Chapter 1

The joint time-space H^∞-calculus

One of the most successful approaches in the theory of partial differential equations is based on the Fourier transform and the symbols of differential operators and their inverses. The Fourier transform maps differentiation into multiplication by the covariable and thus directly leads to a representation of the inverse (i.e., solution) operator. In particular in the case of boundary value problems, non-polynomial symbols like

$$P(\lambda, z) := \lambda - (z_1^2 + \cdots + z_n^2)\sqrt{\lambda - z_1^2 - \cdots - z_n^2} \tag{1.1}$$

appear. Here the related operator is, at least symbolically, given by $P(\partial_t, \nabla_x) = \partial_t - \Delta\sqrt{\partial_t - \Delta}$. The symbol (1.1) appears in the analysis of the Stefan problem with Gibbs-Thomson correction as the determinant of the Lopatinskii matrix.

The aim of this chapter is to give a precise definition of operators like $P(\partial_t, \nabla_x)$. For this, one can apply the joint H^∞-calculus for sectorial and bisectorial operators as developed, e.g., by G. Dore and A. Venni in [DV05]. Note that P is a holomorphic function of λ and $z = (z_1, \ldots, z_n)$ for

$$|\arg \lambda| + 2 \max_{j=1,\ldots,n} |\arg(z_j) - \tfrac{\pi}{2}| < \pi.$$

As we will see below, the tuple $(\partial_t, \nabla_x) = (\partial_t, \partial_1, \ldots, \partial_n)$ has a bounded joint H^∞-calculus. Here the operators $\partial_j = \partial_{x_j}$ act on the whole real line, have their spectra on the imaginary axis and are bisectorial operators with bisectorial angle 0. In contrast to this, the time derivative ∂_t acts on the half-axis $(0, \infty)$. Therefore, we include zero initial values in the domain of the corresponding operator, the spectrum of which coincides with the complex half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq 0\}$. This leads to a sectorial operator with sectorial angle $\frac{\pi}{2}$.

The present chapter answers the following questions:

- How can one define (holomorphic) functions of ∂_t and ∇_x (Section 1.1)?

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- What are appropriate Sobolev-type spaces (Section 1.2)?
- What results follow for the joint H^∞-calculus for the time-space operator (∂_t, ∇_x) (Section 1.3)?

1.1 The joint H^∞-calculus for tuples of operators

Motivation. The aim of this section is to define functions of operator tuples where the operators are either sectorial or bisectorial and where the function is holomorphic and bounded in corresponding sectors or bisectors. The main result states that the resulting operator is well-defined and bounded. As the proof is based on a Kalton-Weis type theorem, the concept of $\mathcal{R}$-boundedness will play a role. Therefore, we will include a short summary of $\mathcal{R}$-bounded operator families. For the proofs, we will mainly restrict ourselves to giving some references.

Throughout this section, let X and Y be complex Banach spaces.

a) Sectorial and bisectorial operators, $\mathcal{R}$-boundedness

For a linear operator $T\colon X \supseteq D(T) \to X$, the resolvent set $\rho(T)$ is given as the set of all complex numbers λ for which $\lambda - T\colon D(T) \to X$ is bijective and $(\lambda - T)^{-1} \in L(X)$, where $L(X)$ stands for the space of all bounded linear operators in X. Note that $\rho(T) \neq \emptyset$ implies that T is closed. By $R(T)$ and $\ker T$ we denote the range and the kernel of T, respectively.

Definition 1.1. For $\theta \in (0, \pi)$ and $\delta \in (0, \frac{\pi}{2})$ we define the *sector* S_θ and the *bisector* Σ_δ by

$$S_\theta := \{r \exp(i\alpha)\colon r \in \mathbb{R}_+,\ \alpha \in (-\theta, \theta)\},$$
$$\Sigma_\delta := \{r \exp(i\alpha)\colon r \in \mathbb{R} \setminus \{0\},\ \alpha \in (\pi/2 - \delta, \pi/2 + \delta)\}.$$

Here, $\mathbb{R}_+ := (0, \infty)$. Let Γ_φ be the curve which is parametrized by $\gamma_\varphi\colon \mathbb{R} \setminus \{0\} \to \mathbb{C},\ r \mapsto |r| \exp(-i\varphi \operatorname{sgn}(r))$ for $\varphi \in (0, \pi)$. The curve Γ_φ is called *admissible* for S_θ if $0 < \theta < \varphi$. An admissible curve for Σ_δ is a curve of the form $\Gamma_\varphi \cup (-\Gamma_\varphi)$ with $\delta + \frac{\pi}{2} < \varphi < \pi$, see Figures 1.1 and 1.2.

Definition 1.2. (i) An operator $T\colon X \supseteq D(T) \to X$ is said to be *sectorial* if

(I) $D(T)$ and $R(T)$ are dense in X,

(II) there exists $\theta \in (0, \pi)$ such that $\rho(T) \supseteq \mathbb{C} \setminus \overline{S}_\theta = -S_{\pi - \theta}$ and

$$\sup_{\lambda \in \mathbb{C} \setminus \overline{S}_\theta} \|\lambda(\lambda - T)^{-1}\|_{L(X)} < \infty. \tag{1.2}$$

If T is sectorial, we define the *spectral angle* φ_T as the infimum of all angles $\theta \in (0,\pi)$ such that (1.2) holds.

(ii) An operator $T\colon X \supseteq D(T) \to X$ is said to be *bisectorial* if

(I) $D(T)$ and $R(T)$ are dense in X,

(II) there exists $\delta \in (0,\pi/2)$ such that $\rho(T) \supseteq \mathbb{C} \setminus \overline{\Sigma}_\delta = (-S_{\pi/2-\delta}) \cup S_{\pi/2-\delta}$ and

$$\sup_{\lambda \in \mathbb{C}\setminus\overline{\Sigma}_\delta} \|\lambda(\lambda - T)^{-1}\|_{L(X)} < \infty. \tag{1.3}$$

If T is bisectorial, we define the *spectral angle* $\varphi_T^{(\mathrm{bi})}$ as the infimum of all angles $\delta \in (0, \frac{\pi}{2})$ such that (1.3) holds.

Example 1.3. (i) Let $X = L_2(\mathbb{R}^n)$ and $T := -\Delta$ with domain $D(T) := H^2(\mathbb{R}^n) \subseteq L_2(\mathbb{R}^n)$. Then T is sectorial with spectral angle 0. This can easily be seen from the fact that T is a selfadjoint operator with spectrum $\sigma(T) = [0,\infty)$.

(ii) Let $X = L_2(\mathbb{R})$ and $T := \partial_x$ with domain $D(T) := H^1(\mathbb{R})$. Then T is skew-selfadjoint, i.e., $T^* = -T$, and the spectrum of T coincides with the imaginary axis. From this we obtain that T is bisectorial with spectral angle 0.

(iii) The results from (i) and (ii) hold also in the case when L_2 is replaced by L_p with $1 < p < \infty$. This follows from an application of Michlin's theorem.

Remark 1.4. (i) Note that every bisectorial operator T is also sectorial with $\frac{\pi}{2} - \varphi_T^{(\mathrm{bi})} \le \varphi_T \le \frac{\pi}{2} + \varphi_T^{(\mathrm{bi})}$.

(ii) In reflexive Banach spaces, the density of $R(T)$ follows from the injectivity of T. This is an immediate consequence of the following result (see [KW04, Proposition 15.2 c)]): Let T be a linear operator on a reflexive Banach space X with domain $D(T)$ such that there exists $\theta \in (0,\pi)$ for which (1.2) holds. Then we can decompose X into $X = \ker(T) \oplus \overline{R(T)}$.

The following approximation property will be useful later.

Lemma 1.5 ([KW04, Proposition 15.2 a)]). *Let $A\colon X \supseteq D(A) \to X$ be a linear operator with $(-\infty,0) \subseteq \rho(A)$ satisfying*

$$\sup_{\lambda \in (-\infty,0)} \|\lambda(\lambda - A)^{-1}\|_{L(X)} < \infty. \tag{1.4}$$

Then we have

$$\begin{aligned} \overline{R(A)} &= \{x \in X\colon \lim_{n\to\infty} -\tfrac{1}{n}(-\tfrac{1}{n} - A)^{-1}x = 0\}, \\ \overline{D(A)} &= \{x \in X\colon \lim_{n\to\infty} -n(-n - A)^{-1}x = x\}. \end{aligned} \tag{1.5}$$

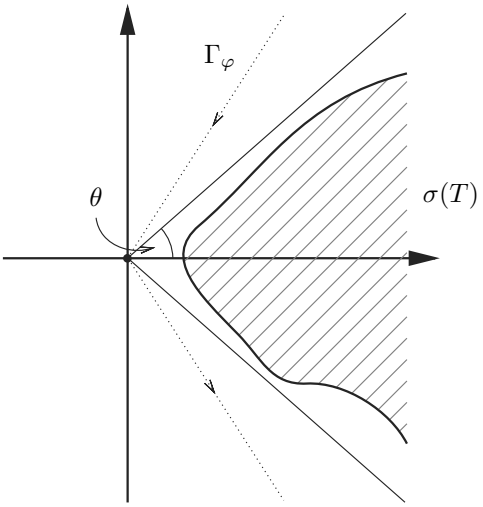


Figure 1.1: Spectrum, S_θ, and admissible curve Γ_φ

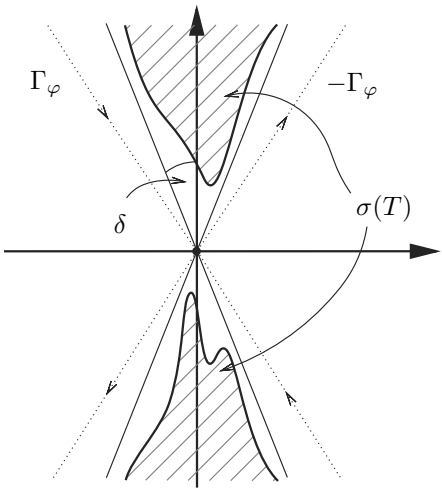


Figure 1.2: Spectrum, Σ_δ, and admissable curve $\Gamma_\varphi \cup (-\Gamma_\varphi)$

In the definition of a sectorial operator, the operator family $\{\lambda(\lambda - T)^{-1} : \lambda \in \mathbb{C}\setminus\overline{S}_\theta\}$ is bounded in the operator norm $\|\cdot\|_{L(X)}$. To show maximal regularity in L_p-Sobolev spaces or the existence of a bounded H^∞-calculus, the concept of $\mathcal{R}$-boundedness turns out to be more appropriate. In the following, we restrict ourselves to the main definitions and results. For a more detailed discussion of $\mathcal{R}$-bounded operators, we refer to [DHP03] and [KW04].

In the following definition, $L(X,Y)$ denotes the space of all bounded linear operators from X to Y, and $L_p([0,1],X)$ stands for the standard Bochner space of X-valued L_p-functions.

Definition 1.6 ($\mathcal{R}$**-boundedness**)**.** Let $\mathcal{T} \subseteq L(X,Y)$ and $p \in [1,\infty)$. Then $\mathcal{T}$ is said to be $\mathcal{R}$*-bounded* if there exists a constant $C > 0$ such that for all $m \in \mathbb{N}$, $(T_k)_{k=1,\dots,m} \subseteq \mathcal{T}$, and all $(x_k)_{k=1,\dots,m} \subseteq X$ we have

$$\Big\| \sum_{k=1}^m r_k T_k x_k \Big\|_{L_p([0,1],Y)} \leq C \Big\| \sum_{k=1}^m r_k x_k \Big\|_{L_p([0,1],X)}. \tag{1.6}$$

In this case, $\mathcal{R}_p(\mathcal{T}) := \inf\{C > 0\colon (1.6) \text{ is satisfied}\}$ is said to be the $\mathcal{R}$*-bound* of $\mathcal{T}$. Here the *Rademacher functions* r_k, $k \in \mathbb{N}$, are given by $r_k\colon [0,1] \to \{-1,1\}, t \mapsto \operatorname{sign}(\sin(2^k \pi t))$.

Remark 1.7. The proofs of the following results can be found in [DHP03] or [KW04].

(i) If $\mathcal{T} \subseteq L(X,Y)$ is $\mathcal{R}$-bounded for one $p \in [1,\infty)$, then (1.6) holds for all $p \in [1,\infty)$ where the $\mathcal{R}$-bounds fulfill $C\mathcal{R}_1(\mathcal{T}) \leq \mathcal{R}_p(\mathcal{T}) \leq C'\mathcal{R}_1(\mathcal{T})$ for some $C, C' > 0$.

(ii) If $\mathcal{T}, \mathcal{S} \subseteq L(X,Y)$ are $\mathcal{R}$-bounded, then $\mathcal{T} + \mathcal{S} := \{T + S \colon T \in \mathcal{T}, S \in \mathcal{S}\}$ is also $\mathcal{R}$-bounded with $\mathcal{R}_p(\mathcal{T} + \mathcal{S}) \leq \mathcal{R}_p(\mathcal{T}) + \mathcal{R}_p(\mathcal{S})$.

(iii) For two given $\mathcal{R}$-bounded families $\mathcal{T}_1 \subseteq L(Z,Y)$ and $\mathcal{T}_2 \subseteq L(X,Z)$ we obtain the $\mathcal{R}$-boundedness of $\mathcal{T}_1\mathcal{T}_2 := \{T_1T_2 \colon T_k \in \mathcal{T}_k, k = 1,2\} \subseteq L(X,Y)$ with $\mathcal{R}_p(\mathcal{T}_1\mathcal{T}_2) \leq \mathcal{R}_p(\mathcal{T}_1) \cdot \mathcal{R}_p(\mathcal{T}_2)$.

(iv) If $\mathcal{T} \subseteq L(X,Y)$ is $\mathcal{R}$-bounded, then $\mathcal{T}$ is also uniformly bounded. The converse is only true if X and Y are Hilbert spaces.

(v) Let $\mathcal{T} \subseteq L(X,Y)$ be $\mathcal{R}$-bounded. Then the closure in the strong operator topology of the convex hull of $\mathcal{T}$ is also $\mathcal{R}$-bounded with

$$\mathcal{R}_p(\overline{\mathrm{co}(\mathcal{T})}^{\mathrm{strong}}) \leq \mathcal{R}_p(\mathcal{T}).$$

(vi) If $\mathcal{T} \subseteq L(X,Y)$ is $\mathcal{R}$-bounded and $\mathcal{S} \subseteq \mathcal{T}$, then $\mathcal{S}$ is also $\mathcal{R}$-bounded. If $\mathcal{T} \subseteq L(X,Y)$ is finite, then $\mathcal{T}$ is $\mathcal{R}$-bounded.

To show $\mathcal{R}$-boundedness of an operator family, the following results are convenient and, in most cases, can be applied better than directly showing the conditions of the definition.

Theorem 1.8 (**Kahane's contraction principle,** cf. [KW04, Proposition 2.5])**.** *Let $p \in (1,\infty)$ and $(a_j)_{j\in\mathbb{N}}, (b_j)_{j\in\mathbb{N}} \subseteq \mathbb{C}$ with $|a_j| \leq |b_j|$ for all $j \in \mathbb{N}$. Then we have*

$$\Big\| \sum_{k=1}^n r_k a_k x_k \Big\|_{L_p([0,1],X)} \leq 2 \Big\| \sum_{k=1}^n r_k b_k x_k \Big\|_{L_p([0,1],X)}$$

for all $n \in \mathbb{N}$ and $(x_j)_{j\in\mathbb{N}} \subseteq X$.

Theorem 1.9 (**Square function estimate,** see [DHP03, Remarks 3.2] or [KW04, Remark 2.9])**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space and*

$$\mathcal{T} \subseteq L(L_p(\Omega,\mu,\mathbb{C}))$$

with $1 < p < \infty$. The family $\mathcal{T}$ is $\mathcal{R}$-bounded if and only if there exists $C > 0$ such that

$$\Big\| \Big(\sum_{k=1}^N |T_k f_k|^2 \Big)^{1/2} \Big\|_{L_p(\Omega,\mu,\mathbb{C})} \leq C \Big\| \Big(\sum_{k=1}^N |f_k|^2 \Big)^{1/2} \Big\|_{L_p(\Omega,\mu,\mathbb{C})} \tag{1.7}$$

for all $N \in \mathbb{N}$, $(f_k)_k \subseteq L_p(\Omega,\mu,\mathbb{C})$, and $(T_k)_k \subseteq \mathcal{T}$.

For the study of singular integral operators the following result is very helpful because it transfers results on scalar-valued kernels to operator-valued kernels.

Proposition 1.10 ([DHP03, Prop. 4.12], [KW04, Section 5.5]). *Let $1 < p < \infty$, and let I be an arbitrary index set. Let $\mathcal{K} = \{k_i \colon \mathbb{R}_+ \times \mathbb{R}_+ \to L(X) \,|\, i \in I\}$ be a family of kernel functions such that for each $i \in I$ the integral operator*

$$(K_i f)(x_n) := \int_0^\infty k_i(x_n, y_n) f(y_n) dy_n, \quad f \in C^\infty(\mathbb{R}_+, X) \cap L_p(\mathbb{R}_+, X), \quad x_n > 0$$

extends to a bounded operator on $L_p(\mathbb{R}_+, X)$. Suppose that there exists $k_0 \colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{C}$ such that

(i) *$(K_0 f)(x_n) := \int_0^\infty k_0(x_n, y_n) f(y_n) dy_n$ ($f \in C^\infty(\mathbb{R}_+) \cap L_p(\mathbb{R}_+)$, $x_n > 0$) extends to a bounded operator on $L_p(\mathbb{R}_+)$,*

(ii) *for each $x_n, y_n > 0$, the operator family $\{k_i(x_n, y_n) \colon i \in I\} \subseteq L(X)$ is $\mathcal{R}$-bounded with*

$$\mathcal{R}_p(\{k_i(x_n, y_n) \colon i \in I\}) \leq k_0(x_n, y_n), \quad x_n, y_n > 0.$$

Then $\{K_i \colon i \in I\} \subseteq L_p(\mathbb{R}_+, X)$ is also $\mathcal{R}$-bounded with $\mathcal{R}_p(\{K_i \colon i \in I\}) \leq \|K_0\|_{L(L_p(\mathbb{R}_+))}$.

For the handling of operator-valued L_p-Fourier multipliers, several results demand geometric properties of the underlying Banach space. We introduce two main properties.

Definition 1.11 (**Banach space of class $\mathcal{HT}$**). The *Hilbert transform* Hf of a function $f \in \mathscr{S}(\mathbb{R}, X)$, the Schwartz space of rapidly decreasing X-valued functions, is defined by

$$(Hf)(x) := \tfrac{1}{\pi} \left[PV(\tfrac{1}{x}) * f\right](x) := \tfrac{1}{\pi} \lim_{\varepsilon \searrow 0} \int_{|x-y| \geq \varepsilon} \frac{f(y)}{x - y} \, dy, \quad x \in \mathbb{R}.$$

The Banach space X is called of *class* $\mathcal{HT}$ if there exists $p \in (1, \infty)$ such that $Hf \in L_p(\mathbb{R}, X)$ for all $f \in \mathscr{S}(\mathbb{R}, X)$ and

$$\|Hf\|_{L_p(\mathbb{R}, X)} \leq C \|f\|_{L_p(\mathbb{R}, X)}, \quad f \in \mathscr{S}(\mathbb{R}, X).$$

In this case we can extend H to a bounded operator on $L_p(\mathbb{R}, X)$.

Remark 1.12. Banach spaces of class $\mathcal{HT}$ are also called *UMD spaces*. Here UMD stands for the probabilistic property of 'unconditional martingal differences'. A Banach space is of class $\mathcal{HT}$ if and only if it is a UMD space. This equivalence can be found in [Bur81] and [Bou83].

Remark 1.13. The following assertions can be found in [RdF86, Proposition 2] and [Ama95, Theorem 4.5.2], for example.

(i) Banach spaces of class $\mathcal{HT}$ are reflexive.

(ii) If X is of class $\mathcal{HT}$ and there exists $T \in L_{\mathrm{Isom}}(X, Y)$, then Y is also of class $\mathcal{HT}$.

