Chapter 5
Controllability, Lyapunov Exponents, and Upper Bounds

In this chapter, we restrict our attention to smooth systems given by differential equations. Under additional controllability assumptions, we derive upper bounds for the invariance entropy in terms of Lyapunov exponents. These numbers measure the exponential rates of divergence for nearby trajectories, and hence are indicators for stability or instability of the system. In the entropy theory of classical dynamical systems, several relations between entropy and Lyapunov exponents are known. A classic result in this direction is *Pesin's formula* [90] which says that the metric entropy of a $\mathscr{C}^2$-diffeomorphism $f : M \to M$ on a compact Riemannian manifold M with respect to a smooth invariant probability measure μ is given by the μ-integral over the sum of the positive Lyapunov exponents which are defined almost everywhere.[1] Liu [77] generalized this result to the case of (not necessarily invertible) $\mathscr{C}^2$-maps. Ruelle [94] (and independently, Margulis) showed that without the assumption of μ being equivalent to the Riemannian volume and only assuming that f is a $\mathscr{C}^1$-map, the expression in Pesin's formula is still an upper bound for the entropy. The crowning achievement finally is a result by Ledrappier and Young [74] which provides a formula for the metric entropy of a $\mathscr{C}^2$-diffeomorphism which involves a weighted sum of positive Lyapunov exponents, where the weights are certain dimension-like characteristics of the conditional measures on unstable manifolds.

In Chap. 3, we have already seen relations between invariance entropy and Lyapunov exponents for (bi-)linear systems (cf. Theorems 3.1, 3.2, and Corollary 3.2). In this chapter, we use controllability assumptions to obtain further relations of this kind for nonlinear systems. The key idea stems from the paper of Nair et al. [85], who show that the infimal data rate for local uniform asymptotic stabilization of a discrete-time nonlinear system at an equilibrium pair (u_0, x_0) is given by the sum of the logarithms of the unstable eigenvalues associated with the linearization

[1] If the invariant measure is ergodic, the Lyapunov exponents are constant almost everywhere, and hence the integral in Pesin's formula can be replaced by the integrand, that is, the sum of those (almost everywhere constant) Lyapunov exponents which are positive. Moreover, the assumption of f being $\mathscr{C}^2$ can be weakened to $\mathscr{C}^{1+\alpha}$.

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at (u_0, x_0). An essential assumption needed for the proof of this result is that the linearization be controllable. This guarantees that appropriate coder–controllers can be constructed that achieve stabilization with data rates arbitrarily close to the sum of the unstable eigenvalues.

In this chapter, we are going to exploit this idea to obtain upper estimates for the invariance entropy in terms of Lyapunov exponents under appropriate infinitesimal and global controllability assumptions.

5.1 The Upper Bound Theorem for Control Sets

Controllable Topological Systems

Let $\Sigma = (\mathbb{T}, X, U, \mathcal{U}, \varphi)$ be a topological time-invariant system such that X has no isolated points. Recall from Sect. 1.4 that a set $Q \subset X$ has the *no-return property* if $x \in Q$, $\tau \in \mathbb{T}_+$ and $\omega \in \mathcal{U}$ with $\varphi(\tau, x, \omega) \in Q$ implies $\varphi([0,\tau], x, \omega) \subset Q$. That is, trajectories cannot leave the set Q and then return. In particular, all control sets with nonempty interior have this property (see Corollary 1.1). The following proposition contains the key observation which makes it possible to use the ideas of Nair et al. [85] to derive upper bounds for the invariance entropy.

Proposition 5.1. *Let $Q \subset X$ be a set with the no-return property. Assume that (K_1, Q) and (K_2, Q) are two admissible pairs for Σ such that K_2 has nonempty interior, and that for every $x \in K_1$ there exist $\omega_x \in \mathcal{U}$ and $\tau_x \in \mathbb{T}_+$ with $\varphi(\tau_x, x, \omega_x) \in \operatorname{int} K_2$. Then*

$$h_{\mathrm{inv}}(K_1, Q) \le h_{\mathrm{inv}}(K_2, Q).$$

Proof. If $r_{\mathrm{inv}}(\tau, K_2, Q) = \infty$ for all τ greater than some τ_0, we have $h_{\mathrm{inv}}(K_2, Q) = \infty$ and the assertion becomes trivial. If this is not the case, there exists a sequence $\tau_k \to \infty$ such that $r_{\mathrm{inv}}(\tau_k, K_2, Q)$ is finite for every k, which implies that $r_{\mathrm{inv}}(\tau, K_2, Q)$ is finite for all τ. In this case, for every $x \in K_1$ let $\omega_x \in \mathcal{U}$ and $\tau_x \in \mathbb{T}_+$ be as in the assumption. Since $\varphi(\tau_x, \cdot, \omega_x)$ is continuous, we find for every $x \in K_1$ an open neighborhood V_x of x such that $\varphi(\tau_x, V_x, \omega_x) \subset \operatorname{int} K_2$. By the no-return property we have $\varphi([0,\tau_x], y, \omega_x) \subset Q$ for all $y \in K_1 \cap V_x$. The family $\{V_x\}_{x \in K_1}$ is an open cover of K_1 and by compactness there exist $x_1, \ldots, x_n \in K_1$ with $K_1 \subset \bigcup_{i=1}^n V_{x_i}$. Now let $\mathcal{S} = \{\mu_1, \ldots, \mu_k\}$ be a minimal (τ, K_2, Q)-spanning set for some τ. For every index pair (i,j) with $1 \le i \le n$ and $1 \le j \le k$ such that there exists $x \in K_1$ with $y_x := \varphi(\tau_{x_i}, x, \omega_{x_i}) \in \operatorname{int} K_2$ and $\varphi([0,\tau], y_x, \mu_j) \subset Q$, we can define a control function $\nu_{ij} \in \mathcal{U}$ which satisfies

$$\nu_{ij}(t) = \begin{cases} \omega_{x_i}(t) & \text{for } t \in [0, \tau_{x_i}], \\ \mu_j(t - \tau_{x_i}) & \text{for } t > \tau_{x_i}. \end{cases}$$

The set $\tilde{\mathscr{S}}$ of all these control functions has cardinality $\leq nk$. Let $\tilde{\tau} := \tau + \min_{1\leq i\leq n}\tau_{x_i}$. Then, by construction, $\tilde{\mathscr{S}}$ is a $(\tilde{\tau}, K_1, Q)$-spanning set and consequently

$$r_{\mathrm{inv}}(\tau, K_1, Q) \leq r_{\mathrm{inv}}(\tilde{\tau}, K_1, Q) \leq n \cdot r_{\mathrm{inv}}(\tau, K_2, Q).$$

By sending τ to infinity, the assertion follows. □

From the properties of control sets (namely, approximate controllability, controlled invariance, and the no-return property), the next corollary immediately follows.

Corollary 5.1. *Let $D \subset X$ be a control set of Σ. Further let $K_1, K_2 \subset D$ be two compact sets with nonempty interior. Then (K_1, D) and (K_2, D) are admissible and*

$$h_{\mathrm{inv}}(K_1, D) = h_{\mathrm{inv}}(K_2, D).$$

With similar arguments as above, the next result follows.

Proposition 5.2. *Let (K, D) be an admissible pair for Σ such that D is a control set. Assume that there exists a nonempty set $V \subset D$ which is open in X and $\mu \in \mathscr{U}$ such that for every $x \in V$ there is $y \in \mathrm{int}\, D$ and a sequence $t_k \in \mathbb{T}_+$, $t_k \to \infty$, with $\varphi(t_k, x, \mu) \to y$. Then $h_{\mathrm{inv}}(K, D) = 0$.*

Proof. By approximate controllability on D, for every $x \in K$ there exist $\omega_x \in \mathscr{U}$ and $t_x \geq 0$ with $\varphi(t_x, x, \omega_x) \in V$. By continuity of $\varphi(t_x, \cdot, \omega_x)$, there is a neighborhood W_x of x with $\varphi(t_x, W_x, \omega_x) \subset V$. Since K is compact, finitely many of these neighborhoods are sufficient to cover K, say $W_{x_1}, \ldots, W_{x_n}$. We define n control functions by

$$\mu_i(t) := \begin{cases} \omega_{x_i}(t) & \text{for } t \in [0, t_{x_i}], \\ \mu(t - t_{x_i}) & \text{for } t > t_{x_i}. \end{cases}$$

Then for every $x \in K$ there exists $i \in \{1, \ldots, n\}$ and a sequence $t_k \in \mathbb{T}_+$, $t_k \to \infty$, such that $\varphi(t_k, x, \mu_i) \in \mathrm{int}\, D$ for all $k \in \mathbb{N}$. By the no-return property of control sets with nonempty interior, this implies $\varphi(\mathbb{T}_+, x, \mu_i) \subset D$. It follows that $r_{\mathrm{inv}}(\tau, K, D) \leq n$ for all τ and hence $h_{\mathrm{inv}}(K, D) = 0$. □

The assumptions of the proposition are in particular satisfied if there exists a constant control function $\mu \in \mathscr{U}$ such that the classical dynamical system associated with μ, that is, the semigroup action $\mathbb{T}_+ \times X \to X$, $(t, x) \mapsto \varphi(t, x, \mu)$, has a compact attractor A in int D. Then V can be chosen as an open neighborhood of A such that A attracts all trajectories with initial values in V.

Controllable Continuous-Time Smooth Systems

Now we consider a smooth system $\Sigma = (\mathbb{R}, M, \mathbb{R}^m, \mathscr{U}, \varphi)$ given by differential equations

$$\dot{x}(t) = F(x(t), \omega(t)), \quad \omega \in \mathscr{U},$$

with compact control range $\Omega \subset \mathbb{R}^m$ satisfying $\operatorname{int} \Omega \neq \emptyset$. Moreover, we assume that M is a $\mathscr{C}^3$-manifold and $F \in \mathscr{C}^1(M \times \mathbb{R}^m, TM)$.

First we show that under mild assumptions finiteness of $h_{\text{inv}}(K, D)$ holds for a control set D.

Proposition 5.3. *If D is a control set of Σ with nonempty interior such that local accessibility holds on* $\operatorname{int} D$, *then $h_{\text{inv}}(K, D) < \infty$ for every compact set $K \subset D$.*

Proof. Any compact subset of D is contained in a compact subset with nonempty interior. Hence, by Proposition 2.1, we may assume that K has nonempty interior. Using local accessibility, we can construct a periodic controlled trajectory with period $\tau^* > 0$ in D corresponding to some $(x^*, \omega^*) \in \operatorname{int} D \times \mathscr{U}$, and by Proposition 1.23 (iv) it holds that $\varphi(\mathbb{R}_+, x^*, \omega^*) \subset \operatorname{int} D$. Since $\varphi(\mathbb{R}_+, x^*, \omega^*) = \varphi([0, \tau^*], x^*, \omega^*)$ is compact, we find a compact set $\tilde{K} \subset \operatorname{int} D$ with nonempty interior and $\varphi(\mathbb{R}_+, x^*, \omega^*) \subset \operatorname{int} \tilde{K}$. By Corollary 5.1 we may assume that $K = \tilde{K}$. For every $x \in K \subset \operatorname{int} D$ we can find a control function $\omega_x \in \mathscr{U}$ and a time $t_x \geq 0$ with $\varphi(t_x, x, \omega_x) = x^*$ by exact controllability in the interior of D (see Proposition 1.23 (iii)). By Proposition 1.23 (v) we may assume that $t_x \leq T_0$ for all $x \in K$ for some $T_0 > 0$. By switching to the control function ω^* after time t_x we can assume that

$$y_x := \varphi(T_0, x, \omega_x) \in \operatorname{int} K \quad \text{for all } x \in K.$$

Let V_x be a neighborhood of y_x with $V_x \subset \operatorname{int} K$. By continuity there exists a neighborhood W_x of x with $\varphi(T_0, W_x, \omega_x) \subset V_x \subset \operatorname{int} K$. Since $\{W_x\}_{x \in K}$ covers the compact set K, we find $x_1, \ldots, x_n \in K$ with $K \subset \bigcup_{j=1}^n W_{x_j}$. Consequently, the set $\mathscr{S} := \{\omega_{x_1}, \ldots, \omega_{x_n}\}$ is (T_0, K, D)-spanning (by the no-return property). Obviously, one can construct (kT_0, K, D)-spanning sets $\mathscr{S}_k$ for all $k \in \mathbb{N}$ from $\mathscr{S}$ such that $\#\mathscr{S}_k \leq n^k$. This proves that $h_{\text{inv}}(K, D) \leq (\log n)/T_0 < \infty$. □

In the following, we provide a characterization of the interior of $\mathscr{U}$ as a subset of the Banach space $L^\infty(\mathbb{R}, \mathbb{R}^m)$. We denote the L^∞-norm by $\|\cdot\|_\infty$.

