Chapter 1
Prerequisites

In this chapter, ten notions or results are gathered, which we assume as background for the remainder of this monograph.

1.1 Brownian Motion

It is not difficult to show the existence of a probability space on which one can construct a Gaussian family $\{B(f); f \in L^2(\mathbb{R}_+, dt)\}$, such that

$$\text{(i)}\ \ E[B(f)] = 0; \qquad \text{(ii)}\ \ E[(B(f))^2] = \int f^2(t)dt.$$

Indeed, from a functional viewpoint, B is a Hilbert space isomorphism

$$\begin{aligned} B : L^2(\mathbb{R}_+, dt) &\to \mathcal{B}(\subset L^2(\Omega)) \\ f &\to B(f) \end{aligned}$$

so that:

$$B(f) = \sum_{n \geq 1} (f, e_n)_{L^2} G_n$$

where $(e_n; n \geq 1)$ is an orthonormal basis of $L^2(\mathbb{R}_+, dt)$, and $(G_n; n \geq 1)$ is a sequence of centered, reduced independent Gaussian variables.

We shall call Brownian motion, BM in brief, a continuous modification $\{B_t, t \geq 0\}$ of the Gaussian family

$$(B(\mathbf{1}_{[0,t]}); t \geq 0)$$

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Once the existence of this continuous modification is established (using Kolmogorov's continuity criterion; see Sect. 1.10), it is natural to use (Wiener) integral notation

$$\int_0^\infty f(t)dB_t$$

instead of $B(f)$ since, in the particular case

$$f(t) = \sum \lambda_i \mathbf{1}_{(t_i, t_i+1]}(t)$$

one has

$$B(f) = \sum \lambda_i (B_{t_{i+1}} - B_{t_i})$$

1.2 Some Extensions

Given any measurable space $(T, \mathcal{T})$ equipped with a positive σ-finite measure μ, one can, just as in the previous case, define a so-called Gaussian measure $(B(f) \equiv \int f(t)B(\mu(dt)); f \in L^2(T, \mathcal{T}; \mu)$ such that

$$\text{(i)}\ \ E[B(f)] = 0; \qquad \text{(ii)}\ \ E[(B(f))^2] = \int f^2(t)\mu(dt).$$

The Brownian sheet corresponds to $T = \mathbb{R}_+^2$ (more generally, $\mathbb{R}_+^n$) and $\mu(dsdt) = dsdt$ the Lebesgue measure.

One can also construct important Gaussian families from a Gaussian measure, by considering:

$$\int \Phi(t, s)B(\mu(ds)).$$

The Lévy's n-parameter Brownian motions and fractional Brownian motions may be defined in this way.

1.3 BM as a Continuous Martingale

The following theorem presents Brownian motion as a prototype for continuous martingales.

Theorem 1.3.1 (Dubins–Schwarz). *Let M be a continuous local martingale such that $M_0 = 0$ and $\langle M \rangle_\infty = \infty$. There exists a BM $(B_u; u \geq 0)$ such that*

$$M_t = B_{\langle M \rangle_t}.$$

Next, here is a partial extension of the preceding theorem to multidimensional continuous martingales.

Theorem 1.3.2 (Knight). *Let $M^{(1)}, M^{(2)}, \dots, M^{(k)}$ be k continuous local martingales with $M_0^{(i)} = 0$, $\langle M^{(i)} \rangle_\infty = \infty$ and $\langle M^{(i)}, M^{(j)} \rangle_t = 0$ for $i \neq j$; then there exist k independent BM's $(B_u^{(i)}; u \geq 0)$, $i = 1, \dots, k$ such that*

$$M_t^{(i)} = B^{(i)}_{\langle M^{(i)} \rangle_t}.$$

If moreover $\langle M^{(i)} \rangle_t \equiv \langle M \rangle_t$ for $i = 1, \dots, k$, i.e., there is a common time change, Theorem 1.3.2 implies that $M_t = B_{\langle M \rangle_t}$ where $B = (B^{(1)}, \dots, B^{(k)})$ and $B^{(i)}$ are independent BM's. Such multidimensional martingales are called *conformal martingales* (in particular in the case $k = 2$).

Examples of Conformal Martingales.