(iii) If X is a Banach space of class $\mathcal{HT}$ and (Ω, μ) is a σ-finite measure space, then
$$L_p(\Omega, \mu, X), \quad 1 < p < \infty,$$
is also of class $\mathcal{HT}$.

(iv) If X is of class $\mathcal{HT}$, then we have $\|Hf\|_{L_p(\mathbb{R},X)} \le C_p \|f\|_{L_p(\mathbb{R},X)}$ for all $f \in \mathscr{S}(\mathbb{R}, X)$ and $p \in (1, \infty)$.

(v) Every Hilbert space is of class $\mathcal{HT}$.

Definition 1.14 (**Banach space with property** (α))**.** The Banach space X has *property* (α) if there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$, $(\alpha_{ij})_{i,j=1,\dots,n} \subseteq \mathbb{C}$, $|\alpha_{ij}| \le 1$, and all $(x_{ij})_{i,j=1,\dots,n} \subseteq X$ we have
$$\int_0^1 \int_0^1 \Big\| \sum_{i,j=1}^n r_i(u) r_j(v) \alpha_{ij} x_{ij} \Big\|_X du\, dv \le C \int_0^1 \int_0^1 \Big\| \sum_{i,j=1}^n r_i(u) r_j(v) x_{ij} \Big\|_X du\, dv.$$

Remark 1.15. (i) Let X and Y be Banach spaces. If X has property (α) and there exists $T \in L_{\mathrm{Isom}}(X, Y)$, then Y also has property (α).

(ii) Let X be a Banach space with property (α), and let (Ω, μ) be a σ-finite measure space. Then
$$L_p(\Omega, \mu, X), \quad 1 \le p < \infty$$
also has property (α), cf. [KW04, Remark 4.10].

(iii) Every Hilbert space has property (α).

b) Joint H^∞-calculus

For the analysis of the time-space derivative, it is essential to consider not a single (sectorial or bisectorial) operator but tuples of operators and the joint H^∞-calculus for such tuples. The concept of H^∞-calculus goes back to A. McIntosh, see for example [McI86] and [CDMY96]. Another comprehensive and more recent work on the H^∞-calculus can be found in [Haa06]. Here we present the basic notation and results by G. Dore and A. Venni [DV05] on the joint H^∞-calculus for sectorial and bisectorial operators.

In the following, X and Y are complex Banach spaces.

Definition 1.16. Let $T_k \colon X \supseteq D(T_k) \to X$, $k = 1, \dots, N$, be linear operators. The operator tuple $\mathbf{T} := (T_1, \dots, T_N)$ is called *admissible* if

(i) each of the operators T_k $(k = 1, \dots, N)$ is sectorial or bisectorial,

(ii) for every $j, k = 1, \dots, N$ the resolvents of T_j and T_k commute.

Note that (ii) means that $(\lambda - T_j)^{-1}(\mu - T_k)^{-1} = (\mu - T_k)^{-1}(\lambda - T_j)^{-1}$ holds for all $\lambda \in \rho(T_j)$ and $\mu \in \rho(T_k)$. For this, it is sufficient that this equality holds for one $\lambda \in \rho(T_j)$ and one $\mu \in \rho(T_k)$ (see [Kat76, Theorem 6.5]).

Let us fix the situation for the remainder of this subsection. Let $\mathbf{T} = (T_1, \dots, T_N)$ be an admissible operator tuple. If T_k is sectorial, we fix a sector $\Omega_k := S_{\theta_k}$ with $\theta_k > \varphi_{T_k}$ and a curve $\Gamma^{(k)} \subseteq \Omega_k$ of the form $\Gamma^{(k)} = \Gamma_{\varphi_k}$ (see Definition 1.1) with $\varphi_{T_k} < \varphi_k < \theta_k$. Similarly, if T_k is bisectorial, we fix a bisector Σ_{δ_k} with $\delta_k > \varphi_{T_k}^{(\mathrm{bi})}$ and a curve $\Gamma^{(k)} \subseteq \Sigma_k$ of the form $\Gamma^{(k)} = \Gamma_{\varphi_k} \cup (-\Gamma_{\varphi_k})$ with $\varphi_{T_k}^{(\mathrm{bi})} < \varphi_k < \delta_k$. We set $\Omega := \prod_{k=1}^N \Omega_k \subseteq \mathbb{C}^N$ and $\Gamma := \prod_{k=1}^N \Gamma^{(k)}$. Further, let $\mathcal{B}_{\mathbf{T}}$ be the commutator of $\{(\lambda - T_k)^{-1} \colon \lambda \in \rho(T_k),\ k \in \{1, \dots, N\}\} \subseteq L(X)$, that is, the closed subalgebra consisting of all bounded operators that commute with all resolvents $(\lambda - T_k)^{-1}$, $\lambda \in \rho(T_k)$, $k \in \{1, \dots, N\}$.

Definition 1.17. We define the following spaces of holomorphic functions:

(i) $H(\Omega, Y)$, the vector space of all Y-valued holomorphic functions on Ω,

(ii) $H^\infty(\Omega, Y)$, the Banach space of all Y-valued bounded holomorphic functions on Ω,

(iii) $H^\infty_{\mathcal{R}}(\Omega, L(X)) := \{f \in H^\infty(\Omega, L(X)) \colon f(\Omega) \subseteq L(X) \text{ is } \mathcal{R}\text{-bounded}\}$,

(iv) $H_0^\infty(\Omega, Y) := \Big\{ f \in H(\Omega, Y) \colon \exists\, C, s > 0\, \forall\, z \in \Omega \colon \|f(z)\|_Y \le C \prod_{k=1}^N \big(\min\{|z_k|, |z_k|^{-1}\}\big)^s \Big\}$,

(v) $H_P(\Omega, Y) := \Big\{ f \in H(\Omega, Y) \colon \exists\, C, s > 0\, \forall z \in \Omega \colon \|f(z)\|_Y \le C \prod_{k=1}^N \big(\max\{|z_k|, |z_k|^{-1}\}\big)^s \Big\}$.

Note that functions in $H_0^\infty(\Omega, Y)$ show an additional decay at zero and at infinity while functions in $H_P(\Omega, Y)$ are polynomially bounded at zero and infinity.

We will now define $f(\mathbf{T})$ for functions $f \in H_0^\infty(\Omega, Y)$. In many cases, the functions are scalar-valued, i.e., $Y = \mathbb{C}$ (in this case, we will omit Y in the notation). However, if one considers an iterative calculus, Y has to be more general, and therefore we formulate the following definitions and results for closed subalgebras of $\mathcal{B}_{\mathbf{T}}$. Of course, the scalar-valued setting fits into this generalization via the identification $z \mapsto z \,\mathrm{id}_Y$.

Definition 1.18 (Operator-valued H^∞-calculus). Let $\mathcal{A} \subseteq \mathcal{B}_{\mathbf{T}}$ be a closed subalgebra and $f \in H_0^\infty(\Omega, \mathcal{A})$. Then we set

$$f(\mathbf{T}) := \frac{1}{(2\pi i)^N} \int_\Gamma f(z) \prod_{k=1}^N (z_k - T_k)^{-1} dz \in \mathcal{A}.$$

Note that the integral converges in the operator norm $\|\cdot\|_{L(X)}$, and $f(\mathbf{T})$ does not depend on Γ by Cauchy's integral theorem.

Lemma 1.19 ([DV05, Lemma 2.7]). *For $n \in \mathbb{N}$ the function*

$$\psi_{n,N}\colon (\mathbb{C} \setminus \{-n, -\tfrac{1}{n}\})^N \to \mathbb{C}, \quad z \mapsto \prod_{k=1}^{N} \frac{n^2 z_k}{(1+nz_k)(n+z_k)}$$

has the following properties:

(i) *We have $\psi_{n,N} \in H_0^\infty(\Omega)$.*

(ii) *For all $f \in H_P(\Omega, Y)$ there exists $m \in \mathbb{N}_0$ such that $\psi_{n,N}^m f \in H_0^\infty(\Omega, Y)$ for all $n \in \mathbb{N}$.*

(iii) *The operator $\psi_{n,N}(\mathbf{T})$ is injective.*

Definition 1.20 (H_P**-calculus**). Let $f \in H_P(\Omega, \mathcal{B}_\mathbf{T})$ and $m \in \mathbb{N}_0$ such that $\psi^m f \in H_0^\infty(\Omega, \mathcal{B}_\mathbf{T})$ where $\psi := \psi_{1,N}$. Then we set

$$f(\mathbf{T})x := \psi(\mathbf{T})^{-m}(\psi^m f)(\mathbf{T})x, \quad x \in D(f(\mathbf{T}))$$

with domain $D(f(\mathbf{T})) := \{x \in X\colon (\psi^m f)(\mathbf{T})x \in R(\psi(\mathbf{T})^m)\}$.

Remark 1.21. (i) Every bisectorial operator A is also sectorial. Therefore, we can define $f(A)$ by using the two different curves Γ_s and Γ_bi but both interpretations are equal, i.e., $f(A_\mathrm{s}) = f(A_\mathrm{bi})$ for all $f \in H^\infty(S_\theta, \mathcal{B}_\mathbf{T})$. This can be easily proved by Cauchy's integral theorem.

(ii) The definition of the domain in Definition 1.20 is not constructive and it is not easy to determine $D(f(\mathbf{T}))$ in a concrete situation.

(iii) Let $f \in H^\infty(\Omega, \mathcal{B}_\mathbf{T})$ with $f(\mathbf{T}) \in L(X)$. Then we have $\psi_{n,N} f \in H_0^\infty(\Omega, \mathcal{B}_\mathbf{T})$ for all $n \in \mathbb{N}$. For all $x \in X$ we can represent $f(\mathbf{T})x$ by

$$f(\mathbf{T})x = \lim_{n\to\infty} (\psi_{n,N} f)(\mathbf{T})x.$$

This is the so-called convergence lemma, cf. [DV05, Theorem 4.7], [Haa06, Proposition 5.1.4].

(iv) Setting $f(\lambda) = \mathrm{id}_X$, $\lambda \in \Omega$, in (iii), we see that for all $x \in X$ we have $\psi_{n,N}(\mathbf{T})x \to x$, $n \to \infty$, in X.

Definition 1.22. Let $\mathcal{A}$ be a closed subalgebra of $\mathcal{B}_\mathbf{T}$. We say that the operator tuple $\mathbf{T} = (T_1, \dots, T_N)$ admits a *bounded joint $H^\infty(\Omega, \mathcal{A})$-calculus* if there exists $C > 0$ such that $\|f(\mathbf{T})\|_{L(X)} \le C\|f\|_\infty$ for all $f \in H_0^\infty(\Omega, \mathcal{A})$. The tuple $\mathbf{T}$ admits an *$\mathcal{R}$-bounded joint $H^\infty(\Omega, \mathcal{A})$-calculus* if

$$\big\{f(\mathbf{T})\colon f \in H_0^\infty(\Omega, \mathcal{A}),\ \|f\|_\infty \le 1\big\} \subseteq L(X)$$

is $\mathcal{R}$-bounded. In the case $\mathcal{A} = \mathbb{C}$, we speak of an *($\mathcal{R}$-)bounded joint $H^\infty(\Omega)$-calculus*. In the case of one operator, i.e., for $N = 1$, we speak of an *($\mathcal{R}$-)bounded $H^\infty(\Omega, \mathcal{A})$-calculus*.

Lemma 1.23. *Let $\mathcal{A}$ be a closed subalgebra of $\mathcal{B}_{\mathbf{T}}$. If $\mathbf{T}$ admits an $\mathcal{R}$-bounded joint $H^\infty(\Omega,\mathcal{A})$-calculus, then*

$$\big\{f(\mathbf{T})\colon f\in H^\infty(\Omega,\mathcal{A}),\ \|f\|_\infty\le C\big\}\subseteq L(X)$$

is also $\mathcal{R}$-bounded for all $C>0$ and we have

$$\begin{aligned}\mathcal{R}_p\big(\{f(\mathbf{T})\colon f\in H^\infty(\Omega,\mathcal{A}),\ \|f\|_\infty\le C\}\big)\\ \le C\cdot\mathcal{R}_p\big(\{g(\mathbf{T})\colon g\in H_0^\infty(\Omega,\mathcal{A}),\ \|g\|_\infty\le 1\}\big).\end{aligned}$$

Proof. We trivially have

$$\begin{aligned}M&:=\big\{(1/C\cdot f)(\mathbf{T})\colon f\in H_0^\infty(\Omega,\mathcal{A}),\ \|f\|_\infty\le C\big\}\\ &\subseteq\big\{g(\mathbf{T})\colon g\in H_0^\infty(\Omega,\mathcal{A}),\ \|g\|_\infty\le 1\big\}.\end{aligned}$$

Hence M is also $\mathcal{R}$-bounded with $\mathcal{R}_p(M)\le\mathcal{R}_p(\{g(\mathbf{T})\colon g\in H_0^\infty(\Omega,\mathcal{A}),\|g\|_\infty\le 1\})$. Remark 1.7 (iii) and (v) and Remark 1.21 (iii) then yield the claim. □

Definition 1.24. Let T be a sectorial operator. Then the *H^∞-angle* φ_T^∞ is defined as the infimum of the set of all φ such that T admits a bounded $H^\infty(S_\varphi)$-calculus (if this set is non-empty). In the same way, the *$\mathcal{R}$-H^∞-angle* $\varphi_T^{\mathcal{R},\infty}$ is defined as the infimum over all φ such that T admits an $\mathcal{R}$-bounded $H^\infty(S_\varphi)$-calculus.

Analogously, the angles $\varphi_T^{\infty,(\mathrm{bi})}$ and $\varphi_T^{\mathcal{R},\infty,(\mathrm{bi})}$ are defined for a bisectorial operator T.

Remark 1.25. (i) In general we cannot define an analog to the spectral angle or H^∞-angle in the case of more than one operator (i.e., $N>1$), because the angles of the sectors or bisectors can be correlated.

(ii) We have $\varphi_T\le\varphi_T^\infty\le\varphi_T^{\mathcal{R},\infty}$ (respectively, $\varphi_T^{(\mathrm{bi})}\le\varphi_T^{\infty,(\mathrm{bi})}\le\varphi_T^{\mathcal{R},\infty,(\mathrm{bi})}$).

In the following, the notation $A\subseteq B$ for two operators stands for $D(A)\subseteq D(B)$ and $B|_{D(A)}=A$.

Theorem 1.26 ([DV05, Theorem 4.5])**.** *Let $f,g\in H_P(\Omega,\mathcal{B}_{\mathbf{T}})$. Then the following assertions hold:*

(i) *$f(\mathbf{T})$ is a closed operator with dense domain.*

(ii) *We have*

$$\begin{aligned}f(\mathbf{T})g(\mathbf{T})&\subseteq(fg)(\mathbf{T}),\\ f(\mathbf{T})+g(\mathbf{T})&\subseteq(f+g)(\mathbf{T})\end{aligned}$$

where the domains are given by

$$\begin{aligned}D(f(\mathbf{T})g(\mathbf{T}))&:=\{x\in D(g(\mathbf{T}))\colon g(\mathbf{T})x\in D(f(\mathbf{T}))\},\\ D(f(\mathbf{T})+g(\mathbf{T}))&:=D(f(\mathbf{T}))\cap D(g(\mathbf{T})).\end{aligned}$$

(iii) *If* $g(\mathbf{T}) \in L(X)$, *then* $(fg)(\mathbf{T}) = f(\mathbf{T})g(\mathbf{T})$ *and* $(f+g)(\mathbf{T}) = f(\mathbf{T}) + g(\mathbf{T})$.

Theorem 1.27 ([DV05, Theorem 4.9]). *Let* $\mathcal{A}$ *be a closed subalgebra of* $\mathcal{B}_{\mathbf{T}}$. *Then the following statements are equivalent:*

(i) $f(\mathbf{T}) \in L(X)$ *for all* $f \in H^\infty(\Omega, \mathcal{A})$,

(ii) $\mathbf{T}$ *has a bounded joint* $H^\infty(\Omega, \mathcal{A})$*-calculus.*

The following theorem is a generalization of a result of N.J. Kalton and L. Weis [KW01, Theorem 4.4] to tuples of operators due to G. Dore and A. Venni [DV05]. Kalton-Weis type results are one of the main ingredients for the functional calculus and state that if an operator tuple admits a bounded scalar H^∞-calculus, then we have also a functional calculus for vector-valued functions provided that the range of the functions are not only bounded in operator norm but $\mathcal{R}$-bounded.

Theorem 1.28 ([DV05, Theorem 6.7]). *Let* $\widetilde{\Omega}$ *be a set of the same type as* Ω *but with smaller angles. If* $\mathbf{T}$ *admits a bounded scalar* $H^\infty(\widetilde{\Omega})$*-calculus, then we have* $g(\mathbf{T}) \in L(X)$ *for all* $g \in H^\infty_{\mathcal{R}}(\Omega, \mathcal{B}_{\mathbf{T}})$. *In particular, there exists* $C = C(\mathbf{T}) > 0$ *with*

$$\|g(\mathbf{T})\|_{L(X)} \leq C \cdot \mathcal{R}_2(g(\Omega)).$$

One motivation for the development of a joint H^∞-calculus is given by the possibility of an iteration. The following result shows some compatibility properties.

Lemma 1.29 (Iterative calculus). (i) *For* $f \in H^\infty_0(\Omega)$ *define* $g(z') := f(T_1, z')$ *for* $z' \in \Omega' := \prod_{k=2}^N \Omega_k$. *Then* $g \in H^\infty_0(\Omega', \mathcal{B}_{\mathbf{T}})$ *and* $f(\mathbf{T}) = g(\mathbf{T}')$ *with* $\mathbf{T}' := (T_2, \ldots, T_N)$.

(ii) *Let* $J \in \{1, \ldots, N-1\}$, $\Omega' := \prod_{k=1}^J \Omega_k$, *and* $f \in H_P(\Omega', \mathcal{B}_{\mathbf{T}})$. *For* $g\colon \Omega \to \mathcal{B}_{\mathbf{T}}$, $z \mapsto f(z_1, \ldots, z_J)$ *we have* $g \in H_P(\Omega, \mathcal{B}_{\mathbf{T}})$ *and* $g(\mathbf{T}) = f(\mathbf{T}')$ *with* $\mathbf{T}' := (T_1, \ldots, T_J)$.

Proof. (i) We obviously have $f(\cdot, z') \in H^\infty_0(\Omega_1)$ for all $z' \in \Omega'$ and $g(z') := f(T_1, z') \in \mathcal{B}_{\mathbf{T}}$ due to Definition 1.18. Then Morera's theorem and Fubini's theorem yield the holomorphy of the function g and even

$$g \in H^\infty_0(\Omega', \mathcal{B}_{\mathbf{T}}).$$

Hence $g(\mathbf{T}')$ is meaningful. Due to the boundedness of the resolvents we obtain

$$\begin{aligned} g(\mathbf{T}') &= \frac{1}{(2\pi i)^{N-1}} \int_{\Gamma'} g(z') \prod_{k=2}^N (z_k - T_k)^{-1} dz' \\ &= \frac{1}{(2\pi i)^N} \int_{\Gamma'} \left(\int_{\Gamma_1} f(z)(z_1 - T_1)^{-1} dz_1 \right) \prod_{k=2}^N (z_k - T_k)^{-1} dz' \end{aligned}$$

$$= \frac{1}{(2\pi i)^N} \int_{\Gamma'} \int_{\Gamma_1} f(z)(z_1 - T_1)^{-1} \prod_{k=2}^{N} (z_k - T_k)^{-1} dz_1 dz' = f(\mathbf{T})$$

where all integrals converge with respect to $\|\cdot\|_{L(X)}$.

(ii) The easy proof for $J = 1$ can be found in [DV05, Theorem 4.12]. It is trivial to extend this to get the result of (ii).

□

One disadvantage of the H_P-calculus is the fact that in general no explicit description of the domain of $f(\mathbf{T})$ is available. The following two results show that at least domains can be compared if estimates between the functions are known.