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^m$ be a compact set, $(X, \mathscr{A})$ a measurable space, and $f : X \to \mathbb{R}^m$ a measurable function whose image is contained in Ω. Further assume that* $\operatorname{dist}(f(x), \Omega^c) < \varepsilon/3$ *for all $x \in X$ and some $\varepsilon > 0$. Then there exists a measurable function $g : X \to \mathbb{R}^m$ such that $|f(x) - g(x)| < \varepsilon$ and $g(x) \in \Omega^c$ for all $x \in X$.*

Proof. By translation of the set Ω, we may assume that all coordinate functions $f_i : X \to \mathbb{R}$, $i = 1, \dots, m$, are nonnegative measurable functions. It is well-known that such a function can be approximated by a (monotonically increasing) sequence of nonnegative simple functions. In particular, there are simple functions

$$s_i : X \to \mathbb{R}, \quad s_i(x) = \sum_{j=1}^{n_i} a^i_j \mathbb{1}_{A^i_j}(x), \quad i = 1, \dots, m,$$

with $X = \bigcup_{j=1}^{n_i} A^i_j$ for each i, $A^i_j \subset X$ measurable, such that

$$|s_i(x) - f_i(x)| < \frac{\varepsilon}{3\sqrt{m}} \quad \text{for all } x \in X,\ i = 1, \dots, m.$$

Here we used that f is a bounded function, and hence the sequences of simple functions can be chosen such that the convergence is uniform. By adding sets of measure zero, we may assume that the numbers n_i, $i = 1, \dots, m$, are all equal to each other, say $n_i = n$. Now define the sets

$$A(j_1, \dots, j_m) := A^1_{j_1} \cap \dots \cap A^m_{j_m}, \quad j_k \in \{1, \dots, n\}.$$

These sets are obviously measurable and their union is equal to X. We define a measurable function

$$s(x) := \sum_{(j_1, \dots, j_m)} (s_1(x), \dots, s_m(x))^T \mathbb{1}_{A(j_1, \dots, j_m)}(x), \quad s : X \to \mathbb{R}^m.$$

Taking the standard Euclidean norm $|\cdot|$ on $\mathbb{R}^m$, we find that

$$|f(x) - s(x)| < \frac{\varepsilon}{3} \quad \text{for all } x \in X.$$

The assumption that $\operatorname{dist}(f(x), \Omega^c) < \varepsilon/3$ implies

$$\operatorname{dist}(s(x), \Omega^c) = \inf_{u \in \Omega^c} |s(x) - u| \le |s(x) - f(x)| + \operatorname{dist}(f(x), \Omega^c) < \frac{2\varepsilon}{3}$$

for all $x \in X$. By construction, the values of s are the vectors $a(j_1, \dots, j_m) := (a^1_{j_1}, \dots, a^m_{j_m})^T$. Therefore, for each $(j_1, \dots, j_m)$, there exists $b(j_1, \dots, j_m) := (b^1_{j_1}, \dots, b^m_{j_m})^T \in \Omega^c$ with $|a(j_1, \dots, j_m) - b(j_1, \dots, j_m)| < (2\varepsilon)/3$. Define the desired function g as

$$g(x) := \sum_{(j_1, \dots, j_m)} b(j_1, \dots, j_m) \mathbb{1}_{A(j_1, \dots, j_m)}(x), \quad g : X \to \mathbb{R}^m.$$

This gives

$$|f(x) - g(x)| \leq |f(x) - s(x)| + |s(x) - g(x)| < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon,$$

which concludes the proof. □

Proposition 5.4. *For a function $\omega \in L^\infty(\mathbb{R}, \mathbb{R}^m)$ it holds that $\omega \in \operatorname{int} \mathscr{U}$ if and only if there exists a compact set $K \subset \operatorname{int} \Omega$ with $\omega(t) \in K$ for almost all $t \in \mathbb{R}$.*

Proof. We start with the easier direction: Assume that $\omega(t) \in K$ for almost all $t \in \mathbb{R}$ and a compact set $K \subset \operatorname{int} \Omega$. Then, by compactness, we find $\varepsilon > 0$ such that the ε-neighborhood of K is contained in Ω. Hence, if $\|\mu - \omega\|_\infty < \varepsilon$ for some $\mu \in L^\infty(\mathbb{R}, \mathbb{R}^m)$, then $\mu(t) \in \Omega$ almost everywhere, that is, $\mu \in \mathscr{U}$. This shows that $\omega \in \operatorname{int} \mathscr{U}$.

Now, conversely, assume that $\omega \in \operatorname{int} \mathscr{U}$. Then there exists $\varepsilon > 0$ such that $\|\omega - \mu\|_\infty < \varepsilon$ with $\mu \in L^\infty(\mathbb{R}, \mathbb{R}^m)$ implies $\mu \in \mathscr{U}$, that is, if $|\omega(t) - \mu(t)| < \varepsilon$ for almost all $t \in \mathbb{R}$, then $\mu(t) \in \Omega$ for almost all $t \in \mathbb{R}$.

By a general fact in real analysis, $\operatorname{int} \Omega$ can be written as the countable union of the elements of an increasing sequence of compact sets, that is, $\operatorname{int} \Omega = \bigcup_{n \geq 1} K_n$, K_n compact with $K_n \subset K_{n+1}$. Indeed, such a sequence can be constructed as follows: Let $\{u_k\}$ be a countable dense subset of $\operatorname{int} \Omega$ and consider for each u_k all compact balls centered at u_k of rational radius which are contained in $\operatorname{int} \Omega$. The family of all these balls is countable and its union is easily seen to be $\operatorname{int} \Omega$. Enumerate the members of this family and define K_n to be the union of the first n members. This gives the desired increasing sequence of compact sets. Moreover, from this construction it can easily be seen that every $u \in \operatorname{int} \Omega$ is contained in the interior of one of the sets K_n.

This construction also implies that there is $n_0 \geq 1$ such that

$$u \in \operatorname{int} \Omega \backslash K_{n_0} \quad \Rightarrow \quad \operatorname{dist}(u, \mathbb{R}^m \backslash \Omega) < \frac{\varepsilon}{3}. \tag{5.1}$$

We prove this by contradiction: Assume that such n_0 does not exist. Then for every $n \geq 1$ there is $v_n \in \operatorname{int} \Omega \backslash K_n$ with $\operatorname{dist}(v_n, \mathbb{R}^m \backslash \Omega) \geq \varepsilon/3$, that is, $|v_n - w| \geq \varepsilon/3$ for all $w \notin \Omega$. By compactness of Ω we may assume that $v_n \to v \in \Omega$. The limit v on the one hand satisfies $|v - w| \geq \varepsilon/3$ for all $w \notin \Omega$. On the other hand, $v \in \partial\Omega$, since $v \in \operatorname{int} \Omega$ implies $v \in \operatorname{int} K_{n_1}$ for some n_1 which gives $v_n \in K_{n_1}$ for all sufficiently large n, contradicting the definition of the sequence v_n.

Now consider the compact set $K := K_{n_0} \subset \operatorname{int} \Omega$ which satisfies (5.1). We claim that $\omega(t) \in K$ for almost all $t \in \mathbb{R}$. Indeed, if this was not true, there would be a set $I \subset \mathbb{R}$ of positive measure with $|\omega(t) - w| < \varepsilon/3$ for all $t \in I$ and all $w \notin \Omega$. By Lemma 5.1 there exists a measurable function $\mu : I \to \mathbb{R}^m \backslash \Omega$ with $|\mu(t) - \omega(t)| < \varepsilon$ for all $t \in I$. We can extend this function to a measurable function $\mu : \mathbb{R} \to \mathbb{R}^m$ by putting $\mu(t) := \omega(t)$ for all $t \in \mathbb{R} \backslash I$. This gives $\|\omega - \mu\|_\infty < \varepsilon$ which is a contradiction to the choice of ε. □

Given a Riemannian metric g on M, to every trajectory $\varphi(\cdot, x, \omega)$ of the smooth system Σ we can associate a finite set of Lyapunov exponents. For the control function ω, the Lyapunov exponent at x in direction $v \in T_x M$, $v \neq 0_x$, is given by

$$\lambda(v) = \lambda(v; x, \omega) := \limsup_{t \to \infty} \frac{1}{t} \log |\mathrm{d}_x \varphi_{t,\omega}(v)| \in \mathbb{R} \cup \{-\infty, +\infty\}.$$

We also call these numbers the *Lyapunov exponents at* (ω, x). Some basic and well-known properties are summarized in the following proposition (see also Arnold [4, Sect. 3.2.1]).[2]

Proposition 5.5. *The following assertions hold:*

(i) $\lambda(\alpha v) = \lambda(v)$ *for all nonzero* $v \in T_x M$ *and* $\alpha \in \mathbb{R} \backslash \{0\}$.
(ii) $\lambda(v + w) \leq \max\{\lambda(v), \lambda(w)\}$ *for all nonzero* $v, w \in T_x M$ *with* $w \neq -v$, *with equality if* $\lambda(v) \neq \lambda(w)$.
(iii) The number of different Lyapunov exponents $\lambda(v; x, \omega)$, $v \in T_x M \backslash \{0_x\}$, *is bounded by* $d = \dim M$.
(iv) If (u, x) *is an equilibrium pair, the Lyapunov exponents* $\lambda(v; x, u)$ *are the real parts of the eigenvalues of* $\nabla F_u(x) : T_x M \to T_x M$.
(v) If there is a compact set $K \subset M$ *with* $\varphi(\mathbb{R}_+, x, \omega) \subset K$, *then the Lyapunov exponents* $\lambda(v; x, \omega)$ *are all* $< \infty$.
(vi) If two Riemannian metrics are equivalent on the image of a trajectory $\varphi(\cdot, x, \omega)$, *then the Lyapunov exponents with respect to these two metrics are the same. In particular, if* M *is compact, the Lyapunov exponents of a trajectory are independent of the metric.*
(vii) For a periodic trajectory, the Lyapunov exponents are independent of the metric.

Remark 5.1. From the statements of Proposition 5.5 we mainly use the fourth and the seventh. The proof of statement (vii) is contained in the proof of the next theorem. Statement (iv) is an easy consequence of the Riemannian variational equation (see Proposition A.3). Indeed, for an equilibrium pair (ω, x) the variational equation becomes an autonomous linear equation on $T_x M$ whose solutions have the form $z(t) = \exp(t \nabla F_\omega(x))v$, $v \in T_x M$, which immediately implies the assertion.

Each Lyapunov exponent has a *multiplicity* which can be defined as follows. For every (ω, x) let $\lambda_1(\omega, x) < \lambda_2(\omega, x) < \cdots < \lambda_{s(\omega,x)}(\omega, x)$ be the associated Lyapunov exponents. Then there exists a filtration

$$\{0_x\} = V_0(\omega, x) \subsetneq V_1(\omega, x) \subsetneq \cdots \subsetneq V_{s(\omega,x)}(\omega, x) = T_x M,$$

[2]In the dynamical systems literature, usually the notion of *Lyapunov exponents* refers to the Lyapunov exponents associated with an invariant measure. Sometimes, the Lyapunov exponents as we define them are called *upper Lyapunov exponents* because of the upper limit in their definition.

such that

$$V_i(\omega, x) = \{0_x\} \cup \{v \in T_x M \backslash \{0_x\} \,:\, \lambda(v; x, \omega) \leq \lambda_i(\omega, x)\}\,.$$

The multiplicity of the Lyapunov exponent $\lambda_i(\omega, x)$ is defined as the natural number $\dim V_i(\omega, x) - \dim V_{i-1}(\omega, x)$.

Before we state the main result of this section, let us recall the fundamental lemma of *Floquet theory*. A proof can be found, for instance, in Chicone [17, Theorem 2.47].

Lemma 5.2 (Fundamental Lemma of Floquet Theory). *Let C be a nonsingular real $n \times n$-matrix. Then there exists a (possibly complex) $n \times n$-matrix A with* $\exp(A) = C$. *Moreover, there exists a real $n \times n$-matrix B with* $\exp(B) = C^2$.

In the formulation of our theorem we already use the knowledge that the Lyapunov exponents of a periodic trajectory are metric-independent, as asserted in statement (vii) of Proposition 5.5. This fact also becomes clear in the first step of the proof.

