Preface

> Man kann mit der Frage beginnen: Was ist Abstraktion, und welche Rolle spielt sie im begrifflichen Denken? Als Antwort kann man etwa formulieren: Abstraktion bezeichnet die Möglichkeit, einen Gegenstand oder eine Gruppe von Gegenstanden unter einem Gesichtspunkt unter Absehen von allen anderen Gegenstandseigenschaften zu betrachten.
> (Heisenberg 1960)

In the 1890s, D. Hilbert, in connection with investigations on the irreducibility of rational functions, proved a Ramsey-type result nowadays known as Hilbert's cube lemma. Some 25 years later, I. Schur showed in reproving a theorem of Dickson on a modular version of Fermat's conjecture that if the positive integers are finitely colored, one color class contains x, y, and z with $x+y = z$. A conjecture concerning the distribution of quadratic residues, respectively nonresidues modulo p, led Schur to a question on arithmetic progressions. This problem was solved in 1927 by B. L. van der Waerden and the corresponding theorem became famous as van der Waerden's theorem on arithmetic progressions.

Around the same time the English mathematician F. P. Ramsey tried to give a decision procedure for propositional logic. The need for such procedures, we would say algorithms in the present-day terminology, arose with the crisis of the foundations of mathematics around 1900. It was more or less the theory of sets and the arithmetization of analysis which led to this crisis. In response, the programs of Russell and Whitehead, of Hilbert, and of Brouwer called for a new foundation trying to overwhelm the doubtful principles of mathematics of that time.

It is a kind of irony that a purely mathematical result from Ramsey's paper, an astonishing generalization of the pigeonhole principle, has proved to be of so much greater consequence than the metamathematical investigations for which they were made as tools. Even more, for this result Ramsey became eponymous for a part of discrete mathematics known as Ramsey theory.

These four roots of Ramsey theory were established for different reasons, unaware of the other. A first culmination point, then, was obtained with the work of R. Rado, a doctoral student of Schur. In his *Studien zur Kombinatorik* and several subsequent papers, Rado unified and extended the results of Hilbert, Schur, and van der Waerden giving a complete characterization of those systems of linear equations which are partition regular.

Quite independently from this direction of research, there was a profound development based on Ramsey's theorem, which is closely connected with the name of P. Erdős. A kind of first step in popularizing Ramsey's theorem was an application to a combinatorial problem in geometry due to Erdős and Szekeres: "I am sure that this paper had a strong influence on both of us. Paul with his deep insight recognized the possibilities of a vast unexplored territory and opened up a new world of combinatorial set theory and combinatorial geometry" (Szekeres 1973). But it took until the middle of the 1960s when A. W. Hales and R. I. Jewett revealed the combinatorial core of van der Waerden's theorem on arithmetic progressions proving a kind of pigeonhole principle for parameter sets. Some years later, R. L. Graham and B. L. Rothschild extended Hales-Jewett's result in a remarkable way. They established a complete analogue to Ramsey's theorem for the structure of parameter sets and, as it turns out, Ramsey's theorem itself is an immediate consequence of the Graham-Rothschild theorem. But the concept of parameter sets does not only glue together arithmetic progressions and finite sets. It also provides a natural framework for seemingly different structures like Boolean lattices, partition lattices, hypergraphs and Deuber's (m, p, c)-sets, just to mention a few. So, to a certain extent the Graham-Rothschild theorem can be viewed as a starting point of Ramsey theory for discrete structures:

> Dies kann also beim Vorgang der Abstraktion geschehen: Der im Prozeß der Abstraktion gebildete Begriff gewinnt ein eigenes Leben, er läßt eine unerwartete Fülle von Formen oder ordnenden Strukturen aus sich entstehen, die sich später auch beim Verständnis der uns umgebenden Erscheinungen in irgendeiner Weise bewähren können (Heisenberg 1960).

The present work is organized as follows. In the first part, we give a more detailed discussion of the roots of Ramsey theory. Thereafter, we focus on three discrete structures: sets, parameter sets, and graphs.