Lemma 1.30. *Let $\mathbf{T}$ admit a bounded $H^\infty(\Omega)$-calculus, and let $f_1, f_2 \in H_P(\Omega)$.*

(i) *If there exists $C > 0$ with $|f_1(z)| \leq C\,|f_2(z)| \neq 0$ for all $z \in \Omega$, then we have $D(f_2(\mathbf{T})) \subseteq D(f_1(\mathbf{T}))$, $R(f_1(\mathbf{T})) \subseteq R(f_2(\mathbf{T}))$, and*

$$\|f_1(\mathbf{T})x\|_X \leq C' \,\|f_2(\mathbf{T})x\|_X, \quad x \in D(f_2(\mathbf{T}))$$

for some $C' > 0$.

(ii) *If there exist $C_1, C_2 > 0$ such that $C_1|f_2(z)| \leq |f_1(z)| \leq C_2|f_2(z)|$ and $f_1(z), f_2(z) \neq 0$ for all $z \in \Omega$, then we have $D(f_2(\mathbf{T})) = D(f_1(\mathbf{T}))$, $R(f_1(\mathbf{T})) = R(f_2(\mathbf{T}))$ and*

$$C_1' \,\|f_2(\mathbf{T})x\|_X \leq \|f_1(\mathbf{T})x\|_X \leq C_2' \,\|f_2(\mathbf{T})x\|_X, \quad x \in D(f_2(\mathbf{T}))$$

for some $C_1', C_2' > 0$.

Proof. (i) We have $f_1/f_2 \in H^\infty(\Omega)$ by assumption. Hence we get

$$f_1(\mathbf{T}) = \left(\frac{f_1}{f_2} \cdot f_2\right)(\mathbf{T}) \supseteq \left(\frac{f_1}{f_2}\right)(\mathbf{T}) f_2(\mathbf{T})$$

(cf. Theorem 1.26 (ii)) and $(f_1/f_2)(\mathbf{T}) \in L(X)$. So we obtain the claimed relation between the domains as well as the claimed estimate. On the other hand we have

$$f_1(\mathbf{T}) = \left(f_2 \cdot \frac{f_1}{f_2}\right)(\mathbf{T}) = f_2(\mathbf{T}) \left(\frac{f_1}{f_2}\right)(\mathbf{T})$$

(cf. Theorem 1.26 (iii)), which yields the claimed relation between the ranges.

(ii) This follows immediately from (i).

□

Lemma 1.31. *Let $g, h \in H_P(\Omega)$ with $g(z), h(z) \neq 0$ for all $z \in \Omega$ and $g^{-1}, h^{-1} \in H^\infty(\Omega)$. If we define $f(z) := g(z) \cdot h(z)$, $z \in \Omega$, we can conclude*

$$f(\mathbf{T}) = g(\mathbf{T})h(\mathbf{T}) = h(\mathbf{T})g(\mathbf{T}),$$

which in particular yields $D(f(\mathbf{T})) = D(g(\mathbf{T})h(\mathbf{T})) = D(h(\mathbf{T})g(\mathbf{T}))$.

Proof. It suffices to show $f(\mathbf{T}) = g(\mathbf{T})h(\mathbf{T})$. From Theorem 1.26 we already know

$$f(\mathbf{T}) \supseteq g(\mathbf{T})h(\mathbf{T}), \quad f^{-1}(\mathbf{T}) = g^{-1}(\mathbf{T})h^{-1}(\mathbf{T}) = h^{-1}(\mathbf{T})g^{-1}(\mathbf{T}),$$

which yields $D(g(\mathbf{T})h(\mathbf{T})) \subseteq D(f(\mathbf{T})) = R(f^{-1}(\mathbf{T})) = R(g^{-1}(\mathbf{T})h^{-1}(\mathbf{T})) = R(h^{-1}(\mathbf{T})g^{-1}(\mathbf{T}))$. One can easily show $R(h^{-1}(\mathbf{T})g^{-1}(\mathbf{T})) \subseteq D(g(\mathbf{T})h(\mathbf{T}))$, which finishes the proof. □

At the end of this subsection on the joint H^∞-calculus, we formulate some results on the behaviour with respect to isomorphisms and with respect to shifted operators which will be useful in the applications.

Proposition 1.32 (H^∞**-calculus and isomorphisms**). *Let X, Y be complex Banach spaces such that there exists an isomorphism $\Phi \in L_{\mathrm{Isom}}(Y, X)$. For each $k = 1, \dots, N$ let*

$$T_k \colon X \supseteq D(T_k) \to X$$

be a linear operator. Define

$$S_k \colon Y \supseteq D(S_k) \to Y, \quad y \mapsto \Phi^{-1} T_k \Phi y$$

where $D(S_k) := \Phi^{-1}(D(T_k))$. Defining $\mathbf{T} := (T_1, \dots, T_N)$ and $\mathbf{S} := (S_1, \dots, S_N)$, we get the following assertions:

(i) *If T_k is sectorial, then the operator S_k is also sectorial with $\varphi_{S_k} = \varphi_{T_k}$. In particular, we have $\rho(S_k) = \rho(T_k)$ and $(\lambda - S_k)^{-1} = \Phi^{-1}(\lambda - T_k)^{-1}\Phi$ for all $\lambda \in \rho(T_k)$. The analog results hold in the case of T_k being bisectorial. If $\mathbf{T}$ is admissible, then $\mathbf{S}$ is also admissible.*

(ii) *If $\mathbf{T}$ has a bounded joint $H^\infty(\Omega)$-calculus, then $\mathbf{S}$ also has a bounded joint $H^\infty(\Omega)$-calculus and the representation*

$$f(\mathbf{S}) = \Phi^{-1} f(\mathbf{T}) \Phi$$

holds for all $f \in H_P(\Omega)$.

(iii) *If $\mathbf{T}$ has an $\mathcal{R}$-bounded joint $H^\infty(\Omega)$-calculus, then $\mathbf{S}$ also has an $\mathcal{R}$-bounded joint $H^\infty(\Omega)$-calculus.*

Proof. (i) Without loss of generality let T_k be a sectorial operator. Then we obviously have $\rho(T_k) = \rho(S_k)$ and $(\lambda - S_k)^{-1} = \Phi^{-1}(\lambda - T_k)^{-1}\Phi$ for all $\lambda \in \rho(T_k)$. From this we easily obtain for $\theta \in (0, \pi)$ with $\rho(T_k) = \rho(S_k) \supseteq \mathbb{C} \setminus \overline{S}_\theta$

$$\sup_{\lambda \in \overline{S}_{\theta'}} \|\lambda(\lambda - S_k)^{-1}\|_{L(Y)} \leq C \sup_{\lambda \in \overline{S}_{\theta'}} \|\lambda(\lambda - T_k)^{-1}\|_{L(X)} < \infty$$

for all $\theta' \in (\theta, \pi)$. The density of $D(S_k)$ and $R(S_k)$ is obvious. Therefore, S_k is also sectorial with $\varphi_{T_k} = \varphi_{S_k}$. The claimed admissibility of $\mathbf{S}$ is obvious, too.

(ii) For $f \in H_0^\infty(\Omega)$ we directly derive from (i)

$$\begin{aligned} f(\mathbf{S})y &= \frac{1}{(2\pi i)^N} \int_\Gamma f(z) \prod_{k=1}^N (z_k - S_k)^{-1} y \, dz \\ &= \frac{1}{(2\pi i)^N} \int_\Gamma f(z) \prod_{k=1}^N \Phi^{-1}(\lambda - T_k)^{-1} \Phi y \, dz \\ &= \frac{1}{(2\pi i)^N} \Phi^{-1} \left[\int_\Gamma f(z) \prod_{k=1}^N (\lambda - T_k)^{-1} dz \right] \Phi y = \Phi^{-1} f(\mathbf{T}) \Phi y \end{aligned}$$

for all $y \in Y$. Hence we have $\|f(\mathbf{S})\|_{L(Y)} \le C\|f(\mathbf{T})\|_{L(X)} \le C\|f\|_\infty$ for all $f \in H_0^\infty(\Omega)$. This proves that $\mathbf{S}$ admits a bounded joint $H^\infty(\Omega)$-calculus.

Let $f \in H_P(\Omega)$ and $m \in \mathbb{N}_0$ such that $\psi^m f \in H_0^\infty(\Omega)$. Then we have $\psi(\mathbf{S}) = \Phi^{-1}\psi(\mathbf{T})\Phi$ and $(\psi^m f)(\mathbf{S}) = \Phi^{-1}(\psi^m f)(\mathbf{T})\Phi$ due to $\psi \in H_0^\infty(\Omega)$. With this we obtain

$$\begin{aligned} f(\mathbf{S}) &= \psi(\mathbf{S})^{-m}(\psi^m f)(\mathbf{S}) = \Phi^{-1}\psi(\mathbf{T})^{-m}\Phi\Phi^{-1}(\psi^m f)(\mathbf{T})\Phi \\ &= \Phi^{-1} f(\mathbf{T})\Phi. \end{aligned}$$

(iii) This easily follows from the representation in (ii) and Remark 1.7 (iii). □

Lemma 1.33 (H^∞**-calculus of a shifted operator**). *Let $T\colon X \supseteq D(T) \to X$ be a sectorial operator. Then for all $\sigma \ge 0$ the operator $S := \sigma + T$ is also sectorial with $\varphi_S \le \varphi_T$. In this situation we also have*

$$f(S) = [f(\sigma + \cdot)](T) \tag{1.8}$$

for all $f \in H_P(S_\theta)$ with $\theta > \varphi_T$. If T admits a bounded H^∞-calculus, then S also admits a bounded H^∞-calculus with $\varphi_S^\infty \le \varphi_T^\infty$. If T even admits an $\mathcal{R}$-bounded H^∞-calculus, then S also admits an $\mathcal{R}$-bounded H^∞-calculus with $\varphi_S^{\mathcal{R},\infty} \le \varphi_T^{\mathcal{R},\infty}$.

Proof. The sectoriality of S and $\varphi_S \le \varphi_T$ are obvious. Let $\theta > \varphi_T$ and $g(z) := \sigma + z$ for $z \in S_\theta$. Then we have $g(T) = \sigma + T$ by Theorem 1.26 (iii). Now we directly obtain (1.8) by [KW04, Proposition 15.11], where the authors prove the transformation formula $(f \circ g)(T) = f(g(T))$ for a more general situation. Trivially S also admits a bounded H^∞-calculus since

$$\|f(S)\|_{L(X)} \le C\|f(\sigma + \cdot)\|_\infty \le C\|f\|_\infty$$

for all $f \in H_0^\infty(S_\theta)$. The claimed $\mathcal{R}$-boundedness is obvious by $\|f(\sigma+\cdot)\|_\infty \le \|f\|_\infty$ and Remark 1.7 (vi). □

Lemma 1.34 (**H^∞-calculus for a shifted operator tuple**). *Let* $\mathbf{T}$ *be a tuple of operators with a bounded* $H^\infty(\Omega)$*-calculus and* $\mathbf{S} := (\sigma + T_1, T_2, \dots, T_N)$ *for* $\sigma > 0$*. If* T_1 *is a sectorial operator, then* $\mathbf{S}$ *also admits a bounded* $H^\infty(\Omega)$*-calculus, and*

$$f(\mathbf{S}) = f_\sigma(\mathbf{T}), \quad f_\sigma(z) := f(\sigma + z_1, z_2, \dots, z_N)$$

for all $f \in H_P(\Omega)$*.*

Proof. (I) First, we prove the assertion for $f \in H_0^\infty(\Omega)$. For all $z' \in \Omega' := \prod_{k=2}^N \Omega_k$ we define $g(z') := f(\sigma + T_1, z')$. Then Lemma 1.33 and Lemma 1.29 yield $g(z') = f_\sigma(T_1, z')$ and

$$f(\mathbf{S}) = g(\mathbf{T}') = f_\sigma(\mathbf{T})$$

where $\mathbf{T}' := (T_2, \dots, T_N)$. Obviously, $\|f(\mathbf{S})\|_{L(X)} \le C\|f_\sigma\|_\infty \le C\|f\|_\infty$ for all $f \in H_0^\infty(\Omega)$ and thus $\mathbf{S}$ also admits a bounded $H^\infty(\Omega)$-calculus.

(II) Let $f \in H_P(\Omega)$. Then we get

$$\begin{aligned} f(\mathbf{S}) &= \psi(\mathbf{S})^{-m}(\psi^m f)(\mathbf{S}) = \psi_\sigma(\mathbf{T})^{-m}(\psi_\sigma^m f_\sigma)(\mathbf{T}) \\ &\supseteq \psi_\sigma(\mathbf{T})^{-m}\psi_\sigma(\mathbf{T})^m f_\sigma(\mathbf{T}) = f_\sigma(\mathbf{T}) \end{aligned}$$

due to Theorem 1.26 and the result of part (I). The converse inclusion can be obtained in the same way by

$$\begin{aligned} f_\sigma(\mathbf{T}) &= \psi(\mathbf{T})^{-m}(\psi^m f_\sigma)(\mathbf{T}) = \psi(\mathbf{T})^{-m}\left(\left(\frac{\psi^m}{\psi_\sigma^m}\right)\cdot(\psi_\sigma^m f_\sigma)\right)(\mathbf{T}) \\ &\supseteq \psi(\mathbf{T})^{-m}\left(\frac{\psi^m}{\psi_\sigma^m}\right)(\mathbf{T})\,(\psi_\sigma^m f_\sigma)\,(\mathbf{T}) \\ &= \psi(\mathbf{T})^{-m}\left(\frac{\psi^m}{\psi_\sigma^m}\right)(\mathbf{T})\,(\psi^m f)\,(\mathbf{S}) \\ &\supseteq \psi(\mathbf{T})^{-m}\psi(\mathbf{T})^m\psi_\sigma^{-m}(\mathbf{T})\,(\psi^m f)\,(\mathbf{S}) \\ &= \psi(\mathbf{S})^{-m}\,(\psi^m f)\,(\mathbf{S}) = f(\mathbf{S}). \end{aligned}$$

□

In the following, the notation $Y \hookrightarrow X$ means that Y is continuously embedded into X, i.e., there exists a continuous and injective linear mapping from Y to X.

Proposition 1.35 (**Compatibility of the H^∞-calculus on spaces of higher regularity**). *Let* X, Y *be complex Banach spaces with* $Y \hookrightarrow X$*. Let*

$$\begin{aligned} T_k &\colon X \supseteq D(T_k) \to X, \\ S_k &\colon Y \supseteq D(S_k) \to Y, \quad k = 1, \dots, N \end{aligned}$$

be sectorial or bisectorial operators (of the same type) with $D(S_k) \subseteq D(T_k)$ *(with respect to* $Y \hookrightarrow X$*) such that the tuples* $\mathbf{T} := (T_1, \dots, T_N)$ *and* $\mathbf{S} := (S_1, \dots, S_N)$

are admissible and $T_k y = S_k y$ for all $y \in D(S_k)$ and $k \in \{1, \dots, N\}$. Then we obtain

$$f(\mathbf{S}) \subseteq f(\mathbf{T})$$

for all $f \in H_P(\Omega)$ where Ω is admissible for $\mathbf{T}$ and $\mathbf{S}$.

Proof. (I) Let $f \in H_0^\infty(\Omega)$ and $y \in Y$. Then we have

$$\begin{aligned} f(\mathbf{S})y &= \frac{1}{(2\pi i)^N} \Big[\int_\Gamma f(z) \prod_{k=1}^N (z_k - S_k)^{-1} dz \Big] y \\ &= \frac{1}{(2\pi i)^N} \int_\Gamma f(z) \prod_{k=1}^N (z_k - S_k)^{-1} y \, dz \\ &= \frac{1}{(2\pi i)^N} \int_\Gamma f(z) \prod_{k=1}^N (z_k - T_k)^{-1} y \, dz \\ &= \frac{1}{(2\pi i)^N} \Big[\int_\Gamma f(z) \prod_{k=1}^N (z_k - T_k)^{-1} dz \Big] y \\ &= f(\mathbf{T})y \end{aligned}$$

because of $(z_k - T_k)^{-1}|_Y = (z_k - S_k)^{-1}$ for all $z_k \in \rho(T_k) \cap \rho(S_k)$ and $Y \hookrightarrow X$.

(II) Let $f \in H_P(\Omega)$ and $m \in \mathbb{N}_0$ with $(\psi^m f) \in H_0^\infty(\Omega)$. Using (I) we obtain

$$\begin{aligned} D(f(\mathbf{S})) &= \{y \in Y \colon (\psi^m f)(\mathbf{S})y \in R(\psi(\mathbf{S})^m)\} \\ &= \{y \in Y \colon (\psi^m f)(\mathbf{T})y \in R(\psi(\mathbf{S})^m)\} \\ &\subseteq \{y \in Y \colon (\psi^m f)(\mathbf{T})y \in R(\psi(\mathbf{T})^m)\} \\ &= Y \cap D(f(\mathbf{T})). \end{aligned}$$

This yields

$$D(f(\mathbf{S})) \subseteq D(f(\mathbf{T})).$$

It is easy to prove that $\psi(\mathbf{T})^{-m} v = \psi(\mathbf{S})^{-m} v$ for all $v \in R(\psi(\mathbf{S})^m)$. For $y \in D(f(\mathbf{S}))$ we then get

$$f(\mathbf{T})y = \psi(\mathbf{T})^{-m} (\psi^m f)(\mathbf{T})y = \psi(\mathbf{T})^{-m} \underbrace{(\psi^m f)(\mathbf{S})y}_{\in R(\psi(\mathbf{S})^m)} = f(\mathbf{S})y$$

by (I). So we have proved the assertion. □

1.2 Vector-valued Sobolev spaces

Motivation. In this section, we will provide the main spaces for the right-hand sides and the solutions of the general parabolic equations considered in later chapters. In the theory of nonlinear parabolic differential equations, the maximal regularity approach for the linearization turns out to be particularly useful in the setting of L_p-Sobolev spaces. On the other hand, following a standard approach, it is advantageous to consider functions in time and space as vector-valued functions in time (where the values are in an appropriate function class with respect to the space variables). Therefore, we have to consider vector-valued L_p-Sobolev spaces. Here non-integer orders of differentiability appear naturally in the context of boundary value problems and traces with respect to the boundary and with respect to time.

In contrast to the L_2-setting, for $p \neq 2$ there are different types of L_p-Sobolev spaces, the main classes being Besov spaces, Bessel potential spaces, and Triebel-Lizorkin spaces. In the present section we will consider Besov and Bessel potential spaces while Triebel-Lizorkin spaces will be discussed in Chapter 3.

Modern Sobolev space theory is based on two ingredients:

- interpolation theory for Banach spaces,
- and the concept of retractions and coretractions.

Interpolation theory allows us to define and analyze Sobolev spaces of non-integer order, while the concept of retractions is a key method for the step from $\mathbb{R}^n$ to domains. Therefore, this section starts with the discussion of interpolations spaces in Subsection a) and of retractions and coretractions in Subsection b) before we define the Sobolev spaces which will be relevant for our purpose in Subsection c).

a) Interpolation of Banach spaces

We start with the main definitions and results on interpolation spaces, where we refer the reader to the standard references [Tri78], [BL76] for details and proofs.

An interpolation couple $\{X_0, X_1\}$ consists of two complex Banach spaces X_0 and X_1, both linearly and continuously embedded into the same linear complex Hausdorff space. For an interpolation couple, the Banach spaces $X_0 + X_1$ (sum) and $X_0 \cap X_1$ (intersection) are defined in a natural way.

In the following, for interpolation couples $\{X_0, X_1\}$ and $\{Y_0, Y_1\}$ we will write

$$L(\{X_0, X_1\}, \{Y_0, Y_1\}) := \{T\colon X_0 + X_1 \to Y_0 + Y_1 \,\big|\, T|_{X_k} \in L(X_k, Y_k),\ k = 0, 1\}.$$

Roughly speaking, an interpolation functor $\mathcal{F}$ maps the interpolation couple $\{X_0, X_1\}$ to an intermediate Banach space $\mathcal{F}(\{X_0, X_1\})$ satisfying

$$X_0 \cap X_1 \hookrightarrow \mathcal{F}(\{X_0, X_1\}) \hookrightarrow X_0 + X_1.$$

A precise definition in the language of categories and (covariant) functors can be found in [Tri78, Subsection 1.2.2].