Let $Z_t \equiv B_t^{(1)} + iB_t^{(2)}$ be a complex BM. If $f \in \mathcal{H}(\mathbb{C})$ is an entire function, which is not constant, then $(M_t = f(Z_t); t \geq 0)$ is a conformal (local) martingale. Then

$$\langle M \rangle_t = \int_0^t ds |f'(Z_s)|^2$$

and Theorem 1.3.2 implies that there exists a $\mathbb{C}$ valued BM $(\hat{Z}_u; u \geq 0)$ such that

$$M_t = \hat{Z}_{\int_0^t ds |f'(Z_s)|^2}.$$

In a general case (i.e. $f \in \mathcal{C}^2(\mathbb{R}^2)$), Itô's "complex" formula may be written as:

$$f(Z_t) = f(Z_0) + \int_0^t \frac{\partial f}{\partial z}(Z_s) dZ_s + \int_0^t \frac{\partial f}{\partial \overline{z}}(Z_s) d\overline{Z}_s + \int_0^t \frac{\partial^2 f}{\partial z \partial \overline{z}}(Z_s) d\langle Z, \overline{Z} \rangle_s$$

and if f is holomorphic, then:

$$f(Z_t) = f(Z_0) + \int_0^t f'(Z_s) dZ_s.$$

More generally again, let $X = M + V$ be a continuous semimartingale in $\mathbb{R}^n$ and $f \in C^2(\mathbb{R}^n)$; then Itô's formula is

$$f(X_t) = f(X_0) + \int_0^t (\nabla f)(X_s) \cdot dX_s + \frac{1}{2}\int_0^t \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s) d\langle X^{(i)}, X^{(j)}\rangle_s.$$

For a detailed exposition see [4].

1.4 Girsanov's Theorem

This fundamental theorem often allows to extend theorems known to be valid for BM to "mild perturbations of BM".

On the canonical space $C(\mathbb{R}_+, \mathbb{R})$, we consider the canonical process $X_t(\omega) = \omega(t)$ and the canonical filtration $\mathcal{F}_t \equiv \sigma\{X_s; s \le t\}$.

For every $x \in \mathbb{R}$, $\mathbf{W}_x$ will denote the Wiener measure on $\mathcal{F}_\infty$ such that $\mathbf{W}_x(X_0 = x) = 1$.

We shall say that a process Y is a mild perturbation of BM if its law P_Y has the same null sets as $\mathbf{W}$ on each σ-field $\mathcal{F}_t$, i.e. the measure P_Y is such that

$$P_Y|_{\mathcal{F}_t} \sim \mathbf{W}|_{\mathcal{F}_t}; \quad t \ge 0.$$

Example 1.4.1.

(a) Brownian motion with drift μ.
Let $B_t^{(\mu)} = B_t + \mu t,\ t \ge 0$; then the associated measure $\mathbf{W}^{(\mu)}$ satisfies

$$\mathbf{W}^{(\mu)}|_{\mathcal{F}_t} = \exp\left(\mu X_t - \frac{\mu^2}{2}t\right)\mathbf{W}|_{\mathcal{F}_t}.$$

(b) The Cameron–Martin formula.
Let $B_t^{(f)} = B_t + \int_0^t ds\ f(s)$ where $f \in L^2_{loc}(\mathbb{R}_+)$; then the corresponding measure $\mathbf{W}^{(f)}$ satisfies

$$\mathbf{W}^{(f)}|_{\mathcal{F}_t} = \exp\left(\int_0^t f(s)dX_s - \frac{1}{2}\int_0^t f^2(s)ds\right)\mathbf{W}|_{\mathcal{F}_t}.$$

(c) Girsanov's formula.
Let $Z_t = B_t + \int_0^t ds\, \varphi(Z_s)$ where φ is a bounded Borel function. The associated measure $P^{(\varphi)}$ satisfies

$$P^{(\varphi)}|_{\mathcal{F}_t} = \exp\left(\int_0^t \varphi(X_s)dX_s - \frac{1}{2}\int_0^t \varphi^2(X_s)ds\right)\mathbf{W}|_{\mathcal{F}_t}.$$

All these examples are particular cases of Girsanov's theorem, of which we now present the continuous martingale version.

