

# Contents

|          |   |    |
|----------|---|----|
| <b>1</b> | <b>Prerequisites</b>  | 1  |
| 1.1      | Brownian Motion   | 1  |
| 1.2      | Some Extensions   | 2  |
| 1.3      | BM as a Continuous Martingale   | 2  |
| 1.4      | Girsanov's Theorem  | 4  |
| 1.5      | Brownian Bridge   | 5  |
| 1.6      | The BES(3) Process as a Doob $h$ -Transform of BM   | 7  |
| 1.7      | The Beta–Gamma Algebra  | 7  |
| 1.8      | The Law of the Maximum of a Positive Continuous<br>Local Martingale, Which Converges to 0 | 8  |
| 1.9      | A First Taste of Enlargement Formulae   | 9  |
| 1.10     | Kolmogorov's Continuity Criterion   | 9  |
|          | References  | 10 |

## Part I Local Times of Continuous Semimartingales

|          |   |    |
|----------|---|----|
| <b>2</b> | <b>The Existence and Regularity of Semimartingale Local Times</b>                       | 13 |
| 2.1      | From Itô's Formula to the Occupation Time Formula                                       | 13 |
| 2.2      | Regularity of Occupation Times  | 14 |
| 2.3      | Occupation Times Are Local Times  | 18 |
| 2.4      | Local Times and the Balayage Formula  | 19 |
| 2.5      | Some Simple Martingales   | 22 |
| 2.6      | The Existence of Principal Values Related to Brownian<br>Local Times                    | 23 |
| 2.7      | Some Extensions of Itô's Formula  | 24 |
|          | References  | 27 |
| <b>3</b> | <b>Lévy's Representation of Reflecting BM and Pitman's<br/>Representation of BES(3)</b> | 29 |
| 3.1      | Lévy's Identity in Law: The Local Time<br>as a Supremum Process                         | 29 |
| 3.2      | A Solution to Skorokhod's Embedding Problem   | 30 |

|  |  |    |
|--|--|----|
| 3.3  | Pitman's Representation of BES(3) .....  | 33 |
| 3.4  | A Relation Between (The Above Solution to)<br>Skorokhod's Problem and the Balayage Formula ..... | 35 |
| 3.5  | An Extension of Pitman's Theorem to Brownian<br>Motion with Drift .....                          | 36 |
| 3.6  | Skorokhod's Lemma and the Balayage Formula .....   | 38 |
| 3.7  | Seshadri's Remark on the Joint Law of $(S_t, B_t)$ .....   | 40 |
| 3.8  | A Combination of Skorokhod's Lemma and Time-Substitution ...                                     | 41 |
|  | References .....   | 41 |
| <b>4</b>   | <b>Paul Lévy's Arcsine Laws</b> .....  | 43 |
| 4.1  | Two Brownian Functionals with the Arcsine Distribution .....                                     | 43 |
| 4.2  | Two Independent Reflected Brownian Motions.....  | 44 |
| 4.3  | Random Brownian Scaling and Absolute Continuity Properties...                                    | 45 |
| 4.4  | The Second Arcsine Law .....   | 48 |
| 4.5  | The Time Spent in $\mathbb{R}_+$ by a Brownian Bridge .....                                      | 50 |
| 4.6  | The Law of $A_T^+$ for More Random Times $T$ and Other<br>Processes than BM .....                | 51 |
|  | References .....   | 53 |
| <br><b>Part II Excursion Theory for Brownian Paths</b> |  |    |
| <b>5</b>   | <b>Brownian Excursion Theory: A First Approach</b> .....   | 57 |
| 5.1  | Some Motivations .....   | 57 |
| 5.2  | Itô's Theorem on Excursions .....  | 59 |
| 5.3  | Two Master Formulae (A) and (M).....   | 60 |
| 5.4  | Relationship Between Certain Lévy Measures<br>and Itô Measure $\mathbf{n}$ .....                 | 61 |
| 5.5  | Two Applications of (A) and (M) .....  | 62 |
|  | References .....   | 64 |
| <b>6</b>   | <b>Two Descriptions of <math>\mathbf{n}</math>: Itô's and Williams'</b> .....                    | 65 |
| 6.1  | Statements .....   | 65 |
| 6.2  | An Agreement Formula .....   | 67 |
| 6.3  | $\mathbf{n}$ is a Markovian Measure .....  | 68 |
| 6.4  | Proof of Itô's Disintegration (b) in Sect. 6.1 .....   | 68 |
| 6.5  | Proof of the Formula (6.4.4) for $\Pi^r(\Gamma)$ .....   | 70 |
| 6.6  | Proof of the Markovianity of $\mathbf{n}$ .....  | 72 |
| 6.7  | The Formula for Entrance Laws .....  | 74 |
| 6.8  | A (Partial) Proof of Williams' Representation of $\mathbf{n}$ .....                              | 75 |
|  | References .....   | 77 |
| <b>7</b>   | <b>A Simple Path Decomposition of Brownian Motion<br/>Around Time <math>t = 1</math></b> .....   | 79 |
| 7.1  | Another Representation of the Brownian Bridge .....  | 79 |
| 7.2  | The Normalized Brownian Excursion .....  | 80 |