Theorem 5.1. *Let $D \subset M$ be a control set with nonempty interior and compact closure. Let $(\varphi(\cdot, x_0, \omega_0), \omega_0(\cdot))$ be a τ_0-periodic controlled trajectory which is regular on $[0, \tau_0]$ such that $(x_0, \omega_0) \in \operatorname{int} D \times \operatorname{int} \mathscr{U}$. Moreover, let $\rho_1, \ldots, \rho_r$ be the different Lyapunov exponents at (ω_0, x_0) with corresponding multiplicities $d_1, \ldots, d_r$. Then for every compact subset $K \subset D$ and every superset $Q \supset D$ the pair (K, Q) is admissible and*

$$h_{\mathrm{inv}}(K, Q) \leq \sum_{j=1}^{r} \max\{0, d_j \rho_j\}. \tag{5.2}$$

The basic idea of the proof of Theorem 5.1 is to steer close to the point x_0 on the periodic trajectory and then use local controllability along the trajectory to stay in a neighborhood of the periodic orbit for arbitrary future times, that is, to stabilize the system at the periodic trajectory. This can be done by using a collection of control functions whose cardinality is arbitrarily close to the sum of the positive Lyapunov exponents (up to log and dividing by the time), which can be regarded as a measure for how fast one is driven away from the periodic trajectory on average without applying controls. The actual proof is quite lengthy and technical, so we give a short overview of the main ideas involved before we start: We proceed in three steps. In the first step, we use the fundamental lemma of Floquet theory in order to write the solutions of the linearization along the controlled trajectory $(\varphi(\cdot, x_0, \omega_0), \omega_0(\cdot))$ in terms of the matrix exponential of an endomorphism R of $T_{x_0} M$. Then we construct an adapted Riemannian metric, which yields an orthonormal Jordan basis for R. In the second step, we define several constants. In particular, a (large) time step $\tau \in \tau_0 \mathbb{N}$ and a (small) radius $b_0 > 0$ are defined such that the controllability of the linearization can be used in order to steer the system from the ball $B(x_0, b_0)$ to itself in time τ, using a finite number of control functions that is related to the eigenvalues of R and hence to the Lyapunov exponents $\rho_1, \ldots, \rho_r$. This is done in Step 3 by subdividing a cube of side length $2b_0$ centered at the origin of $T_{x_0}M$ into an appropriate number of subcuboids whose midpoints are steered to $0_{x_0} \in T_{x_0}M$ in time τ via the linearization. Using the Riemannian exponential map at x_0, it is shown that the corresponding control functions also work for the nonlinear system in order to get back to $B(x_0, b_0)$ in time τ. This process can be repeated and thus yields $(k\tau, B(x_0, b_0), Q)$-spanning sets for all $k \in \mathbb{N}$. By choosing τ big enough and b_0 small enough, the corresponding cardinality growth rate of these sets comes arbitrarily close to $\sum_j \max\{0, d_j\rho_j\}$. Since $h_{\mathrm{inv}}(K, Q)$ does not depend on the set K as long as it has a nonempty interior, this proves the assertion.

Proof (of Theorem 5.1). By controlled invariance of D, it is clear that every pair (K, Q) with $K \subset D$ and $Q \supset D$ is admissible. For brevity in notation, the map ϕ^{x_0,ω_0} associated with the linearization along $(\varphi(\cdot, x_0, \omega_0), \omega_0(\cdot))$ is simply denoted by ϕ (cf. Sect. 1.5). The proof of estimate (5.2) now proceeds in three steps.

Step 1. Let M be endowed with an arbitrary Riemannian metric and consider the automorphism

$$A := \mathrm{D}\varphi_{2\tau_0}(x_0, \omega_0)(\cdot, \mathbf{0}) \overset{(1.6)}{=} \phi(2\tau_0, \cdot, \mathbf{0}) : T_{x_0}M \to T_{x_0}M.$$

From Proposition 1.26 (iv) it follows that $A = \phi(\tau_0, \cdot, \mathbf{0})^2$, and hence from Lemma 5.2 it follows that there exists $R \in \mathscr{L}(T_{x_0}M, T_{x_0}M)$ with

$$A = \exp(2\tau_0 R).$$

From Proposition 1.26 (iv) we get

$$\phi(2\tau_0 k, \lambda, \mathbf{0}) = A^k\lambda = \exp(2\tau_0 kR)\,\lambda \quad \text{for all } \lambda \in T_{x_0}M,\ k \in \mathbb{Z}_+. \tag{5.3}$$

We claim that the real parts of the eigenvalues of R coincide with the Lyapunov exponents at (ω_0, x_0). To show this, we write every $t > 0$ as $t = 2\tau_0 k + s$ with $k \in \mathbb{Z}_+$ and $s \in [0, 2\tau_0)$. Then for all $\lambda \in T_{x_0}M$ we obtain

$$\phi(t, \lambda, \mathbf{0}) = \phi(s, \phi(k(2\tau_0), \lambda, \mathbf{0}), \mathbf{0}) \overset{(5.3)}{=} \phi(s, \cdot, \mathbf{0}) \exp(2\tau_0 kR)\,\lambda.$$

Hence, it follows that

$$l_1 \left|\exp(2k\tau_0 R)\,\lambda\right| \le |\phi(t, \lambda, \mathbf{0})| \le l_2 \left|\exp(2k\tau_0 R)\,\lambda\right|$$

with the positive constants

$$l_1 := \min_{s\in[0,2\tau_0]} \left\|\phi(s, \cdot, \mathbf{0})^{-1}\right\|^{-1}, \quad l_2 := \max_{s\in[0,2\tau_0]} \|\phi(s, \cdot, \mathbf{0})\|.$$

By Proposition 1.26 (ii) we have

$$\mathrm{d}_{x_0}\varphi_{t,\omega_0}(\lambda) = \phi(t, \lambda, \mathbf{0}),$$

and hence the exponential growth rate of $|\mathrm{d}_{x_0}\varphi_{t,\omega_0}(\lambda)|$ for $t \to \infty$ equals the growth rate of $|\exp(2\tau_0\lfloor t/(2\tau_0)\rfloor R)\lambda|$ for all nonzero $\lambda \in T_{x_0}M$, which implies the claim.

Now choose a basis B_{x_0} of $T_{x_0}M$ adapted to the real Jordan structure of R and let $L_1(R), \dots, L_r(R)$ be the different Lyapunov spaces of R, that is, the sums of the generalized eigenspaces corresponding to eigenvalues with the same real part. Then we have the decomposition

$$T_{x_0}M = L_1(R) \oplus \cdots \oplus L_r(R).$$

Let $d_j = \dim L_j(R)$ and denote by $\lambda^{(j)} \in L_j(R)$ the j-th component of a vector $\lambda \in T_{x_0}M$ with respect to this decomposition. Moreover, denote by ρ_j the common real part of the eigenvalues corresponding to $L_j(R)$. The restriction of R to $L_j(R)$ is denoted by R_j. Now let g be a Riemannian metric on M of class $\mathscr{C}^2$ such that the basis B_{x_0} is orthonormal with respect to g_{x_0}, and let ϱ denote the Riemannian distance induced by g. In order to obtain a metric with this property, one can start with an arbitrary $\mathscr{C}^2$-metric $\tilde{g}$ on M. Then one takes a chart (ψ, V) around x_0 and an inner product $(\cdot,\cdot)$ on $\mathbb{R}^d$ such that B_{x_0} is orthonormal with respect to the induced inner product $(\mathrm{d}_{x_0}\psi(\cdot), \mathrm{d}_{x_0}\psi(\cdot))$ on $T_{x_0}M$. On V consider the pullback $\hat{g}$ of $(\cdot,\cdot)$ by ψ, that is,

$$\hat{g}(x)(v, w) := (\mathrm{d}_x\psi(v), \mathrm{d}_x\psi(w)) \quad \text{for all } x \in V,\ v, w \in T_xM.$$

Let $\theta : M \to [0, 1]$ be a cut-off function of class $\mathscr{C}^2$ such that $\operatorname{supp}\theta \subset V$ and $\theta(x) \equiv 1$ on a compact neighborhood W of x_0 (see Proposition A.6). Define g by

$$g(x) := \begin{cases} \theta(x)\hat{g}(x) + (1 - \theta(x))\tilde{g}(x) & \text{for all } x \in V, \\ \tilde{g}(x) & \text{for all } x \in M \backslash V. \end{cases}$$

It can easily be seen that g is a Riemannian metric on M with g_{x_0} having the desired property.

Step 2. We fix some constants: Let S_0 be a real number which satisfies

$$S_0 > \sum_{j=1}^{r} \max\{0, d_j\rho_j\}.$$

Choose $\xi = \xi(S_0) > 0$ such that

$$0 < d\xi < S_0 - \sum_{j=1}^{r} \max\{0, d_j\rho_j\}. \tag{5.4}$$

Let $\delta \in (0, \xi)$ be chosen small enough such that $\rho_j < 0$ implies $\rho_j + \delta < 0$ for all $j \in \{1, \ldots, r\}$. From Lemma B.2 it follows that there exists a constant $c = c(\delta) \geq 1$ such that

$$\forall j \in \{1, \ldots, r\}\ \forall k \in \mathbb{Z}_+ : \ \left\|\exp(k\tau_0 R_j)\right\| \leq c\mathrm{e}^{(\rho_j+\delta)k\tau_0}, \tag{5.5}$$

where $\|\cdot\|$ denotes the operator norm on $\mathscr{L}(T_{x_0}M, T_{x_0}M)$ induced by g_{x_0}. For every $t > 0$ we define positive integers

$$M_j(t) := \begin{cases} \lfloor \mathrm{e}^{(\rho_j+\xi)t} \rfloor + 1 & \text{if } \rho_j \geq 0 \\ 1 & \text{if } \rho_j < 0 \end{cases}, \quad j = 1, \ldots, r. \tag{5.6}$$

Moreover, we define a function $\beta : (0, \infty) \to (0, \infty)$ by

$$\beta(t) := c\sqrt{r} \max_{1 \leq j \leq r} \left[\mathrm{e}^{(\rho_j+\delta)t} \frac{\sqrt{d_j}}{M_j(t)} \right]. \tag{5.7}$$

If $\rho_j < 0$, then (by definition) $\rho_j + \delta < 0$ and $M_j(t) \equiv 1$. This implies that $\mathrm{e}^{(\rho_j+\delta)t}(\sqrt{d_j}/M_j(t))$ converges to zero for $t \to \infty$. If $\rho_j \geq 0$, we have $M_j(t) \geq \mathrm{e}^{(\rho_j+\xi)t}$ by (5.6) and hence

$$\mathrm{e}^{(\rho_j+\delta)t} \frac{\sqrt{d_j}}{M_j(t)} \leq \mathrm{e}^{(\rho_j+\delta)t} \frac{\sqrt{d_j}}{\mathrm{e}^{(\rho_j+\xi)t}} = \sqrt{d_j}\mathrm{e}^{(\delta-\xi)t}.$$

Since $\delta \in (0, \xi)$, we have $\delta - \xi < 0$ and hence the term above converges to zero for $t \to \infty$. Thus, also $\beta(t) \to 0$ for $t \to \infty$. This implies that for given $\varepsilon > 0$ we can choose a number $\tau = 2k\tau_0$ with $k \in \mathbb{N}$ big enough such that

$$\beta(\tau) < 1 \quad \text{and} \quad \frac{d}{\tau}\log(2) < \varepsilon. \tag{5.8}$$

Since we assume regularity of $(\varphi(\cdot, x_0, \omega_0), \omega_0(\cdot))$ on $[0, \tau_0]$, by Proposition 1.30 there exists a constant $C > 0$ with the following property (note that regularity on $[0, \tau_0]$ implies regularity on $[0, \tau]$):

$$\forall \lambda \in T_{x_0}M\ \exists \mu \in L^\infty([0, \tau], \mathbb{R}^m) : \begin{cases} \phi(\tau, \lambda, \mu) = 0_{x_0} \\ \text{and} \\ \|\mu\|_{[0,\tau]} \leq C|\lambda|. \end{cases} \tag{5.9}$$

Let $W_1 \subset T_{x_0}M$ and $W_2 \subset M$ be open neighborhoods of 0_{x_0} and x_0, respectively, such that $\exp_{x_0} : W_1 \to W_2$ is a $\mathscr{C}^1$-diffeomorphism. The inverse of $\exp_{x_0}|_{W_1}$ is simply denoted by $\exp_{x_0}^{-1}$. Now choose $b_0 > 0$ small enough such that the following conditions are satisfied:

$$\left.\begin{cases} \operatorname{cl} B(0_{x_0}, b_0) \subset W_1, \\ \operatorname{cl} B(x_0, b_0) \subset D, \\ \operatorname{cl} B(\omega_0(t), C\sqrt{d}b_0) \subset \Omega & \text{for almost all } t \in [0, \tau_0], \\ \varphi(\tau, \operatorname{cl} B(x_0, b_0), \omega) \subset W_2 & \text{if } \|\omega - \omega_0\|_{[0,\tau]} \le C\sqrt{d}b_0. \end{cases}\right\} \tag{5.10}$$