Theorem 1.4.2 (Girsanov–Wong–Van Schuppen). *Given a probability measure P and a $(P, \mathcal{F}_t)$-local martingale M such that Q can be defined with the property*

$$Q|_{\mathcal{F}_t} = \exp\left(M_t - \frac{1}{2}\langle M\rangle_t\right) P|_{\mathcal{F}_t}.$$

Then, if N is a $(P, \mathcal{F}_t)$-local martingale, $N_t - \langle N, M\rangle_t$ is a $(Q, \mathcal{F}_t)$-local martingale.

Corollary 1.4.3. *If N is a $(\mathcal{F}_t)$-BM under P, then $(\tilde{N} \equiv N_t - \langle N, M\rangle_t;\ t \geq 0)$ is a BM under Q.*

Corollary 1.4.3 holds since $\langle \tilde{N}\rangle_t = \langle N\rangle_t = t$.

Example 1.4.4. If $N = M$ then $M_t = \tilde{M}_t + \langle M\rangle_t$ where $(\tilde{M}_t; t \geq 0)$ is a Q-local martingale.

Let us see how Girsanov theorem applies to Example 1.4.1(c). Let $(X_t; t \geq 0)$ be a BM, i.e. a **W**-martingale, then $M_t = \int_0^t \varphi(X_s)dX_s$ is a **W**-local martingale. The theorem implies that $\tilde{X}_t \equiv X_t - \langle X, M\rangle_t$ is a $P^{(\varphi)}$-local martingale, whence $\tilde{X}_t$ is a $P^{(\varphi)}$-BM, since $\langle \tilde{X}\rangle_t = t$.

Note that

$$\langle X, M\rangle_t = \int_0^t \varphi(X_s)ds.$$

The other examples can be treated similarly.

1.5 Brownian Bridge

The Brownian bridge $b = \{b_u; 0 \leq u \leq 1\}$ is defined as the conditioned process $\{(B_u; u \leq 1) | B_1 = 0\}$.

We shall use the fact that $B_t = (B_t - tB_1) + tB_1$ is the orthogonal decomposition of B_t with respect to $L^2(\sigma(B_1))$, since:

$$E[(B_t - tB_1)B_1] = 0.$$

Now, the Gaussian property implies that $(B_t - tB_1; t \leq 1)$ is independent of B_1, hence:

$$(B_t, t \leq 1 | B_1 = y) \overset{(\text{law})}{=} (B_t - tB_1 + ty).$$

We can thus represent the bridge between 0 and y during the time interval [0, 1] as

$$(B_t - tB_1 + ty; t \leq 1)$$

and we denote by $\mathbf{W}^{(1)}_{0\to y}$ the associated measure. In general, $\mathbf{W}^{(t)}_{x\to y}$ denotes the measure associated to the bridge between x and y during the time interval $[0, t]$, which may be realized as

$$\left(x + \left(B_u - \frac{u}{t}B_t\right) + \frac{u}{t}(y - x); u \leq t\right),$$

where $(B_u; u \leq t)$ is a standard BM starting from 0.

Theorem 1.5.1. $\mathbf{W}^{(t)}_{x\to y}$ *is equivalent to* $\mathbf{W}_x$ *on* $\mathcal{F}_s$ *for* $s < t$.

Proof. Let $F_s \geq 0$ be an $\mathcal{F}_s$-measurable functional, then

$$E_x[F_s f(X_t)] = E_x[E_x(F_s|X_t) f(X_t)] = E_x[F_s P_{t-s} f(X_s)]$$

where $(X_t; t \geq 0)$ is a Markov process with semigroup

$$P_t(x, dy) = p_t(x, y)dy.$$

On the other hand,

$$E_x[F_s P_{t-s} f(X_s)] = E_x[F_s \int f(y) p_{t-s}(X_s, y)dy] = \int f(y) E_x[F_s p_{t-s}(X_s, y)]dy$$

and also

$$E_x[E_x[F_s|X_t] f(X_t)] = \int dy f(y) p_t(x, y) E^{(t)}_{x\to y}(F_s)$$

whence

$$E^{(t)}_{x\to y}(F_s) = \frac{E_x[F_s p_{t-s}(X_s, y)]}{p_t(x, y)}.$$

Thus

$$P^{(t)}_{x\to y}|_{\mathcal{F}_s} = \frac{p_{t-s}(X_s, y)}{p_t(x, y)} P_x|_{\mathcal{F}_s}.$$

