

Chapter 2

Structural Limit States and Reliability Measures

Abstract In the present chapter the different levels of structural reliability methods are first summarized. Subsequently, the concept of the failure function is introduced and computation of the associated failure probability is considered. The interpretation of this probability as a volume is highlighted. Structures with time-varying load and resistance properties are next addressed. The simplified situation where both the random variable representing the load effect and the resistance are Gaussian is discussed. The resulting failure probability and the associated reliability index (beta-index) are elaborated.

Keywords Reliability methods • Failure function • Probability of failure • Reliability index

2.1 Introduction

Failure of a structure generally designates the event that the structure does not satisfy a specific set of functional requirements. Hence, it is a fairly wide concept which comprises such diversified phenomena as loss of stability, excessive response levels in terms of displacements, velocities or accelerations, as well as plastic deformations or fracture e.g. due to overload or fatigue.

The consequences of different types of failure also vary significantly. Collapse of a single sub-component does not necessarily imply that the structure as a system immediately loses the ability to carry the applied loads. At the other extreme, a sudden loss of stability is frequently accompanied by a complete and catastrophic collapse of the structure. Failure can also consist of a complex sequence of unfortunate events, possibly due to a juxtaposition of low-probability external or man-made actions and internal defects.

In engineering design, distinction is typically made between different categories of design criteria. These are frequently also referred to as *limit states*. The three most common categories are the Serviceability Limit State (SLS), the Ultimate

Limit State (ULS) and the Fatigue Limit State (FLS). Many design documents also introduce the so-called Accidental Limit State and the Progressive Limit State in order to take care of unlikely but serious structural conditions.

Engineering design rules are generally classified as Level I reliability methods. These design procedures apply point values for the various design parameters and also introduce specific codified safety factors (also referred to as partial coefficients) which are intended to reflect the inherent statistical scatter associated in the parameters.

At the next level, second-order statistical information (i.e. information on variances and correlation properties in addition to mean values) can be applied if such is available. The resulting reliability measure and analysis method are then referred to as a Level II reliability method. At Level III, it is assumed that a complete set of probabilistic information (i.e. in the form of joint density and distribution functions) is at hand.

2.2 Failure Function and Probability of Failure

The common basis for the different levels of reliability methods is the introduction of a so-called failure function (or limit state function, or g-function) which gives a mathematical definition of the failure event in mechanical terms. In order to be able to estimate the failure probability, it is necessary to know the difference between the maximum load a structure is able to withstand, R (often referred to as *resistance*), the loads it will be exposed to, Q , and the associated *load effects* S . The latter are typically obtained by means of (more or less) conventional structural analysis methods. For this “generic” case, the “g-function” is then expressed as:

$$g(R, S) = R - S \quad (2.1)$$

For positive values of this function (i.e. for $R > S$), the structure is in a safe state. Hence, the associated parameter region is referred to as the *safe domain*. For negative values (i.e. $R < S$), the structure is in a failed condition. The associated parameter region is accordingly referred to as the *failure domain*. The boundary between these two regions is the failure surface (i.e. $R = S$). The reason for application of these generalized terms is that the scalar quantities R and S in most cases are functions of a number of more basic design parameters. This implies that the simplistic two-dimensional formulation in reality involves a much larger number of such parameters corresponding to a reliability formulation of (typically) high dimension.

Here, a brief introduction is given to the basis for the Level III structural reliability methods which are required in the subsequent sections. Further details of these methods are found e.g. in [1, 2]. When concerned with waves, wind and dynamic structural response, it is common to assume that the statistical parameters are constant over a time period with a duration of (at least) 1 h. This is frequently

referred to as a *short term statistical analysis*. A further assumption is typically that the stochastic dynamic excitation processes (i.e. the surface elevation or the wind turbulence velocity) are of the Gaussian type.

If the joint probability density function (or distribution function) of the strength and the load effect, i.e. $f_{RS}(r,s)$ is known, the probability of failure can generally be expressed as

$$p_f = P(Z = R - S \leq 0) = \iint_{R \leq S} f_{R,S}(r,s) dr ds \quad (2.2)$$

where the integration is to be performed over the failure domain, i.e. the region where the strength is smaller than or equal to the load effect.