The second and third inclusion are possible, since $x_0 \in \operatorname{int} D$ and, by Proposition 5.4, $\omega_0(t)$ is contained in a compact subset of $\operatorname{int}\Omega$ for almost all $t \in [0, \tau_0]$. The last one is possible by continuity of $(x, \omega) \mapsto \varphi(\tau, x, \omega)$. By Proposition 1.29 there exists a function $\zeta = \zeta_{\tau,\sqrt{d}C} : [0, \alpha) \to \mathbb{R}_+$ ($\alpha > 0$) with

$$\left|\exp_{x_0}^{-1}(\varphi(\tau, x, \omega)) - \phi(\tau, \exp_{x_0}^{-1}(x), \omega - \omega_0)\right| \le \zeta(b)b \tag{5.11}$$

for all $(x, \omega) \in M \times \mathscr{U}$ with $\varrho(x, x_0) \le b \le b_0$ and $\|\omega - \omega_0\|_{[0,\tau]} \le C\sqrt{d}b$, and $\zeta(b) \to 0$ for $b \to 0$. We can assume that $b_0 < \alpha$ and hence $\zeta(b_0)$ is defined. Because of the strict inequality $\beta(\tau) < 1$ we can also assume that b_0 is chosen small enough such that

$$\sqrt{r}\zeta(b_0) + \beta(\tau) \le 1. \tag{5.12}$$

Step 3. By Corollary 5.1 and (5.10) we can assume that $K = \operatorname{cl} B(x_0, b_0)$. Consider a d-dimensional compact cube $\mathscr{C}$ in $T_{x_0}M$ centered at the origin with sides of length $2b_0$ parallel to the vectors of the basis B_{x_0}. Then $\exp_{x_0}^{-1}(K) = \operatorname{cl} B(0_{x_0}, b_0) \subset T_{x_0}M$, since $\exp_{x_0}$ is a radial isometry, and hence $\exp_{x_0}^{-1}(K) \subset \mathscr{C}$. Partition $\mathscr{C}$ by dividing each coordinate axis corresponding to a component of the j-th Lyapunov space of R into $M_j(\tau)$ intervals of equal length. The total number of subcuboids in this partition is $\prod_{j=1}^{r} M_j(\tau)^{d_j}$. Now pick an arbitrary $x \in \operatorname{cl} B(x_0, b_0)$. Let $\gamma_0 : [0, 1] \to M$ be a shortest geodesic from x_0 to x and let $\lambda_x \in \mathscr{C}$ be the center of a subcuboid which contains $\exp_{x_0}^{-1}(x) = \dot{\gamma}_0(0)$. (Note that $|\dot{\gamma}_0(0)| = \mathscr{L}(\gamma_0) = \varrho(x_0, x) \le b_0$.) Then the following estimate holds, where the additional superscripts denote components of vectors within the corresponding Lyapunov spaces of R:

$$\begin{aligned} \left|\dot{\gamma}_0(0)^{(j)} - \lambda_x^{(j)}\right| &= \left[\sum_{l=1}^{d_j} \left(\dot{\gamma}_0(0)^{(j,l)} - \lambda_x^{(j,l)}\right)^2\right]^{1/2} \\ &\le \left[\sum_{l=1}^{d_j} \left(\frac{b_0}{M_j(\tau)}\right)^2\right]^{1/2} = \frac{\sqrt{d_j}}{M_j(\tau)} b_0. \end{aligned} \tag{5.13}$$

By (5.9) there exists $\omega_x \in L^\infty([0, \tau], \mathbb{R}^m)$ such that $\phi(\tau, \lambda_x, \omega_x - \omega_0) = 0_{x_0}$ or equivalently,

$$\phi(\tau, \lambda_x, \omega_x) = \phi(\tau, 0_{x_0}, \omega_0) \tag{5.14}$$

and

$$\|\omega_x - \omega_0\|_{[0,\tau]} \leq C|\lambda_x| \leq C\left[\sum_{j=1}^{r}\sum_{l=1}^{d_j}\left|\lambda_x^{(j,l)}\right|^2\right]^{1/2} \leq C\sqrt{d}b_0,$$

since $\lambda_x \in \mathscr{C}$ implies $|\lambda_x^{(j,l)}| \leq b_0$ for each component. By (5.10) it holds that $\omega_x \in \mathscr{U}$ and

$$\varphi(\tau, x, \omega_x) \in W_2.$$

Let $\gamma_1 : [0, 1] \to M$ be a shortest geodesic from x_0 to $\varphi(\tau, x, \omega_x)$. Then

$$\varrho\left(\varphi(\tau, x, \omega_x), x_0\right) = \mathscr{L}(\gamma_1) = \int_0^1 \underbrace{|\dot{\gamma}_1(t)|}_{=\text{ constant}}\, \mathrm{d}t = |\dot{\gamma}_1(0)|\,.$$

By the triangle inequality we have

$$\begin{aligned}\left|\dot{\gamma}_1(0)^{(j)}\right| \leq &\left|\dot{\gamma}_1(0)^{(j)} - \phi\left(\tau, \dot{\gamma}_0(0), \omega_x - \omega_0\right)^{(j)}\right| \\ &+ \left|\phi(\tau, \dot{\gamma}_0(0), \omega_x - \omega_0)^{(j)}\right|.\end{aligned}$$

Since g is chosen such that the Lyapunov spaces of R are orthogonal, for the first term we obtain

$$\begin{aligned}&\left|\dot{\gamma}_1(0)^{(j)} - \phi(\tau, \dot{\gamma}_0(0), \omega_x - \omega_0)^{(j)}\right| \\ &= \left|[\dot{\gamma}_1(0) - \phi(\tau, \dot{\gamma}_0(0), \omega_x - \omega_0)]^{(j)}\right| \\ &\leq |\dot{\gamma}_1(0) - \phi(\tau, \dot{\gamma}_0(0), \omega_x - \omega_0)| \\ &= \left|\exp_{x_0}^{-1}(\varphi(\tau, x, \omega_x)) - \phi(\tau, \exp_{x_0}^{-1}(x), \omega_x - \omega_0)\right| \\ &\overset{(5.11)}{\leq} \zeta(b_0)b_0.\end{aligned}$$

By linearity of $\phi(\tau, \cdot, \cdot)$, for the second term we obtain

$$\begin{aligned}\left|\phi(\tau, \dot{\gamma}_0(0), \omega_x - \omega_0)^{(j)}\right| &= \left|\phi(\tau, \dot{\gamma}_0(0), \omega_x)^{(j)} - \phi(\tau, 0_{x_0}, \omega_0)^{(j)}\right| \\ &\overset{(5.14)}{=} \left|\phi(\tau, \dot{\gamma}_0(0), \omega_x)^{(j)} - \phi(\tau, \lambda_x, \omega_x)^{(j)}\right| \\ &= \left|\phi(\tau, \dot{\gamma}_0(0) - \lambda_x, \mathbf{0})^{(j)}\right| \\ &\overset{(5.3)}{=} \left|[\exp(2k\tau_0 R)(\dot{\gamma}_0(0) - \lambda_x)]^{(j)}\right| \\ &= \left|[\exp(\tau R)(\dot{\gamma}_0(0) - \lambda_x)]^{(j)}\right|.\end{aligned}$$

By invariance of the Lyapunov spaces of R under $\exp(\tau R)$, we get

$$\begin{aligned}\left|\phi(\tau, \dot{\gamma}_0(0), \omega_x - \omega_0)^{(j)}\right| &= \left|\exp(\tau R)(\dot{\gamma}_0(0) - \lambda_x)^{(j)}\right| \\ &\leq \left\|\exp(\tau R_j)\right\| \left|(\dot{\gamma}_0(0) - \lambda_x)^{(j)}\right| \\ &\overset{(5.5)}{\leq} c\mathrm{e}^{(\rho_j+\delta)\tau} \left|(\dot{\gamma}_0(0) - \lambda_x)^{(j)}\right|.\end{aligned}$$

Altogether, we have

$$\begin{aligned}\left|\dot{\gamma}_1(0)^{(j)}\right| &\leq \zeta(b_0)b_0 + c\mathrm{e}^{(\rho_j+\delta)\tau} \left|(\dot{\gamma}_0(0) - \lambda_x)^{(j)}\right| \\ &\overset{(5.13)}{\leq} \zeta(b_0)b_0 + c\mathrm{e}^{(\rho_j+\delta)\tau} \frac{\sqrt{d_j}}{M_j(\tau)} b_0.\end{aligned}$$

By orthogonality of the Lyapunov spaces of R, it follows that

$$\begin{aligned}\varrho\left(\varphi(\tau, x, \omega_x), x_0\right) &= |\dot{\gamma}_1(0)| = \left(\sum_{j=1}^{r} \left|\dot{\gamma}_1(0)^{(j)}\right|^2\right)^{1/2} \\ &\leq \left(\sum_{j=1}^{r} \left(\zeta(b_0)b_0 + c\mathrm{e}^{(\rho_j+\delta)\tau} \frac{\sqrt{d_j}}{M_j(\tau)} b_0\right)^2\right)^{1/2} \\ &\overset{(\Delta)}{\leq} \sqrt{r}\zeta(b_0)b_0 + \left(\sum_{j=1}^{r} \left(c\mathrm{e}^{(\rho_j+\delta)\tau} \frac{\sqrt{d_j}}{M_j(\tau)} b_0\right)^2\right)^{1/2} \\ &\leq \sqrt{r}\zeta(b_0)b_0 + c\sqrt{r} \max_{1\leq j\leq r} \left[\mathrm{e}^{(\rho_j+\delta)\tau} \frac{\sqrt{d_j}}{M_j(\tau)}\right] b_0 \\ &\overset{(5.7)}{=} \left[\sqrt{r}\zeta(b_0) + \beta(\tau)\right] b_0 \overset{(5.12)}{\leq} b_0.\end{aligned}$$

The estimate (Δ) follows from the triangle inequality in $\mathbb{R}^r$. Hence, we have proved that $\prod_{j=1}^{r} M_j(\tau)^{d_j}$ admissible control functions are sufficient to steer the system from all states in K back to K in time τ. By the no-return property of control sets it follows that the trajectories do not leave D within the time interval $(0, \tau)$. By iterated concatenation of these control functions we can construct an $(n\tau, K, D)$-spanning set for each $n \in \mathbb{N}$ with $(\prod_{j=1}^{r} M_j(\tau)^{d_j})^n$ elements and hence we obtain

$$r_{\mathrm{inv}}(n\tau, K, D) \leq \left(\prod_{j=1}^{r} M_j(\tau)^{d_j}\right)^n = \left(\prod_{j:\,\rho_j\geq 0} \left(\lfloor \mathrm{e}^{(\rho_j+\xi)\tau} \rfloor + 1\right)^{d_j}\right)^n,$$

which implies

$$\begin{aligned}
h_{\mathrm{inv}}(K,Q) &\le h_{\mathrm{inv}}(K,D) = \limsup_{n\to\infty} \frac{1}{n\tau} \log r_{\mathrm{inv}}(n\tau,K,D) \\
&\le \frac{1}{\tau} \sum_{j:\,\rho_j \ge 0} \log \left(\lfloor \mathrm{e}^{(\rho_j+\xi)\tau} \rfloor + 1\right)^{d_j} \\
&= \sum_{j:\,\rho_j \ge 0} d_j \frac{1}{\tau} \log \left(\lfloor \mathrm{e}^{(\rho_j+\xi)\tau} \rfloor + 1\right) \\
&\le \sum_{j:\,\rho_j \ge 0} d_j \frac{1}{\tau} \log \left(2\mathrm{e}^{(\rho_j+\xi)\tau}\right) \\
&= \sum_{j:\,\rho_j \ge 0} d_j \left(\frac{\log(2)}{\tau} + (\rho_j + \xi)\right) \\
&\le \frac{d}{\tau} \log(2) + d\xi + \sum_{j=1}^{r} \max\{0, d_j \rho_j\} \\
&\overset{(5.4)}{<} \frac{d}{\tau} \log(2) + S_0 \overset{(5.8)}{<} S_0 + \varepsilon.
\end{aligned}$$

The first equality follows from Proposition 2.6. Since ε can be chosen arbitrarily small and S_0 arbitrarily close to $\sum_{j=1}^{r} \max\{0, d_j \rho_j\}$, the assertion of the theorem follows. □

Remark 5.2. It is clear that the above theorem implies the estimate

$$h_{\mathrm{inv}}(K,Q) \le \inf_{(\omega,x)} \sum_{j=1}^{r(\omega,x)} \max\{0, d_j(\omega,x)\rho_j(\omega,x)\}, \tag{5.15}$$

where the infimum is taken over all $(\omega,x) \in \mathcal{U} \times M$ such that the controlled trajectory $(\varphi(\cdot,x,\omega),\omega(\cdot))$ is periodic and regular with $x \in \operatorname{int} D$ and $\omega \in \operatorname{int} \mathcal{U}$. In general, it is not clear if any such trajectory exists. However, in many cases we can guarantee their existence. A quite general approach in this direction is worked out in Sect. 5.2.