If $x = y = 0$, we have

$$P^{(t)}_{0\to 0}|_{\mathcal{F}_s} = \left(\frac{t}{t-s}\right)^{n/2} \exp\left(\frac{-|X_s|^2}{2(t-s)}\right) P_0|_{\mathcal{F}_s}.$$

As a consequence, we can write the canonical decomposition of the standard Brownian bridge (under $P^{(t)}_{0\to 0}$) as:

$$X_s = B_s - \int_0^s du \frac{X_u}{t-u}, \quad s \leq t,$$

where $(B_s, s \leq t)$ is a BM under $P^{(t)}_{0\to 0}$. $\square$

1.6 The BES(3) Process as a Doob h-Transform of BM

We use the notation $\mathrm{BES}_a(3)$ for the three-dimensional Bessel process starting from a, and $P_a^{(3)}$ for its law.

Using Girsanov theorem (see Sect. 1.4) one can show the following absolute continuity relation

$$P_a^{(3)}|_{\mathcal{F}_t} = \left(\frac{X_{t\wedge T_0}}{a}\right) \mathbf{W}_a|_{\mathcal{F}_t}.$$

As an important consequence, if $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a harmonic space-time function, then $\left(\frac{1}{X_t} f(t, X_t); t \geq 0\right)$ is a $(P_a^{(3)}, \mathcal{F}_t)$ local martingale. The absolute continuity relation, or h-process relation, between a BES(3) and BM is a key property to the proof of Williams' time-reversal theorem.

Theorem 1.6.1 (Williams' time reversal). *Let $(B_t; t \leq T_1)$ be a BM starting at 0 and considered up to time $T_1 \equiv \inf\{t \geq 0 : B_t = 1\}$, then*

$$(1 - B_{T_1 - t}; t \leq T_1) \overset{(\,law)}{=} (R_t; t \leq \gamma^1)$$

where $(R_t; t \leq \gamma^1)$ denotes a BES(3) process starting at 0 considered up to time $\gamma^1 \equiv \sup\{t \geq 0 : R_t = 1\}$.

1.7 The Beta–Gamma Algebra

Let Z_a be a random variable having Gamma density $h_a(t) = \frac{t^{a-1}e^{-t}}{\Gamma(a)}$ on $\mathbb{R}_+$ and $Z_{a,b}$ a variable with Beta density $\tilde{h}_{a,b}(t) = \frac{t^{a-1}(1-t)^{b-1}}{\beta(a,b)}$ on $[0,1]$.

If Z_a and Z_b are independent, then

(i) $Z_a + Z_b \overset{(\mathrm{law})}{=} Z_{a+b}$
(ii) $Z_{a,b} \overset{(\mathrm{law})}{=} \frac{Z_a}{Z_a + Z_b}$

From (i) and (ii), one gets $Z_a \overset{(\mathrm{law})}{=} Z_{a,b} Z_{a+b}$, which implies

(iii) $(Z_a, Z_b) \stackrel{\text{(law)}}{=} Z_{a+b}(Z_{a,b}, 1 - Z_{a,b})$
As an application of (iii), one can show that

$$(N^2, N'^2) \stackrel{\text{(law)}}{=} 2T(Z, 1 - Z)$$

where N and N' are two independent standard Gaussian r.v.'s, T is an exponential r.v. with parameter 1 and is independent of $Z \stackrel{\text{(law)}}{=} Z_{1/2,1/2}$, a so-called arc-sine variable.

1.8 The Law of the Maximum of a Positive Continuous Local Martingale, Which Converges to 0

The following universal result for such a local martingale is:

$$\sup_{t \geq 0} M_t \stackrel{\text{(law)}}{=} \frac{M_0}{U},$$

where U is uniform and independent from M_0.

