + 2)]$, and each trajectory-control pair

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$

which satisfies

$$\min\{I^{f_r}(T_1, T_2, x, u), \ I^g(T_1, T_2, x, u)\} \leq 16(\Lambda_0 + a_0 + 4)(2b_* + 6) \tag{2.119}$$

the inequality

$$|I^g(T_1, T_2, x, u) - I^{f_r}(T_1, T_2, x, u)| \leq 1 \tag{2.120}$$

holds.

Fix a number

$$S_1 > 2S + 1 + 2[1 + \Lambda_0(4 + b_*) + 4a_0 + b_* + 1] + 2a_0(1 + b_*). \tag{2.121}$$

By Lemma 2.17 there exist $\Delta_1 \geq 1$ and a neighborhood $\mathcal{U}$ of f_r in $\mathfrak{M}$ such that $\mathcal{U} \subset \mathcal{U}_1$ and for each $g \in \mathcal{U}$, each $T_1 \in R^1$, each $T_2 \geq T_1 + \Delta_1$, and each trajectory-control pair

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$

which satisfies

$$|x(t) - x_*(t)| \geq \delta \text{ for all } t \in [T_1, T_2] \tag{2.122}$$

the following inequality holds:

$$I^g(T_1, T_2, x, u) > I^g(T_1, T_2, x_*, u_*) + S_1. \tag{2.123}$$

Fix a number

$$\Delta > \Delta_1 + 4(4 + 4b_*) + 16. \tag{2.124}$$

Assume that $g \in \mathcal{U}$, $T_1 \in R^1$, each $T_2 \geq T_1 + \Delta$ and a trajectory-control pair

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$

satisfies (2.118). We show that

$$I^g(T_1, T_2, x, u) > U^g(T_1, T_2, x(T_1), x(T_2)) + S. \tag{2.125}$$

By (2.118), the choice of γ and property (iv) there exist

$$\tau_1 \in (T_1, T_1 + b_*] \text{ and } \tau_2 \in [T_2 - b_*, T_2) \tag{2.126}$$

and trajectory-control pairs

$$x_1 : [T_1, \tau_1] \to R^n, \ u_1 : [T_1, \tau_1] \to R^m,$$
$$x_2 : [\tau_2, T_2] \to R^n, \ u_2 : [\tau_2, T_2] \to R^m$$

such that

$$\begin{aligned}
x_i(T_i) &= x(T_i), \ i = 1, 2, \\
x_i(\tau_i) &= x_*(\tau_i), \ i = 1, 2, \\
|x_1(t) - x_*(t)| &\leq 1 \text{ for all } t \in [T_1, \tau_1], \\
|x_2(t) - x_*(t)| &\leq 1 \text{ for all } t \in [\tau_2, T_2], \\
I^f(T_1, \tau_1, x_1, u_1) &\leq I^f(T_1, \tau_1, x_*, u_*) + 1, \\
I^f(\tau_2, T_2, x_2, u_2) &\leq I^f(\tau_2, T_2, x_*, u_*) + 1.
\end{aligned} \tag{2.127}$$

Define a trajectory-control pair

$$x_3 : [T_1, T_2] \to R^n,\ u_3 : [T_1, T_2] \to R^m$$

by

$$\begin{aligned} x_3(t) &= x_1(t),\ u_3(t) = u_1(t) \text{ for all } t \in [T_1, \tau_1], \\ x_3(t) &= x_*(t),\ u_3(t) = u_*(t) \text{ for all } t \in (\tau_1, \tau_2], \\ x_3(t) &= x_2(t),\ u_3(t) = u_2(t) \text{ for all } t \in (\tau_2, T_2]. \end{aligned} \tag{2.128}$$

Relations (2.126)–(2.128) imply that

$$U^g(T_1, T_2, x(T_1), x(T_2)) \le I^g(T_1, T_2, x_3, u_3). \tag{2.129}$$

By (2.124), (2.127), (2.128), (2.126), (2.38), and (2.39),

$$\begin{aligned} I^{f_r}(T_1, T_1 + 1 + b_*, x_3, u_3) &\le I^f(T_1, T_1 + 1 + b_*, x_3, u_3) + b_* \\ &\le I^f(T_1, T_1 + b_* + 1, x_*, u_*) + b_* + 1 \end{aligned}$$

and

$$\begin{aligned} I^{f_r}(T_2 - 1 - b_*, T_2, x_3, u_3) &\le I^f(T_2 - 1 - b_*, T_2, x_3, u_3) + b_* \\ &\le I^f(T_2 - 1 - b_*, T_2, x_*, u_*) + b_* + 1. \end{aligned}$$

By these relations, (2.40) and the choice of $\mathcal{U}_1$ [see (2.119), (2.120)]

$$\begin{aligned} \max\{I^g(T_1, T_1 + 1 + b_*, x_3, u_3),\ I^g(T_2 - 1 - b_*, T_2, x_3, u_3)\} \\ \le 1 + \Lambda_0(4 + b_*) + 4a_0 + b_* + 1. \end{aligned} \tag{2.130}$$

It follows from (2.118), (2.124), and the choice of $\mathcal{U}$ and Δ_1 [see (2.122), (2.123)] that

$$I^g(T_1 + 1 + b_*, T_2 - 1 - b_*, x, u) > I^g(T_1 + 1 + b_*, T_2 - 1 - b_*, x_*, u_*) + S_1.$$

It follows from this relation, (2.128), (2.126), (2.130), (2.121), and assumption (A) that

$$\begin{aligned} & I^g(T_1, T_2, x, u) - I^g(T_1, T_2, x_3, u_3) \\ & > S_1 - 2[1 + \Lambda_0(4 + b_*) + 4a_0 + b_* + 1] - 2a_0(1 + b_*) > 2S + 1. \end{aligned}$$

Together with (2.129) this implies (2.125). This completes the proof of Lemma 2.18. □

Lemma 2.19. *Let $\epsilon \in (0, \min\{1, d_0\})$. Then there exist $\delta \in (0, \epsilon)$, $\Delta \geq 1$ and a neighborhood $\mathcal{U}$ of f_r in $\mathfrak{M}$ such that for each $g \in \mathcal{U}$, each $T_1 \in R^1$, each $T_2 \geq T_1 + \Delta$, and each trajectory-control pair*

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$

which satisfies

$$|x(T_i) - x_*(T_i)| \leq \delta, \ i = 1, 2,$$
$$I^g(T_1, T_2, x, u) \leq U^g(T_1, T_2, x(T_1), x(T_2)) + \delta \tag{2.131}$$

the following inequality holds:

$$|x(t) - x_*(t)| \leq \epsilon \textit{ for all } t \in [T_1, T_2].$$

Proof. There exist $\Delta_1 \geq 1$ and $\delta_0 \in (0, \epsilon)$ such that Lemma 2.16 holds with $\delta = \delta_0$.

Let a number $\gamma > 0$ be as guaranteed in Lemma 2.18. Choose a number

$$\delta \in (0, 8^{-1} \min\{\delta_0, \gamma, 1\}). \tag{2.132}$$

By Lemma 2.18 there exist $\Delta_2 \geq \Delta_1 + 1$ and a neighborhood $\mathcal{U}_1$ of f_r in $\mathfrak{M}$ such that for each $g \in \mathcal{U}_1$, each $T_1 \in R^1$, each $T_2 \geq T_1 + \Delta_2$, and each trajectory-control pair

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$

which satisfies

$$|x(T_i) - x_*(T_i)| \leq \gamma, \ i = 1, 2,$$
$$|x(t) - x_*(t)| > \delta \text{ for all } t \in [T_1, T_2] \tag{2.133}$$

the inequality

$$I^g(T_1, T_2, x, u) > U^g(T_1, T_2, x(T_1), x(T_2)) + 1 \tag{2.134}$$

holds.