Remark 5.3. Estimates for the topological entropy of diffeomorphisms, which are formally similar to (5.15), can be found in the work of Catalan and Tahzibi [16]. However, these results are of generic nature and use the variational principle.

Since an equilibrium pair is a τ-periodic controlled trajectory for every $\tau > 0$, the following result immediately follows (using Proposition 5.5 (iv)).

Corollary 5.2. *Let $D \subset M$ be a control set with nonempty interior and compact closure. Let $(\omega_0, x_0) \in \operatorname{int} \Omega \times \operatorname{int} D$ be a regular equilibrium pair. Then for every compact set $K \subset D$ and every superset $Q \supset D$ we have*

$$h_{\mathrm{inv}}(K, Q) \leq \sum_{\lambda \in \sigma(\nabla F_{\omega_0}(x_0))} \max\{0, n_\lambda \operatorname{Re}(\lambda)\}.$$

Corollary 5.3. *Consider a linear system $\Sigma = (\mathbb{R}, \mathbb{R}^d, \mathbb{R}^m, \mathscr{U}, \varphi)$ given by differential equations associated with a controllable matrix pair (A, B) such that A is hyperbolic (that is, A has no eigenvalues on the imaginary axis). Further assume that the control range Ω is a compact and convex set with $0 \in \operatorname{int} \Omega$. Let $D \subset \mathbb{R}^d$ be the unique control set of Σ with nonempty interior. Then for every compact set $K \subset D$ it holds that*

$$h_{\mathrm{inv}}(K, D) \leq \sum_{\lambda \in \sigma(A)} \max\{0, n_\lambda \operatorname{Re}(\lambda)\}. \tag{5.16}$$

If, additionally, K has positive Lebesgue measure and $Q = \operatorname{cl} D$, then

$$h_{\mathrm{inv}}(K, Q) = h_{\mathrm{inv,out}}(K, Q) = \sum_{\lambda \in \sigma(A)} \max\{0, n_\lambda \operatorname{Re}(\lambda)\}. \tag{5.17}$$

Proof. As noted in Remark 3.1, the assumptions about the matrix pair (A, B) and the control range Ω guarantee the existence of a unique control set $D = \operatorname{cl} \mathscr{O}^+(0) \cap \mathscr{O}^-(0)$ with nonempty interior and compact closure. In particular, $0 \in \operatorname{int} D$. Then the pair $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^d$ is an equilibrium pair which is regular by the controllability assumption. Hence, Corollary 5.2 implies (5.16). Formula (5.17) follows from the combination of Theorem 3.1 with (5.16) and the fact that $h_{\mathrm{inv,out}}(K, Q) \leq h_{\mathrm{inv}}(K, Q)$. □

Recall the definition of inner control sets (Definition 2.6). For such sets, the estimate of Theorem 5.1 holds for the outer invariance entropy without the assumption that the periodic trajectory is contained in the interior.

Corollary 5.4. *Let D be an inner control set of Σ with closure $Q = \operatorname{cl} D$. Let $(\varphi(\cdot, x_0, \omega_0), \omega_0(\cdot))$ be a regular τ_0-periodic controlled trajectory with $x_0 \in Q$ and $\omega_0 \in \mathscr{U}_1$. Then*

$$h_{\mathrm{inv,out}}(Q) \leq \sum_{j=1}^{r} \max\{0, d_j \rho_j\}$$

holds, where $\lambda_1, \ldots, \lambda_r$ are the different Lyapunov exponents at (ω_0, x_0) with corresponding multiplicities $d_1, \ldots, d_r$.

Proof. Note that the definition of inner control sets implies that Q is compact. From Theorem 5.1 it follows that

$$h_{\mathrm{inv}}(Q, \mathrm{cl}\, D_\rho; \Sigma_\rho) \leq \sum_{j=1}^{r} \max\{0, d_j \lambda_j\} \quad \text{for all } \rho \in [0, 1).$$

Now for given $\varepsilon > 0$ choose $\rho \in [0, 1)$ such that $\mathrm{cl}\, D_\rho \subset N_\varepsilon(Q)$. Then

$$\begin{aligned} h_{\mathrm{inv}}(Q, N_\varepsilon(Q); \Sigma_0) &\leq h_{\mathrm{inv}}(Q, N_\varepsilon(Q); \Sigma_\rho) \\ &\leq h_{\mathrm{inv}}(Q, \mathrm{cl}\, D_\rho; \Sigma_\rho) \leq \sum_{j=1}^{r} \max\{0, d_j \lambda_j\}. \end{aligned}$$

The first two inequalities follow from $\mathscr{U}_\rho \subset \mathscr{U}_0$ and Proposition 2.1. Since $h_{\mathrm{inv,out}}(Q) = \lim_{\varepsilon \searrow 0} h_{\mathrm{inv}}(Q, N_\varepsilon(Q); \Sigma_0)$, the assertion follows. □

Remark 5.4. For discrete-time smooth systems given by difference equations $x_{k+1} = F(x_k, u_k)$ it is no problem to prove the analog of Theorem 5.1. In fact, the proof of Theorem 5.1 has been developed using a discrete-time blueprint which can be found in Nair et al. [85, Theorem 3]. As mentioned in the beginning of this chapter, this result of Nair et al. asserts that the infimal data rate for local uniform asymptotic stabilization of a discrete-time nonlinear system at a regular equilibrium pair (u_0, x_0) is given by the sum of the logarithms of the unstable eigenvalues associated with the linearization at (u_0, x_0). These numbers are identical with the positive Lyapunov exponents at (u_0, x_0). Essentially, all the arguments needed for a discrete-time version of Theorem 5.1 are contained in the proof of [85, Theorem 3].

5.2 Approximation Results for Lyapunov Exponents

The main result of the preceding section, Theorem 5.1, naturally leads to the following questions:

1. Are there easy-to-verify conditions which guarantee that a regular periodic controlled trajectory as required exists?
2. Can the assumptions of regularity and periodicity be weakened?

In this section, we show that there are indeed conditions which imply the existence of plenty of regular periodic trajectories in the interior of a control set, and which in many cases are relatively easy to check. Under a weak hyperbolicity assumption these trajectories then can be used to weaken the assumptions of regularity and periodicity in the upper estimate of Theorem 5.1. To this end, we first have to introduce the notion of *strong accessibility*. A well-known result of Sontag asserts that real-analytic systems with this property possess so-called *universally regular control functions*. These can be used to construct regular periodic trajectories as required.

Strong Accessibility

Assume that $\Sigma = (\mathbb{R}, M, \mathbb{R}^m, \mathscr{U}, \varphi)$ is a smooth system given by differential equations

$$\dot{x}(t) = F(x(t), \omega(t)), \quad \omega \in \mathscr{U},$$

where M is a real-analytic manifold of dimension d and $F : M \times \mathbb{R}^m \to TM$ is a real-analytic map. Moreover, assume that the control range $\Omega \subset \mathbb{R}^m$ is a compact, locally path-connected[3] set with nonempty and connected interior such that $\Omega = \operatorname{cl}\operatorname{int}\Omega$. We also consider the associated system $\Sigma^0 = (\mathbb{R}, M, \mathbb{R}^m, \mathscr{U}^0, \varphi^0)$ with control range $\Omega^0 := \operatorname{int}\Omega$ and the same right-hand side F. Then $\varphi^0(t,x,\omega) = \varphi(t,x,\omega)$ for all $(t,x,\omega) \in \mathbb{R} \times M \times \mathscr{U}^0$.

Definition 5.1. A topological time-invariant system is called *strongly accessible* if for each $x \in M$ there is some $\tau > 0$ such that $\operatorname{int}\mathscr{O}_\tau(x) \neq \emptyset$.

Recall from Sect. 1.5 that we call a control function ω regular for a state x on a time interval $[0,\tau]$ if the linearization along $(\varphi(\cdot,x,\omega), \omega(\cdot))$ is controllable on $[0,\tau]$.

Definition 5.2. A control function $\omega \in \mathscr{U}$ is said to be *universally regular* if it is regular for every $x \in M$ on some time interval $[0,\tau]$, $\tau = \tau(x) > 0$.

The following proposition summarizes some well-known results about strong accessibility.

Proposition 5.6. *The following assertions hold:*

(i) Let $\mathscr{L}$ denote the Lie subalgebra of vector fields on M generated by the vector fields F_u, $u \in \operatorname{int}\Omega$. Then Σ^0 is strongly accessible if and only if the ideal $\mathscr{L}_0$ in $\mathscr{L}$ generated by the vector fields

$$F_{u,v} := F_u - F_v, \quad u, v \in \operatorname{int}\Omega,$$

satisfies $\dim\mathscr{L}_0(x) = d$ for all $x \in M$, where $\mathscr{L}_0(x) := \{f(x)\}_{f \in \mathscr{L}_0}$. (See Sussmann and Jurdjevic [106, Corollary 4.7].)

(ii) System Σ^0 is strongly accessible if and only if for every $x \in M$ there is some $\omega \in \mathscr{U}^0$ which is regular for x on some time interval $[0,\tau]$, $\tau > 0$. (See Sontag [100] and [101, Sect. 1].)

(iii) If $\omega \in \mathscr{U}^0$ is an analytic control function, then ω is regular for $x \in M$ on some time interval $[0,\tau]$, $\tau > 0$, if and only if it is regular for x on every interval of this form. (See Sontag [101, Sect. 1].)

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[3]Recall that a topological space X is called *locally path-connected* if every neighborhood of a point $x \in X$ contains a path-connected neighborhood of x.

(iv) Assume that Σ^0 is strongly accessible. Then there exists an analytic universally regular control function $\omega \in \mathscr{U}^0$. (See Sontag [101, Theorem 1].)[4]

(v) If the universal covering space of M is compact, then strong accessibility of Σ^0 is equivalent to local accessibility. (See Sussmann and Jurdjevic [106, Theorem 4.9].)

(vi) If Σ is control-affine with right-hand side $F(x,u) = f_0 + \sum_{i=1}^m u_i f_i$, then Σ is strongly accessible if and only if Σ^0 is strongly accessible if and only if the ideal $\mathscr{L}_0$ generated by the vector fields $f_1, \dots, f_m$ satisfies $\dim \mathscr{L}_0(x) = d$ *for all $x \in M$.*

Remark 5.5. Statement (iv) is proved in Sontag [101] for systems whose state space is an open subset of $\mathbb{R}^d$, but can easily be generalized to systems on arbitrary real-analytic manifolds as noted in [101, Remark 2.3]. Its proof is based on Sussmann's theorem about the existence of universally distinguishing control functions (cf. Sussmann [105, Theorem 2.1]).

Lemma 5.3. *Let $D \subset M$ be a control set of Σ with nonempty interior. If Σ^0 is strongly accessible, then for every $x \in \operatorname{int} D$ there exist $\tau > 0$ and $\omega \in \operatorname{int} \mathscr{U}$ such that $(\varphi(\cdot, x, \omega), \omega(\cdot))$ is τ-periodic and regular on* $[0, \tau]$.

Proof. By Proposition 5.6 (iii) and (iv) we can apply a universally regular control function $\omega_* \in \mathscr{U}^0$ to x and obtain a trajectory $\varphi(\cdot, x, \omega_*)$ which is regular on every nontrivial interval of the form $[0, \tau_1]$. For τ_1 chosen sufficiently small we have $\varphi([0,\tau_1], x, \omega_*) \subset \operatorname{int} D$. Let $y := \varphi(\tau_1, x, \omega_*)$. Since $\omega_*(t) \in \operatorname{int} \Omega$ and ω_* is continuous, $\omega_*([0,\tau_1])$ is a compact subset of $\operatorname{int} \Omega$. Hence, by Proposition 5.4, we can assume that $\omega_* \in \operatorname{int} \mathscr{U}$. Strong accessibility implies local accessibility and the latter implies exact controllability on $\operatorname{int} D$ by Proposition 1.23 (iii). Hence, we find an admissible control function $\mu \in \mathscr{U}$ and a time $\tau_2 \geq 0$ with $\varphi(\tau_2, y, \mu) = x$. This gives the desired periodic trajectory with corresponding period $\tau := \tau_1 + \tau_2$ and control function $\omega := \omega_*|_{[0,\tau_1]} \mu^{\tau_1}$. By Proposition 1.28 this periodic trajectory is regular on $[0, \tau]$. To conclude the proof, we have to show that μ can be chosen such that $\mu \in \operatorname{int} \mathscr{U}$. In fact, we can assume that μ is piecewise constant with values in $\operatorname{int} \Omega$ which by Proposition 5.4 guarantees that $\mu \in \operatorname{int} \mathscr{U}$. This easily follows from the fact that local accessibility and approximate controllability on D also hold for the class of piecewise constant control functions with values in $\operatorname{int} \Omega$. $\square$

The First Approximation Result

The aim of this subsection is to prove an approximation result, which shows that the sum of positive Lyapunov exponents of an arbitrary periodic trajectory in the

[4] Sontag also proves a stronger result which asserts that the set of smooth universally regular control functions is generic in $\mathscr{C}^\infty([0,T], \operatorname{int} \Omega)$ for all $T > 0$.

interior of a control set can be approximated by the corresponding sums for regular periodic trajectories. Let the following assumptions be satisfied:

(a) There is a control set D of Σ with nonempty interior and compact closure;
(b) System Σ^0 is strongly accessible.