This is illustrated in Fig. 2.1a, where both the joint density function and the two marginal density functions $f_R(r)$ and $f_S(s)$ are shown (the latter are obtained by a one-dimensional integration of the joint density function with respect to each of the variables from minus to plus infinity). The joint density function can then be split into two separate pieces as shown in part (b) and (c) of the same figure. The failure probability can now be interpreted in a geometric sense as the volume of the joint density function which is located in the failure domain, i.e. the part of the plane to the right of the line $R = S$ (i.e. the region for which $S > R$). This corresponds to the slice of the volume of the joint pdf which is shown in Fig. 2.1c.

For the case of independent variables, the joint density function is just expressed as the product of the two marginal density functions. The resulting expression for the failure probability then becomes:

$$p_f = P(Z = R - S \leq 0) = \iint_{R \leq S} f_R(r) \cdot f_S(s) dr ds \quad (2.3)$$

where it is assumed that R and S are independent. By performing the integration with respect to the resistance variable, this can also be expressed as

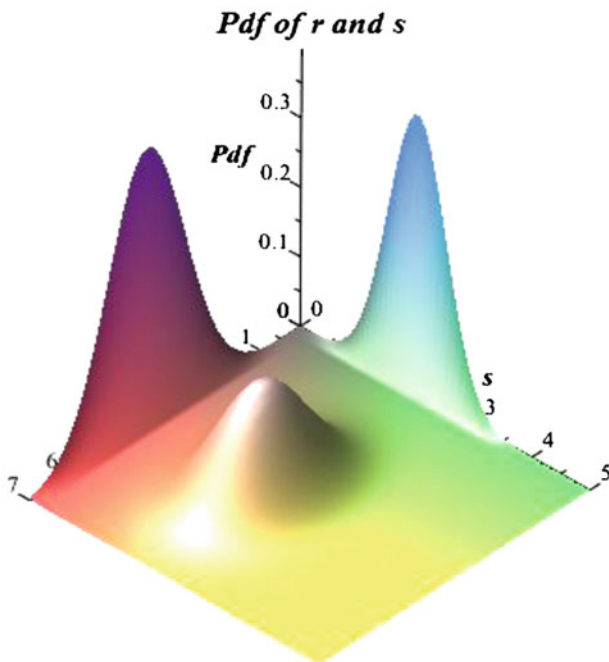
$$p_f = P(Z = R - S \leq 0) = \int_{-\infty}^{+\infty} F_R(s) f_S(s) ds \quad (2.4)$$

where

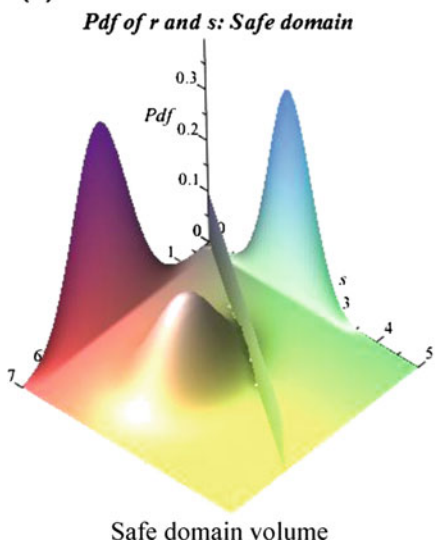
$$F_R(s) = P(R \leq s) = \int_{-\infty}^s f_R(r) dr \quad (2.5)$$

This situation is illustrated in Fig. 2.2 where the two marginal density functions now are shown in the same plane. The interval which contributes most to the failure probability is where both of the density functions have non-vanishing values (i.e. in the range between 1 and 3.5 for this particular example).

(a)



(b)



(c)