|   |  |            |
|---|--|------------|
| 7.3   | The Brownian Meander .....   | 81         |
| 7.4   | The Brownian Co-meander .....  | 83         |
| 7.5   | Two Isolation Formulae .....   | 85         |
| 7.6   | Azéma's Martingale and the Brownian Meander .....  | 88         |
|   | References .....   | 91         |
| <b>8</b>  | <b>The Laws of, and Conditioning with Respect to,<br/>Last Passage Times</b> .....         | <b>93</b>  |
| 8.1   | The Bessel Case .....  | 93         |
| 8.2   | General Transient Diffusions .....   | 93         |
| 8.3   | Absolute Continuity Relationships up to $\gamma_y$ .....                                   | 96         |
| 8.4   | Applications .....   | 97         |
| 8.4.1   | BM with drift considered up to last passage time .....                                     | 97         |
| 8.4.2   | BES process up to last passage time .....  | 98         |
| 8.4.3   | First hit of 0 by Ornstein–Uhlenbeck process .....   | 99         |
|   | References .....   | 100        |
| <b>9</b>  | <b>Integral Representations Relating <math>W</math> and <math>n</math></b> .....           | <b>101</b> |
| 9.1   | Statement of the Main Theorem .....  | 101        |
| 9.2   | Proof of the Theorem .....   | 102        |
| 9.3   | Proof of (9.2.1) .....   | 103        |
|   | References .....   | 104        |
| <br><b>Part III Some Applications of Excursion Theory</b> |  |            |
| <b>10</b>   | <b>The Feynman–Kac Formula and Excursion Theory</b> .....                                  | <b>107</b> |
| 10.1  | Statement of the FK Formula .....  | 107        |
| 10.2  | Proof of FK via Excursion Theory .....   | 108        |
|   | References .....   | 110        |
| <b>11</b>   | <b>Some Identities in Law</b> .....  | <b>111</b> |
| 11.1  | On Linear Combinations of Reflected BM and Its Local Time ....                             | 111        |
| 11.2  | On the Joint Laws of $(S_b, I_b, L_b)$ and $(S_1, I_1, L_1)$ .....                         | 114        |
| 11.3  | Knight's Identity in Law .....   | 118        |
| 11.4  | The Földes–Révész Identity .....   | 120        |
| 11.5  | Cauchy Principal Value of Brownian Local Times .....                                       | 122        |
| 11.6  | The Agreement Formula and the Functional Equation<br>of the Riemann $\zeta$ Function ..... | 123        |
| 11.7  | On Ranked Lengths of Excursions .....  | 126        |
|   | References .....   | 130        |
|   | <b>General References</b> .....  | <b>133</b> |
|   | <b>Index</b> .....   | <b>135</b> |

Local Times and Excursion Theory for Brownian Motion  
A Tale of Wiener and Itô Measures

Yen, J.-Y.; Yor, M.

2013, IX, 135 p. 9 illus., 8 illus. in color., Softcover

ISBN: 978-3-319-01269-8