We remark that in many situations one has $X_1 \hookrightarrow X_0$. In this case, $X_1 \hookrightarrow \mathcal{F}(\{X_0, X_1\}) \hookrightarrow X_0$. A typical example is $X_0 = L_p(\Omega)$ and $X_1 = W_p^k(\Omega)$, the L_p-Sobolev space of order k. Here interpolation leads to Besov or Bessel potential spaces as will be discussed below.

Definition 1.36 (Exact interpolation functor). An interpolation functor $\mathcal{F}$ is called *exact of type* $\theta \in (0,1)$ if for all interpolation couples $\{X_0, X_1\}$, $\{Y_0, Y_1\}$ we have

$$\big\|T|_{\mathcal{F}(\{X_0,X_1\})}\big\|_{L(\mathcal{F}(\{X_0,X_1\}),\mathcal{F}(\{Y_0,Y_1\}))} \le \big\|T|_{X_0}\big\|^{1-\theta}_{L(X_0,Y_0)} \big\|T|_{X_1}\big\|^{\theta}_{L(X_1,Y_1)}.$$

The most important interpolation functors are the real and the complex interpolation functors. There are several equivalent definitions for them, but we restrict ourselves to one variant. In the following, let $\{X_0, X_1\}$ be an interpolation couple.

Definition 1.37 (Real interpolation functor). For $\theta \in (0,1)$ and $p \in (1,\infty)$ we define

$$K(t,u) := \inf\big\{\|u_0\|_{X_0} + t\|u_1\|_{X_1} : u_0 + u_1 = u,\ u_i \in X_i\big\}, \quad t \in \mathbb{R}_+,\ u \in X_0 + X_1.$$

Then the *real interpolation space* $(X_0, X_1)_{\theta,p}$ is defined as the space of all $u \in X_0 + X_1$ satisfying

$$(t \mapsto t^{-\theta} K(t,u)) \in L_p(\mathbb{R}_+, \tfrac{dt}{t}).$$

The norm in $(X_0, X_1)_{\theta,p}$ is defined by

$$\|u\|_{\theta,p} := \Big(\int_0^\infty \big(t^{-\theta}K(t,u)\big)^p \tfrac{dt}{t}\Big)^{1/p}.$$

Definition 1.38 (Complex interpolation functor). Let $\theta \in (0,1)$, and set $S := (0,1) + i\mathbb{R} \subseteq \mathbb{C}$. Then $F(X_0, X_1)$ is defined as the set of all functions $f\colon \overline{S} \to X_0 + X_1$ satisfying

(i) f is $(X_0 + X_1)$-continuous in $\overline{S}$,

(ii) f is $(X_0 + X_1)$-holomorphic in S,

(iii) $f(k + it) \in X_k$ $(t \in \mathbb{R})$ and $(t \mapsto f(k+it)) \in C(\mathbb{R}, X_k)$ for $k = 0, 1$,

(iv) $\lim\limits_{|t|\to\infty} \|f(k+it)\|_{X_k} = 0$ for $k = 0, 1$.

We endow $F(X_0, X_1)$ with the norm

$$\|f\|_{F(X_0,X_1)} := \max\Big\{\sup_{t\in\mathbb{R}} \|f(it)\|_{X_0}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{X_1}\Big\}.$$

Then the *complex interpolation space* $[X_0, X_1]_\theta$ is defined as the set of all $u \in X_0 + X_1$ for which there exists $f \in F(X_0, X_1)$ with $f(\theta) = u$. The norm on $[X_0, X_1]_\theta$ is given by

$$\|u\|_{[X_0,X_1]_\theta} := \inf\big\{\|f\|_{F(X_0,X_1)} \colon f \in F(X_0, X_1),\ f(\theta) = u\big\}.$$

Theorem 1.39 ([Tri78, Sections 1.3.3, 1.6.2, 1.9.3]). *Let $\theta \in (0,1)$ and $p \in (1,\infty)$. Then the real and the complex interpolation functors $(\cdot,\cdot)_{\theta,p}$ and $[\cdot,\cdot]_\theta$ are both exact of type θ. We have*

$$X_0 \cap X_1 \overset{d}{\hookrightarrow} (X_0, X_1)_{\theta,p} \hookrightarrow X_0 + X_1,$$
$$X_0 \cap X_1 \overset{d}{\hookrightarrow} [X_0, X_1]_\theta \hookrightarrow X_0 + X_1$$

where $\overset{d}{\hookrightarrow}$ stands for dense continuous embedding.

Theorem 1.40 (**Reiteration theorem,** cf. [Tri78, Theorem 1.10.2]). *Let $0 < \theta_0 < \theta_1 < 1$, $\lambda \in (0,1)$, and $p \in (1,\infty)$. Then for $\theta := (1-\lambda)\theta_0 + \lambda\theta_1$ we have*

$$\big((X_0, X_1)_{\theta_0,p}, (X_0, X_1)_{\theta_1,p}\big)_{\lambda,p} = (X_0, X_1)_{\theta,p},$$
$$\big([X_0, X_1]_{\theta_0}, [X_0, X_1]_{\theta_1}\big)_{\lambda,p} = (X_0, X_1)_{\theta,p}$$

with equivalent norms. The same holds if $[X_0, X_1]_{\theta_0}$ is replaced by X_0 (then $\theta_0 := 0$) and if $[X_0, X_1]_{\theta_1}$ is replaced by X_1 (then $\theta_1 := 1$).

Theorem 1.41 (**Interpolation of L_p-spaces,** cf. [Tri78, Theorem 1.18.4]). *Let $\Omega \subseteq \mathbb{R}^n$ be open and $1 < p_0, p_1 < \infty$. Then we have*

$$\big(L_{p_0}(\Omega, X_0), L_{p_1}(\Omega, X_1)\big)_{\theta,p} = L_p(\Omega, (X_0, X_1)_{\theta,p}),$$
$$\big[L_{p_0}(\Omega, X_0), L_{p_1}(\Omega, X_1)\big]_\theta = L_p(\Omega, [X_0, X_1]_\theta)$$

where $1/p = (1-\theta)/p_0 + \theta/p_1$ and $\theta \in (0,1)$.

Theorem 1.42 (**Interpolation of ℓ_p^s-spaces,** cf. [BL76, Theorem 5.6.3]). *We define for a Banach space X, $p \in [1,\infty)$, and $s \in \mathbb{R}$ the weighted ℓ_p-space*

$$\ell_p^s(X) := \Big\{(x_k)_{k\in\mathbb{N}_0} \subseteq X \colon \sum_{k=0}^{\infty} 2^{skp}\|x_k\|_X^p < \infty\Big\},$$
$$\|(x_k)_{k\in\mathbb{N}_0}\|_{\ell_p^s(X)} := \Big(\sum_{k=0}^{\infty} 2^{skp}\|x_k\|_X^p\Big)^{1/p}.$$

Let $\{X_0, X_1\}$ be an interpolation couple. Then we have

$$[\ell_{p_0}^{s_0}(X_0), \ell_{p_1}^{s_1}(X_1)]_\theta = \ell_p^s([X_0, X_1]_\theta)$$

for all $s_0, s_1 \in \mathbb{R}$, $p_0, p_1 \in (1,\infty)$, $\theta \in (0,1)$ with $1/p = (1-\theta)/p_0 + \theta/p_1$, and $s = (1-\theta)s_0 + \theta s_1$.

Real and complex interpolation is compatible with the geometric properties for Banach spaces as introduced above. More precisely, we have:

Lemma 1.43. *Let $\theta \in (0,1)$ and $p \in (1,\infty)$.*

(i) *If X_0 and X_1 are Banach spaces of class $\mathcal{HT}$, then $(X_0,X_1)_{\theta,p}$ and $[X_0,X_1]_\theta$ are also of class $\mathcal{HT}$.*

(ii) *If X_0 and X_1 are Banach spaces of class $\mathcal{HT}$ with property (α), then the spaces $(X_0,X_1)_{\theta,p}$ and $[X_0,X_1]_\theta$ are also Banach spaces of class $\mathcal{HT}$ with property (α).*

Proof. Part (i) can be found, e.g., in [Ama95, Theorem 4.5.2], part (ii) in [KS12, Theorem 4.5]. □

Theorem 1.44 ($\mathcal{R}$**-boundedness and interpolation,** cf. [KKW06, Prop. 3.7], [KS12, Cor. 3.19]**).** *Let $\{X_0,X_1\}$ and $\{Y_0,Y_1\}$ be interpolation couples of Banach spaces of class $\mathcal{HT}$. For a given family $\mathcal{T} \subseteq L(\{X_0,X_1\},\{Y_0,Y_1\})$, $p \in (1,\infty)$, and $0<\theta<1$ we have:*
If $\mathcal{T}|_{X_k} \subseteq L(X_k,Y_k)$ is $\mathcal{R}$-bounded for $k=0,1$, then

$$\begin{aligned}\mathcal{T}|_{(X_0,X_1)_{\theta,p}} &\subseteq L((X_0,X_1)_{\theta,p},(Y_0,Y_1)_{\theta,p}),\\ \mathcal{T}|_{[X_0,X_1]_\theta} &\subseteq L([X_0,X_1]_\theta,[Y_0,Y_1]_\theta)\end{aligned}$$

are also $\mathcal{R}$-bounded and

$$\begin{aligned}\mathcal{R}_p\left(\mathcal{T}|_{(X_0,X_1)_{\theta,p}}\right) &\le C\cdot[\mathcal{R}_p(\mathcal{T}|_{X_0})]^{1-\theta}[\mathcal{R}_p(\mathcal{T}|_{X_1})]^\theta,\\ \mathcal{R}_p(\mathcal{T}|_{[X_0,X_1]_\theta}) &\le C'\cdot[\mathcal{R}_p(\mathcal{T}|_{X_0})]^{1-\theta}[\mathcal{R}_p(\mathcal{T}|_{X_1})]^\theta,\end{aligned}$$

with constants $C,C'>0$.

Finally, real and complex interpolation is also compatible with the joint H^∞-calculus:

Theorem 1.45 (H^∞**-calculus and interpolation**)**.** *Let $\mathcal{F} \in \{(\cdot,\cdot)_{\theta,p},[\cdot,\cdot]_\theta\}$, $\theta \in (0,1)$, $p\in(1,\infty)$, and*

$$\begin{aligned}T_k &\colon X_0 \supseteq D(T_k) \to X_0,\\ S_k &\colon X_1 \supseteq D(S_k) \to X_1,\end{aligned}$$

$k=1,\dots,N$, be linear operators satisfying the compatibility conditions $T_kx = S_kx$ for all $x \in D(T_k)\cap D(S_k)$ and

$$(\lambda - T_k)^{-1}x = (\lambda - S_k)^{-1}x, \quad \lambda \in \rho(T_k)\cap\rho(S_k), \quad x \in X_0 \cap X_1. \tag{1.9}$$

Moreover we define for $k=1,\dots,N$ the interpolated operators

$$A_k \colon \mathcal{F}(\{X_0,X_1\}) \supseteq D(A_k) \to \mathcal{F}(\{X_0,X_1\}), \quad D(A_k) := \mathcal{F}(\{D(T_k),D(S_k)\})$$

with $A_kx := T_kx_0 + S_kx_1$ for $x = x_0+x_1 \in D(A_k) \hookrightarrow D(T_k)+D(S_k)$ with $x_0 \in D(T_k)$, $x_1 \in D(S_k)$. Note that we equip $D(T_k)$ and $D(S_k)$ with the graph norm. Then we have the following results:

(i) *Let $k = 1, \dots, N$. If T_k and S_k are both sectorial, then A_k is also sectorial with $\varphi_{A_k} \le \max\{\varphi_{T_k}, \varphi_{S_k}\}$. The analog result holds for bisectorial operators. If $\mathbf{T} := (T_1, \dots, T_N)$ and $\mathbf{S} := (S_1, \dots, S_N)$ are both admissible, then $\mathbf{A} := (A_1, \dots, A_N)$ is also admissible.*

(ii) *If in the situation of* (i) *both $\mathbf{T}$ and $\mathbf{S}$ admit a bounded joint $H^\infty(\Omega)$-calculus, then the interpolated tuple $\mathbf{A}$ also admits a bounded joint $H^\infty(\Omega)$-calculus, and we have the representation*

$$f(\mathbf{A})x = f(\mathbf{T})x_0 + f(\mathbf{S})x_1, \qquad x \in \mathcal{F}(\{X_0, X_1\}), \quad x = x_0 + x_1, \quad x_i \in X_i$$

for all $f \in H^\infty(\Omega)$. For $N = 1$ we even have $\varphi_{A_1}^\infty \le \max\{\varphi_{T_1}^\infty, \varphi_{S_1}^\infty\}$ (respectively, $\varphi_{A_1}^{\infty,(\mathrm{bi})} \le \max\{\varphi_{T_1}^{\infty,(\mathrm{bi})}, \varphi_{S_1}^{\infty,(\mathrm{bi})}\}$).

(iii) *Let $\mathbf{T}$ and $\mathbf{S}$ be the same operators as in (ii) but with an $\mathcal{R}$-bounded joint $H^\infty(\Omega)$-calculus. If X_0, X_1 are of class $\mathcal{HT}$, then $\mathbf{A}$ also has an $\mathcal{R}$-bounded joint $H^\infty(\Omega)$-calculus. In particular for $N = 1$ we obtain $\varphi_{A_1}^{\mathcal{R},\infty} \le \max\{\varphi_{T_1}^{\mathcal{R},\infty}, \varphi_{S_1}^{\mathcal{R},\infty}\}$ (respectively, $\varphi_{A_1}^{\mathcal{R},\infty,(\mathrm{bi})} \le \max\{\varphi_{T_1}^{\mathcal{R},\infty,(\mathrm{bi})}, \varphi_{S_1}^{\mathcal{R},\infty,(\mathrm{bi})}\}$).*

Proof. (i) We have to consider the relation between the resolvents of T_k, S_k and A_k. Without loss of generality we assume that T_k and S_k are sectorial. Let $\lambda \in -S_{\pi-\theta'} \subseteq \rho(T_k) \cap \rho(S_k)$ with $\theta' > \max\{\varphi_{T_k}, \varphi_{S_k}\}$. Then we define the bounded operator

$$\begin{aligned} R_\lambda^{(k)} \colon X_0 + X_1 &\to D(T_k) + D(S_k), \\ x_0 + x_1 &\mapsto (\lambda - T_k)^{-1} x_0 + (\lambda - S_k)^{-1} x_1. \end{aligned}$$

Note that $R_\lambda^{(k)}$ is well defined by (1.9). Hence,

$$\begin{aligned} (R_\lambda^{(k)})|_{X_0} &= (\lambda - T_k)^{-1}, \\ (R_\lambda^{(k)})|_{X_1} &= (\lambda - S_k)^{-1}, \\ (R_\lambda^{(k)})|_{\mathcal{F}(\{X_0,X_1\})} &\in L(\mathcal{F}(\{X_0, X_1\}), D(A_k)). \end{aligned}$$

With this we can easily show $\lambda \in \rho(A_k)$ and $(\lambda - A_k)^{-1} = (R_\lambda^{(k)})|_{\mathcal{F}(\{X_0,X_1\})}$, which already yields $-S_{\pi-\theta'} \subseteq \rho(A_k)$. Thus, we derive

$$\|(\lambda - A_k)^{-1}\|_{L(\mathcal{F}(\{X_0,X_1\}))} \le \|(\lambda - T_k)^{-1}\|_{L(X_0)}^{1-\theta} \|(\lambda - S_k)^{-1}\|_{L(X_1)}^{\theta}$$

for all $\lambda \in -S_{\pi-\theta'}$ due to Theorem 1.39. We still have to show the density of $R(A_k)$ and $D(A_k)$ in $\mathcal{F}(\{X_0, X_1\})$. Here we use the characterizations

$$\begin{aligned} \overline{R(A_k)} &= \big\{x \in \mathcal{F}(\{X_0, X_1\}) \colon \lim_{n\to\infty} -\tfrac{1}{n}(-\tfrac{1}{n} - A_k)^{-1} x = 0 \text{ in } \mathcal{F}(\{X_0, X_1\})\big\}, \\ \overline{D(A_k)} &= \big\{x \in \mathcal{F}(\{X_0, X_1\}) \colon \lim_{n\to\infty} -n(-n - A_k)^{-1} x = x \text{ in } \mathcal{F}(\{X_0, X_1\})\big\} \end{aligned}$$

given in Lemma 1.5. Let $x \in X_0 \cap X_1$ be arbitrary. Then we get

$$-\tfrac{1}{n}(-\tfrac{1}{n} - A_k)^{-1}x = -\tfrac{1}{n}(-\tfrac{1}{n} - T_k)^{-1}x = -\tfrac{1}{n}(-\tfrac{1}{n} - S_k)^{-1}x,$$
$$-n(-n - A_k)^{-1}x = -n(-n - T_k)^{-1}x = -n(-n - S_k)^{-1}x.$$

Due to $\overline{R(T_k)} = \overline{D(T_k)} = X_0$, $\overline{R(S_k)} = \overline{D(S_k)} = X_1$, and Lemma 1.5 we have

$$\lim_{n\to\infty} -\tfrac{1}{n}(-\tfrac{1}{n} - T_k)^{-1}x = 0 \text{ in } X_0, \quad \lim_{n\to\infty} -\tfrac{1}{n}(-\tfrac{1}{n} - S_k)^{-1}x = 0 \text{ in } X_1,$$
$$\lim_{n\to\infty} -n(-n - T_k)^{-1}x = x \text{ in } X_0, \quad \lim_{n\to\infty} -n(-n - S_k)^{-1}x = x \text{ in } X_1,$$

which yields $-\frac{1}{n}(-\frac{1}{n} - A_k)^{-1}x \to x$ in $X_0 \cap X_1$ and, by $X_0 \cap X_1 \hookrightarrow \mathcal{F}(\{X_0, X_1\})$, also in $\mathcal{F}(\{X_0, X_1\})$. Thus we have proved $X_0 \cap X_1 \subseteq \overline{R(A_k)}$, $\overline{D(A_k)}$. Therefore, $X_0 \cap X_1 \overset{d}{\hookrightarrow} \mathcal{F}(\{X_0, X_1\})$ (cf. Theorem 1.39) yields

$$\overline{R(A_k)} = \overline{D(A_k)} = \mathcal{F}(\{X_0, X_1\}).$$

Hence we have shown that A_k is sectorial with $\varphi_{A_k} \le \max\{\varphi_{T_k}, \varphi_{S_k}\}$. The claimed admissibility of $\mathbf{A}$ can easily be shown with the representation of the resolvents by $R_\lambda^{(k)}$.

(ii) Let $f \in H_0^\infty(\Omega)$ and define the operator

$$F\colon X_0 + X_1 \to X_0 + X_1,$$
$$x_0 + x_1 \mapsto f(\mathbf{T})x_0 + f(\mathbf{S})x_1,$$

which is well-defined due to (1.9). For $x \in \mathcal{F}(\{X_0, X_1\})$ we obtain

$$f(\mathbf{A})x = \frac{1}{(2\pi i)^N}\Big(\int_\Gamma f(z)\prod_{k=1}^N (z_k - A_k)^{-1}dz\Big)x$$
$$= \frac{1}{(2\pi i)^N}\int_\Gamma f(z)\prod_{k=1}^N (z_k - A_k)^{-1}x\,dz.$$

Here the integral converges in $\mathcal{F}(\{X_0, X_1\})$ and, due to $\mathcal{F}(\{X_0, X_1\}) \hookrightarrow X_0 + X_1$, also in $X_0 + X_1$. In the same way we can show

$$Fx = \frac{1}{(2\pi i)^N}\int_\Gamma f(z)\prod_{k=1}^N R_{z_k}^{(k)}x\,dz, \quad x \in X_0 + X_1$$

(convergence of the integral in $X_0 + X_1$) due to $X_i \hookrightarrow X_0 + X_1$ $(i = 0, 1)$. So we derive $F|_{\mathcal{F}(\{X_0,X_1\})} = f(\mathbf{A})$. Additionally, we have $F|_{X_0} = f(\mathbf{T})$ and $F|_{X_1} = f(\mathbf{S})$. Using Theorem 1.39, we deduce

$$\|f(\mathbf{A})\|_{L(\mathcal{F}(\{X_0,X_1\}))} \le \|f(\mathbf{T})\|_{L(X_0)}^{1-\theta}\|f(\mathbf{S})\|_{L(X_1)}^{\theta} \le C\|f\|_\infty$$

for all $f \in H_0^\infty(\Omega)$. So we have proved that $\mathbf{A}$ admits a bounded joint $H^\infty(\Omega)$-calculus and that the claimed representation of $f(\mathbf{A})$ holds.