Since Lemma 2.16 holds with $\delta = \delta_0$ and Δ_1 there exists a neighborhood $\mathcal{U}$ of f_r in $\mathfrak{M}$ such that

$$\mathcal{U} \subset \mathcal{U}_1$$

and for each $g \in \mathcal{U}$, each $T_1 \in R^1$, each

$$T_2 \in [T_1 + \Delta_1, T_1 + 2\Delta_1 + 2\Delta_2 + 8]$$

and each trajectory-control pair

$$x : [T_1, T_2] \to R^n, \; u : [T_1, T_2] \to R^m$$

which satisfies

$$|x(T_i) - x_*(T_i)| \le \delta_0, \; i = 1, 2,$$
$$I^g(T_1, T_2, x, u) \le U^g(T_1, T_2, x(T_1), x(T_2)) + \delta_0 \tag{2.135}$$

the following inequality holds:

$$|x(t) - x_*(t)| \le \epsilon \text{ for all } t \in [T_1, T_2]. \tag{2.136}$$

Choose a number

$$\Delta > 8\Delta_2 + 8\Delta_1 + 8. \tag{2.137}$$

Assume that

$$g \in \mathcal{U}, \; T_1 \in R^1, \; T_2 \ge T_1 + \Delta$$

and that a trajectory-control pair

$$x : [T_1, T_2] \to R^n, \; u : [T_1, T_2] \to R^m$$

satisfies (2.131). Let

$$t_1 \in [T_1, T_2 - \Delta_1 - \Delta_2 - 1]. \tag{2.138}$$

We show that there exists a number t_2 such that

$$t_2 \in [t_1 + \Delta_1, t_1 + \Delta_1 + \Delta_2],$$
$$|x(t_2) - x_*(t_2)| \le \delta. \tag{2.139}$$

Assume the contrary and set

$$\tilde{t}_1 = \sup\{t \in [T_1, t_1 + \Delta_1] : \; |x(t) - x_*(t)| \le \delta\},$$
$$\tilde{t}_2 = \inf\{t \in [t_1 + \Delta_1, T_2] : \; |x(t) - x_*(t)| \le \delta\}. \tag{2.140}$$

Then

$$\tilde{t}_1 < t_1 + \Delta_1,\ \tilde{t}_2 > t_1 + \Delta_1 + \Delta_2,$$
$$|x(\tilde{t}_i) - x_*(\tilde{t}_i)| = \delta,\ i = 1, 2.$$

There exist

$$b_1 \in (\tilde{t}_1, t_1 + \Delta_1) \text{ and } b_2 \in (t_1 + \Delta_1 + \Delta_2, \tilde{t}_2) \tag{2.141}$$

such that

$$|x(b_i) - x_*(b_i)| < \gamma,\ i = 1, 2. \tag{2.142}$$

It is easy to see that

$$|x(t) - x_*(t)| > \delta \text{ for all } t \in [b_1, b_2]. \tag{2.143}$$

By (2.141)–(2.143) and the choice of $\mathcal{U}_1$, Δ_2 [see (2.133), (2.134)],

$$I^g(b_1, b_2, x, u) > U^g(b_1, b_2, x(b_1), x(b_2)) + 1.$$

This contradicts (2.131). The contradiction we have reached proves that for each number t_1 satisfying (2.138) there exists a number t_2 which satisfies (2.139). Therefore there exists a sequence of numbers $\{t_j\}_{j=1}^{Q}$ such that

$$t_1 = T_1,\ t_Q = T_2,$$
$$t_{j+1} - t_j \in [\Delta_1, 2\Delta_1 + 2\Delta_2 + 4],\ j = 1, \dots, Q - 1,$$
$$|x(t_j) - x_*(t_j)| \le \delta,\ j = 1, \dots, Q. \tag{2.144}$$

In view of (2.131),

$$I^g(t_j, t_{j+1}, x, u) \le U^g(t_j, t_{j+1}, x(t_j), x(t_{j+1})) + 2\delta.$$

It follows from this inequality, (2.132), (2.144) and the choice of $\mathcal{U}$ [see (2.133), (2.136)] that for all $j = 1, \dots, Q - 1$,

$$|x(t) - x_*(t)| \le \epsilon \text{ for all } t \in [t_j, t_{j+1}].$$

This completes the proof of Lemma 2.19. □

Lemma 2.20. *Let $S > 0$ and $\epsilon \in (0, 1)$. Then there exist $\Delta \ge 1$, $\delta \in (0, \epsilon)$ and a neighborhood $\mathcal{U}$ of f_r in $\mathfrak{M}$ such that for each $g \in \mathcal{U}$, each $T_1 \in R^1$, each $T_2 \ge T_1 + 2\Delta$, and each trajectory-control pair*

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

which satisfies

$$I^g(T_1, T_2, x, u) \le \inf\{\sigma^g(T_1, T_2, y) : \ (T_1, y) \in A\} + S,$$
$$I^g(T_1, T_2, x, u) \le U^g(T_1, T_2, x(T_1), x(T_2)) + \delta \tag{2.145}$$

the following inequality holds:

$$|x(t) - x_*(t)| \le \epsilon \textit{ for all } t \in [T_1 + \Delta, T_2 - \Delta]. \tag{2.146}$$

Moreover, if

$$|x(T_1) - x_*(T_1)| \le \delta,$$

then

$$|x(t) - x_*(t)| \le \epsilon \textit{ for all } t \in [T_1, T_2 - \Delta]$$

and if

$$|x(T_2) - x_*(T_2)| \le \delta,$$

then

$$|x(t) - x_*(t)| \le \epsilon \textit{ for all } t \in [T_1 + \Delta, T_2].$$

Proof. By property (ii) (see the definition of f), assumption (A) and Proposition 2.9, there exists a neighborhood $\mathcal{U}_0$ of f_r in $\mathfrak{M}$ such that for each $g \in \mathcal{U}_0$, each $T_1 \in R^1$, each $T_2 \ge T_1 + 8^{-1}$,

$$\inf\{\sigma^g(T_1, T_2, y) : \ (T_1, y) \in A\} < \infty. \tag{2.147}$$

There exists $\delta_1 > 0$ such that property (iv) (see the definition of f) holds with $\epsilon = 1$, $\delta = \delta_1$. By Proposition 2.9 there exists a neighborhood $\mathcal{U}_1$ of f_r in $\mathfrak{M}$ such that

$$\mathcal{U}_1 \subset \mathcal{U}_0$$

and for each $g \in \mathcal{U}_1$, each $T_1 \in R^1$, each $T_2 \in [T_1 + 4^{-1}b_*, T_1 + 4b_*]$, and each trajectory-control pair

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$

which satisfies

$$\min\{I^{f_r}(T_1,T_2,x,u),\ I^g(T_1,T_2,x,u)\}$$
$$\le 4+4b_*+4\Lambda_0(b_*+1)+4a_0 \tag{2.148}$$

the inequality

$$|I^{f_r}(T_1,T_2,x,u)-I^g(T_1,T_2,x,u)|\le 1 \tag{2.149}$$

holds.

By Lemma 2.19 there exist

$$\delta\in(0,\min\{1,\ 2^{-1}\delta_1,\ d_0,\ \epsilon\}),\ \Delta_0\ge 1 \tag{2.150}$$

and a neighborhood $\mathcal{U}_2$ of f_r in $\mathfrak{M}$ such that

$$\mathcal{U}_2\subset\mathcal{U}_1$$

and for each $g\in\mathcal{U}_2$, each $T_1\in R^1$, each $T_2\ge T_1+\Delta_0$ and each trajectory-control pair

$$x:[T_1,T_2]\to R^n,\ u:[T_1,T_2]\to R^m$$

which satisfies

$$|x(T_i)-x_*(T_i)|\le\delta,\ i=1,2,$$
$$I^g(T_1,T_2,x,u)\le U^g(T_1,T_2,x(T_1),x(T_2))+\delta \tag{2.151}$$

the following inequality holds:

$$|x(t)-x_*(t)|\le\epsilon \text{ for all } t\in[T_1,T_2]. \tag{2.152}$$

By Lemma 2.17, there exist $\Delta_1\ge 1$ and a neighborhood $\mathcal{U}$ of f_r in $\mathfrak{M}$ such that

$$\mathcal{U}\subset\mathcal{U}_2$$

and for each $g\in\mathcal{U}$, each $T_1\in R^1$, each $T_2\ge T_1+\Delta_1$, and each trajectory-control pair

$$x:[T_1,T_2]\to R^n,\ u:[T_1,T_2]\to R^m$$

which satisfies

$$|x(t)-x_*(t)|\ge\delta \text{ for all } t\in[T_1,T_2] \tag{2.153}$$

the following inequality holds:

$$I^g(T_1, T_2, x, u) > I^g(T_1, T_2, x_*, u_*)$$
$$+S + 10 + (b_* + 4)(4 + 4\Lambda_0 + 4a_0) + 4a_0 + 4.$$

Fix a number

$$\Delta > 128(b_* + 1 + \Delta_0 + \Delta_1). \tag{2.154}$$

Assume that

$$g \in \mathcal{U},\ T_1 \in R^1,\ T_2 \geq T_1 + 2\Delta$$

and that a trajectory-control pair

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

satisfies (2.145).