This is a simple consequence of the optional stopping theorem. Precisely:

Lemma 1.8.1. *Let M be a local continuous martingale with $M_0 = a$, $M_t \geq 0$ and $\lim_{t \to \infty} M_t = 0$. Then*

$$\sup_{t \geq 0} M_t \stackrel{\textit{(law)}}{=} \frac{a}{U}$$

where U is a uniform variable on $[0, 1]$.

Proof. Let $y > a$, then

$$a = E[M_{T_y}] = yP(T_y < \infty) = yP\big(\sup_{t \geq 0} M_t \geq y\big),$$

thus

$$P\big(\sup_t M_t \geq y\big) = \frac{a}{y} = P\Big(\frac{a}{U} \geq y\Big).$$

□

Exercise 1.8.2. The aim of this exercise is to show the identity:
for $F_t \geq 0$, $\mathcal{F}_t$-measurable

$$E\Big[F_t\Big(1 - \frac{M_t}{a}\Big)^+\Big] = E\big[F_t \mathbf{1}_{(g_\infty^{(a)} \leq t)}\big], \tag{1.8.1}$$

where $M_t \geq 0$, is a continuous local martingale, and $M_t \underset{t\to\infty}{\longrightarrow} 0$, and $g_\infty^{(a)} = \sup\{t : M_t = a\}$.

(a) Note that (1.8.1) is equivalent to:

$$P\big(g_\infty^{(a)} \leq t|\mathcal{F}_t\big) = \Big(1 - \frac{M_t}{a}\Big)^+.$$

(b) Deduce (1.8.1) from $\big(g_\infty^{(a)} \leq t\big) = \big(\sup_{u\geq t} M_u \leq a\big)$, then apply Lemma 1.8.1.

1.9 A First Taste of Enlargement Formulae

We are concerned here with the following theorem.

Theorem 1.9.1. *(a) If $L \equiv \sup\{t : (t,\omega) \in \Gamma\}$, where Γ is a set belonging to the predictable σ-field of $(\mathcal{F}_t)$, a given filtration, then all $(\mathcal{F}_t)$ martingales remain $(\mathcal{F}_t^L)$ semimartingales, where $(\mathcal{F}_t^L \equiv \mathcal{F}_t \vee \sigma(L \wedge t))$ is the smallest filtration containing $(\mathcal{F}_t)$ and making L a stopping time.*

(b) If we define $Z_t \equiv Z_t^L = P(L > t|\mathcal{F}_t)$, then a generic $(\mathcal{F}_t)$ martingale (M_t) becomes a semimartingale in $(\mathcal{F}_t^L)$, with canonical decomposition:

$$M_t = \tilde{M}_t + \int_0^{L\wedge t} \frac{d\langle M, Z^L\rangle_s}{Z_s^L} + \int_L^t \frac{d\langle M, 1 - Z^L\rangle_s}{1 - Z_s^L}.$$

We have assumed the hypothesis:
(CA): every $(\mathcal{F}_t)$ martingale is continuous and, for any $(\mathcal{F}_t)$ stopping time T, $P(L = T) = 0$.

Such formulae shall be useful when we shall enlarge a given filtration with, say: $\Lambda_a = \sup\{t : R_t = a\}$ for some transient process R.

A number of computations of Z^L are presented in [3].

1.10 Kolmogorov's Continuity Criterion

This important lemma allows to construct continuous modification of a process which satisfies a simple criterion.

Theorem 1.10.1. *Let $X = (X_x)_{x\in I}$ be a random process indexed by a bounded interval I of $\mathbb{R}$, and taking values in a complete metric space (M, d). Assume the existence of three reals $p, \epsilon, C > 0$ such that for every $x, y \in I$:*

$$E[(d(X_x, X_y))^p] \leq C|x - y|^{1+\epsilon}.$$

Then, there exists a modification $\tilde{X}$ of this process X whose trajectories are Hölder with exponent α, for any $\alpha \in]0, \frac{\epsilon}{p}[$. This means that for any $\alpha \in]0, \frac{\epsilon}{p}[$, there exists a constant $C_\alpha(\omega)$ such that for all $x, y \in I$:

$$d(\tilde{X}_x(\omega), \tilde{X}_y(\omega)) \leq C_\alpha(\omega)|x - y|^\alpha.$$

In particular, $\tilde{X}$ is a continuous modification of X.

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