Now we choose $f \in H^\infty(\Omega)$. According to Remark 1.21 (iii), we have $\mathcal{F}(\{X_0, X_1\}) \hookrightarrow X_0 + X_1$ and $X_i \hookrightarrow X_0 + X_1$, and we get

$$\begin{aligned} f(\mathbf{A})x &= \lim_{n\to\infty} \underbrace{(\psi_{n,N} f)}_{\in H_0^\infty(\Omega)}(\mathbf{A})x = \lim_{n\to\infty} (\psi_{n,N} f)(\mathbf{T})x_0 + \lim_{n\to\infty} (\psi_{n,N} f)(\mathbf{S})x_1 \\ &= f(\mathbf{T})x_0 + f(\mathbf{S})x_1 \end{aligned}$$

for all $x = x_0 + x_1 \in \mathcal{F}(\{X_0, X_1\})$, $x_i \in X_i$, $i = 0, 1$. Note that here the first limit holds in the topology of $\mathcal{F}(\{X_0, X_1\})$, the second in X_0, and the third in X_1. This yields the asserted representation for $f(\mathbf{A})$ for all functions $f \in H^\infty(\Omega)$.

For $N = 1$ it is obvious that $\varphi_{A_1}^\infty \le \max\{\varphi_{T_1}^\infty, \varphi_{S_1}^\infty\}$.

(iii) Part (ii) already yields that $\mathbf{A}$ admits a bounded joint $H^\infty(\Omega)$-calculus. By assumption the families

$$\begin{aligned} \mathcal{T}_1 &:= \{h(\mathbf{T})\colon h \in H_0^\infty(\Omega),\ \|h\|_\infty \le 1\} \subseteq L(X_0), \\ \mathcal{T}_2 &:= \{h(\mathbf{S})\colon h \in H_0^\infty(\Omega),\ \|h\|_\infty \le 1\} \subseteq L(X_1) \end{aligned}$$

are both $\mathcal{R}$-bounded. Thus, the representation of $f(\mathbf{A})$ in part (ii) and Theorem 1.44 yield the $\mathcal{R}$-boundedness of

$$\{h(\mathbf{A})\colon h \in H_0^\infty(\Omega),\ \|h\|_\infty \le 1\} \subseteq L(\mathcal{F}(\{X_0, X_1\})).$$

Therefore $\mathbf{A}$ admits an $\mathcal{R}$-bounded joint $H^\infty(\Omega)$-calculus.

Finally, the assertion on the angle in the case $N = 1$ is obvious. □

Remark 1.46. This theorem about H^∞-calculus and interpolation is essential for our purpose and therefore we want to make some remarks for the special case if X_1 embeds into X_0:

(i) The compatibility conditions for the resolvents in (1.9) are always fulfilled in case of $X_1 \hookrightarrow X_0$. This is due to the fact that we have $(\lambda - S_k)^{-1} = (\lambda - T_k)^{-1}|_{X_1}$ for all $\lambda \in \rho(T_k) \cap \rho(S_k)$ in this case.

(ii) For $X_1 \hookrightarrow X_0$ the representation of $f(\mathbf{A})$ in Theorem 1.45 (ii) is given by

$$f(\mathbf{A}) = f(\mathbf{T})|_{\mathcal{F}(\{X_0, X_1\})}, \quad f \in H^\infty(\Omega).$$

This can be easily seen by $(\lambda - A_k)^{-1} = (\lambda - T_k)^{-1}|_{\mathcal{F}(\{X_0, X_1\})}$ for $\lambda \in \rho(T_k) \cap \rho(S_k)$.

The full strength of Theorem 1.45 will become much clearer in Chapter 3. There we develop a bounded H^∞-calculus of $\nabla_+ = (\partial_t, \nabla)$ on Triebel-Lizorkin spaces. Except for Chapter 3 we always have $X_1 \hookrightarrow X_0$ in every situation.

b) Retractions and coretractions

To define function spaces on domains, the concept of retractions and coretractions can be used. Here the main properties of the function spaces can easily be transferred from the whole space to domains. For our purpose, it is important that the concept is compatible with the real and complex interpolation functors. We will mainly consider the half-space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$.

Definition 1.47 (**Retraction and coretraction,** cf. [Ama95, I.2.3], [Ama09, p. 4])**.** Let X and Y be locally convex spaces. A mapping $r \in L(X, Y)$ is called a *retraction* if there exists $e \in L(Y, X)$ such that $(re)y = y$ for all $y \in Y$. The mapping e is then called a corresponding *coretraction*.

Definition 1.48. Let X be a Banach space and let Y be a locally convex space. For a mapping $\varphi \in L(X, Y)$ we endow the *image space* $\varphi X := \varphi(X)$ with the quotient norm

$$\|u\|_{\varphi X} := \inf \left\{ \|f\|_X \colon f \in \varphi^{-1}(\{u\}) \right\}, \quad u \in \varphi X.$$

Remark 1.49. In the situation of Definition 1.48, $(\varphi X, \|\cdot\|_{\varphi X})$ is a Banach space satisfying $\varphi X \hookrightarrow Y$.

Lemma 1.50. *Let X and S be locally convex spaces, and let $r \in L(X, S)$ be a retraction with coretraction $e \in L(S, X)$. Let X_1 and X_2 be Banach spaces with $X_i \subseteq X$. Assume that $r|_{X_i} \in L(X_i, rX_i)$ is a retraction with coretraction $e|_{rX_i} \in L(rX_i, X_i)$, $i = 1, 2$. Then we have*

$$r(X_1 \cap X_2) = (rX_1) \cap (rX_2)$$

with equivalent norms.

Proof. We trivially have $r(X_1 \cap X_2) \subseteq (rX_1) \cap (rX_2)$. For $u \in (rX_1) \cap (rX_2)$ we write $u = (re)u$. As by assumption on e we have $eu \in X_1 \cap X_2$, we see that $r(X_1 \cap X_2) = (rX_1) \cap (rX_2)$ as an equality of sets. Therefore, we only have to show the equivalence of the norms. For $u \in r(X_1 \cap X_2)$ we derive

$$\begin{aligned}
\|u\|_{r(X_1 \cap X_2)} &= \inf\{\|f\|_{X_1 \cap X_2} \colon f \in X_1 \cap X_2,\, rf = u\} \\
&= \inf\{\|f\|_{X_1} + \|f\|_{X_2} \colon f \in X_1 \cap X_2,\, rf = u\} \\
&\geq \inf\{\|f\|_{X_1} \colon f \in X_1 \cap X_2,\, rf = u\} + \inf\{\|g\|_{X_2} \colon g \in X_1 \cap X_2,\, rg = u\} \\
&\geq \inf\{\|f\|_{X_1} \colon f \in X_1,\, rf = u\} + \inf\{\|g\|_{X_2} \colon g \in X_2,\, rg = u\} \\
&= \|u\|_{rX_1} + \|u\|_{rX_2} = \|u\|_{(rX_1) \cap (rX_2)}.
\end{aligned}$$

Using the bounded inverse theorem and Remark 1.49 we conclude the claimed equivalence of norms. □

Lemma 1.51 (**Interpolation and retraction**)**.** *Let A_0, A_1, Z_0, Z_1 be Banach spaces and $r_k \in L(A_k, Z_k)$, $k = 0, 1$. If r_k is a retraction in $L(A_k, r_k A_k)$ for $k = 0, 1$ with*

corresponding coretraction $e_k \in L(r_k A_k, A_k)$ *and* $r_0 x = r_1 x$ *for all* $x \in A_0 \cap A_1$, *then we have*

$$\begin{aligned} r\,[A_0, A_1]_\theta &= [r_0 A_0, r_1 A_1]_\theta\,, \quad \theta \in (0,1), \\ r\,(A_0, A_1)_{\theta,p} &= (r_0 A_0, r_1 A_1)_{\theta,p}\,, \quad \theta \in (0,1),\ 1 < p < \infty \end{aligned}$$

with equivalent norms where $r\colon A_0 + A_1 \to r_0 A_0 + r_1 A_1, \quad a_0 + a_1 \mapsto r_0 a_0 + r_1 a_1$.

Proof. We only consider the complex interpolation functor as the proof for the real method is essentially the same. We have

$$r \in L([A_0, A_1]_\theta, [r_0 A_0, r_1 A_1]_\theta),$$

which yields

$$r[A_0, A_1]_\theta \hookrightarrow [r_0 A_0, r_1 A_1]_\theta \tag{1.10}$$

according to Remark 1.49. Next, we apply [Tri78, Theorem 1.2.4] with $B_k := r_k A_k$, $k = 0, 1$. This gives

$$\Phi := e|_{[B_0, B_1]_\theta} \in L_{\mathrm{Isom}}\left([B_0, B_1]_\theta\,,\ \left(er\left([A_0, A_1]_\theta\right), \|\cdot\|_{[A_0, A_1]_\theta}\right)\right) \tag{1.11}$$

and therefore $e([B_0, B_1]_\theta) = er([A_0, A_1]_\theta)$ as an equality of sets. Due to (1.10) and (1.11) we also obtain $[B_0, B_1]_\theta = r([A_0, A_1]_\theta)$ as an equality of sets. The bounded inverse theorem and (1.10) then yield the assertion. □

Lemma 1.52 (Interpolation and isomorphism). *Let* A_0, A_1, B_0, *and* B_1 *be Banach spaces. If there exist* $r_k \in L_{\mathrm{Isom}}(A_k, B_k)$, $k = 0, 1$, *with* $r_0 x = r_1 x$ *for all* $x \in A_0 \cap A_1$, *then we have* $B_k = r_k A_k$ *with equivalent norms and*

$$\begin{aligned} r\,[A_0, A_1]_\theta &= [B_0, B_1]_\theta\,, \quad \theta \in (0,1), \\ r\,(A_0, A_1)_{\theta,p} &= (B_0, B_1)_{\theta,p}\,, \quad \theta \in (0,1),\ 1 < p < \infty \end{aligned}$$

with equivalent norms where $r\colon A_0 + A_1 \to r_0 A_0 + r_1 A_1, \quad a_0 + a_1 \mapsto r_0 a_0 + r_1 a_1$.

Proof. Again, we only consider the complex interpolation functor. Due to $r_k \in L_{\mathrm{Isom}}(A_k, B_k)$ we have

$$\|u\|_{r_k A_k} = \|r_k^{-1} u\|_{A_k} \le C \|u\|_{B_k}.$$

This yields $r_k A_k = B_k$. Lemma 1.51 then already shows

$$r\,[A_0, A_1]_\theta = [B_0, B_1]_\theta\,, \quad \theta \in (0,1). \qquad \square$$

Lemma 1.53. *Let* X, Z *be Banach spaces and* $r \in L(X, Z)$. *If* $Y \overset{\iota}{\hookrightarrow} X$ *for a Banach space* Y, *then we also have*

$$(r \circ \iota) Y \hookrightarrow rX$$

where the injection is given by the identity.

Proof. It is trivial that $(r \circ \iota)Y \subseteq rX$. For $u \in Y$ we get

$$\|(r \circ \iota)u\|_{rX} = \inf\{\|f\|_X \colon f \in r^{-1}(\{(r \circ \iota)u\})\} \leq \|\iota(u)\|_X \leq C\|u\|_Y,$$

which yields $r \circ \iota \in L(Y, rX)$. This directly implies $(r \circ \iota)Y \hookrightarrow rX$ due to Remark 1.49. □

Lemma 1.54. *Let X, Y_0, and Y_1 be Banach spaces with retractions and corresponding coretractions*

$$r_i \colon X \to Y_i, \quad e_i \colon Y_i \to X, \quad i = 0, 1$$

and let $D \subseteq X$ be a dense subspace such that $r_0 f = r_1 f$ for all $f \in D$. Then $Y_0 = Y_1$ with equivalence of norms.

Proof. The retractions r_0 and r_1 are continuous and therefore we obtain $r_0 u = r_1 u$ for all $u \in X$. This yields the equality of the sets Y_0 and Y_1 because the mappings r_0 and r_1 are onto. In particular, we get

$$r_1 e_0 u = r_0 e_0 u = u, \quad u \in Y_0$$

by $r_1 f = r_0 f$ for $f \in X$ and $e_0 u \in X$. With this we obtain

$$\|u\|_{Y_1} = \|r_1 e_0 u\|_{Y_1} \leq C\|e_0 u\|_X \leq C'\|u\|_{Y_0}, \quad u \in Y_0.$$

Using $r_0 e_1 u = u$ for $u \in Y_1$ we can show analogously $\|u\|_{Y_0} \leq C\|u\|_{Y_1}$. So the equivalence of the norms $\|\cdot\|_{Y_0}$ and $\|\cdot\|_{Y_1}$ is proved. □

c) Definition of Sobolev spaces

To define Sobolev spaces, we start with the whole space $\mathbb{R}^n$. In the following, let X be a complex Banach space of class $\mathcal{HT}$ with property (α).

As usual, we denote by $\mathscr{S}(\mathbb{R}^n, X)$ the Schwartz space of smooth rapidly decreasing X-valued functions, equipped with the canonical locally convex topology induced by the family of seminorms

$$p_{\alpha,k}(\varphi) := \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^{k/2} \|\partial^\alpha \varphi(x)\|_X, \quad \varphi \in \mathscr{S}(\mathbb{R}^n, X),$$

with $\alpha \in \mathbb{N}_0^n$, $k \in \mathbb{N}_0$. The space of X-valued tempered distributions is defined by $\mathscr{S}'(\mathbb{R}^n, X) := L(\mathscr{S}(\mathbb{R}^n), X)$. Again this is a locally convex space where the topology is induced by the family of seminorms

$$p_\varphi(f) := \|f(\varphi)\|_X, \quad f \in \mathscr{S}'(\mathbb{R}^n, X)$$

for all $\varphi \in \mathscr{S}(\mathbb{R}^n)$. As usual, we write $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$ for $X = \mathbb{C}$. Note that in this case the above topology on $\mathscr{S}'(\mathbb{R}^n)$ coincides with the weak-$*$-topology.

In the same way as in the scalar case, one sees that the Fourier transform

$$(\mathscr{F} f)(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\xi} dx, \quad \xi \in \mathbb{R}^n, \ f \in \mathscr{S}(\mathbb{R}^n, X),$$

defines an isomorphism $\mathscr{F}\colon \mathscr{S}(\mathbb{R}^n, X) \to \mathscr{S}(\mathbb{R}^n, X)$. Here and in the following, we write $x\xi$ for the standard scalar product of x and ξ in $\mathbb{R}^n$. Therefore, the Fourier transform can be extended by duality to an isomorphism of $\mathscr{S}'(\mathbb{R}^n, X)$, setting $(\mathscr{F}f)(\varphi) := f(\mathscr{F}\varphi)$ for $f \in \mathscr{S}'(\mathbb{R}^n, X)$ and $\varphi \in \mathscr{S}(\mathbb{R}^n)$.

For the definition of Sobolev spaces, we will frequently use the symbol

$$\Lambda_r(z) := \Big(1 - \sum_{k=1}^{n} z_k^2\Big)^{r/2}, \quad z \in ((\mathbb{C} \setminus \mathbb{R}) \cup \{0\})^n$$

for $r \in \mathbb{R}$. In general, for a scalar-valued symbol $p(z)$ being defined at least on the set $(i\mathbb{R})^n \subseteq \mathbb{C}^n$, we define $\mathrm{op}[p] := \mathscr{F}^{-1}p(i\xi)\mathscr{F}$, i.e.,

$$(\mathrm{op}[p]f)(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} p(i\xi)(\mathscr{F}f)(\xi)d\xi, \quad f \in \mathscr{S}(\mathbb{R}^n, X),$$

provided the integral exists for all $f \in \mathscr{S}(\mathbb{R}^n, X)$ (this is the case, for instance, if p is continuous and polynomially bounded on $(i\mathbb{R})^n$). If $\xi \mapsto p(i\xi)$ is smooth and polynomially bounded, $\mathrm{op}[p]$ is well-defined on $\mathscr{S}'(\mathbb{R}^n, X)$ by duality. In particular, we have

$$\mathrm{op}[\Lambda_r] = \mathscr{F}^{-1}\langle\xi\rangle^r\mathscr{F} \quad \text{with } \langle\xi\rangle := \sqrt{1 + |\xi|^2}.$$

We start with Sobolev type spaces on $\mathbb{R}^n$.

Definition 1.55 (Sobolev spaces on $\mathbb{R}^n$). Let $p, q \in (1, \infty)$.

(i) For $k \in \mathbb{N}_0$, the (classical) *Sobolev space* is defined as

$$W_p^k(\mathbb{R}^n, X) := \{f \in L_p(\mathbb{R}^n, X) : \partial^\alpha f \in L_p(\mathbb{R}^n, X) \text{ for all } |\alpha| \le k\}$$

with norm

$$\|f\|_{W_p^k(\mathbb{R}^n, X)} := \Big(\sum_{|\alpha| \le k} \|\partial^\alpha f\|^p_{L_p(\mathbb{R}^n, X)}\Big)^{1/p}, \quad f \in W_p^k(\mathbb{R}^n, X).$$

(ii) For $r \in \mathbb{R}$, the *Bessel potential space* is defined as

$$H_p^r(\mathbb{R}^n, X) := \{f \in \mathscr{S}'(\mathbb{R}^n, X) : \mathrm{op}[\Lambda_r]f \in L_p(\mathbb{R}^n, X)\}$$

with norm

$$\|f\|_{H_p^r(\mathbb{R}^n, X)} := \|\mathrm{op}[\Lambda_r]f\|_{L_p(\mathbb{R}^n, X)}, \quad f \in H_p^r(\mathbb{R}^n, X).$$

(iii) Let $\psi \in \mathscr{D}(\mathbb{R}^n)$ with $\psi(x) \in [0, 1]$, $x \in \mathbb{R}^n$, $\psi(x) = 1$ if $|x| \le A$, and $\psi(x) = 0$ if $|x| > B$ for $0 < A < B < \infty$. We define the smooth *dyadic decomposition* of unity by

$$\varphi_0 := \psi, \quad \varphi_1(x) := \varphi_0(x/2) - \varphi_0(x), \quad \varphi_j(x) := \varphi_1(2^{-j+1}x), \quad j \ge 2.$$

For $r \in \mathbb{R}$, the *Besov space* $B^r_{pq}(\mathbb{R}^n, X)$ is defined as the set of all $u \in \mathscr{S}'(\mathbb{R}^n, X)$ with

$$\|u\|_{B^r_{pq}(\mathbb{R}^n,X)} := \left\| \left(2^{rj} \|\mathrm{op}[\varphi_j]u\|_{L_p(\mathbb{R}^n,X)}\right)_{j\in\mathbb{N}_0} \right\|_{\ell_q} < \infty.$$

Theorem 1.56 (Sobolev spaces and interpolation). *Let $p, q \in (1,\infty)$.*

(i) *For $k \in \mathbb{N}_0$ we have $H^k_p(\mathbb{R}^n, X) = W^k_p(\mathbb{R}^n, X)$ with equivalent norms.*

(ii) *For all $r_0, r_1 \in \mathbb{R}$, $r_0 \neq r_1$ and $\theta \in (0,1)$ we have*

$$\left[H^{r_0}_p(\mathbb{R}^n, X), H^{r_1}_p(\mathbb{R}^n, X)\right]_\theta = H^r_p(\mathbb{R}^n, X)$$

with equivalent norms where $r := (1-\theta)r_0 + \theta r_1$.