By the definition of Δ_1, $\mathcal{U}$ [see (2.153)], (2.145) and (2.154) there exists $\tilde{t} \in [T_1, T_2]$ which satisfies

$$|x(\tilde{t}) - x_*(\tilde{t})| \leq \delta.$$

Set

$$t_1 = \inf\{t \in [T_1, T_2] :\ |x(t) - x_*(t)| \leq \delta\},$$
$$t_2 = \sup\{t \in [T_1, T_2] :\ |x(t) - x_*(t)| \leq \delta\}. \tag{2.155}$$

It follows from (2.155), property (iv), and the choice of δ_1 that there exist

$$\tau_2 \in (t_2, t_2 + b_*] \text{ and } \tau_1 \in [t_1 - b_*, t_1), \tag{2.156}$$

and trajectory-control pairs

$$x_1 : [\tau_1, t_1] \to R^n,\ u_1 : [\tau_1, t_1] \to R^m,$$
$$x_2 : [t_2, \tau_2] \to R^n,\ u_2 : [t_2, \tau_2] \to R^m$$

such that

$$x_i(t_i) = x(t_i),\ i = 1, 2,$$
$$x_i(\tau_i) = x_*(\tau_i),\ i = 1, 2,$$
$$|x_1(t) - x_*(t)| \leq 1 \text{ for all } t \in [\tau_1, t_1],$$

$$|x_2(t) - x_*(t)| \le 1 \text{ for all } t \in [t_2, \tau_2],$$
$$I^f(\tau_1, t_1, x_1, u_1) \le I^f(\tau_1, t_1, x_*, u_*) + 1,$$
$$I^f(t_2, \tau_2, x_2, u_2) \le I^f(t_2, \tau_2, x_*, u_*) + 1. \tag{2.157}$$

We may assume without loss of generality that

$$\tau_2 = t_2 + b_*, \ \tau_1 = t_1 - b_*. \tag{2.158}$$

By (2.158), (2.157), (2.38), and (2.39),

$$I^{f_r}(\tau_1, t_1, x_1, u_1) \le I^f(\tau_1, t_1, x_1, u_1) + b_* \le I^f(\tau_1, t_1, x_*, u_*) + b_* + 1,$$
$$I^{f_r}(t_2, \tau_2, x_2, u_2) \le I^f(t_2, \tau_2, x_2, u_2) + b_* \le I^f(t_2, \tau_2, x_*, u_*) + b_* + 1. \tag{2.159}$$

By these relations, (2.158) and the choice of $\mathcal{U}_1$ [see (2.148), (2.149)],

$$|I^g(\tau_1, t_1, y, v) - I^{f_r}(\tau_1, t_1, y, v)| \le 1, \ (y, v) \in \{(x_1, u_1), \ (x_*, u_*)\},$$
$$|I^g(t_2, \tau_2, y, v) - I^{f_r}(t_2, \tau_2, y, v)| \le 1, \ (y, v) \in \{(x_2, u_2), \ (x_*, u_*)\}.$$

Together with (2.159) these relations imply that

$$I^g(\tau_1, t_1, x_1, u_1) \le I^g(\tau_1, t_1, x_*, u_*) + b_* + 3,$$
$$I^g(t_2, \tau_2, x_2, u_2) \le I^g(t_2, \tau_2, x_*, u_*) + b_* + 3. \tag{2.160}$$

We show that

$$t_1 - T_1 \le 8^{-1}\Delta.$$

Assume the contrary. Then

$$t_1 - T_1 > 8^{-1}\Delta. \tag{2.161}$$

Define a trajectory-control pair

$$x_3 : [T_1, T_2] \to R^n, \ u_3 : [T_1, T_2] \to R^m$$

by

$$x_3(t) = x_*(t), \ u_3(t) = u_*(t) \text{ for all } t \in [T_1, \tau_1],$$
$$x_3(t) = x_1(t), \ u_3(t) = u_1(t) \text{ for all } t \in (\tau_1, t_1],$$
$$x_3(t) = x(t), \ u_3(t) = u(t) \text{ for all } t \in (t_1, T_2]. \tag{2.162}$$

Relations (2.161), (2.158), (2.154) and (2.155) imply that

$$\tau_1 - T_1 \geq 2\Delta_0 + 2\Delta_1 + 2,$$
$$|x(t) - x_*(t)| \geq \delta \text{ for all } t \in [T_1, \tau_1].$$

By these relations and the choice of $\mathcal{U}$ and Δ_1 [see (2.153)]

$$I^g(T_1, \tau_1, x, u) > I^g(T_1, \tau_1, x_*, u_*)$$
$$+S + 10 + (b_* + 4)(4 + 4\Lambda_0 + 4a_0) + 4a_0 + 4.$$

It follows from this inequality, (2.162), (2.160), (2.158), and assumption (A) that

$$I^g(T_1, T_2, x, u) - I^g(T_1, T_2, x_3, u_3) > S + 10.$$

This contradicts (2.145). The contradiction we have reached proves that

$$t_1 - T_1 \leq 8^{-1}\Delta. \tag{2.163}$$

Analogously we can show that

$$T_2 - t_2 \leq 8^{-1}\Delta. \tag{2.164}$$

It follows from the choice of $\mathcal{U}_2$ and δ [see (2.150)–(2.152)], (2.163), (2.164), (2.154), (2.145), and (2.155) that

$$|x(t) - x_*(t)| \leq \epsilon \text{ for all } t \in [t_1, t_2].$$

This completes the proof of Lemma 2.20. □

Proposition 2.9, property (ii) (see the definition of f), and assumption (A) imply the following result.

Lemma 2.21. *There exists a neighborhood $\mathcal{U}$ of f_r in $\mathfrak{M}$ such that for each $g \in \mathcal{U}$, each $T_1 \in R^1$, each $T_2 > T_1$,*

$$\inf\{U^g(T_1, T_2, y_1, y_2) : \ (T_i, y_i) \in A, \ i = 1, 2\} \leq I^g(T_1, T_2, x_*, u_*) < \infty. \tag{2.165}$$

Lemma 2.22. *Let $S > 0$. Then there exist a neighborhood $\mathcal{U}$ of f_r in $\mathfrak{M}$ and numbers $\delta, Q > 0$ such that for each $g \in \mathcal{U}$, each $T_1 \in R^1$, each $T_2 \geq T_1 + 1$ and each trajectory-control pair*

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$

which satisfies

$$I^g(T_1, T_2, x, u) \le \inf\{U^g(T_1, T_2, y_1, y_2) : \ (T_i, y_i) \in A, \ i = 1, 2\} + S,$$
$$I^g(T_1, T_2, x, u) \le U^g(T_1, T_2, x(T_1), x(T_2)) + \delta \tag{2.166}$$

the following inequality holds:

$$|x(t)| \le Q \textit{ for all } t \in [T_1, T_2]. \tag{2.167}$$

Proof. By Lemma 2.21 there exists a neighborhood $\mathcal{U}_0$ of f_r in $\mathfrak{M}$ such that (2.165) holds for each $g \in \mathcal{U}_0$, each $T_1 \in R^1$, and each $T_2 > T_1$. There exists a number

$$\delta_0 \in (0, 8^{-1} \min\{1, d_0\}) \tag{2.168}$$

such that property (iv) (see the definition of f) holds with $\epsilon = 1$, $\delta = \delta_0$. Fix

$$S_1 > S + 1. \tag{2.169}$$

There exist

$$\Delta_0 \ge 16 + 8b_*, \ \delta_1 \in (0, \delta_0)$$

and a neighborhood $\mathcal{U}_1$ of f_r in $\mathcal{M}$ such that Lemma 2.20 holds with

$$S = 4S_1 + 8, \ \epsilon = \delta_0, \ \Delta = \Delta_0,$$
$$\delta = \delta_1, \ \mathcal{U} = \mathcal{U}_1. \tag{2.170}$$

By Proposition 2.9 there exists a neighborhood $\mathcal{U}$ of f_r in $\mathfrak{M}$ such that

$$\mathcal{U} \subset \mathcal{U}_1 \cap \mathcal{U}_2$$

and for each $g \in \mathcal{U}$, each $T_1 \in R^1$, each

$$T_2 \in [T_1 + 8^{-1} \min\{1, b_*\}, \ T_1 + 2\Delta_0 + 8]$$

and each trajectory-control pair

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$

which satisfies

$$\min\{I^g(T_1, T_2, x, u), \ I^{f_r}(T_1, T_2, x, u)\}$$
$$\le 4 + 4b_* + (4\Lambda_0 + 4 + 4a_0)(3\Delta_0 + 4 + 4b_*) \tag{2.171}$$

the inequality

$$|I^g(T_1,T_2,x,u) - I^{f_r}(T_1,T_2,x,u)| \le 1 \tag{2.172}$$

holds.