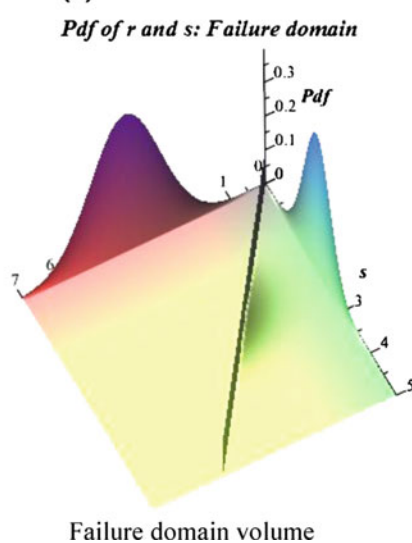
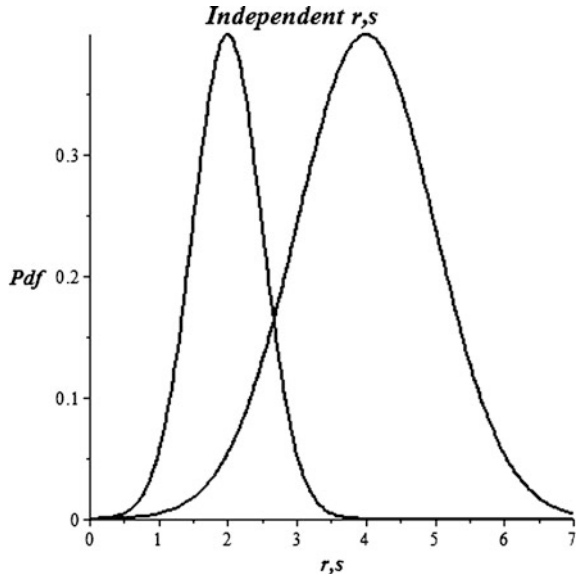


Fig. 2.1 Interpretation of failure probability as a volume. **a** Joint Pdf of r and s . **b** Safe domain volume. **c** Failure domain volume

Fig. 2.2 Marginal densities projected into same plane for the case of independent variables



The integral in (2.4) is known as a convolution integral, where $F_R(r)$ denotes the cumulative distribution function of the mechanical resistance variable R . Closed-form expressions for this integral can be obtained for certain distributions, such as the Gauss distribution as will be discussed below. The resistance probability density function in Eq. (2.3), $f_R(r)$, is frequently represented as a Gaussian or Lognormal variable. The density function of the load-effect, $f_S(s)$, typically corresponds to extreme environmental conditions (such as wind and waves) and is frequently assumed to be described by a Gumbel distribution, see e.g. [3].

2.3 Time-Varying Load and Capacity

As the loads on marine structures are mainly due to wave-, wind and current, the statistical properties will fluctuate with time. The resistance will also in general be a function of time e.g. due to deterioration processes such as corrosion (this can clearly be counteracted by repair or other types of strength upgrading). Furthermore, a typical situation is that the extreme load effects increase with the duration of the time interval (i.e. the 20 year extreme value is higher than the 10 year extreme value, and the 3-h extreme load-effect during a storm is higher than the 1-h extreme load-effect).

This situation is illustrated by Fig. 2.3 for a relatively long time horizon. Here, t denotes time, and $t_0 = 0$ is the start time. The second “time slice” is at 10 years, and the third slice is at 20 years. The corresponding probability density functions

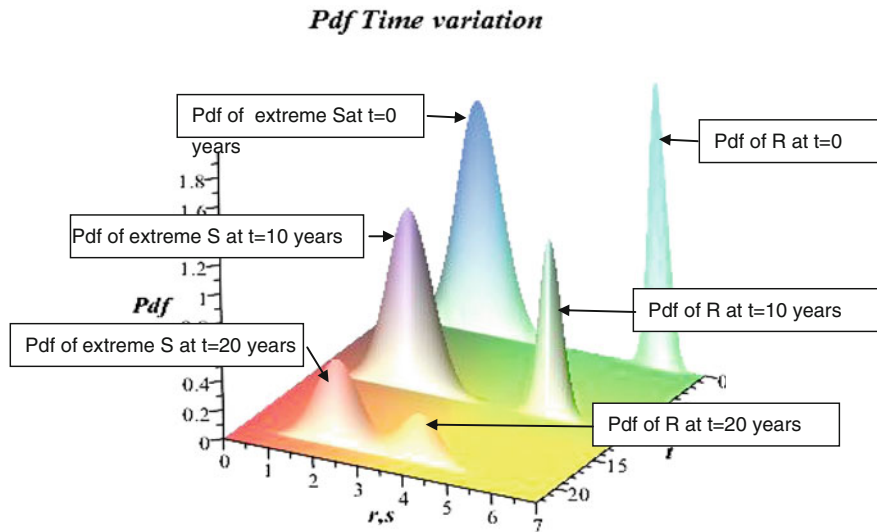


Fig. 2.3 Illustration of the time-varying marginal Pdfs of resistance, $r(t)$, and extreme load-effect $s(t)$

of the mechanical resistance and the load effect are also shown for each of the three slices.