(iii) *For all $r_0, r_1 \in \mathbb{R}$, $r_0 \neq r_1$ and $\theta \in (0,1)$ we have*

$$\left(H^{r_0}_p(\mathbb{R}^n, X), H^{r_1}_p(\mathbb{R}^n, X)\right)_{\theta,q} = B^r_{pq}(\mathbb{R}^n, X)$$

with equivalent norms where $r := (1-\theta)r_0 + \theta r_1$.

Proof. See, e.g., [Ama09, Theorem 3.7.1]. □

Remark 1.57. (i) The above statements describe only some properties of the spaces which will be useful for our later purposes. A very detailed and comprehensive work about Banach space valued function spaces can be found in [Ama09]. The author even considers the anisotropic case which is, however, not necessary for our purposes.

(ii) We remark that Theorem 1.56 leads to an alternative but equivalent definition for H^r_p and B^r_{pq}: Starting with the classical spaces $W^k_p(\mathbb{R}^n, X)$ with $k \in \mathbb{N}_0$, the Bessel potential spaces and Besov spaces of order $r > 0$ can be defined by complex and real interpolation, respectively. The spaces with negative order may then be defined by duality.

Lemma 1.58. *Let $p, q \in (1,\infty)$.*

(i) *For every $r \in \mathbb{R}$ we have*

$$\mathscr{S}(\mathbb{R}^n, X) \overset{d}{\hookrightarrow} H^r_p(\mathbb{R}^n, X),$$
$$\mathscr{S}(\mathbb{R}^n, X) \overset{d}{\hookrightarrow} B^r_{pq}(\mathbb{R}^n, X).$$

(ii) *For every $r, r' \in \mathbb{R}$ we have*

$$\mathrm{op}[\Lambda_r]|_{H^{r'+r}_p(\mathbb{R}^n,X)} \in L_{\mathrm{Isom}}(H^{r'+r}_p(\mathbb{R}^n, X), H^{r'}_p(\mathbb{R}^n, X)),$$
$$\mathrm{op}[\Lambda_r]|_{B^{r'+r}_{pq}(\mathbb{R}^n,X)} \in L_{\mathrm{Isom}}(B^{r'+r}_{pq}(\mathbb{R}^n, X), B^{r'}_{pq}(\mathbb{R}^n, X)).$$

Proof. (i) See [Ama09, Theorem 2.3.2].

(ii) The first assertion easily follows from the definition of the Bessel potential spaces. The second assertion is then obtained from Theorem 1.56. □

Now we want to define the corresponding Sobolev spaces in the half-space $\mathbb{R}^n_+$. The analog of the Schwartz functions can be defined explicitly:

Definition 1.59. We define the *Schwartz functions* $\mathscr{S}(\mathbb{R}^n_+, X)$ on $\mathbb{R}^n_+$ as the space of all $f \in C^\infty(\mathbb{R}^n_+, X)$ such that

$$\sup_{x\in\mathbb{R}^n_+} (1+|x|^2)^{k/2} \|\partial^\alpha f(x)\|_X < \infty$$

for all $k \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^n$. Furthermore, we define

$${}_0\mathscr{S}(\mathbb{R}^n_+, X) := \{f \in \mathscr{S}(\mathbb{R}^n_+, X) \colon (\partial^\alpha f)|_{\partial\mathbb{R}^n_+} = 0,\ \alpha \in \mathbb{N}_0^n\}.$$

All spaces are equipped with the canonical locally convex topology.

Lemma 1.60 ([Ama09, Lemma 4.1.1])**.** *There exists a function $h \in C^\infty(\mathbb{R}_+, \mathbb{R})$ with $h(1/t) = -th(t)$, $t > 0$, and*

$$\int_0^\infty t^s |h(t)|dt < \infty, \quad (-1)^k \int_0^\infty t^k h(t)dt = 1$$

for all $s \in \mathbb{R}$ and $k \in \mathbb{Z}$.

Definition 1.61. (i) The *extension operator* $e^+ \colon \mathscr{S}(\mathbb{R}^n_+, X) \to \mathscr{S}(\mathbb{R}^n, X)$ is defined by

$$[e^+(u)](x) := \begin{cases} u(x), & x \in \overline{\mathbb{R}^n_+}, \\ \int_0^\infty h(t)u(x', -tx_n)dt, & x = (x', x_n) \in \mathbb{R}^n_- \end{cases}$$

and the *pointwise restriction operator* $r^+ \colon \mathscr{S}(\mathbb{R}^n, X) \to \mathscr{S}(\mathbb{R}^n_+, X)$ is defined by $r^+u := u|_{\mathbb{R}^n_+}$.

(ii) We define the *trivial extension operator* $e_0^+ \colon {}_0\mathscr{S}(\mathbb{R}^n_+, X) \to \mathscr{S}(\mathbb{R}^n, X)$ by

$$[e_0^+(u)](x) := \begin{cases} u(x), & x \in \overline{\mathbb{R}^n_+}, \\ 0, & x \in \mathbb{R}^n_- \end{cases}$$

and the operator $r_0^+ \colon \mathscr{S}(\mathbb{R}^n, X) \to {}_0\mathscr{S}(\mathbb{R}^n_+, X),\ u \mapsto r^+(1 - e^-r^-)u$. Here r^- and e^- denote the corresponding restriction and extension operators from (i) with respect to $\mathbb{R}^n_-$.

Remark 1.62. The restriction operator $r^+ \in L(\mathscr{S}(\mathbb{R}^n, X), \mathscr{S}(\mathbb{R}^n_+, X))$ is a retraction with coretraction $e^+ \in L(\mathscr{S}(\mathbb{R}^n_+, X), \mathscr{S}(\mathbb{R}^n, X))$. In the same way, $r_0^+ \in L(\mathscr{S}(\mathbb{R}^n, X),\ {}_0\mathscr{S}(\mathbb{R}^n_+, X))$ is a retraction with coretraction $e_0^+ \in L({}_0\mathscr{S}(\mathbb{R}^n_+, X), \mathscr{S}(\mathbb{R}^n, X))$ (see [Ama09, Lemma 4.1.2, Theorem 4.1.3], [Ama09, (4.1.11), Theorem 4.1.7]).

Next, we define two classes of tempered distributions on half spaces, and by duality we then extend the operators r^+, e^+, r_0^+, and e_0^+ canonically.

Definition 1.63. We define

$$\mathscr{S}'(\mathbb{R}^n_+, X) := L(\mathscr{S}(\mathbb{R}^n_+), X) \quad \text{and} \quad {}_0\mathscr{S}'(\mathbb{R}^n_+, X) := L({}_0\mathscr{S}(\mathbb{R}^n_+), X).$$

Then we lift the operators r^+, e^+, r_0^+, and e_0^+ by

$$\begin{aligned} e^+ &\colon {}_0\mathscr{S}'(\mathbb{R}^n_+, X) \to \mathscr{S}'(\mathbb{R}^n, X), & e^+ &:= (r_0^+)', \\ r^+ &\colon \mathscr{S}'(\mathbb{R}^n, X) \to {}_0\mathscr{S}'(\mathbb{R}^n_+, X), & r^+ &:= (e_0^+)', \\ e_0^+ &\colon \mathscr{S}'(\mathbb{R}^n_+, X) \to \mathscr{S}'(\mathbb{R}^n, X), & e_0^+ &:= (r^+)', \\ r_0^+ &\colon \mathscr{S}'(\mathbb{R}^n, X) \to \mathscr{S}'(\mathbb{R}^n_+, X), & r_0^+ &:= (e^+)', \end{aligned}$$

where $(\cdot)'$ stands for the adjoint operator, e.g., we have $(e^+u)(\varphi) := u(r_0^+\varphi)$ for $\varphi \in \mathscr{S}(\mathbb{R}^n)$ and $u \in {}_0\mathscr{S}'(\mathbb{R}^n_+, X)$.

Remark 1.64. (i) Note that this definition is consistent with Definition 1.61 (i) and (ii) when we keep in mind the embeddings

$$\begin{aligned} &\mathscr{S}(\mathbb{R}^n, X) \hookrightarrow \mathscr{S}'(\mathbb{R}^n, X), \qquad \mathscr{S}(\mathbb{R}^n_+, X) \hookrightarrow {}_0\mathscr{S}'(\mathbb{R}^n_+, X), \\ &{}_0\mathscr{S}(\mathbb{R}^n_+, X) \hookrightarrow \mathscr{S}'(\mathbb{R}^n_+, X). \end{aligned}$$

(ii) We have

$$\begin{aligned} e^+ &\in L({}_0\mathscr{S}'(\mathbb{R}^n_+, X), \mathscr{S}'(\mathbb{R}^n, X)), & r^+ &\in L(\mathscr{S}'(\mathbb{R}^n, X), {}_0\mathscr{S}'(\mathbb{R}^n_+, X)), \\ e_0^+ &\in L(\mathscr{S}'(\mathbb{R}^n_+, X), \mathscr{S}'(\mathbb{R}^n, X)), & r_0^+ &\in L(\mathscr{S}'(\mathbb{R}^n, X), \mathscr{S}'(\mathbb{R}^n_+, X)) \end{aligned}$$

and r_0^+ and r^+ are retractions with corresponding coretractions e_0^+ and e^+, respectively. For this result and further details we refer to [Ama09, Theorem 4.2.2].

The following result shows that e_0^+ and r_0^+ are connected with the support of the distribution.

Lemma 1.65. *For $u \in \mathscr{S}'(\mathbb{R}^n, X)$ the equality*

$$e_0^+ r_0^+ u = u$$

holds if and only if $\operatorname{supp} u \subseteq \overline{\mathbb{R}^n_+}$.

Proof. Analogously to Definition 1.61 and Definition 1.63, we define the operators r^-, r_0^-, e^-, and e_0^- with respect to the half space $\mathbb{R}^n_-$. In [Ama09, p. 81] one can find the decomposition

$$\mathscr{S}'(\mathbb{R}^n, X) = e_0^+ \mathscr{S}'(\mathbb{R}^n_+, X) \oplus e^- {}_0\mathscr{S}'(\mathbb{R}^n_-, X).$$

Let $u \in \mathscr{S}'(\mathbb{R}^n, X)$. Then there exist $u_1 \in \mathscr{S}'(\mathbb{R}^n_+, X)$ and $u_2 \in {}_0\mathscr{S}'(\mathbb{R}^n_-, X)$ such that $u = e_0^+ u_1 + e^- u_2$. With this decomposition we obtain

$$r_0^+ u = u_1 + r_0^+ e^- u_2, \quad r^- u = r^- e_0^+ u_1 + u_2.$$

It is elementary to show $r_0^+ e^- u_2 = 0$ and $r^- e_0^+ u_1 = 0$ (see [Ama09, Corollary 4.1.9]). Hence we have

$$u = e_0^+ r_0^+ u + e^- r^- u. \tag{1.12}$$

Now assume $\operatorname{supp} u \subseteq \overline{\mathbb{R}^n_+}$. For all $\varphi \in \mathscr{D}(\mathbb{R}^n_-)$ we have $\operatorname{supp}(e_0^- \varphi) \subseteq \mathbb{R}^n_-$ and therefore $(r^- u)\varphi = u(e_0^- \varphi) = 0$ since $\operatorname{supp} u \cap \operatorname{supp}(e_0^- \varphi) = \emptyset$. From this we derive $e^- r^- u = 0$. Then we obtain $u = e_0^+ r_0^+ u$ from (1.12).

On the other hand, assume $e_0^+ r_0^+ u = u$. Then we have $e^- r^- u = 0$ according to (1.12). So we have $u(e_0^- r_0^- \varphi) = 0$ for all $\varphi \in \mathscr{D}(\mathbb{R}^n_-)$. For all $\varphi \in \mathscr{D}(\mathbb{R}^n_-)$ we derive $e_0^- r_0^- \varphi = \varphi$ and therefore $\operatorname{supp} u \subseteq \overline{\mathbb{R}^n_+}$. □

With the help of the retractions and coretractions above, we can define the Sobolev spaces on $\mathbb{R}^n_+$. For simplicity, we do not consider negative orders of differentiation, so all spaces will be subspaces of $L_p(\mathbb{R}^n_+, X)$. We remark that $H_p^0(\mathbb{R}^n_+, X) = L_p(\mathbb{R}^n_+, X)$ but $B_{pp}^0(\mathbb{R}^n, X) \neq L_p(\mathbb{R}^n_+, X)$ in general, and therefore we will not consider $B_{pp}^0(\mathbb{R}^n_+, X)$ in the following.

Definition 1.66 (Sobolev spaces on the half-space). Let $p, q \in (1, \infty)$ and $s > 0$, and let $\mathcal{F} \in \{H_p^s, B_{pq}^s\}$. Then we define

$$\mathcal{F}(\mathbb{R}^n_+, X) := r^+ \mathcal{F}(\mathbb{R}^n, X),$$
$${}_0\mathcal{F}(\mathbb{R}^n_+, X) := r_0^+ \mathcal{F}(\mathbb{R}^n, X)$$

with the canonical norms

$$\|u\|_{\mathcal{F}(\mathbb{R}^n_+, X)} := \inf \big\{ \|f\|_{\mathcal{F}(\mathbb{R}^n, X)} : f \in \mathcal{F}(\mathbb{R}^n, X) \text{ with } u = r^+ f \big\},$$
$$\|u\|_{{}_0\mathcal{F}(\mathbb{R}^n_+, X)} := \inf \big\{ \|f\|_{\mathcal{F}(\mathbb{R}^n, X)} : f \in \mathcal{F}(\mathbb{R}^n, X) \text{ with } u = r_0^+ f \big\}.$$

We also set $H_p^0(\mathbb{R}^n_+, X) := L_p(\mathbb{R}^n, X)$.

Remark 1.67. (i) In the situation of Definition 1.66, we have

$$e^+ \in L(\mathcal{F}(\mathbb{R}^n_+, X), \mathcal{F}(\mathbb{R}^n, X)), \qquad r^+ \in L(\mathcal{F}(\mathbb{R}^n, X), \mathcal{F}(\mathbb{R}^n_+, X)),$$
$$e_0^+ \in L({}_0\mathcal{F}(\mathbb{R}^n_+, X), \mathcal{F}(\mathbb{R}^n, X)), \qquad r_0^+ \in L(\mathcal{F}(\mathbb{R}^n, X), {}_0\mathcal{F}(\mathbb{R}^n_+, X)).$$

Hence r^+ and r_0^+ are retractions on the spaces above with corresponding coretractions e^+ and e_0^+, respectively (see [Ama09, Section 4.4]).

(ii) The results on interpolation of Sobolev spaces carry over to spaces on $\mathbb{R}^n_+$. More precisely, the statements of Theorem 1.56 hold for all $r_0, r_1 \geq 0$ with $H_p^r(\mathbb{R}^n, X)$ being replaced by $H_p^r(\mathbb{R}^n_+, X)$ and $B_{pq}^r(\mathbb{R}^n, X)$ being replaced by $B_{pq}^r(\mathbb{R}^n_+, X)$. In the same way, we may replace $H_p^r(\mathbb{R}^n, X)$ and $B_{pq}^r(\mathbb{R}^n, X)$ by ${}_0H_p^r(\mathbb{R}^n_+, X)$ and ${}_0B_{pq}^r(\mathbb{R}^n_+, X)$, respectively.

(iii) The spaces ${}_0B^s_{pq}(\mathbb{R}_+, X)$ and ${}_0H^s_p(\mathbb{R}_+, X)$ can also be characterized by vanishing traces, i.e., we have

$$ {}_0H^s_p(\mathbb{R}_+, X) = \begin{cases} \{u \in H^s_p(\mathbb{R}_+, X) \colon u^{(j)}(0) = 0,\ j = 0, \dots, k\}, \\ \qquad\qquad s \in (k + \frac{1}{p}, k + 1 + \frac{1}{p}), k \in \mathbb{N}_0, \\ H^s_p(\mathbb{R}_+, X), \qquad s \in [0, \frac{1}{p}), \end{cases} $$

$$ {}_0B^s_{pq}(\mathbb{R}_+, X) = \begin{cases} \{u \in B^s_{pq}(\mathbb{R}_+, X) \colon u^{(j)}(0) = 0,\ j = 0, \dots, k\}, \\ \qquad\qquad s \in (k + \frac{1}{p}, k + 1 + \frac{1}{p}), k \in \mathbb{N}_0, \\ B^s_{pq}(\mathbb{R}_+, X), \qquad s \in (0, \frac{1}{p}), \end{cases} $$

cf. [Ama09, Theorem 4.7.1].

(iv) Due to [Ama09, p. 100] we have

$$ \mathscr{D}(\mathbb{R}_+, X) \overset{d}{\hookrightarrow} {}_0H^s_p(\mathbb{R}_+, X) \text{ for } s \geq 0, $$
$$ \mathscr{D}(\mathbb{R}_+, X) \overset{d}{\hookrightarrow} {}_0B^s_{pq}(\mathbb{R}_+, X) \text{ for } s > 0. $$

(v) (Sobolev embedding theorem) Let $s > k + \frac{1}{p}$ with $k \in \mathbb{N}_0$. Then the embeddings

$$ H^s_p(\mathbb{R}_+, X) \hookrightarrow C^k_b([0, \infty), X), $$
$$ B^s_{pq}(\mathbb{R}_+, X) \hookrightarrow C^k_b([0, \infty), X) $$

hold. These vector-valued variants of the well-known embedding theorems can be found in [Ama09, Theorem 3.9.1]. Therefore, the traces in (iii) are classical.

For the application of L_p-Fourier multiplier theorems we often need the Bessel potential spaces and Besov spaces to be of class $\mathcal{HT}$ and have property (α). For clarity we state these common facts in the next remark.

Remark 1.68. Let X be a Banach space of class $\mathcal{HT}$, $r \in \mathbb{R}$, and $1 < p, q < \infty$.

(i) The spaces $H^r_p(\mathbb{R}^n, X)$ and $B^r_{pq}(\mathbb{R}^n, X)$ are also of class $\mathcal{HT}$. Using Remark 1.13 (iii), Lemma 1.43 (i), and Lemma 1.58 (ii), this can be easily verified.

(ii) If X additionally has property (α), then $H^r_p(\mathbb{R}^n, X)$ and $B^r_{pq}(\mathbb{R}^n, X)$ also have property (α). This can be seen by Lemma 1.58 (ii), Remark 1.15 (ii) and Lemma 1.43 (ii). For the Besov spaces we can also use a retraction argument instead of the interpolation result in Lemma 1.43 (ii).

Proposition 1.69. *Let $\{X_0, X_1\}$ be an interpolation couple of Banach spaces of class $\mathcal{HT}$. Then we have*

$$\begin{aligned}
\big[H^s_{p_0}(\mathbb{R}^n_+, X_0), H^s_{p_1}(\mathbb{R}^n_+, X_1)\big]_\theta &= H^s_p(\mathbb{R}^n_+, [X_0, X_1]_\theta),\\
\big[{}_0H^s_{p_0}(\mathbb{R}^n_+, X_0), {}_0H^s_{p_1}(\mathbb{R}^n_+, X_1)\big]_\theta &= {}_0H^s_p(\mathbb{R}^n_+, [X_0, X_1]_\theta),\\
(H^s_{p_0}(\mathbb{R}^n_+, X_0), H^s_{p_1}(\mathbb{R}^n_+, X_1))_{\theta,p} &= H^s_p(\mathbb{R}^n_+, (X_0, X_1)_{\theta,p}),\\
({}_0H^s_{p_0}(\mathbb{R}^n_+, X_0), {}_0H^s_{p_1}(\mathbb{R}^n_+, X_1))_{\theta,p} &= {}_0H^s_p(\mathbb{R}^n_+, (X_0, X_1)_{\theta,p})
\end{aligned}$$

for all $s \geq 0$, $p_0, p_1 \in (1, \infty)$, $\theta \in (0,1)$, and $1/p = (1-\theta)/p_0 + \theta/p_1$.