By Proposition 2.7 there exists

$$D_1 > 100 + \sup\{|x_*(t)| : \ t \in R^1\} \tag{2.173}$$

such that for each $g \in \mathfrak{M}$, each $T_1 \in R^1$, each

$$T_2 \in [T_1 + 8^{-1}, T_1 + 100(\Delta_0 + 1)]$$

and each trajectory-control pair

$$x : [T_1,T_2] \to R^n, \ u : [T_1,T_2] \to R^m$$

which satisfies

$$I^g(T_1,T_2,x,u) \le S + 2b_* + (\Lambda_0 + 4)(8\Delta_0 + 8) + 4a_0 \tag{2.174}$$

the following relation holds:

$$|x(t)| \le D_1 \text{ for all } t \in [T_1,T_2]. \tag{2.175}$$

Choose numbers

$$\delta \in (0, 4^{-1}\delta_1), \ Q > 4D_1 + 4. \tag{2.176}$$

Assume that

$$g \in \mathcal{U}, \ T_1 \in R^1, \ T_2 \ge T_1 + 1$$

and that a trajectory-control pair

$$x : [T_1,T_2] \to R^n, \ u : [T_1,T_2] \to R^m$$

satisfies (2.166). We show that (2.167) holds. There are two cases:

$$(1)\ T_2 - T_1 \ge 4\Delta_0 + 4;$$

$$(2)\ T_2 - T_1 < 4\Delta_0 + 4.$$

Consider the case (1). It follows from Lemma 2.20 and the definition of $\Delta_0 \ge 1$, $\delta_1 \in (0, \delta_0)$ [see (2.170)], (2.166), (2.176), and (2.169) that

$$|x(t) - x_*(t)| \le \delta_0 \text{ for all } t \in [T_1 + \Delta_0, T_2 - \Delta_0]. \tag{2.177}$$

By property (iv) and the choice of δ_0 there exist

$$\tau_1 \in [T_1 + \Delta_0 - b_*, T_1 + \Delta_0) \text{ and } \tau_2 \in (T_2 - \Delta_0, T_2 - \Delta_0 + b_*] \tag{2.178}$$

and trajectory-control pairs

$$x_1 : [\tau_1, T_1 + \Delta_0] \to R^n,\ u_1 : [\tau_1, T_1 + \Delta_0] \to R^m,$$
$$x_2 : [T_2 - \Delta_0, \tau_2] \to R^n,\ u_2 : [T_2 - \Delta_0, \tau_2] \to R^m$$

such that

$$\begin{gathered} x_i(\tau_i) = x_*(\tau_i),\ i = 1, 2, \\ x_1(T_1 + \Delta_0) = x(T_1 + \Delta_0), \\ x_2(T_2 - \Delta_0) = x(T_2 - \Delta_0), \\ I^f(\tau_1, T_1 + \Delta_0, x_1, u_1) \le I^f(\tau_1, T_1 + \Delta_0, x_*, u_*) + 1, \\ I^f(T_2 - \Delta_0, \tau_2, x_2, u_2) \le I^f(T_2 - \Delta_0, \tau_2, x_*, u_*) + 1, \\ |x_1(t) - x_*(t)| \le 1 \text{ for all } t \in [\tau_1, T_1 + \Delta_0], \\ |x_2(t) - x_*(t)| \le 1 \text{ for all } t \in [T_2 - \Delta_0, \tau_2]. \end{gathered} \tag{2.179}$$

We may assume that

$$\tau_1 = T_1 + \Delta_0 - b_*,\ \tau_2 = T_2 - \Delta_0 + b_*. \tag{2.180}$$

Define a trajectory-control pair

$$x_3 : [T_1, T_2] \to R^n,\ u_3 : [T_1, T_2] \to R^m$$

and

$$x_4 : [T_1, T_2] \to R^n,\ u_4 : [T_1, T_2] \to R^m$$

by

$$\begin{gathered} x_3(t) = x_*(t),\ u_3(t) = u_*(t) \text{ for all } t \in [T_1, \tau_1], \\ x_3(t) = x_1(t),\ u_3(t) = u_1(t) \text{ for all } t \in (\tau_1, T_1 + \Delta_0], \\ x_3(t) = x(t),\ u_3(t) = u(t) \text{ for all } t \in (T_1 + \Delta_0, T_2], \\ x_4(t) = x(t),\ u_4(t) = u(t) \text{ for all } t \in [T_1, T_2 - \Delta_0], \\ x_4(t) = x_2(t),\ u_4(t) = u_2(t) \text{ for all } t \in (T_2 - \Delta_0, \tau_2], \\ x_4(t) = x_*(t),\ u_4(t) = u_*(t) \text{ for all } t \in (\tau_2, T_2]. \end{gathered} \tag{2.181}$$

Relations (2.180), (2.179), and (2.181) imply that

$$\begin{aligned} I^{f_r}(T_1, T_1 + \Delta_0, x_3, u_3) &\le I^f(T_1, T_1 + \Delta_0, x_3, u_3) + b_* \\ &\le I^f(T_1, T_1 + \Delta_0, x_*, u_*) + b_* + 1, \\ I^{f_r}(T_2 - \Delta_0, T_2, x_4, u_4) &\le I^f(T_2 - \Delta_0, , T_2, x_4, u_4) + b_* \\ &\le I^f(T_2 - \Delta_0, T_2, x_*, u_*) + b_* + 1. \end{aligned} \tag{2.182}$$

By these relations, the choice of $\mathcal{U}$ [see (2.171), (2.172)], (2.180),

$$\begin{gathered} |I^{f_r}(T_1, T_1 + \Delta_0, y, v) - I^g(T_1, T_1 + \Delta_0, y, v)| \le 1, \\ (y, v) \in \{(x_3, u_3),\ (x_*, u_*)\}, \\ |I^{f_r}(T_2 - \Delta_0, T_2, y, v) - I^g(T_2 - \Delta_0, T_2, y, v)| \le 1, \\ (y, v) \in \{(x_4, u_4),\ (x_*, u_*)\}. \end{gathered}$$

Together with (2.182) this implies that

$$\begin{aligned} I^g(T_1, T_1 + \Delta_0, x_3, u_3) &\le I^f(T_1, T_1 + \Delta_0, x_*, u_*) + b_* + 3, \\ I^g(T_2 - \Delta_0, T_2, x_4, u_4) &\le I^f(T_2 - \Delta_0, T_2, x_*, u_*) + b_* + 3. \end{aligned} \tag{2.183}$$

It follows from (2.166), (2.181) and (2.183) that

$$\begin{aligned} S &\ge I^g(T_1, T_2, x, u) - I^g(T_1, T_2, x_3, u_3) \\ &= I^g(T_1, T_1 + \Delta_0, x, u) - I^g(T_1, T_1 + \Delta_0, x_3, u_3) \\ &\ge I^g(T_1, T_1 + \Delta_0, x, u) - b_* - 3 - \Lambda_0(\Delta_0 + 2) - 2a_0 \end{aligned}$$

and

$$I^g(T_1, T_1 + \Delta_0, x, u) \le S + b_* + 3 + \Lambda_0(\Lambda_0 + 2) + 2a_0. \tag{2.184}$$

Analogously,

$$I^g(T_2 - \Delta_0, T_2, x, u) \le S + b_* + 3 + \Lambda_0(\Delta_0 + 2) + 2a_0. \tag{2.185}$$

By (2.184), (2.185) and the choice of D_1 [see (2.173)–(2.175)]

$$|x(t)| \le D_1 \text{ for all } t \in [T_1, T_1 + \Delta_0] \cup [T_2 - \Delta_0, T_2].$$

Together with (2.177), (2.173), (2.168), and (2.176) this implies (2.167).

Consider the case (2). By (2.166), the choice of $\mathcal{U}$ [see (2.171), (2.172)], and assumption (A),

$$I^g(i, i+1, x_*, u_*) \le \Lambda_0 + 1,\ i = 0, 1, \dots$$

and

$$I^g(T_1, T_2, x, u) \le I^g(T_1, T_2, x_*, u_*) + S$$
$$\le S + 2a_0 + (\Lambda_0 + 1)(4\Delta_0 + 6).$$

Relation (2.167) follows from the relation above, the choice of D_1, (2.175) and (2.176). This completes the proof of Lemma 2.22. □

Lemma 2.23. *Let $\epsilon_0, S > 0$. Then there exist $\Delta > 0$ and a neighborhood $\mathcal{U}$ of f_r in $\mathfrak{M}$ such that for each $g \in \mathcal{U}$, each $T_1 \in R^1$, each $T_2 \ge T_1 + \Delta$, and each trajectory-control pair*

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

which satisfies

$$I^g(T_1, T_2, x, u) \le \inf\{\sigma^g(T_1, T_2, y) :\ (T_1, y) \in A\} + S, \tag{2.186}$$

each $\tau \in [T_1, T_2 - \Delta]$,

$$\min\{|x(t) - x_*(t)| :\ t \in (\tau, \tau + \Delta)\} < \epsilon_0. \tag{2.187}$$

Proof. We may assume without loss of generality that

$$\epsilon_0 < 2^{-1} \min\{1, d_0, b_*\}. \tag{2.188}$$

We may also assume without loss of generality that property (iv) holds with $\epsilon = 1$, $\delta = \epsilon_0$.

By property (ii) and assumption (A) there is $\tilde{\Lambda} > 0$ such that

$$I^f(T, T+b, x_*, u_*) \le \tilde{\Lambda} \text{ for all } T \in R^1$$
$$\text{and all } b \in [2^{-1}\min\{b_*, 1\},\ 2b_* + 2]. \tag{2.189}$$

By Proposition 2.9 there exists a neighborhood $\mathcal{U}_1$ of f_r in $\mathfrak{M}$ such that for each $g \in \mathcal{U}_1$, each pair of numbers

$$S_1 \in R^1,\ S_2 \in [S_1 + 2^{-1}\min\{b_*, 1\},\ S_1 + 2b_* + 2]$$

and each trajectory-control pair

$$x : [S_1, S_2] \to R^n,\ u : [S_1, S_2] \to R^m$$

which satisfies

$$\min\{I^g(S_1, S_2, x, u),\ I^{f_r}(S_1, S_2, x, u)\}$$
$$\leq \tilde{2}\Lambda + 2 + 2b_* \tag{2.190}$$

the inequality

$$|I^g(S_1, S_2, x, u) - I^{f_r}(S_1, S_2, x, u)| \leq 1 \tag{2.191}$$

holds.