Furthermore, let g be an arbitrary $\mathscr{C}^\infty$-Riemannian metric on M.

In the following, we speak of subadditive cocycles over the control flow $\Phi : \mathbb{R}\times(\mathscr{U}\times M) \to \mathscr{U}\times M$ of Σ. However, note that we do not impose any continuity assumptions here (neither on the control flow nor on the cocycles). In particular, we do not assume that Σ is control-affine.

Proposition 5.7. *Let $(\varphi(\cdot,x,\omega),\omega(\cdot))$ be a τ-periodic controlled trajectory with $(x,\omega) \in \operatorname{int} D\times\operatorname{int}\mathscr{U}$. Moreover, let $a : \mathbb{R}\times(\mathscr{U}\times M) \to \mathbb{R}$, $(t,(\omega,x)) \mapsto a_t(\omega,x)$, be a subadditive cocycle over the control flow which satisfies the following two assumptions:*

(a) $a_\tau(\omega,x) \geq 0$;
(b) For all $T > 0$, $y \in M$, and $\omega_1,\omega_2 \in \mathscr{U}$ it holds that

$$\omega_1(t) = \omega_2(t) \text{ a.e. on } [0,T] \quad \Rightarrow \quad a_T(\omega_1,y) = a_T(\omega_2,y). \tag{5.18}$$

Then for every $\varepsilon > 0$ there exists a regular periodic controlled trajectory $(\varphi(\cdot,x,\omega_*),\omega_*(\cdot))$ with $\omega_* \in \operatorname{int}\mathscr{U}$ and period $\tau_* > 0$ such that

$$\frac{1}{\tau_*}a_{\tau_*}(\omega_*,x) \leq \frac{1}{\tau}a_\tau(\omega,x) + \varepsilon.$$

Proof. For the given periodic trajectory $\varphi(\cdot,x,\omega)$ we construct a family of approximating trajectories as follows. By Lemma 5.3 there exists a regular periodic trajectory $\varphi(t,x,\mu)$, $t \in [0,\rho]$. For every $N \in \mathbb{N}$ we define

$$\omega_N(t) := \begin{cases} \omega(t) & \text{for } t \in [0,N\tau) \\ \mu(t-N\tau) & \text{for } t \in [N\tau,N\tau+\rho] \end{cases},$$

and we extend ω_N $(N\tau+\rho)$-periodically. By construction and Proposition 5.4, ω_N is an admissible control function in $\operatorname{int}\mathscr{U}$. Moreover, from Proposition 1.28 it follows that ω_N is regular for x on $[0,N\tau+\rho]$. Using subadditivity of a, we obtain

$$a_{N\tau+\rho}(\omega_N,x) \leq a_\rho(\Theta_{N\tau}\omega_N,\varphi_{N\tau,\omega_N}(x)) + \sum_{i=0}^{N-1} a_\tau(\Theta_{i\tau}\omega_N,\varphi_{i\tau,\omega_N}(x)).$$

By construction we have $\varphi_{i\tau,\omega_N}(x) = x$ for $i = 0,1,\ldots,N$. Moreover, we have $\Theta_{i\tau}\omega_N(t) = \omega(t)$ for all $t \in [0,\tau]$ and $i = 0,\ldots,N-1$. By assumption (5.18) this implies

$$a_{N\tau+\rho}(\omega_N, x) \le a_\rho(\mu, x) + N a_\tau(\omega, x).$$

Hence, for given $\varepsilon > 0$ we can choose N sufficiently large so that

$$\begin{aligned}\frac{1}{N\tau+\rho} a_{N\tau+\rho}(\omega_N, x) &\le \frac{N}{N\tau+\rho} a_\tau(\omega, x) + \frac{1}{N\tau+\rho} a_\rho(\mu, x) \\ &\le \frac{1}{\tau + \frac{\rho}{N}} a_\tau(\omega, x) + \varepsilon \le \frac{1}{\tau} a_\tau(\omega, x) + \varepsilon.\end{aligned}$$

In the last inequality we used that $a_\tau(\omega, x) \ge 0$. Consequently, the desired estimate follows with $\omega_* = \omega_N$ and $\tau_* = N\tau + \rho$. □

Next we introduce some notation. For given $(t, x, \omega) \in \mathbb{R} \times M \times \mathcal{U}$, the derivative

$$\mathrm{d}_x \varphi_{t,\omega} : T_x M \to T_{\varphi(t,x,\omega)} M$$

is a linear isomorphism between d-dimensional Euclidean spaces, and hence has well-defined (positive) singular values, which we denote by

$$\sigma_1(t, x, \omega) \ge \cdots \ge \sigma_d(t, x, \omega) > 0.$$

For $0 \le k \le d$, the singular value function of order k of $\mathrm{d}_x \varphi_{t,\omega}$ is denoted by

$$\alpha_k(t, x, \omega) = \begin{cases} \sigma_1(t, x, \omega)\sigma_2(t, x, \omega) \cdots \sigma_k(t, x, \omega) & \text{for } k > 0, \\ 1 & \text{for } k = 0. \end{cases}$$

Proposition 5.8. *For every $k \in \{0, 1, \ldots, d\}$ the function*

$$a_t^k(\omega, x) := \log \alpha_k(t, x, \omega), \quad a^k : \mathbb{R} \times (\mathcal{U} \times M) \to \mathbb{R},$$

is a subadditive cocycle over the control flow which satisfies assumption (5.18).

Proof. To prove subadditivity, let $t, s \in \mathbb{R}_+$. Then, using Horn's inequality (cf. Sect. A.1), we find

$$\begin{aligned} a_{t+s}^k(\omega, x) &= \log \alpha_k(\mathrm{d}_x \varphi_{t+s,\omega}) \\ &= \log \alpha_k \left(\mathrm{d}_{\varphi(t,x,\omega)} \varphi_{s,\Theta_t \omega} \circ \mathrm{d}_x \varphi_{t,\omega}\right) \\ &\le \log \alpha_k \left(\mathrm{d}_{\varphi(t,x,\omega)} \varphi_{s,\Theta_t \omega}\right) + \log \alpha_k \left(\mathrm{d}_x \varphi_{t,\omega}\right) \\ &= a_t^k(\omega, x) + a_s^k(\Phi_t(\omega, x)). \end{aligned}$$

Finally, assumption (5.18) is satisfied. Indeed, $\omega_1(t) = \omega_2(t)$ almost everywhere on $[0, \tau]$ implies $\varphi(t, x, \omega_1) = \varphi(t, x, \omega_2)$ for all $t \in [0, \tau]$ and $x \in M$. In particular, $\varphi_{\tau,\omega_1} = \varphi_{\tau,\omega_2}$ and hence $\mathrm{d}_x \varphi_{\tau,\omega_1} \equiv \mathrm{d}_x \varphi_{\tau,\omega_2}$. □

Lemma 5.4. *For every $k \in \{1,\dots,d\}$ and all $t \geq 0$, $(\omega, x) \in \mathcal{U} \times M$, the following estimate holds:*

$$a_t^k(\omega, x) \leq k \int_0^t \lambda_{\max}\left(S\nabla F_{\omega(s)}(\varphi(s,x,\omega))\right) \mathrm{d}s.$$

Therefore, if $\varphi(t,x,\omega)$ is contained in a compact set for all $t \geq 0$, there is a constant $C \geq 0$ (which does not depend on (ω, x)) with

$$a_t^k(\omega, x) \leq Ct \quad \text{for all } t \geq 0. \tag{5.19}$$

Proof. First note that $\sigma_1(t,x,\omega)$ equals the operator norm of $\mathrm{d}_x\varphi_{t,\omega}$. Hence,

$$\alpha_k(t,x,\omega) = \sigma_1(t,x,\omega)\cdots\sigma_k(t,x,\omega) \leq \sigma_1(t,x,\omega)^k = \|\mathrm{d}_x\varphi_{t,\omega}\|^k.$$

Using the Wazewski inequality (Proposition A.4) gives

$$a_t^k(\omega,x) \leq k \log\|\mathrm{d}_x\varphi_{t,\omega}\| \leq k \int_0^t \lambda_{\max}\left(S\nabla F_{\omega(s)}(\varphi(s,x,\omega))\right) \mathrm{d}s.$$

If $\varphi(t,x,\omega)$ is contained in a compact set K, then $C := k \max_{(z,u)\in K\times\Omega} \lambda_{\max}$ $(S\nabla F_u(z))$ gives $a_t^k(\omega,x) \leq Ct$ for all $t \geq 0$. □

We introduce the *local Lyapunov exponents* at (ω, x),[5] defined recursively by

$$\nu_1(\omega,x) + \cdots + \nu_k(\omega,x) := \limsup_{t\to\infty} \frac{1}{t} a_t^k(\omega,x), \quad k = 1,2,\dots,d.$$

Then we obtain the first improvement over Theorem 5.1 which shows that under the assumption that all periodic trajectories have the same number of positive Lyapunov exponents, the condition of regularity is no longer necessary.

Lemma 5.5. *If the controlled trajectory $(\varphi(\cdot,x,\omega),\omega(\cdot))$ in $\mathcal{Q}$ (the forward lift of $Q = \mathrm{cl}\, D$) is periodic, then for every $k \in \{1,\dots,d\}$ the identities*

$$\begin{aligned}\nu_1(\omega,x) + \cdots + \nu_k(\omega,x) &= \lim_{t\to\infty} \frac{1}{t} a_t^k(\omega,x)\\ &= \lambda_1(\omega,x) + \cdots + \lambda_k(\omega,x)\end{aligned}$$

hold, where $\lambda_1(\omega,x) \geq \cdots \geq \lambda_k(\omega,x)$ denote the k largest Lyapunov exponents at (ω,x). In particular, $\nu_i(\omega,x) = \lambda_i(\omega,x)$ for $i = 1,\dots,d$.

[5] See, for instance, Boichenko et al. [9, Chap. IV, Sect. 8.1].