Proof. For simplicity we only consider the first case, the other results can be obtained by minor modifications. First, we show $\big[H^s_{p_0}(\mathbb{R}^n, X_0), H^s_{p_1}(\mathbb{R}^n, X_1)\big]_\theta = H^s_p(\mathbb{R}^n, [X_0, X_1]_\theta)$. According to Lemma 1.58 we have the isomorphisms

$$\begin{aligned}
r_0 &:= (\mathrm{op}_{X_0}[\Lambda_{-s}])|_{L_{p_0}(\mathbb{R}^n, X_0)} \in L_{\mathrm{Isom}}(L_{p_0}(\mathbb{R}^n, X_0), H^s_{p_0}(\mathbb{R}^n, X_0)),\\
r_1 &:= (\mathrm{op}_{X_1}[\Lambda_{-s}])|_{L_{p_1}(\mathbb{R}^n, X_1)} \in L_{\mathrm{Isom}}(L_{p_1}(\mathbb{R}^n, X_1), H^s_{p_1}(\mathbb{R}^n, X_1)).
\end{aligned}$$

Lemma 1.52 then yields

$$r\,[L_{p_0}(\mathbb{R}^n, X_0), L_{p_1}(\mathbb{R}^n, X_1)]_\theta = \big[H^s_{p_0}(\mathbb{R}^n, X_0), H^s_{p_1}(\mathbb{R}^n, X_1)\big]_\theta,\ \theta \in (0,1). \tag{1.13}$$

The interpolation result of Theorem 1.41 implies

$$[L_{p_0}(\mathbb{R}^n, X_0), L_{p_1}(\mathbb{R}^n, X_1)]_\theta = L_p(\mathbb{R}^n, [X_0, X_1]_\theta). \tag{1.14}$$

Due to

$$rf = r_0 f_0 + r_1 f_1 = \mathscr{F}^{-1}_{X_0} \Lambda_{-s} \mathscr{F}_{X_0} f_0 + \mathscr{F}^{-1}_{X_1} \Lambda_{-s} \mathscr{F}_{X_1} f_1 = \mathscr{F}^{-1}_{X_0+X_1} \Lambda_{-s} \mathscr{F}_{X_0+X_1} f$$

we have

$$\begin{aligned}
r|_{L_p(\mathbb{R}^n, [X_0,X_1]_\theta)} &= (\mathrm{op}_{[X_0,X_1]_\theta}[\Lambda_{-s}])|_{L_p(\mathbb{R}^n, [X_0,X_1]_\theta)}\\
&\in L_{\mathrm{Isom}}(L_p(\mathbb{R}^n, [X_0, X_1]_\theta), H^s_p(\mathbb{R}^n, [X_0, X_1]_\theta)).
\end{aligned}$$

Using (1.13) and (1.14) we obtain

$$\big[H^s_{p_0}(\mathbb{R}^n, X_0), H^s_{p_1}(\mathbb{R}^n, X_1)\big]_\theta = H^s_p(\mathbb{R}^n, [X_0, X_1]_\theta). \tag{1.15}$$

Applying Lemma 1.51, Remark 1.67 (ii), and (1.15) we then obtain

$$\big[H^s_{p_0}(\mathbb{R}^n_+, X_0), H^s_{p_1}(\mathbb{R}^n_+, X_1)\big]_\theta = H^s_p(\mathbb{R}^n_+, [X_0, X_1]_\theta). \qquad \square$$

Remark 1.70 (Spaces with exponential weights). In many applications, the operators are sectorial or $\mathcal{R}$-sectorial only after a shift, i.e., if one replaces T by $T + \varrho$. Instead of shifting the operator, one can also consider the original operator in spaces with exponential weight. The above properties of Sobolev spaces carry

over to their exponentially weighted versions. Here we only give the definition and some remarks on these spaces.

For $\varrho \in \mathbb{R}$ and $u \in L_p(\mathbb{R}_+, X)$, we define $\mathscr{M}_\varrho u$ by

$$(\mathscr{M}_\varrho u)(t) := e^{-\varrho t} u(t), \quad t \in \mathbb{R}_+. \tag{1.16}$$

For $\varrho \geq 0$, the spaces with exponential weight are then defined by

$$\begin{aligned} {}_0H^s_{p,\varrho}(\mathbb{R}_+, X) &:= \mathscr{M}_{-\varrho}\left({}_0H^s_p(\mathbb{R}_+, X)\right), \quad s \geq 0, \\ {}_0B^s_{pq,\varrho}(\mathbb{R}_+, X) &:= \mathscr{M}_{-\varrho}\left({}_0B^s_{pq}(\mathbb{R}_+, X)\right), \quad s > 0, \end{aligned}$$

with norms

$$\|u\|_{{}_0H^s_{p,\varrho}(\mathbb{R}_+,X)} := \|\mathscr{M}_\varrho u\|_{{}_0H^s_p(\mathbb{R}_+,X)}, \quad u \in {}_0H^s_{p,\varrho}(\mathbb{R}_+, X),\ s \geq 0, \tag{1.17}$$

$$\|f\|_{{}_0B^s_{pq,\varrho}(\mathbb{R}_+,X)} := \|\mathscr{M}_\varrho f\|_{{}_0B^s_{pq}(\mathbb{R}_+,X)}, \quad f \in {}_0B^s_{pq,\varrho}(\mathbb{R}_+, X),\ s > 0. \tag{1.18}$$

It is easily seen that for $\mathcal{F} \in \{H^s_p, B^s_{pq}\}$ we have

$$\mathscr{M}_\varrho|_{\mathcal{F}_\varrho(\mathbb{R}_+,X)} \in L_{\mathrm{Isom}}\left({}_0\mathcal{F}_\varrho(\mathbb{R}_+, X), {}_0\mathcal{F}(\mathbb{R}_+, X)\right)$$

(see [DSS08, Lemma 2.2]), and therefore the interpolation results above also hold for weighted spaces.

In the following, we will frequently use Sobolev spaces in time and space variables which belong to different types, e.g., ${}_0B^s_{pp}(\mathbb{R}_+, H^r_p(\mathbb{R}^n))$. Therefore, we introduce an abbreviation for these spaces.

Definition 1.71 (Spaces of mixed scales). Let $1 < p_0, q_0, p_1, q_1 < \infty$, $s \geq 0$, $\varrho \geq 0$, and $r \in \mathbb{R}$. Let X be a Banach space of class $\mathcal{HT}$ with property (α). For $\mathcal{F} \in \{B_{p_0q_0}, H_{p_0}\}$ (with $s > 0$ if $\mathcal{F} = B_{p_0q_0}$) and for $\mathcal{K} \in \{B_{p_1q_1}, H_{p_1}\}$ we define

$$\begin{aligned} {}_0\mathcal{F}^s_\varrho(\mathcal{K}^r) &:= {}_0\mathcal{F}^s_\varrho\big(\mathbb{R}_+, \mathcal{K}^r(\mathbb{R}^n, X)\big), \\ {}_0\mathcal{F}^s(\mathcal{K}^r) &:= {}_0\mathcal{F}^s\big(\mathbb{R}_+, \mathcal{K}^r(\mathbb{R}^n, X)\big). \end{aligned}$$

Note that we always include zero initial values at $t = 0$ and that we do not indicate the dependence on the space X which will be fixed in most situations.

Remark 1.72. In the definition above, for simplicity we always assume that X is a Banach space of class $\mathcal{HT}$ with property (α). This will be the case for all applications. We use the abbreviations ${}_0\mathcal{F}^s(\mathcal{K}^r)$ and ${}_0\mathcal{F}^s_\varrho(\mathcal{K}^r)$ for better readability but one should keep in mind that we have X-valued spaces. In all cases, the proofs of the results below for $\varrho > 0$ will follow from the case $\varrho = 0$ by the isomorphism of Remark 1.70. Therefore, we will formulate most proofs without explicit reference to the case $\varrho > 0$.

1.3 The time-space derivative

Motivation. After introducing function spaces in the previous section, we are now going to show that the time-space derivative $(\partial_t, \nabla_x) = (\partial_t, \partial_{x_1}, \dots, \partial_{x_n})$ admits a bounded joint H^∞-calculus in these spaces. This means that we may replace the symbols $\lambda, z_1, \dots, z_n$ by derivatives and obtain a well-defined and (in case of a bounded holomorphic symbol) bounded operator.

The key step to show this is the application of a vector-valued Michlin-type theorem. For this, the notion of a Fourier multiplier will be defined and discussed in Subsection a). The operators ∂_t on one hand and ∂_{x_j}, $j = 1, \dots, n$, on the other hand, have a different meaning: while ∇_x is considered in the whole space $\mathbb{R}^n$, the time derivative acts on functions defined for $t \in \mathbb{R}_+$, and therefore we include zero initial conditions at $t = 0$ in the domain of the related operator. This implies a change in the spectrum; and in fact, ∂_t is sectorial with angle $\frac{\pi}{2}$ while ∂_{x_j} is bisectorial with angle 0.

The properties of these operators are discussed in Subsection b), while in Subsection c) we will show that (∂_t, ∇_x) has a bounded joint H^∞-calculus in all types of Sobolev spaces.

Throughout this section, let X and Y be complex Banach spaces of class $\mathcal{HT}$ with property (α). We remark that some results below also hold in more general cases.

a) Fourier multipliers

The notion of an L_p-Fourier multiplier is usually connected with operators of the form $\mathscr{F}^{-1} m(\xi) \mathscr{F}$ with $m \in L_\infty(\mathbb{R}^n, L(X,Y))$. As we are considering holomorphic symbols, we will replace ξ by $i\xi$ in our definition.

Definition 1.73 (L_p**-Fourier multiplier).** Let $(i\mathbb{R})^n \subseteq \Omega \subseteq \mathbb{C}^n$, $1 < p < \infty$, and let $m \colon \Omega \to L(X,Y)$ be a function satisfying $(\xi \mapsto m(i\xi)) \in L_\infty(\mathbb{R}^n, L(X,Y))$. Then we define

$$\operatorname{op}[m] \colon \mathscr{S}(\mathbb{R}^n, X) \to L_\infty(\mathbb{R}^n, Y), \ f \mapsto \mathscr{F}^{-1} m(i\xi) \mathscr{F} f.$$

The function m is called an L_p-*Fourier multiplier* if there exists $C_p > 0$ such that

(i) $\operatorname{op}[m] f \in L_p(\mathbb{R}^n, Y)$ for all $f \in \mathscr{S}(\mathbb{R}^n, X)$,

(ii) $\|\operatorname{op}[m] f\|_{L_p(\mathbb{R}^n, Y)} \le C_p \|f\|_{L_p(\mathbb{R}^n, X)}$ for all $f \in \mathscr{S}(\mathbb{R}^n, X)$.

In this case we have a unique continuous extension of $\operatorname{op}[m]$ to an operator in $L(L_p(\mathbb{R}^n, X), L_p(\mathbb{R}^n, Y))$ which will also be denoted by $\operatorname{op}[m]$.

The following theorem is the key result to prove that a given symbol is a Fourier multiplier. In the scalar case $X = Y = \mathbb{C}$, the well-known Michlin theorem (see, e.g., [Tri78, Section 2.2]) gives conditions on m to be a Fourier multiplier. In

the vector-valued case, the essential condition is not the norm-boundedness of the symbols and their derivatives but the $\mathcal{R}$-boundedness. The following result due to Weis is the vector-valued analog of Michlin's theorem.

Theorem 1.74 (**Theorem of Weis,** see [KW04, 5.2 b)]). *Let $1 < p < \infty$ and let $\{m_j : j \in I\}$ be a family of functions $m_j \colon (i\mathbb{R})^n \setminus \{0\} \to L(X,Y)$ satisfying*

$$\xi \mapsto m_j(i\xi) \in C^n\big(\mathbb{R}^n \setminus \{0\}, L(X,Y)\big), \quad j \in I.$$

If the set

$$\mathcal{T} := \big\{\xi^\alpha \partial^\alpha m_j(i\xi) : \xi \in \mathbb{R}^n \setminus \{0\},\, \alpha \in \{0,1\}^n,\, j \in I\big\}$$

is $\mathcal{R}$-bounded, then each m_j is a Fourier multiplier, and

$$\big\{\mathrm{op}[m_j] : j \in I\big\} \subseteq L(L_p(\mathbb{R}^n, X), L_p(\mathbb{R}^n, Y))$$

is also $\mathcal{R}$-bounded with $\mathcal{R}_p$-bound not greater than $C(p,n,X,Y)\mathcal{R}_p(\mathcal{T})$ for some constant $C(p,n,X,Y) > 0$.

From this we easily obtain:

Proposition 1.75. *Let $m \in H^\infty(\Omega)$ with $\Omega := \prod_{k=1}^n \Sigma_{\delta_k}$. Then $m = m\,\mathrm{id}_X$ is an $L_p(\mathbb{R}^n, X)$-Fourier multiplier for all $p \in (1,\infty)$.*

Proof. According to [DV02b, Lemma 6.29] we have

$$|\xi^\alpha \partial^\alpha m(i\xi)| \le \alpha! \prod_{k=1}^n (\sin(\delta_k))^{-\alpha_k} \|m\|_\infty, \quad \xi \in \mathbb{R}^n \setminus \{0\},\, \alpha \in \mathbb{N}_0^n. \tag{1.19}$$

With Kahane's contraction principle (Theorem 1.8) and (1.19) we obtain the $\mathcal{R}$-boundedness of

$$\mathcal{T} := \{\xi^\alpha \partial^\alpha m(i\xi)\,\mathrm{id}_X \colon \xi \in \mathbb{R}^n \setminus \{0\},\, \alpha \in \{0,1\}^n\} \subseteq L(X).$$

The assertion is then obtained by Theorem 1.74. □

Lemma 1.76. *Let $1 < p < \infty$, and let $m \in H^\infty(\Omega)$ with $\Omega := \prod_{k=1}^n \Sigma_{\delta_k}$. Then the following assertions hold.*

(i) *The mapping $\mathrm{op}[m] \colon \mathscr{S}(\mathbb{R}^n, X) \to L_\infty(\mathbb{R}^n, X)$ extends to an operator in $L(H_p^r(\mathbb{R}^n, X))$ for every $r \in \mathbb{R}$. Denoting by $\mathrm{op}^{(r,p)}[m]$ and $\mathrm{op}^{(0,p)}[m]$ the extension of $\mathrm{op}[m]$ to $H_p^r(\mathbb{R}^n, X)$ and $L_p(\mathbb{R}^n, X)$, respectively, we get*

$$\mathrm{op}[\Lambda_{-r}]\mathrm{op}^{(0,p)}[m]\mathrm{op}[\Lambda_r] f = \mathrm{op}^{(r,p)}[m] f, \quad f \in H_p^r(\mathbb{R}^n, X). \tag{1.20}$$

(ii) *Let $s, r \in \mathbb{R}$ and $p, q \in (1,\infty)$ with $p = q$ or $s = r$. Then the compatibility*

$$\mathrm{op}^{(r,p)}[m] f = \mathrm{op}^{(s,q)}[m] f, \quad f \in H_p^r(\mathbb{R}^n, X) \cap H_q^s(\mathbb{R}^n, X) \tag{1.21}$$

holds.

Proof. (i) Let $f \in \mathscr{S}(\mathbb{R}^n, X)$. Then we have $\operatorname{op}[m]f \in L_p(\mathbb{R}^n, X)$ according to Definition 1.73 (i) and Proposition 1.75. Due to the holomorphy of $\Lambda_{-|r|}$ we have that $\Lambda_{-|r|}m$ is also a Fourier multiplier. This enables us to prove

$$\operatorname{op}[\Lambda_{-r}]\operatorname{op}[m]\operatorname{op}[\Lambda_r]f = \operatorname{op}[m]f, \quad f \in \mathscr{S}(\mathbb{R}^n, X). \tag{1.22}$$

In particular, we deduce $\operatorname{op}[m]f \in H^r_p(\mathbb{R}^n, X)$ for all $f \in \mathscr{S}(\mathbb{R}^n, X)$. So we obtain

$$\begin{aligned}\|\operatorname{op}[m]f\|_{H^r_p(\mathbb{R}^n,X)} &= \|\operatorname{op}[\Lambda_r]\operatorname{op}[m]f\|_{L_p(\mathbb{R}^n,X)} = \|\operatorname{op}[m]\operatorname{op}[\Lambda_r]f\|_{L_p(\mathbb{R}^n,X)} \\ &\le C\|\operatorname{op}[\Lambda_r]f\|_{L_p(\mathbb{R}^n,X)} = C\|f\|_{H^r_p(\mathbb{R}^n,X)}\end{aligned}$$

due to $\operatorname{op}[\Lambda_r]f \in \mathscr{S}(\mathbb{R}^n, X)$ and Definition 1.73 (ii). According to Remark 1.67 (iv) we can now extend $\operatorname{op}[m]$ to a bounded operator in $L(H^r_p(\mathbb{R}^n, X))$. The representation in (1.20) then follows from (1.22).

(ii) First, we consider the case $r = s$. Let $f \in H^r_p(\mathbb{R}^n, X) \cap H^r_q(\mathbb{R}^n, X)$ and define the function $h := \operatorname{op}[\Lambda_r]f \in L_p(\mathbb{R}^n, X) \cap L_q(\mathbb{R}^n, X)$. Let $\psi_n \in \mathscr{D}(\mathbb{R}^n, X)$ be a Dirac sequence. With this we get $\varphi_n := \psi_n * h \in \mathscr{D}(\mathbb{R}^n, X)$ and $\varphi_n \to h$ in $L_u(\mathbb{R}^n, X)$ for all $1 < u < \infty$. Hence $\operatorname{op}[\Lambda_{-r}]\varphi_n \to f$ in $H^r_p(\mathbb{R}^n, X) \cap H^r_q(\mathbb{R}^n, X)$ due to Lemma 1.58 (ii). It is easy to see that $\operatorname{op}[\Lambda_{-r}]\varphi_n \in \mathscr{S}(\mathbb{R}^n, X)$ and therefore

$$\begin{aligned}\operatorname{op}^{(r,p)}[m]f &= \lim_{n\to\infty} \operatorname{op}^{(r,p)}[m]\operatorname{op}[\Lambda_{-r}]\varphi_n = \lim_{n\to\infty} \operatorname{op}^{(r,q)}[m]\operatorname{op}[\Lambda_{-r}]\varphi_n \\ &= \operatorname{op}^{(r,q)}[m]f.\end{aligned}$$

Next, we consider the case $p = q$. Without loss of generality, we assume $r < s$, which implies $H^s_p(\mathbb{R}^n, X) \hookrightarrow H^r_p(\mathbb{R}^n, X)$. Let $f \in H^s_p(\mathbb{R}^n, X)$ and $(f_n)_{n\in\mathbb{N}} \subseteq \mathscr{S}(\mathbb{R}^n, X)$ with $f_n \to f$ in $H^s_p(\mathbb{R}^n, X)$ (cf. Remark 1.67 (iv)). Then we obtain $f_n \to f$ in $H^r_p(\mathbb{R}^n, X)$ and therefore

$$\operatorname{op}^{(r,p)}[m]f = \lim_{n\to\infty} \operatorname{op}^{(r,p)}[m]f_n = \lim_{n\to\infty} \operatorname{op}^{(s,p)}[m]f_n = \operatorname{op}^{(s,p)}[m]f. \qquad \square$$

Remark 1.77. For $1 < p < \infty$ we set $H^{-\infty}_p(\mathbb{R}^n, X) := \bigcup_{r\in\mathbb{R}} H^r_p(\mathbb{R}^n, X)$. Let $m \in H^\infty(\Omega)$ with $\Omega := \prod_{k=1}^n \Sigma_{\delta_k}$. Then Lemma 1.76 (ii) shows that

$$\begin{aligned}\operatorname{op}[m]\colon H^{-\infty}_p(\mathbb{R}^n, X) &\to H^{-\infty}_p(\mathbb{R}^n, X), \\ f &\mapsto \operatorname{op}^{(r,p)}[m]f, \quad f \in H^r_p(\mathbb{R}^n, X),\end{aligned}$$

is well-defined. By Lemma 1.76 (i) we have

$$\operatorname{op}[m]|_{H^r_p(\mathbb{R}^n,X)} \in L(H^r_p(\mathbb{R}^n, X)), \quad r \in \mathbb{R},$$

and by real interpolation (Theorem 1.56 (iii)) we obtain

$$\operatorname{op}[m]|_{B^r_{pq}(\mathbb{R}^n,X)} \in L(B^r_{pq}(\mathbb{R}^n, X)), \quad r \in \mathbb{R}.$$

The proof of the next result includes ideas from [Tri78, p. 232] and [GGHR06, p. 174]. It shows that the existence of a holomorphic extension of a symbol m to a complex half-space is connected with the fact that the support of a function is preserved by $\operatorname{op}[m]$. In the case of functions of time, this can be interpreted as causality of the operator. In the context of pseudodifferential operators, operators with this property are also called Volterra operators, see, e.g., [Kra04], [DS11].