By (2.188) and Lemma 2.17 there exist $\Delta_1 \geq 1$ and a neighborhood $\mathcal{U}$ of f_r in $\mathfrak{M}$ such that

$$\mathcal{U} \subset \mathcal{U}_1$$

and for each $g \in \mathcal{U}$, each $S_1 \in R^1$, each $S_2 \geq S_1 + \Delta_1$, and each trajectory-control pair

$$y : [S_1, S_2] \to R^n,\ v : [S_1, S_2] \to R^m$$

which satisfies

$$|y(t) - x_*(t)| \geq \epsilon_0 \text{ for all } t \in [S_1, S_2] \tag{2.192}$$

the following inequality holds:

$$I^g(S_1, S_2, y, v) > I^g(S_1, S_2, x_*, u_*)$$
$$+S + 2\tilde{\Lambda} + 8 + 2b_*(1 + a_0). \tag{2.193}$$

Choose a number

$$\Delta \geq \Delta_1 + 2b_* + 2. \tag{2.194}$$

Assume that

$$g \in \mathcal{U},\ T_1 \in R^1,\ T_2 \geq T_1 + \Delta,\ \tau \in [T_1, T_2 - \Delta], \tag{2.195}$$

a trajectory-control pair

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

satisfies (2.186) and

$$\tau \in [T_1, T_2 - \Delta].$$

By (2.195), the choice of $\mathcal{U}_1$ and (2.189),

$$I^g(T_1, T_2, x, u) < \infty.$$

In order to complete the proof of the lemma it is sufficient to show that (2.187) holds.

Assume the contrary. Then

$$|x(t) - x_*(t)| \geq \epsilon_0 \text{ for all } t \in (\tau, \tau + \Delta). \tag{2.196}$$

It is not difficult to see that there exist real numbers τ_1, τ_2 such that

$$T_1 \leq \tau_1 < \tau_2 \leq T_2,$$

$$[\tau, \tau + \Delta] \subset [\tau_1, \tau_2], \tag{2.197}$$

$$|x(t) - x_*(t)| \geq \epsilon_0 \text{ for all } t \in [\tau_1, \tau_2] \tag{2.198}$$

and one of the following cases holds:

$$|x(\tau_i) - x_*(\tau_i)| = \epsilon_0, \;\; i = 1, 2; \tag{2.199}$$

$$|x(\tau_1) - x_*(\tau_1)| = \epsilon_0, \;\; \tau_2 = T_2; \tag{2.200}$$

$$|x(\tau_2) - x_*(\tau_2)| = \epsilon_0, \;\; \tau_1 = T_1; \tag{2.201}$$

$$\tau_i = T_i, \;\; i = 1, 2. \tag{2.202}$$

Define $\xi_1, \xi_2 \in R^n$ as follows:

$$\text{if (2.199) holds, then } \xi_i = x(\tau_i), \;\; 1, 2; \tag{2.203}$$

$$\text{if (2.200) holds, then } \xi_1 = x(\tau_1), \;\; \xi_2 = x_*(T_2); \tag{2.204}$$

$$\text{if (2.201) holds, then } \xi_2 = x(\tau_2), \;\; \xi_1 = x_*(T_1); \tag{2.205}$$

$$\text{if (2.202) holds, then } \xi_i = x_*(\tau_i), \;\; i = 1, 2. \tag{2.206}$$

We will define a trajectory-control pair

$$y : [\tau_1, \tau_2] \to R^n, \;\; v : [\tau_1, \tau_2] \to R^m.$$

Since property (iv) holds with $\epsilon = 1$ and $\delta = \epsilon_0$ it follows from (2.199)–(2.205) that there exist trajectory-control pairs

$$y_1 : [\tau_1, \tau_1 + b_*] \to R^n, \;\; v_1 : [\tau_1, \tau_1 + b_*] \to R^m$$

and

$$y_2 : [\tau_2 - b_*, \tau_2] \to R^n,\ v_2 : [\tau_2 - b_*, \tau_2] \to R^m$$

such that

$$y_1(\tau_1) = \xi_1,$$
$$y_1(\tau_1 + b_*) = x_*(\tau_1 + b_*), \tag{2.207}$$
$$|y_1(t) - x_*(t)| \le 1 \text{ for all } t \in [\tau_1, \tau_1 + b_*], \tag{2.208}$$
$$I^f(\tau_1, \tau_1 + b_*, y_1, v_1) \le I^f(\tau_1, \tau_1 + b_*, x_*, u_*) + 1, \tag{2.209}$$
$$y_2(\tau_2 - b_*) = x_*(\tau_2 - b_*),$$
$$y_2(\tau_2) = \xi_2, \tag{2.210}$$
$$|y_2(t) - x_*(t)| \le 1 \text{ for all } t \in [\tau_2 - b_*, \tau_2], \tag{2.211}$$
$$I^f(\tau_2 - b_*, \tau_2, y_2, v_2) \le I^f(\tau_2 - b_*, \tau_2, x_*, u_*) + 1. \tag{2.212}$$

By (2.208), (2.38), (2.39), (2.209), (2.211), and (2.212)

$$\begin{aligned} I^{f_r}(\tau_1, \tau_1 + b_*, y_1, v_1) &\le I^f(\tau_1, \tau_1 + b_*, y_1, v_1) + b_* \\ &\le I^f(\tau_1, \tau_1 + b_*, x_*, u_*) + 1 + b_* \end{aligned} \tag{2.213}$$

and

$$\begin{aligned} I^{f_r}(\tau_2 - b_*, \tau_2, y_2, v_2) &\le I^f(\tau_2 - b_*, \tau_2, y_2, v_2) + b_* \\ &\le I^f(\tau_2 - b_*, \tau_2, x_*, u_*) + 1 + b_*. \end{aligned} \tag{2.214}$$

By (2.197), (2.207), and (2.210) there exists a trajectory-control pair

$$y : [\tau_1, \tau_2] \to R^n,\ v : [\tau_1, \tau_2] \to R^m$$

such that

$$y(t) = y_1(t),\ v(t) = v_1(t) \text{ for all } t \in [\tau_1, \tau_1 + b_*],$$
$$y(t) = x_*(t),\ v(t) = u_*(t) \text{ for all } t \in (\tau_1 + b_*, \tau_2 - b_*),$$
$$y(t) = y_2(t),\ v(t) = v_2(t) \text{ for all } t \in [\tau_2 - b_*, \tau_2]. \tag{2.215}$$

By (2.215), (2.189), (2.195), and the choice of $\mathcal{U}_1$ [see (2.190), (2.191)],

$$I^g(\tau_1 + b_*, \tau_2 - b_*, y, v)$$

is finite.

By (2.215), (2.213), (2.214), and (2.189),

$$I^{f_r}(\tau_1, \tau_1 + b_*, y, v) \le I^f(\tau_1, \tau_1 + b_*, x_*, u_*) + 1 + b_* \le \tilde{\Lambda} + 1 + b_*,$$
$$I^{f_r}(\tau_2 - b_*, \tau_2, y, v) \le I^f(\tau_2 - b_*, \tau_2, x_*, u_*) + 1 + b_* \le \tilde{\Lambda} + 1 + b_*.$$

Together with (2.215) and the choice of $\mathcal{U}_1$ [see (2.190), (2.191)] this implies that

$$I^g(\tau_1, \tau_1 + b_*, y, v) \le \tilde{\Lambda} + 2 + b_*,$$
$$I^g(\tau_2 - b_*, \tau_2, y, v) \le \tilde{\Lambda} + 2 + b_*. \tag{2.216}$$

By the construction of (y, v) [see (2.215)], (2.207) and (2.210),

$$y(\tau_1) = \xi_1,\ y(\tau_2) = \xi_2. \tag{2.217}$$

By (2.186), (2.217) and the choice of ξ_1, ξ_2 [see (2.199)–(2.206)],

$$S \ge I^g(\tau_1, \tau_2, x, u) - I^g(\tau_1, \tau_2, y, v). \tag{2.218}$$

In view of (2.218), (2.215), (2.216), and (A),

$$\begin{aligned} S \ge\ & I^g(\tau_1 + b_*, \tau_2 - b_*, x, u) - I^g(\tau_1 + b_*, \tau_2 - b_*, x_*, u_*) \\ & + I^g(\tau_1, \tau_1 + b_*, x, u) + I^g(\tau_2 - b_*, \tau_2, x, u) \\ & - I^g(\tau_1, \tau_1 + b_*, y, v) - I^g(\tau_2 - b_*, \tau_2, y, v) \\ \ge\ & I^g(\tau_1 + b_*, \tau_2 - b_*, x, u) - I^g(\tau_1 + b_*, \tau_2 - b_*, x_*, u_*) \\ & - 2a_0 b_* - 2\tilde{\Lambda} - 4 - 2b_* \end{aligned}$$

and

$$\begin{aligned} & I^g(\tau_1 + b_*, \tau_2 - b_*, x, u) - I^g(\tau_1 + b_*, \tau_2 - b_*, x_*, u_*) \\ & \le S + 2b_*(1 + a_0) + 4 + 2\tilde{\Lambda}. \end{aligned} \tag{2.219}$$

On the other hand it follows from (2.195), (2.194), (2.197), (2.198) and the choice of $\mathcal{U}$ [see (2.192), (2.193)] that

$$\begin{aligned} I^g(\tau_1 + b_*, \tau_2 - b_*, x, u) >\ & I^g(\tau_1 + b_*, \tau_2 - b_*, x_*, u_*) \\ & + S + 2b_*(1 + a_0) + 8 + 2\tilde{\Lambda}. \end{aligned}$$

This contradicts (2.219). The contradiction we have reached completes the proof of Lemma 2.23. □

We suppose that the sum over empty set is zero.