Proof. Let $\tau > 0$ be the period of $(\varphi(\cdot, x, \omega), \omega(\cdot))$ and fix $k \in \{1, \ldots, d\}$. From the first step of the proof of Theorem 5.1 we know that there exists a linear operator $R : T_x M \to T_x M$ such that

$$\mathrm{d}_x \varphi_{2\tau n,\omega} = \mathrm{e}^{2\tau n R} \quad \text{for all } n \in \mathbb{Z},$$

and that the Lyapunov exponents are the real parts of the eigenvalues of R. Using subadditivity of a^k and writing each $t \geq 0$ as $t = 2\tau n(t) + r(t)$ with $n(t) \in \mathbb{Z}_+$ and $r(t) \in [0, 2\tau)$, we find

$$a_t^k(\omega, x) \leq a_{2\tau n(t)}^k(\omega, x) + a_{r(t)}^k(\omega, x).$$

Since $a_{(\cdot)}^k(\omega, x)$ is bounded on the compact set $[0, 2\tau]$ by Lemma 5.4, we thus obtain

$$\limsup_{t\to\infty} \frac{1}{t} a_t^k(\omega, x) \leq \limsup_{t\to\infty} \frac{1}{t} a_{2\tau n(t)}^k(\omega, x) = \frac{1}{2\tau} \limsup_{\mathbb{N}\ni n\to\infty} \frac{1}{n} a_{2\tau n}^k(\omega, x).$$

On the other hand, for each $t \geq 0$ we find $n(t) \in \mathbb{Z}_+$ and $r(t) \in [0, 2\tau)$ such that $t + r(t) = 2\tau n(t)$. Subadditivity gives $a_{2\tau n(t)}^k(\omega, x) \leq a_t^k(\omega, x) + a_{r(t)}^k(\Phi_t(\omega, x))$. Using that $\varphi(t, x, \omega)$ is contained in the compact set Q for all t, Lemma 5.4 implies boundedness of $a_{r(t)}^k(\Phi_t(\omega, x))$. Hence,

$$\frac{1}{2\tau} \liminf_{n\to\infty} \frac{1}{n} a_{2\tau n}^k(\omega, x) = \liminf_{t\to\infty} \frac{1}{2\tau n(t)} a_{2\tau n(t)}^k(\omega, x) \leq \liminf_{t\to\infty} \frac{1}{t} a_t^k(\omega, x).$$

We have the relations $(\mathrm{e}^{2\tau n R})^{\wedge k} = \mathrm{e}^{2\tau n R_k} = (\mathrm{e}^{2\tau R_k})^n$, where R_k denotes the k-th derivation operator of R. This gives

$$\frac{1}{n} a_{2\tau n}^k(\omega, x) = \frac{1}{n} \log \prod_{i=1}^{k} \sigma_i(\mathrm{e}^{2\tau n R}) = \frac{1}{n} \log \left\| (\mathrm{e}^{2\tau n R})^{\wedge k} \right\| = \frac{1}{n} \log \left\| (\mathrm{e}^{2\tau R_k})^n \right\|.$$

We know that the limit for $n \to \infty$ of the last expression exists and is equal to the logarithm of the spectral radius of $\mathrm{e}^{2\tau R_k}$. The eigenvalues of R_k are the sums $\lambda_{i_1} + \cdots + \lambda_{i_k}$, where $\{\lambda_{i_1}, \ldots, \lambda_{i_k}\}$ is any subset of the spectrum of R consisting of k elements. Since the real parts of these eigenvalues are the Lyapunov exponents $\lambda_1(\omega, x) \geq \cdots \geq \lambda_d(\omega, x)$, it follows that

$$\frac{1}{2\tau} \lim_{n\to\infty} \frac{1}{n} a_{2\tau n}^k(\omega, x) = \lambda_1(\omega, x) + \cdots + \lambda_k(\omega, x).$$

Putting everything together, the proof is finished. □

Proposition 5.9. *Assume that every periodic trajectory corresponding to some* $(x, \omega) \in \operatorname{int} D \times \operatorname{int} \mathscr{U}$ *has exactly* k *positive Lyapunov exponents (counted* *with multiplicities), where $k \in \{0, 1, \dots, d\}$. Then for every periodic controlled trajectory $(\varphi(\cdot, x, \omega), \omega(\cdot))$ with $(x, \omega) \in \operatorname{int} D \times \operatorname{int} \mathscr{U}$ and every compact set $K \subset D$ it holds that*

$$h_{\text{inv}}(K, D) \leq \sum_{j=1}^{r} \max\{0, d_j \lambda_j\},$$

where $\lambda_1, \dots, \lambda_r$ are the different Lyapunov exponents at (ω, x) with corresponding multiplicities $d_1, \dots, d_r$.

Proof. The case $k = 0$ is trivial, since here anyway $h_{\text{inv}}(K, D) = 0$ (by Lemma 5.3 combined with Theorem 5.1). Hence, we may assume that $1 \leq k \leq d$. Given a τ_0-periodic controlled trajectory $(\varphi(\cdot, x, \omega), \omega(\cdot))$ with $(x, \omega) \in \operatorname{int} D \times \operatorname{int} \mathscr{U}$, we write $\lambda_1(\omega, x) \geq \dots \geq \lambda_d(\omega, x)$ for the Lyapunov exponents at (ω, x) (here every Lyapunov exponent can appear several times according to its multiplicity). By assumption, the first k of these Lyapunov exponents are positive. From Lemma 5.5 it follows that

$$\lambda_1(\omega, x) + \dots + \lambda_k(\omega, x) = \nu_1(\omega, x) + \dots + \nu_k(\omega, x) = \lim_{t \to \infty} \frac{1}{t} a_t^k(\omega, x).$$

Now fix some $\varepsilon > 0$ and choose $n_0 \in \mathbb{N}$ sufficiently large such that

$$\left| \frac{1}{n_0 \tau_0} a_{n_0 \tau_0}^k(\omega, x) - \lim_{t \to \infty} \frac{1}{t} a_t^k(\omega, x) \right| \leq \frac{\varepsilon}{2}. \tag{5.20}$$

The limit $\lim_{t \to \infty}(1/t) a_t^k(\omega, x)$ is positive. Hence, we can choose n_0 large enough that also $a_{n_0 \tau_0}^k(\omega, x) > 0$. Applying Proposition 5.7, we obtain a regular periodic trajectory $(\varphi(\cdot, x, \omega_*), \omega_*(\cdot))$ with $\omega_* \in \operatorname{int} \mathscr{U}$ of some period $\tau_* > 0$ such that

$$\frac{1}{\tau_*} a_{\tau_*}^k(\omega_*, x) \leq \frac{1}{n_0 \tau_0} a_{n_0 \tau_0}^k(\omega, x) + \frac{\varepsilon}{2}. \tag{5.21}$$

Now Theorem 5.1 gives

$$h_{\text{inv}}(K, D) \leq \lambda_1(\omega_*, x) + \dots + \lambda_k(\omega_*, x).$$

The sequence $n \mapsto a_{n\tau_*}^k(\omega_*, x)$ is easily seen to be subadditive and hence, the subadditivity Lemma B.3 implies

$$\begin{aligned} \lim_{n \to \infty} \frac{1}{n\tau_*} a_{n\tau_*}^k(\omega_*, x) &= \inf_{n \in \mathbb{N}} \frac{1}{n\tau_*} a_{n\tau_*}^k(\omega_*, x) \leq \frac{1}{\tau_*} a_{\tau_*}^k(\omega_*, x) \\ &\overset{(5.21)}{\leq} \frac{1}{n_0 \tau_0} a_{n_0 \tau_0}^k(\omega, x) + \frac{\varepsilon}{2}. \end{aligned}$$

Using Lemma 5.5 again, we find

$$\begin{aligned}\lambda_1(\omega_*, x) + \cdots + \lambda_k(\omega_*, x) &= \lim_{t\to\infty} \frac{1}{t} a_t^k(\omega_*, x) \\ &= \lim_{n\to\infty} \frac{1}{n\tau_*} a_{n\tau_*}^k(\omega_*, x) \leq \frac{1}{n_0\tau_0} a_{n_0\tau_0}^k(\omega, x) + \frac{\varepsilon}{2}.\end{aligned}$$

Altogether, we obtain

$$\begin{aligned} h_{\mathrm{inv}}(K, D) &\leq \frac{1}{n_0\tau_0} a_{n_0\tau_0}^k(\omega, x) + \frac{\varepsilon}{2} \\ &\overset{(5.20)}{\leq} \lim_{t\to\infty} \frac{1}{t} a_t^k(\omega, x) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \lambda_1(\omega, x) + \cdots + \lambda_k(\omega, x) + \varepsilon. \end{aligned}$$

Since ε can be chosen arbitrarily small, this finishes the proof. □

The Second Approximation Result

Proposition 5.9 shows that under appropriate assumptions we can do without regularity of the periodic trajectory in Theorem 5.1. Let us impose the same assumptions on the system Σ as before (real-analytic, strongly accessible, compact control range). By using a second approximation result for subadditive cocycles, we can also weaken the periodicity assumption.

Proposition 5.10. *Let* $a : \mathbb{R} \times (\mathcal{U} \times M) \to \mathbb{R}$ *be a subadditive cocycle over the control flow satisfying assumption* (5.18) *and the boundedness property* (5.19) *of* a^k. *Furthermore, let* $(x, \omega) \in \operatorname{int} D \times \operatorname{int} \mathcal{U}$ *such that* $\varphi(t, x, \omega)$ *is contained in a compact set* $K \subset \operatorname{int} D$ *for all* $t \geq 0$, *and suppose that there exists* $t_0 \geq 0$ *with* $a_t(\omega, x) \geq 0$ *for all* $t \geq t_0$. *Then for every* $\varepsilon > 0$ *there exists a periodic trajectory with initial state* x *corresponding to a periodic control function* $\omega_* \in \operatorname{int} \mathcal{U}$ *of the same period* $\tau_* > 0$ *such that*

$$\frac{1}{\tau_*} a_{\tau_*}(\omega_*, x) \leq \limsup_{t\to\infty} \frac{1}{t} a_t(\omega, x) + \varepsilon.$$

Proof. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of positive times with $t_n \to \infty$ such that

$$\sigma := \limsup_{t\to\infty} \frac{1}{t} a_t(\omega, x) = \lim_{n\to\infty} \frac{1}{t_n} a_{t_n}(\omega, x).$$

Now define the first hitting time

$$\tau := \inf \left\{ t \geq 0 \;:\; x \in \mathscr{O}^+_{\leq t}(z) \text{ for all } z \in K \right\}.$$

By Proposition 1.23 (v), local accessibility (which follows from strong accessibility) guarantees that $\tau < \infty$. There is $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$ and all $T \in [0, \tau]$ it holds that

$$\frac{1}{t_n + T} \sup_{\substack{(t,z,\nu)\in[0,\tau]\times K\times\mathscr{U} \\ \varphi([0,\tau],z,\nu)\subset Q}} |a_t(\nu, z)| \leq \frac{\varepsilon}{2}. \tag{5.22}$$

Finiteness of the above supremum follows from the boundedness assumption imposed on a. Finally, there is $N \geq n_1$ such that (by assumption)

$$a_{t_N}(\omega, x) \geq 0 \tag{5.23}$$

and such that

$$\left| \frac{1}{t_N} a_{t_N}(\omega, x) - \sigma \right| \leq \frac{\varepsilon}{2}. \tag{5.24}$$

By definition of τ we can choose a control function $\nu \in \mathscr{U}[0, T)$ with $T \leq \tau$ and $\varphi(T, \varphi(t_N, x, \omega), \nu) = x$, and we may assume that ν is piecewise constant taking values in $\operatorname{int} \Omega$. Define the control function ω_* on $[0, t_N + T]$ as

$$\omega_*(t) := \begin{cases} \omega(t) & \text{for } t \in [0, t_N] \\ \nu(t - t_N) & \text{for } t \in (t_N, t_N + T] \end{cases},$$

and extend ω_* $(t_N + T)$-periodically. This yields a $(t_N + T)$-periodic trajectory in $\operatorname{int} D$, and $\omega_* \in \operatorname{int} \mathscr{U}$. Then, with $\tau_* := t_N + T$, we have

$$\begin{aligned} \frac{1}{\tau_*} a_{\tau_*}(\omega_*, x) &\leq \frac{1}{t_N + T} \left(a_{t_N}(\omega_*, x) + a_T(\Theta_{t_N} \omega_*, \varphi(t_N, x, \omega_*)) \right) \\ &= \frac{1}{t_N + T} \left(a_{t_N}(\omega, x) + a_T(\nu, \varphi(t_N, x, \omega)) \right) \\ &\overset{(5.22)}{\leq} \frac{1}{t_N + T} a_{t_N}(\omega, x) + \frac{\varepsilon}{2} \\ &\overset{(5.23)}{\leq} \frac{1}{t_N} a_{t_N}(\omega, x) + \frac{\varepsilon}{2} \overset{(5.24)}{\leq} \sigma + \varepsilon. \end{aligned}$$

This finishes the proof. □

Proposition 5.11. *Let* $(x, \omega) \in \operatorname{int} D \times \operatorname{int} \mathscr{U}$ *such that* $\varphi(t, x, \omega)$ *is contained in a compact subset of* $\operatorname{int} D$ *for all* $t \geq 0$*. Furthermore, assume that there exists* $k \in \{0, 1, \ldots, d\}$ *such that the following assumptions are satisfied:*

(i) Every periodic trajectory corresponding to some $(y, \mu) \in \operatorname{int} D \times \operatorname{int} \mathscr{U}$ *has exactly* k *positive Lyapunov exponents (counted with multiplicities);*
(ii) There exists $t_0 \geq 0$ *such that* $a_t^k(\omega, x) \geq 0$ *for all* $t \geq t_0$*.*

Then for every compact set $K \subset D$ *it holds that*

$$h_{\mathrm{inv}}(K, D) \leq \nu_1(\omega, x) + \cdots + \nu_k(\omega, x).$$

Proof. Note that the assumptions of Proposition 5.10 are satisfied for the subadditive cocycle a^k. Hence, for given $\varepsilon > 0$ we find a periodic controlled trajectory of the form $(\varphi(\cdot, x, \omega_*), \omega_*(\cdot))$ with $\omega_* \in \operatorname{int} \mathscr{U}$ of some period $\tau_* > 0$ such that

$$\begin{aligned} \frac{1}{\tau_*} a_{\tau_*}^k(\omega_*, x) &\leq \limsup_{t \to \infty} \frac{1}{t} a_t^k(\omega, x) + \varepsilon \\ &= (\nu_1(\omega, x) + \cdots + \nu_k(\omega, x)) + \varepsilon. \end{aligned} \tag{5.25}$$