The figure illustrates that the structure will fail if (at any time during the considered time interval)

$$Z(t) = r(t) - s(t) < 0, \quad (2.6)$$

where $Z(t)$ is referred to as the safety margin (which varies with time). The probability that the event described by (2.6) will take place can be evaluated from the amount of overlap by the two probability density functions $f_R(r)$ and $f_S(s)$ at each time step, as shown in Fig. 2.3. At $t = 0$ and 10 years, these two functions barely touch each other, while at $t = 20$ years, they have a significant amount of overlap. The latter case represents a corresponding increase of the failure probability.

If it is chosen to use time-independent values of either R or S (or both), the minimum value of (2.5) during the interval $[0, T]$ must be used, where T denotes the design life time or the duration of a specific operation under consideration. In relation to the maximum load effect, an extreme value distribution, such as the Gumbel distribution, (also referred to as the type I asymptotic form) as mentioned above. The Gumbel distribution may be applied in cases where the initial distribution has an exponentially decaying tail which is the case e.g. for stochastic processes of the Gaussian type. Similarly, the probability density and distribution function of the minimum value is relevant. For durations of the order of a few days

or less, simplifications can typically be introduced, since decrease of the strength properties on such limited time scales can usually be neglected.

The variation of the density functions will furthermore be different for the different types of limit states. For the fatigue limit state, the “resistance” can e.g. correspond to the permissible cumulative damage (i.e. given by a Miner-Palmgren sum equal to 1.0). This is a time-independent quantity which may still be represented by a (time-invariant) random variable. The “load-effect” will now correspond to the (random) cumulative damage which is obtained from the probability distribution of the stress range cycles. If there are other deterioration processes present (such as corrosion), the “resistance” will clearly decrease with time also for this type of limit state.

2.4 Simplified Calculation of Failure Probability for Gaussian Random Variables

As a special (and simplified) case, we next consider the situation when both R and S are Gaussian random variables, with mean values μ_R and μ_S and variances σ_R^2 and σ_S^2 . Furthermore, the two variables are assumed to be uncorrelated. The quantity $Z = R - S$ will then also be Gaussian, with the mean value and variance being given by

$$\text{and } \begin{aligned} \mu_Z &= \mu_R - \mu_S \\ \sigma_Z^2 &= \sigma_R^2 + \sigma_S^2 \end{aligned} \quad (2.7)$$

The probability of failure may then be written as

$$p_f = P(Z = R - S \leq 0) = \Phi\left(\frac{0 - \mu_Z}{\sigma_Z}\right) = \Phi\left(-\frac{\mu_Z}{\sigma_Z}\right) \quad (2.8)$$

where $\Phi(\cdot)$ is the standard normal distribution function (corresponding to a Gaussian variable with mean value 0 and standard deviation of 1.0). By inserting (2.6) into (2.7), we get

$$p_f = \Phi\left(-\frac{\{\mu_R - \mu_S\}}{\sqrt{\sigma_R^2 + \sigma_S^2}}\right) = \Phi(-\beta) \quad (2.9)$$

where

$$\beta = \frac{\{\mu_R - \mu_S\}}{\sqrt{\sigma_R^2 + \sigma_S^2}} \quad (2.10)$$

is defined as the safety index, see [4]. By defining an acceptable failure probability (i.e. $p_f = p_A$) on the left-hand side of this equation, one can find the corresponding value of β , i.e. β_A , that represents an acceptable lower bound on β (since decreasing β results in a higher failure probability). This value can be used to

determine in a probabilistic sense whether the resistance R is within an acceptable range as compared to the load effect, S .

2.5 Calculation of Failure Probability for Non-Gaussian Random Variables

The index above can also be extended to handle reliability formulations which involve random vectors of arbitrary dimensions. Typically, both the load effect term, S , and the resistance term, R , are expressed as functions of a number of basic parameters of a random nature. Assembling these in the respective vectors \mathbf{X}_S and \mathbf{X}_R , the failure function becomes a function of the vector $\mathbf{X} = [\mathbf{X}_S^T, \mathbf{X}_R^T]^T$. The failure surface will accordingly be defined by the equation $g(\mathbf{X}) = 0$.

The safety index can readily be extended to comprise correlated as well as non-gaussian variables. For general types of probability distributions, the failure probability as expressed by the integral in Eq. (2.1) can be computed e.g. by numerical integration. However, there also exist efficient approximate methods based on transformation into uncorrelated and standardized Gaussian variables.