Lemma 2.24. *Let $S, \epsilon > 0$. Then there exist real numbers $l > 0$, $\Delta > 0$, a natural number Q, and a neighborhood $\mathcal{U}$ of f_r in $\mathfrak{M}$ such that for each $g \in \mathcal{U}$, each $T_1 \in R^1$, each $T_2 \geq T_1 + \Delta$, and each trajectory-control pair*

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

which satisfies

$$I^g(T_1, T_2, x, u) \leq \inf\{\sigma^g(T_1, T_2, y) :\ (T_1, y) \in A\} + S \tag{2.220}$$

there exist finite sequences

$$\{a_i\}_{i=1}^q,\ \{b_i\}_{i=1}^q \subset [T_1, T_2],$$

where $q \leq Q$ is a natural number, such that

$$a_i \leq b_i \leq a_i + l \textit{ for all integers } i = 1, \ldots, q$$

and

$$\{t \in [T_1, T_2] :\ |x(t) - x_*(t)| > \epsilon\}$$
$$\subset \cup_{i=1}^q [a_i, b_i].$$

Proof. We may assume without loss of generality that

$$\epsilon < \min\{1, d_0\}. \tag{2.221}$$

By Lemma 2.19, there exist

$$\delta_0 \in (0, \epsilon) \text{ and } \Delta_1 \geq 1,$$

and a neighborhood $\mathcal{U}_1$ of f_r in $\mathfrak{M}$ such that for each $g \in \mathcal{U}_1$, each $T_1 \in R^1$, each $T_2 \geq T_1 + \Delta_1$, and each trajectory-control pair

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

which satisfies

$$|x(T_i) - x_*(T_i)| \leq \delta_0,\ i = 1, 2, \tag{2.222}$$

$$I^g(T_1, T_2, x, u) \leq U^g(T_1, T_2, x(T_1), x(T_2)) + \delta_0 \tag{2.223}$$

the following inequality holds:

$$|x(t) - x_*(t)| \le \epsilon \text{ for all } t \in [T_1, T_2]. \tag{2.224}$$

By Lemma 2.23, there exist $\Delta_2 > 0$ and a neighborhood $\mathcal{U}_2$ of f_r in $\mathfrak{M}$ such that for each $g \in \mathcal{U}_2$, each pair of real numbers

$$T_1 \in R^1,\ T_2 \ge T_1 + \Delta_2,$$

each trajectory-control pair

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

which satisfies (2.220) and each

$$\tau \in [T_1, T_2 - \Delta_2],$$

we have

$$\min\{|x(t) - x_*(t)| :\ t \in (\tau, \tau + \Delta_2)\} < \delta_0. \tag{2.225}$$

Set

$$\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2, \tag{2.226}$$

$$l = 2\Delta_2 + 2\Delta_1 \tag{2.227}$$

and choose

$$\Delta \ge 8\Delta_1 + 8\Delta_2 \tag{2.228}$$

and a natural number

$$Q > 6 + 2\delta_0^{-1} S. \tag{2.229}$$

Assume that

$$g \in \mathcal{U},\ T_1 \in R^1,\ T_2 \ge T_1 + \Delta \tag{2.230}$$

and that a trajectory-control pair

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

satisfies (2.220). By (2.230), (2.226), (2.228), (2.220), and the choice of $\mathcal{U}_2$ [see (2.225)], there exists a strictly increasing sequence of real numbers $\{t_i\}_{i=1}^p$, where p is a natural number, such that

$$\{t_i\}_{i=1}^p \subset [T_1, T_2],$$
$$|x(t_i) - x_*(t_i)| < \delta_0, \ i = 1, \dots, p, \tag{2.231}$$
$$t_1 \in [T_1, T_1 + \Delta_2],$$
$$\Delta_2 \le t_{i+1} - t_i \le 2\Delta_2$$
$$\text{for each integer } i \text{ satisfying } 1 \le i < p,$$
$$t_p \ge T_2 - 2\Delta_2. \tag{2.232}$$

Set

$$S_1 = t_1. \tag{2.233}$$

By induction we construct finite strictly increasing sequences of numbers

$$\{r(i)\}_{i=1}^k \subset \{1, \dots, p\},$$
$$\{S_i\}_{i=1}^k \subset \{t_i : \ i = 1, \dots, p\}$$

such that

$$S_i = t_{r(i)}, \ i = 1, \dots, k, \tag{2.234}$$
$$r(k) = p \text{ and } S_k = t_p \tag{2.235}$$

such that the following two properties hold:

(C1) For each integer i satisfying $1 \le i < k - 1$,

$$I^g(S_i, S_{i+1}, x, u) > U^g(S_i, S_{i+1}, x(S_i), x(S_{i+1})) + \delta_0; \tag{2.236}$$

(C2) If an integer i satisfies $1 \le i \le k - 1$, (2.236) holds and

$$r(i + 1) > r(i) + 1,$$

then

$$I^g(S_i, t_{r(i+1)-1}, x, u) \le U^g(S_i, t_{r(i+1)-1}, x(S_i), x(t_{r(i+1)-1})) + \delta_0. \tag{2.237}$$

Assume that an integer $j \ge 1$ and we have already defined strictly increasing sequences of numbers

$$\{r(i)\}_{i=1}^j \subset \{1, \dots, p\},$$
$$\{S_i\}_{i=1}^j \subset \{t_i : \ i = 1, \dots, p\}$$

such that

$$S_i = t_{r(i)},\ i = 1, \dots, j,$$
$$S_j < t_p,$$

for each integer i satisfying $1 \le i < j$ relation (2.236) holds, and if an integer i satisfies $1 \le i < j$ and

$$r(i+1) > r(i) + 1,$$

then (2.237) is true. (Clearly, for $j = 1$ our assumption holds.)

Let us define

$$r(j+1) \in \{1, \dots, p\} \text{ and } S_{j+1} = t_{r(j+1)}.$$

If

$$I^g(S_j, t_p, x, u) \le U^g(S_j, t_p, x(S_j), x(t_p)) + \delta_0,$$

then we set

$$r(j+1) = p,\ S_{j+1} = t_p,$$

complete our construction and it is easy to see that properties (C1) and (C2) hold.

Assume that

$$I^g(S_j, t_p, x, u) > U^g(S_j, t_p, x(S_j), x(t_p)) + \delta_0. \tag{2.238}$$

Set

$$\begin{aligned} r(j+1) = \min\{i \in \{r(j)+1, \dots, p\} : \\ I^g(S_j, t_i, x, u) > U^g(S_j, t_i, x(S_j), x(t_i)) + \delta_0\}. \end{aligned} \tag{2.239}$$

If $r(j+1) = p$, then we set $k = j+1$, complete our construction and it is easy to see that properties (C1) and (C2) hold.

If $r(j+1) < p$, then we set $S_{j+1} = t_{r(j+1)}$ and it is easy to see that the assumption made for j also holds for $j+1$. Clearly, our construction of the sequences is completed after a finite number of steps. Let $r(k)$ and S_k be their last elements respectively. It follows from our construction that $r(k) = p$, $S_k = t_p$ and that (C1 and (C2) hold.