By Proposition 5.9 we have

$$\begin{aligned} h_{\mathrm{inv}}(K, D) &\leq \lambda_1(\omega_*, x) + \cdots + \lambda_k(\omega_*, x) \\ &= \lim_{t \to \infty} \frac{1}{t} a_t^k(\omega_*, x) \\ &= \lim_{m \to \infty} \frac{1}{m\tau_*} a_{m\tau_*}^k(\omega_*, x) \\ &= \inf_{m \in \mathbb{N}} \frac{1}{m\tau_*} a_{m\tau_*}^k(\omega_*, x) \leq \frac{1}{\tau_*} a_{\tau_*}^k(\omega_*, x). \end{aligned}$$

Here we used that the sequence $m \mapsto a_{m\tau_*}^k(\omega_*, x)$ is subadditive. Combining this inequality with (5.25) gives the desired result. $\square$

Remark 5.6. Notice that the assumption that Σ is real-analytic and strongly accessible has only been used to guarantee that for every point in the interior of the given control set there exists a regular periodic trajectory going through this point. To have that (together with local accessibility) it is sufficient and necessary that there are two points in the interior of the control set which can be joined by a regular trajectory. At first sight, this seems to be a much weaker condition than strong accessibility, but a result of Sontag [100, Proposition 4.2] shows that (under mild assumptions) for real-analytic systems this is equivalent to strong accessibility. However, for control-affine systems there is an easy trick which can be used to show that the assumption of strong accessibility can be weakened to local accessibility. Moreover, using a result of Coron [30, Theorem 1.3 and Corollary 1.8] it can be shown that analyticity can be weakened to smoothness. (In fact, this works not only for control-affine systems, but we do not go into the quite technical details involved here.)

Proposition 5.12. *Assume that Σ is control-affine, $F(x,u) = f_0(x) + \sum_{i=1}^m u_i f_i(x)$ with a (compact and convex) control range with nonempty interior. Then the assertions of Propositions 5.9 and 5.11 also hold if the vector fields $f_0, f_1, \ldots, f_m$ are of class $\mathscr{C}^\infty$ and the Lie algebra rank condition holds on D.*

Proof. The proof proceeds in four steps.

Step 1. We show that if Σ satisfies the particular assumptions of Propositions 5.9 and 5.11, then they are also satisfied for each of the time-transformed systems $\Sigma^\alpha = (\mathbb{R}, M, \mathbb{R}^{m+1}, \mathscr{U}^\alpha, \varphi^\alpha)$, $\alpha > 1$, given by the differential equations

$$\dot{x}(t) = \gamma(t) \cdot F(x(t), \omega(t)), \qquad (\gamma, \omega) \in \mathscr{U}^\alpha = \mathscr{V}^\alpha \times \mathscr{U},$$

where $\mathscr{V}^\alpha = \{\gamma \in L^\infty(\mathbb{R}, \mathbb{R}) : \gamma(t) \in [1/\alpha, \alpha]\}$. First we prove that the trajectories of Σ^α are just time reparametrizations of the trajectories of Σ. To this end, for every $\gamma \in \mathscr{V}^\alpha$ define

$$\sigma(t) := \int_0^t \gamma(s) \mathrm{d}s, \qquad t \geq 0.$$

It is clear that $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is absolutely continuous with $\sigma(0) = 0$. It is bijective, since $\gamma \geq 1/\alpha$ implies that σ is strictly increasing and $\sigma(t) \to \infty$ for $t \to \infty$. We claim that

$$\varphi(\sigma(t), x, \omega) = \varphi^\alpha\left(t, x, (\gamma, \omega \circ \sigma)\right) \tag{5.26}$$

for all $x \in M$, $\omega \in \mathscr{U}$, and $t \geq 0$. Indeed, for almost all $t \geq 0$ we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\varphi(\sigma(t), x, \omega) &= \dot{\sigma}(t) \cdot F\left(\varphi(\sigma(t), x, \omega), \omega(\sigma(t))\right) \\ &= \gamma(t) \cdot F\left(\varphi(\sigma(t), x, \omega), \omega \circ \sigma(t)\right).\end{aligned}$$

By uniqueness of solutions, the identity (5.26) follows. From this identity it can easily be seen that if D is a control set of Σ, then D is a control set of Σ^α. Now assume that every periodic trajectory of Σ corresponding to some $(x, \omega) \in \operatorname{int} D \times \operatorname{int} \mathscr{U}$ has exactly k positive Lyapunov exponents as required in Proposition 5.9. Then the analogous statement for Σ^α is true (with $(x, (\gamma, \omega)) \in \operatorname{int} D \times \operatorname{int}(\mathscr{V}^\alpha \times \mathscr{U})$). Indeed, let $(\varphi^\alpha(\cdot, x, (\gamma, \omega)), (\gamma, \omega))$ be a τ-periodic controlled trajectory with $x \in \operatorname{int} D$ and $(\gamma, \omega) \in \operatorname{int}(\mathscr{V}^\alpha \times \mathscr{U}) = \operatorname{int} \mathscr{V}^\alpha \times \operatorname{int} \mathscr{U}$. The number of positive Lyapunov exponents of the given trajectory is given by the number of eigenvalues of $\mathrm{d}_x \varphi^\alpha_{\tau,(\gamma,\omega)} : T_x M \to T_x M$ of absolute value bigger than one. From (5.26) it follows that $\mathrm{d}_x \varphi^\alpha_{\tau,(\gamma,\omega)} = \mathrm{d}_x \varphi_{\sigma(\tau), \omega \circ \sigma^{-1}}$. From τ-periodicity of γ it follows that

$t + \tau = \sigma^{-1}(\sigma(t) + \sigma(\tau))$ for all $t \geq 0$. This implies $(\omega \circ \sigma^{-1})(t + \sigma(\tau)) = \omega(\sigma^{-1}(t) + \tau) = \omega(\sigma^{-1}(t))$. Hence, $\omega \circ \sigma^{-1}$ is $\sigma(\tau)$-periodic. Thus, $(\varphi(\cdot, x, \omega \circ \sigma^{-1}), \omega \circ \sigma^{-1})$ is a $\sigma(\tau)$-periodic controlled trajectory of Σ with $(x, \omega \circ \sigma^{-1}) \in \operatorname{int} D \times \operatorname{int} \mathscr{U}$ and hence has exactly k positive Lyapunov exponents. This implies the assertion. Analogously, one shows that assumption (ii) in Proposition 5.11 carries over from Σ to Σ^α.

Step 2. We show that the invariance entropies of (K, D) with respect to Σ and Σ^α, respectively, are related by

$$h_{\text{inv}}(K, D; \Sigma) \leq \alpha \cdot h_{\text{inv}}(K, D; \Sigma^\alpha). \tag{5.27}$$

To this end, let $\mathscr{S} \subset \mathscr{V}^\alpha \times \mathscr{U}$ be a (τ, K, D)-spanning set for Σ^α. We claim that

$$\mathscr{S}' := \left\{\omega \circ \sigma^{-1} \mid \exists \gamma \in \mathscr{V}^\alpha : (\gamma, \omega) \in \mathscr{S}\right\}$$

is a $(\tau/\alpha, K, D)$-spanning set for Σ. Indeed, let $x \in K$. Then there is $(\gamma, \omega) \in \mathscr{S}$ with

$$\varphi\left(\sigma(t), x, \omega \circ \sigma^{-1}\right) = \varphi^\alpha\left(t, x, (\gamma, \omega)\right) \in D \quad \text{for all } t \in [0, \tau],$$

which implies $\varphi(t, x, \omega \circ \sigma^{-1}) \in D$ for all $t \in [0, \tau/\alpha]$, since $\sigma(\tau) \geq \int_0^\tau 1/\alpha \mathrm{d}s = \tau/\alpha$. It follows that $r_{\text{inv}}(\tau/\alpha, K, D; \Sigma) \leq r_{\text{inv}}(\tau, K, D; \Sigma^\alpha)$ and hence

$$\begin{aligned} h_{\text{inv}}(K, D; \Sigma) &= \limsup_{\tau \to \infty} \frac{\alpha}{\tau} \log r_{\text{inv}}(\tau/\alpha, K, D; \Sigma) \\ &\leq \limsup_{\tau \to \infty} \frac{\alpha}{\tau} \log r_{\text{inv}}(\tau, K, D; \Sigma^\alpha) = \alpha \cdot h_{\text{inv}}(K, D; \Sigma^\alpha), \end{aligned}$$

which finishes Step 2.

Step 3. We prove the assertion for the case that $f_0, f_1, \ldots, f_m$ are analytic vector fields. Since we assume that the Lie algebra rank condition holds for Σ, the smallest Lie algebra spanned by the vector fields $f_0, f_1, \ldots, f_m$ has full rank at every point (see Proposition 1.8). Note that the strong accessibility algebra of Σ^α, that is, the ideal generated by the differences $v[f_0 + \sum_{i=1}^m u_i f_i] - v'[f_0 + \sum_{i=1}^m u_i' f_i]$, contains the vector fields $f_1, \ldots, f_m$ as well as the vector field f_0 (put $(v, u) := (\alpha, 0) \in \mathbb{R} \times \mathbb{R}^m$ and $(v', u') := (1, 0) \in \mathbb{R} \times \mathbb{R}^m$, then $v[f_0 + \sum_{i=1}^m u_i f_i] - v'[f_0 + \sum_{i=1}^m u_i' f_i] = (\alpha - 1) f_0$). By Proposition 5.6 (vii) this implies that Σ^α is strongly accessible. Hence, we find that the Propositions 5.9 and 5.11 can be applied to the systems Σ^α, if $f_0, f_1, \ldots, f_m$ are analytic. The inequality (5.27) shows that the corresponding estimates for the invariance entropy $h_{\text{inv}}(K, D; \Sigma^\alpha)$ carry over to $h_{\text{inv}}(K, D; \Sigma)$ by letting $\alpha \to 1$.
Step 4. We show that the assumption of analyticity can be weakened to smoothness. Observe that analyticity (in combination with strong accessibility) was only used in the proof of Lemma 5.3 to show the existence of arbitrarily

short regular trajectories in the interior of D. However, this also follows as a consequence of Coron [30, Corollary 1.8] if the right-hand side of the system is of class $\mathscr{C}^\infty$ and polynomial with respect to the control variable, and if the strong accessibility algebra has full rank at every point. Since these assumptions are satisfied for the time-transformed systems Σ^α, if the given system is smooth and satisfies the Lie algebra rank condition, we are done. □

Remark 5.7.

- Of course, one would like to have a third approximation result to get rid of the assumptions that $\varphi(t,x,\omega)$ be contained in a compact subset of $\operatorname{int} D$ and $\omega \in \operatorname{int} \mathscr{U}$. As can be seen in Sect. 7.1, for one-dimensional systems things are easier than in the general case, since here only equilibria instead of arbitrary trajectories have to be considered. The same holds for particular control sets of projective systems, as we show in Sect. 7.4.
- The existence of universally regular control functions and regular periodic trajectories inside of control sets for discrete-time systems has been studied in Wirth [110–112] and Sontag and Wirth [103]. Hence, it should be an easy task to adapt the results of this section to the discrete-time setting.

5.3 Comments and Bibliographical Notes

The main theorem of this chapter, Theorem 5.1, has appeared before in Kawan [62, 64]. All results about the invariance entropy in Sect. 5.2 are new and have not been published before. The methods used in the proofs of the approximation results for subadditive cocycles are basically taken from Colonius and Kliemann [25, Theorem 6.2.17], a result which relates the Lyapunov and Floquet spectra of certain control systems on vector bundles to each other. Further note that the estimate for a^k given in Lemma 5.4 can be improved (see Boichenko et al. [9, Chap. I, Corollary 4.2.1]). Of course, the results of this chapter leave many questions open. For instance, what can be said about the value of $h_{\mathrm{inv}}(Q)$ when Q is the closure of a relatively compact control set D? Is it the same as $h_{\mathrm{inv}}(K,Q)$ for $K \subset D$ or can it be strictly greater? Another question concerns the existence of regular periodic trajectories without the regularity assumptions of Sect. 5.2. One could ask, for instance, if they exist generically. Finally, notice that in this chapter we have seen a second example for the equality $h_{\mathrm{inv,out}}(K,Q) = h_{\mathrm{inv}}(K,Q)$, namely Corollary 5.3.

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