The second part of this work contains a thorough discussion of the role of parameter sets in Ramsey theory. Originally, the idea was to find a combinatorial abstraction of linear and affine spaces over finite fields. This was motivated by a conjecture of G. C. Rota proposing a geometric analogue to Ramsey's theorem. But the impact of parameter sets goes far beyond the proof of Rota's conjecture. In Chap. 3 we present some definitions and several examples of structures which can be interpreted in terms of parameter sets. Chapters 4 and 5, then, contain the most fundamental Ramsey-type results for parameter sets, viz., Hales-Jewett's theorem and Graham-Rothschild's Ramsey theorem for n-parameter sets, as well as several applications thereof. Finally, in Chap. 6 we build upon the Graham-Rothschild theorem to obtain canonical versions of the aforementioned results.

In the third part, we go back to the most basic structure, to sets, and discuss developments which originate in Ramsey's theorem itself. One of the oldest areas in Ramseyean research is the study of Ramsey numbers which essentially starts with the paper of Erdős and Szekeres. We devote Chap. 7 to review old results as well as recent progress on Ramsey numbers and on the asymptotic behavior of the classical Ramsey functions. In Chap. 8 unprovability results are discussed. As it turns out, a slight variation of the finite Ramsey theorem is one of the first mathematical interesting examples for Gödel's incompleteness theorem. Chapter 9 presents product versions of Ramsey's theorem, whereas Chap. 10 covers a result on the necessity of irregularities of set partitions. In the final chapter of this part, we discuss extensions of Ramsey's theorem to larger cardinals, based on the profound work of Erdős, Hajnal, and Rado in this area.

Graphs and hypergraphs seem to be one of the most alive and exciting areas of research in Ramsey theory nowadays. In Chaps. 12 and 14, we present a complete solution of the Ramsey problem for finite graphs, respectively hypergraphs, closely connected with the names of Deuber, Nešetřil, and Rödl. Moreover, we introduce and develop an amalgamation technique for graphs and hypergraphs which is an essential tool in proving sparse and restricted Ramsey theorems. In between, in Chap. 13, we collect some results which are known for infinite graphs, mainly due to Erdős, Hajnal, and Pósa. In Chap. 14, we start to consider graphs and hypergraphs in a broader perspective. Ramsey's theorem for finite hypergraphs can be viewed as an induced version of Ramsey's theorem. Apparently Spencer was the first to consider graphs and hypergraphs which are defined on more complex structures than just sets, proving an induced version of van der Waerden's theorem. In this last part of Chap. 14 we introduce hypergraphs defined on parameter sets and prove an induced Graham-Rothschild theorem.

Sparse Ramsey theorems for graphs originate in investigations of graphs having large chromatic number and high girth. A complete solution to the problem was first given by Erdős using probabilistic means and later by Lovász via an explicit construction. In Chap. 16 we give an account on the probabilistic method for constructing more general sparse Ramsey configurations.

Several areas of Ramsey theory remain uncovered throughout this work, e.g., Euclidean Ramsey theory or topological Ramsey theory. We refer the interested reader to the excellent monograph *Ramsey Theory* of Graham et al. (1980), as well as to the forthcoming volume *Mathematics of Ramsey Theory* edited by Nešetřil and Rödl (1990). We also do not discuss any of the recent applications of Ramsey theory to computer science. Here we refer the reader, for example, to Alon and Maass (1986), Moran et al. (1985), Nešetřil (1984), and Pudlák (1990), just to mention a few.

> Kehren wir zu der am Anfang gestellten Frage zurück. Der Zug zur Abstraktion in der Naturwissenschaft beruht also letzten Endes auf der Notwendigkeit, weiterzufragen, auf dem Streben nach einem einheitlichen Verständnis. ... die Menschen, die über die Natur nachdenken, fragen weiter, weil sie die Welt als Einheit begreifen, ihren einheitlichen Bau verstehen wollen. Sie bilden zu diesem Zweck immer umfassendere Begriffe deren Zusammenhang mit dem unmittelbaren sinnlichen Erlebnis nur schwer zu erkennen ist wobei aber das Bestehen eines solchen Zusammenhangs unabdingbare Voraussetzung dafür ist, daß die Abstraktion überhaupt noch Verständnis der Welt vermittelt. (Heisenberg 1960)

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