By (2.220) and (C1),

$$
\begin{aligned}
S &\geq I^g(T_1, T_2, x, u) - \inf\{\sigma^g(T_1, T_2, y) : \ (T_1, y) \in A\} \\
&\geq \sum \{I^g(S_i, S_{i+1}, x, u) - U^f(S_i, S_{i+1}, x(S_i), x(S_{i+1})) : \\
&\quad i \text{ is an integer such that } 1 \leq i < k-1\} \geq (k-2)\delta_0
\end{aligned}
$$

and

$$k \leq 2 + \delta_0^{-1} S. \tag{2.240}$$

Set

$$
\begin{aligned}
A = \{i \in \{1, \dots, k\} : i < k \text{ and} \\
S_{i+1} - S_i > 2\Delta_2 + 2\Delta_1\}.
\end{aligned} \tag{2.241}
$$

Let

$$i \in A. \tag{2.242}$$

By (2.234), (2.242), and (2.241),

$$t_{r(i+1)} - t_{r(i)} = S_{i+1} - S_i > 2\Delta_2 + 2\Delta_1. \tag{2.243}$$

By (2.243) and (2.232),

$$r(i+1) > r(i) + 1. \tag{2.244}$$

By (2.242), (2.241), (2.244), (C1) and (C2),

$$I^g(t_{r(i)}, t_{r(i+1)-1}, x, u) \leq U^g(t_{r(i)}, t_{r(i+1)-1}, x(t_{r(i)}), x(t_{r(i+1)-1})) + \delta_0. \tag{2.245}$$

By (2.244), (2.232), and (2.243),

$$t_{r(i+1)-1} - t_{r(i)} \geq t_{r(i+1)} - 2\Delta_2 - t_{r(i)} \geq 2\Delta_1. \tag{2.246}$$

By (2.246), (2.245), (2.231), and (2.230), (2.226) and the choice of $\mathcal{U}_1$, δ_0, Δ_1 [see (2.222)–(2.224)],

$$|x(t) - x_*(t)| \leq \epsilon \text{ for all } t \in [t_{r(i)}, t_{r(i+1)-1}]. \tag{2.247}$$

In view of (2.247), (2.234), and (2.232),

$$|x(t) - x_*(t)| \leq \epsilon \text{ for all } t \in [S_i, S_{i+1} - 2\Delta_2] \tag{2.248}$$

for all $i \in A$. By (2.248),

$$\{t \in [T_1, T_2] : \ |x(t) - x_*(t)| > \epsilon\}$$
$$\subset (\cup\{[S_i, S_{i+1}] : \ i \in \{1, \dots, k-1\} \setminus A\}) \cup \{\cup[S_{i+1} - 2\Delta_2, S_{i+1}] : \ i \in A\}$$
$$\cup[T_1, t_1] \cup [t_p, T_2]. \tag{2.249}$$

The right-hand side of (2.249) is a finite union of closed intervals. By (2.240) and (2.229) their number does not exceed

$$2k + 2 \le 6 + 2\delta_0^{-1} S \le Q$$

and in view of (2.241), (2.232), and (2.227) their maximal length does not exceed

$$2\Delta_2 + 2\Delta_1 = l.$$

Lemma 2.24 is proved. □

2.5 Proofs of Theorems 2.1–2.3 and 2.5

Proof of Theorem 2.1. The validity of assertion 1 follows from Lemma 2.10 and assumptions (A) and (B).

We will prove assertion 2. Let $f \in \mathfrak{M}_{reg}$, $s \in R^1$ and

$$x : [s, \infty) \to R^n, \ u : [s, \infty) \to R^m$$

be a trajectory-control pair. Assume that there exists a sequence of numbers $\{t_k\}_{k=1}^{\infty}$ such that

$$t_k \to \infty \text{ as } k \to \infty, \tag{2.250}$$

$$I^f(s, t_k, x, u) - I^f(s, t_k, x_f, u_f) \to \infty \text{ as } k \to \infty. \tag{2.251}$$

We will show that relation (a) holds. It follows from assumption (A) and (B) that there exists a number $S_0 > 0$ such that

$$S_0 > 2|x(s)| + 8,$$

$$\psi(S_0 - 4) - a_0 - 8 \ge 8 \sup\{|I^f(j, j+1, x_f, u_f)| : \ j = 0, \pm 1, \pm 2, \dots\}. \tag{2.252}$$

Let a number $S > 0$ be as guaranteed in assertion 1. By assumption (A) and (2.252) we may assume without loss of generality that

$$\liminf_{t \to \infty} |x(t)| \le S_0 - 1. \tag{2.253}$$

For each integer $k \geq 1$ we set

$$\tau_k = \inf\{t \in [t_k, \infty) : \ |x(t)| \leq S_0\}. \tag{2.254}$$

Let $k \geq 1$ be an integer and $t \geq \tau_k$. It follows from (2.254), (2.251), (2.252), the choice of S, assertion 1, and assumption (A) that

$$\begin{aligned}
& I^f(s,t,x,u) - I^f(s,t,x_f,u_f) \\
& \geq I^f(s,t_k,x,u) - I^f(s,t_k,x_f,u_f) \\
& + I^f(t_k,\tau_k,x,u) - I^f(t_k,\tau_k,x_f,u_f) - S \\
& \geq I^f(s,t_k,x,u) - I^f(s,t_k,x_f,u_f) - S - 4a_0 \\
& -2\sup\{|I^f(j,j+1,x_f,u_f)| : \ j = 0, \pm 1, \pm 2, \dots\} \to \infty \text{ as } k \to \infty.
\end{aligned}$$

Therefore relation (a) holds. Together with assertion 1 and Proposition 2.7 this implies the validity of assertion 2. □

Construction of the set $\mathcal{F}$:

Let $\mathfrak{A}$ be a subset of $\mathfrak{M}_{reg}$ such that $f_r \in \mathfrak{A}$ for each $f \in \mathfrak{A}$ each $r \in (0,1)$. Denote by $\bar{\mathfrak{A}}$ the closure of $\mathfrak{A}$ in the space $\mathfrak{M}$.

It is easy to see that for each $f \in \mathfrak{A}$ and each $r \in (0,1)$ Lemmas 2.12–2.24 hold with

$$x_* = x_f, \ u_* = u_f, \ b_* = b_f, \ d_0 = 1.$$

Relation (2.31) implies that

$$\sup\{|x_f(t)| : \ t \in R^1\} < \infty \text{ for each } f \in \mathfrak{A}. \tag{2.255}$$

Set

$$E = \{f_r : \ f \in \mathfrak{A}, \ r \in (0,1)\}.$$

It follows from Lemma 2.11 that E is an everywhere dense subset of $\bar{\mathfrak{A}}$.

For each $f \in \mathfrak{A}$, each $r \in (0,1)$, and each integer $k \geq 1$ there exist an open neighborhood $V(f,r,k)$ of f_r and numbers

$$\begin{aligned}
& \delta(f,r,k) \in (0,(2k)^{-1}), \\
& \gamma(f,r,k) \in (0,\delta(f,r,k)), \\
& \Delta(f,r,k), \ Q(f,r,k) \geq 1, \\
& l(f,r,k) > 0, \ L(f,r,k) > 0
\end{aligned}$$

and a natural number $p(f,r,k)$ such that:

(a) Lemma 2.20 holds for f and r with

$$S = k,\ \epsilon = (2k)^{-1},\ \mathcal{U} = V(f,r,k),$$
$$\Delta = \Delta(f,r,k),\ \delta = 2\delta(f,r,k),$$
$$x_* = x_f,\ u_* = u_f;$$

(b) Lemma 2.21 holds for f and r with $\mathcal{U} = V(f,r,k)$;
(c) Lemma 2.22 holds for f and r with

$$S = k,\ \mathcal{U} = V(f,r,k),$$
$$\delta = \delta(f,r,k),\ Q = Q(f,r,k);$$

(d) Lemma 2.20 holds for f and r with

$$S = k,\ \epsilon = \delta(f,r,k),\ \mathcal{U} = V(f,r,k),$$
$$\Delta = \Delta(f,r,k),\ \delta = \gamma(f,r,k),$$
$$x_* = x_f,\ u_* = u_f;$$

(e) Lemma 2.24 holds for f and r with

$$S = 4k,\ \epsilon = (4k)^{-1},\ \mathcal{U} = V(f,r,k),$$
$$l = l(f,r,k),\ \Delta = L(f,r,k),\ Q = p(f,r,k),$$
$$x_* = x_f,\ u_* = u_f.$$

We define

$$\mathcal{F} = \bar{\mathfrak{A}} \cap [\cap_{k=1}^{\infty} \cup \{V(f,r,k) :\ f \in \mathfrak{A},\ r \in (0,1)\}]. \tag{2.256}$$

Clearly, $\mathcal{F}$ is a countable intersection of open everywhere dense sets in $\bar{\mathfrak{A}}$.

Theorem 2.2 now follows from the definition of $\mathcal{F}$, properties (b), (c) and Lemmas 2.21 and 2.22.