In the case of non-gaussian and uncorrelated variables this transformation is based on the following expressions:

$$\begin{aligned}\Phi(u_1(t)) &= F_{X_1}(x_1(t)) \\ &\dots \\ \Phi(u_n(t)) &= F_{X_n}(x_n(t))\end{aligned}\tag{2.11}$$

The simplified $g(R,S) = (R-S)$ reliability formulation may serve to illustrate how this transformation works for two different cases. As a first reference case, the two basic variables are taken to be uncorrelated Gaussian variables with mean values ($\mu_R = 3.0, \mu_S = 1.0$) and standard deviations ($\sigma_R = 0.1, \sigma_S = 0.2$). The relationship between the original basic variables and the transformed standardized Gaussian variables are then simply expressed as $R = 0.1 U_1 + 3$ and $S = 0.2 U_2 + 1$. The corresponding failure function in the transformed and normalized (U_1, U_2)-plane is shown as the upper surface in Fig. 2.4.

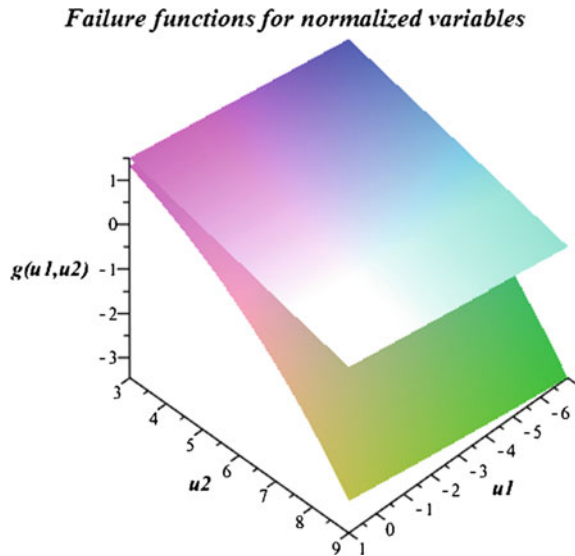
As a second case, the basic random variables R and S are both taken to be uncorrelated lognormal variables with the same mean values and standard deviations as before. The two corresponding transformations based on Eq. (2.11) then are expressed as:

$$\ln(r) = \sigma_{z1}u_1 + \mu_{z1} \quad \ln(s) = \sigma_{z2}u_2 + \mu_{z2}$$

where

$$\left(\sigma_{z1}^2 = \ln \left(1 + \left(\frac{\sigma_R^2}{\mu_R^2} \right) \right) = 0.0011, \sigma_{z2}^2 = \ln \left(1 + \left(\frac{\sigma_S^2}{\mu_S^2} \right) \right) = 0.039 \right)$$

Fig. 2.4 Comparison of failure functions expressed in terms of transformed (normalized) Gaussian variables u_1 and u_2 . *Upper* Gaussian basic random variables. *Lower* Lognormal basic random variables



and

$$(\mu_{z1} = \ln(\mu_R) - 0.5\sigma_{z1}^2 = 1.098, \mu_{z2} = \ln(\mu_S) - 0.5\sigma_{z2}^2 = -0.0196)$$

The corresponding failure function $g(u_1, u_2) = \exp(\sigma_{z1}u_1 + \mu_{z1}) - \exp(\sigma_{z2}u_2 + \mu_{z2})$ is shown in the right part of Fig. 2.4.

Transformation of correlated non-gaussian variables requires that both marginal and conditional distribution functions are applied. For a number of n random variables the expressions become:

$$\begin{aligned} \Phi(u_1(t)) &= F_{X_1}(x_1(t)) \\ &\dots \\ \Phi(u_n(t)) &= F_{X_n|X_1, X_2, \dots, X_{n-1}}(x_n(t)|x_1(t), x_2(t), \dots, x_{n-1}(t)) \end{aligned} \quad (2.12)$$

where the conditional cumulative distribution functions of increasing order are required.

Computation of the failure probability is frequently based on approximating the failure surface by its tangent plane at a proper point, or a second order surface at the same point. This point is identified by means of numerical iteration and is the point which is closest to the origin in the transformed space of standardized Gaussian variables (i.e. all of which have mean value zero and unit standard deviation).

For a more detailed description of these procedures (i.e. related to transformation of the variables and searching for the design point), reference is e.g. made to [1, 2, 5–12].

It is highly relevant to introduce a simplified version of the reliability index in Eq. (2.9) within the context of application within an on-line control algorithm. This is discussed in more detail in relation to the examples of application which are considered below.

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