Proof of Theorem 2.3. Let $f \in \mathcal{F}$. For each integer $k \geq 1$ there exist

$$f_k \in \mathfrak{A} \text{ and } r_k \in (0,1)$$

such that

$$f \in V(f_k, r_k, k). \tag{2.257}$$

Let p, q be natural numbers. We show that

$$|x_{f_p}(t) - x_{f_q}(t)| \le \delta(f_p, r_p, p) + \delta(f_q, r_q, q) \text{ for all } t \in R^1. \tag{2.258}$$

Let N be a natural number. Set

$$\begin{aligned} T_2 &= 4N + 4 + 4\Delta(f_p, r_p, p) + 4\Delta(f_q, r_q, q), \\ T_1 &= -T_2. \end{aligned} \tag{2.259}$$

It follows from Theorem 2.2 that

$$\inf\{U^f(T_1, T_2, y_1, y_2) : \ (T_i, y_i) \in A, \ i = 1, 2\} < \infty. \tag{2.260}$$

Therefore there exists a trajectory-control pair

$$x : [T_1, T_2] \to R^n, \ u : [T_1, T_2] \to R^m$$

such that

$$\begin{aligned} I^f(T_1, T_2, x, u) &\le \inf\{U^f(T_1, T_2, y_1, y_2) : \ (T_i, y_i) \in A, \ i = 1, 2\} \\ &+ 8^{-1} \min\{\gamma(f_p, r_p, p), \ \gamma(f_q, r_q, q)\}. \end{aligned} \tag{2.261}$$

By property (d), for each integer $k \ge 1$, Lemma 2.20 holds with

$$\begin{gathered} f = f_k, \ r = r_k, \ S = k, \\ \epsilon = \delta(f_k, r_k, k), \ \mathcal{U} = V(f_k, r_k, k), \\ \Delta = \Delta(f_k, r_k, k), \ \delta = \gamma(f_k, r_k, k), \\ x_* = x_{f_k}, \ u_* = u_{f_k}. \end{gathered}$$

Together with (2.260), (2.261), (2.259), and (2.257) this implies that

$$\begin{gathered} |x(t) - x_{f_p}(t)| \le \delta(f_p, r_p, p), \ t \in [-N, N], \\ |x(t) - x_{f_q}(t)| \le \delta(f_q, r_q, q), \ t \in [-N, N] \end{gathered}$$

and

$$|x_{f_q}(t) - x_{f_p}(t)| \le \delta(f_p, r_p, p) + \delta(f_q, r_q, q), \ t \in [-N, N].$$

Since this relation holds for any integer $N \ge 1$ we conclude that (2.258) holds. It follows from (2.258) and (2.255) that there exists a bounded continuous function $X_f : R^1 \to R^n$ which satisfies

$$|X_f(t) - x_{f_p}(t)| \le \delta(f_p, r_p, p) \text{ for all } t \in R^1 \text{ and all integers } p \ge 1. \tag{2.262}$$

Let $S, \epsilon > 0$. Fix an integer

$$p > 4 + 4S + 8\epsilon^{-1} \tag{2.263}$$

and set

$$\begin{aligned} \mathcal{U} &= V(f_p, r_p, p), \\ \Delta = \Delta(f_p, r_p, p),\ \delta &= \delta(f_p, r_p, p). \end{aligned} \tag{2.264}$$

Assume that

$$g \in \mathcal{U},\ T_1 \in R^1,\ T_2 \geq T_1 + 2\Delta \tag{2.265}$$

and a trajectory-control pair

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

such that

$$\begin{aligned} I^g(T_1, T_2, x, u) &\leq \inf\{U^g(T_1, T_2, y_1, y_2) :\ (T_i, y_i) \in A,\ i = 1, 2\} + S, \\ I^g(T_1, T_2, x, u) &\leq U^g(T_1, T_2, x(T_1), x(T_2)) + \delta. \end{aligned} \tag{2.266}$$

It follows from (2.264), (2.265), (2.266), (2.263), and Lemma 4.20 which holds with

$$\begin{gathered} f = f_p,\ r = r_p,\ S = p, \\ \epsilon = (2p)^{-1},\ \mathcal{U} = V(f_p, r_p, p), \\ \Delta = \Delta(f_p, r_p, p),\ \delta = 2\delta(f_p, r_p, p), \\ x_* = x_{f_p},\ u_* = u_{f_p} \end{gathered}$$

that

$$|x(t) - x_{f_p}(t)| \leq p^{-1},\ t \in [T_1 + \Delta, T_2 - \Delta].$$

Moreover, if $|x(T_1) - x_{f_p}(T_1)| \leq \delta$, then

$$|x(t) - x_{f_p}(t)| \leq p^{-1},\ t \in [T_1, T_2 - \Delta]$$

and if $|x(T_2) - x_{f_p}(T_2)| \leq \delta$, then

$$|x(t) - x_{f_p}(t)| \leq p^{-1},\ t \in [T_1 + \Delta, T_2].$$

Together with (2.262) and (2.263) this implies the validity of Theorem 2.3. □

Proof of Theorem 2.5. Choose a natural number k such that

$$k > 4 + 4M + 8\epsilon^{-1}. \tag{2.267}$$

By (2.256) there exist

$$f_k \in \mathfrak{A},\ r_k \in (0, 1)$$

such that

$$f \in V(f_k, r_k, k). \tag{2.268}$$

Set

$$l = l(f_k, r_k, k),\ L = L(f_k, r_k, k),\ p = p(f_k, r_k, k),$$
$$\mathcal{U} = V(f_k, r_k, k). \tag{2.269}$$

Assume that

$$g \in \mathcal{U},\ T_1 \in R^1,\ T_2 \geq T_1 + L \tag{2.270}$$

and a trajectory-control pair

$$x : [T_1, T_2] \to R^n,\ u : [T_1, T_2] \to R^m$$

satisfies

$$I^g(T_1, T_2, x, u) \leq \inf\{U^g(T_1, T_2, y_1, y_2) :\ (T_i, y_i) \in A,\ i = 1, 2\} + M. \tag{2.271}$$

By (2.270), (2.271), (2.269), (2.267), condition (e), and Lemma 2.24 which holds with

$$\epsilon = (4k)^{-1},\ S = 4k,$$
$$f = f_k,\ r = r_k,$$
$$\mathcal{U} = V(f_k, r_k, k),$$
$$x_* = x_{f_k},\ u_* = u_{f_k},$$
$$l = l(f_k, r_k, k),\ \Delta = L(f_k, r_k, k),\ Q = p(f_k, r_k, k)$$

that there exist finite sequences of numbers

$$\{a_i\}_{i=1}^q,\ \{b_i\}_{i=1}^q \subset [T_1, T_2],$$

where q is a natural number, such that

$$q \le p(f_k, r_k, k) = p, \tag{2.272}$$

$$a_i \le b_i \le a_i + l \text{ for all integers } i = 1, \dots, q \tag{2.273}$$

and

$$\{t \in [T_1, T_2] : \ |x(t) - x_{f_k}(t)| > (4k)^{-1}\} \\ \subset \cup_{i=1}^{q} [a_i, b_i]. \tag{2.274}$$

By Theorem 2.3 there exist

$$\delta_0 \in (0,1), \ \tau_0 > 0$$

such that for each $\tau \ge \tau_0$ and each trajectory-control pair

$$y : [-\tau, \tau] \to R^n, \ v : [-\tau, \tau] \to R^m$$

satisfying

$$I^f(-\tau, \tau, y, v) \le \inf\{U^f(-\tau, \tau, \xi_1, \xi_2) : \ (-\tau, \xi_1), \ (\tau, \xi_2) \in A\} + \delta_0 \tag{2.275}$$

we have

$$|y(t) - X_f(t)| \le (8k)^{-1}, \ t \in [-\tau + \tau_0, \tau - \tau_0]. \tag{2.276}$$

Let

$$T > \tau_0 + \Delta(f_k, r_k, k) \tag{2.277}$$

and a trajectory-control pair

$$y : [-T, T] \to R^n, \ v : [-T, T] \to R^m$$

satisfy

$$I^f(-T, T, y, v) \le \inf\{U^f(-T, T, \xi_1, \xi_2) : \ (-T, \xi_1), \ (T, \xi_2) \in A\} \\ + \min\{\delta_0, \ \delta(f_k, r_k, k)\}. \tag{2.278}$$

By (2.277), (2.278) and the choice of δ_0, τ_0 [see (2.275), (2.276)],

$$|y(t) - X_f(t)| \le (8k)^{-1}, \ t \in [-T + \tau_0, T - \tau_0]. \tag{2.279}$$

By (2.277), (2.278), (2.268), condition (a), Lemma 2.20 which holds for f_k and r_k with

$$S = k, \ \epsilon = (2k)^{-1}, \ \mathcal{U} = V(f_k, r_k, k),$$
$$\Delta = \Delta(f_k, r_k, k), \ \delta = 2\delta(f_k, r_k, k),$$
$$x_* = x_{f_k}, \ u_* = u_{f_k}$$

we have

$$|y(t) - x_{f_k}(t)| \leq (2k)^{-1} \text{ for all } t \in [-T + \Delta(f_k, r_k, k), T - \Delta(f_k, r_k, k)].$$

Together with (2.279) this implies that

$$|X_f(t) - x_{f_k}(t)| \leq (8k)^{-1} + (2k)^{-1}$$
$$\text{for all } t \in [-T + \tau_0 + \Delta(f_k, r_k, k), T - \Delta(f_k, r_k, k) - \tau_0].$$

Since T is any natural number satisfying (2.277) we conclude that

$$|X_f(t) - x_{f_k}(t)| \leq (8k)^{-1} + (2k)^{-1} \text{ for all } t \in R^1.$$

Together with (2.274) and (2.267) this implies that for all

$$t \in [T_1, T_2] \setminus \cup_{i=1}^{q} [a_i, b_i],$$
$$|x(t) - X_f(t)| \leq |x(t) - x_{f_k}(t)| + |x_{f_k}(t) - X_f(t)|$$
$$\leq (4k)^{-1} + (8k)^{-1} + (2k)^{-1} < k^{-1} < \epsilon.$$

Theorem 2.5 is proved. □

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