Proposition 1.78 (Holomorphic Fourier multiplier and support). *Let $m \in H^\infty(S_\theta)$ with $\theta > \frac{\pi}{2}$, and let $1 < p, q < \infty$, $\mathcal{F} \in \{B_{pq}, H_p\}$, $s \geq 0$ ($s > 0$ if $\mathcal{F} = B_{pq}$). Then m is a Fourier multiplier on $L_p(\mathbb{R}, X)$ and*

$$\left(r_0^+ \operatorname{op}[m] e_0^+\right)\big|_{{}_0\mathcal{F}^s(\mathbb{R}_+,X)} \in L({}_0\mathcal{F}^s(\mathbb{R}_+, X)) \tag{1.23}$$

is a well-defined bounded operator. In particular, we have

$$\operatorname{supp}\left(\operatorname{op}[m] e_0^+ f\right) \subseteq [0, \infty)$$

and therefore $e_0^+ r_0^+ \operatorname{op}[m] e_0^+ f = \operatorname{op}[m] e_0^+ f$ for all $f \in {}_0\mathcal{F}^s(\mathbb{R}_+, X)$.

Proof. The assertion in (1.23) is obvious due to Remark 1.67 (i) and Remark 1.77.

Next, we prove the claimed position of the support. Let $\varphi \in \mathscr{D}(\mathbb{R}_+, X)$. Then there exists $r > 0$ such that $\operatorname{supp}\varphi \subseteq (0, r)$. For simplicity we do not distinguish between φ and $e_0^+\varphi$ in the following. It is obvious that $\mathscr{F}\varphi \in \mathscr{S}(\mathbb{R}, X)$ and therefore $m(i\cdot)(\mathscr{F}\varphi) \in L_1(\mathbb{R}, X)$. Hence, we can use the explicit formula

$$\left[\mathscr{F}^{-1}(m(i\cdot)(\mathscr{F}\varphi))\right](t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} m(i\lambda)(\mathscr{F}\varphi)(\lambda) e^{it\lambda} d\lambda$$

for the inverse Fourier transform for all $t \in \mathbb{R}$. The function $\mathscr{F}\varphi$ can be canonically extended to $\mathbb{C}$ by

$$(\mathscr{F}\varphi)(z) := (2\pi)^{-1/2} \int_{-\infty}^{\infty} \varphi(t) e^{-itz} dt, \quad z \in \mathbb{C}.$$

The complex function $\mathscr{F}\varphi$ is an entire function, and the vector-valued Paley-Wiener Theorem (see [Jef04, Proposition 2.1], for instance) yields the estimate

$$\|(\mathscr{F}\varphi)(z)\|_X \leq C(1 + |z|)^{-1} \exp(r|\operatorname{Im} z|), \quad z \in \mathbb{C}. \tag{1.24}$$

Using Cauchy's integral theorem we will show

$$\left[\mathscr{F}^{-1}(m(i\cdot)(\mathscr{F}\varphi))\right](t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} m(i\lambda + \gamma)(\mathscr{F}\varphi)(\lambda - i\gamma) e^{it(\lambda - i\gamma)} d\lambda \tag{1.25}$$

for all $\gamma > 1$ and $t \in \mathbb{R}$. In order to do this we define the closed curve

$$\begin{aligned}\Gamma(\varepsilon, R) := {} & (-R, -\varepsilon) \cup \{\varepsilon e^{i(\theta+\pi)} \colon \theta \in [0, \pi]\} \cup (\varepsilon, R) \\ & \cup \{R - i\eta \colon \eta \in (0, \gamma)\} \\ & \cup \{-\eta - i\gamma \colon \eta \in [-R, R]\} \\ & \cup \{-R - i(\gamma - \eta) \colon \eta \in (0, \gamma)\} \subseteq \mathbb{C}\end{aligned}$$

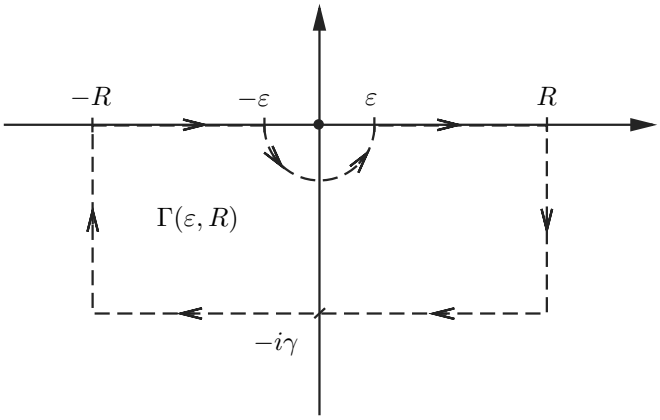


Figure 1.3: Path of integration $\Gamma(\varepsilon, R)$

for $0 < \varepsilon < 1$ and large $1 < R < \infty$ (cf. Figure 1.3).

Due to the holomorphy of all involved functions and Cauchy's integral theorem we obtain

$$\int_{\Gamma(\varepsilon,R)} m(iz)(\mathscr{F}\varphi)(z)e^{itz}dz = 0. \tag{1.26}$$

(I) For the parametrization $\gamma_\varepsilon(\theta) := \varepsilon e^{i(\theta+\pi)}$, $\theta \in [0,\pi]$, we have

$$\begin{aligned}
&\Big\| \int_{\{\varepsilon e^{i(\theta+\pi)}\,:\,\theta\in[0,\pi]\}} m(iz)(\mathscr{F}\varphi)(z)e^{itz}dz \Big\|_X \\
&\qquad = \Big\| \int_0^\pi m(i\gamma_\varepsilon(\theta))(\mathscr{F}\varphi)(\gamma_\varepsilon(\theta))e^{it\gamma_\varepsilon(\theta)}\gamma_\varepsilon'(\theta)d\theta \Big\|_X \\
&\qquad \le \varepsilon \int_0^\pi |m(i\gamma_\varepsilon(\theta))| \cdot \|(\mathscr{F}\varphi)(\gamma_\varepsilon(\theta))\|_X |e^{it\gamma_\varepsilon(\theta)}|d\theta \\
&\qquad \le \varepsilon \|m\|_\infty C(t)
\end{aligned}$$

due to $m \in H^\infty(S_\theta)$ and the boundedness of

$$(\theta,\varepsilon) \mapsto \|(\mathscr{F}\varphi)(\gamma_\varepsilon(\theta))\|_X |e^{it\gamma_\varepsilon(\theta)}|.$$

We get

$$\lim_{\varepsilon\to 0} \int_{\{\varepsilon e^{i(\theta+\pi)}\,:\,\theta\in[0,\pi]\}} m(iz)(\mathscr{F}\varphi)(z)e^{itz}dz = 0.$$

(II) Concerning the parametrization $\gamma_R(\eta) := R - i\eta$, $\eta \in [0,\gamma]$, we have

$$\Big\| \int_{\{R-i\eta\,:\,\eta\in(0,\gamma)\}} m(iz)(\mathscr{F}\varphi)(z)e^{itz}dz \Big\|_X$$

$$\begin{aligned}
&= \left\| \int_0^\gamma m(i\gamma_R(\eta))(\mathscr{F}\varphi)(\gamma_R(\eta)) e^{it\gamma_R(\eta)} d\eta \right\|_X \\
&\le \|m\|_\infty \int_0^\gamma \|(\mathscr{F}\varphi)(\gamma_R(\eta))\|_X e^{t\eta} d\eta \\
&\le C\|m\|_\infty (1+R)^{-1} \int_0^\gamma e^{(r+t)\eta} d\eta,
\end{aligned}$$

where we have used (1.24). With this we obtain

$$\lim_{R\to\infty} \int_{\{R-i\eta\colon\, \eta\in(0,\gamma)\}} m(iz)(\mathscr{F}\varphi)(z) e^{itz} dz = 0.$$

The same arguments also yield

$$\lim_{R\to\infty} \int_{\{-R-i(\gamma-\eta)\colon\, \eta\in(0,\gamma)\}} m(iz)(\mathscr{F}\varphi)(z) e^{itz} dz = 0.$$

With (I), (II) and (1.26) we obtain

$$\begin{aligned}
0 &= \lim_{R\to\infty} \lim_{\varepsilon\to 0} \int_{\Gamma(\varepsilon,R)} m(iz)(\mathscr{F}\varphi)(z) e^{itz} dz \\
&= \int_{-\infty}^{\infty} m(i\lambda)(\mathscr{F}\varphi)(\lambda) e^{it\lambda} d\lambda - \int_{-\infty}^{\infty} m(i\lambda+\gamma)(\mathscr{F}\varphi)(\lambda - i\gamma) e^{it(\lambda - i\gamma)} d\lambda,
\end{aligned}$$

which proves (1.25) for all $\gamma > 0$. For $t < 0$ and all $\gamma > 1$ we obtain

$$\left\| [\mathscr{F}^{-1}(m(i\cdot)(\mathscr{F}\varphi))](t) \right\|_X \le (2\pi)^{-1/2} \|m\|_\infty e^{t\gamma} \int_{-\infty}^{\infty} \|(\mathscr{F}\varphi)(\lambda - i\gamma)\|_X d\lambda.$$

Using integration by parts and $\gamma > 1$ it is easy to see that

$$\begin{aligned}
\|(\mathscr{F}\varphi)(\lambda - i\gamma)\|_X &= (2\pi)^{-1/2} \left\| \int_0^\infty \varphi(x) \partial_x^2 (e^{-ix(\lambda - i\gamma)}) dx \right\|_X \cdot |\lambda - i\gamma|^{-2} \\
&\le (2\pi)^{-1/2} \left\| \int_0^\infty \varphi''(x) e^{-ix(\lambda - i\gamma)} dx \right\|_X \cdot (\lambda^2 + 1)^{-1} \\
&\le (2\pi)^{-1/2} \|\varphi''\|_\infty \int_0^\infty |e^{-ix(\lambda - i\gamma)}| dx \cdot (\lambda^2 + 1)^{-1} \\
&\le (2\pi)^{-1/2} \|\varphi''\|_\infty \cdot (\lambda^2 + 1)^{-1}
\end{aligned}$$

and therefore we derive

$$\left\| [\mathscr{F}^{-1}(m(i\cdot)(\mathscr{F}\varphi))](t) \right\|_X \le \frac{1}{2} \|\varphi''\|_\infty \|m\|_\infty \lim_{\gamma\to\infty} e^{t\gamma} = 0, \quad t < 0,$$

which yields

$$\operatorname{supp}(\operatorname{op}[m]\varphi) \subseteq [0, \infty).$$

Let $f \in {}_0\mathcal{F}^s(\mathbb{R}_+, X)$. By Remark 1.67 (iv) there exists $(\varphi_k)_{k\in\mathbb{N}} \subseteq \mathscr{D}(\mathbb{R}_+, X)$ with

$$f = \lim_{k\to\infty} \varphi_k \text{ in } {}_0\mathcal{F}^s(\mathbb{R}_+, X).$$

Lemma 1.65, Remark 1.67 (i), and Remark 1.77 (ii) yield

$$\operatorname{op}[m]e_0^+ f = \lim_{k\to\infty} \operatorname{op}[m]e_0^+\varphi_k = \lim_{k\to\infty} e_0^+ r_0^+ \operatorname{op}[m]e_0^+\varphi_k = e_0^+ r_0^+ \operatorname{op}[m]e_0^+ f$$

where the limits hold in $\mathcal{F}^s(\mathbb{R}, X)$. We finally derive $\operatorname{supp}(\operatorname{op}[m]f) \subseteq [0,\infty)$ with Lemma 1.65. □

The Fourier multipliers considered above worked in the whole space $\mathbb{R}^n$. With respect to the time variable, however, we will consider the half-line $\mathbb{R}_+$, including vanishing trace condition at $t = 0$. We can easily define Fourier multipliers in $\mathbb{R}_+$ by using the extension and restriction operators e_0^+ and r_0^+. Note that we want to stay in the spaces with zero trace at $t = 0$. For this, the property of the support discussed in Proposition 1.78 will be essential.

For $s \in \mathbb{R}$ and $\lambda \in \mathbb{C} \setminus (-\infty, 0)$, we define the symbol

$$\Psi_s(\lambda) := (1+\lambda)^s.$$

Symbolically, we want to replace λ by ∂_t. But we have to take into account the initial condition at $t = 0$. Therefore, we define

$$\operatorname{op}_+[\Psi_s] := r_0^+ \operatorname{op}[\Psi_s]e_0^+ = r_0^+ \mathscr{F}^{-1}\Psi_s(i\cdot)\mathscr{F}e_0^+$$

on $L_p(\mathbb{R}_+, X)$.

Proposition 1.79 (Properties of $\operatorname{op}_+[\Psi_s]$). *For $1 < p, q < \infty$ and $s \geq 0$ we have*

$$\operatorname{op}_+[\Psi_s]|_{{}_0H_p^{s'+s}(\mathbb{R}_+,X)} \in L_{\mathrm{Isom}}({}_0H_p^{s'+s}(\mathbb{R}_+, X), {}_0H_p^{s'}(\mathbb{R}_+, X)), \quad s' \geq 0,$$
$$\operatorname{op}_+[\Psi_s]|_{{}_0B_{p,q}^{s'+s}(\mathbb{R}_+,X)} \in L_{\mathrm{Isom}}({}_0B_{p,q}^{s'+s}(\mathbb{R}_+, X), {}_0B_{p,q}^{s'}(\mathbb{R}_+, X)), \quad s' > 0.$$

Proof. Similar to the argumentation for Λ_r we obtain

$$\operatorname{op}[\Psi_r]|_{H_p^{r'+r}(\mathbb{R},X)} \in L_{\mathrm{Isom}}(H_p^{r'+r}(\mathbb{R}, X), H_p^{r'}(\mathbb{R}, X)), \quad r', r \in \mathbb{R}. \tag{1.27}$$

From (1.27) and Remark 1.67 (i) we get

$$\operatorname{op}_+[\Psi_s]|_{{}_0H_p^{s'+s}(\mathbb{R}_+,X)} \in L({}_0H_p^{s'+s}(\mathbb{R}_+, X), {}_0H_p^{s'}(\mathbb{R}_+, X)),$$
$$\operatorname{op}_+[\Psi_{-s}]|_{{}_0H_p^{s'+s}(\mathbb{R}_+,X)} \in L({}_0H_p^{s'}(\mathbb{R}_+, X), {}_0H_p^{s'+s}(\mathbb{R}_+, X))$$

for $s, s' \geq 0$. For all $f \in {}_0H_p^{s'}(\mathbb{R}_+, X)$ we then conclude

$$\begin{aligned}\operatorname{op}_+[\Psi_s]\operatorname{op}_+[\Psi_{-s}]f &= r_0^+ \operatorname{op}[\Psi_s]e_0^+ r_0^+ \operatorname{op}[\Psi_{-s}]e_0^+ f = r_0^+ \operatorname{op}[\Psi_s]\operatorname{op}[\Psi_{-s}]e_0^+ f \\ &= r_0^+ e_0^+ f = f\end{aligned}$$

due to Proposition 1.78. For all $g \in {}_0H^{s'+s}_p(\mathbb{R}_+, X)$ we get

$$\mathrm{op}_+[\Psi_{-s}]\mathrm{op}_+[\Psi_s]g = r_0^+\mathrm{op}[\Psi_{-s}]e_0^+ r_0^+\mathrm{op}[\Psi_s]e_0^+ g.$$

In the same way as in the proof of Proposition 1.78, we get $\mathrm{supp}(\mathrm{op}[\Psi_s]e_0^+ g) \subseteq [0,\infty)$. The polynomial boundedness of the symbol Ψ_s can be handled by stronger Paley-Wiener estimates of the form

$$\|(\mathscr{F}\varphi)(z)\|_X \le C(1+|z|)^{\lceil s\rceil+1}\exp(r|\operatorname{Im} z|), \quad z \in \mathbb{C}.$$

Hence, we also obtain $\mathrm{op}_+[\Psi_{-s}]\mathrm{op}_+[\Psi_s]g = g$. This yields the assertion for the Bessel potential spaces. The remaining assertion follows by interpolation. □

Remark 1.80. The above results hold analogously in exponentially weighted spaces. For $\varrho \ge 0$, we define

$$\mathrm{op}_+^{(\varrho)}[\Psi_s] := \mathscr{M}_\varrho^{-1}\mathrm{op}_+[\Psi_s(\cdot+\varrho)]\mathscr{M}_\varrho$$

in $L_{p,\varrho}(\mathbb{R}_+, X)$. Then the statements in Proposition 1.79 hold for $\mathrm{op}_+^{(\varrho)}[\Psi_s]$ if we replace all spaces by their exponentially weighted versions.

b) Vector-valued space and time derivatives

Now we are going to study realizations of the space derivatives $\partial_j = \partial_{x_j}$, $j = 1,\dots,n$, in $\mathbb{R}^n$ and of the time derivative ∂_t in $\mathbb{R}_+$. We will show that the corresponding operators have good mapping properties in all scales of Sobolev spaces considered above. To simplify the notation, we will fix $r \in \mathbb{R}$, $p,q \in (1,\infty)$, and $\mathcal{K} \in \{B_{pq}, H_p\}$. As before, X is assumed to be a Banach space of class $\mathcal{HT}$ with property (α). The $\mathcal{K}^r(\mathbb{R}^n, X)$-realization of ∂_j is defined as the operator $\mathcal{D}_j$ given by

$$\begin{aligned}\mathcal{D}_j\colon \mathcal{K}^r(\mathbb{R}^n,X) \supseteq D(\mathcal{D}_j) &\to \mathcal{K}^r(\mathbb{R}^n,X),\\ u &\mapsto \partial_j u\end{aligned} \tag{1.28}$$

where $D(\mathcal{D}_j) := \{u \in \mathcal{K}^r(\mathbb{R}^n,X)\colon \partial_j u \in \mathcal{K}^r(\mathbb{R}^n,X)\}$.

We are interested in a bounded joint H^∞-calculus for the tuple $\boldsymbol{\mathcal{D}} := (\mathcal{D}_1,\dots,\mathcal{D}_n)$. Note that for better readability, we do not indicate the ground space, in this case $\mathcal{K}^r(\mathbb{R}^n,X)$, in the notation $\boldsymbol{\mathcal{D}}$. Sometimes, we will also use the more precise notation $\boldsymbol{\mathcal{D}}^{\mathcal{N}}$ for the $\mathcal{N}$-realization of $\nabla_x = (\partial_1,\dots,\partial_n)$.

The following result is a slight generalization of [DV02b, Theorem 7.1, Remark 7.5].

Theorem 1.81 (Joint $\mathcal{R}$-bounded H^∞-calculus for $\boldsymbol{\mathcal{D}}$). *Let $\delta_j \in (0,\frac{\pi}{2})$ for $j = 1,\dots,n$ and $\Omega_x := \prod_{j=1}^n \Sigma_{\delta_j}$.*

(i) *The vector-valued partial derivative operator $\mathcal{D}_j$ is bisectorial on $\mathcal{K}^r(\mathbb{R}^n, X)$ with $\varphi^{(\mathrm{bi})}_{\mathcal{D}_j} = 0$ for all $j = 1, \dots, n$. The resolvents are given by*

$$(\lambda - \mathcal{D}_j)^{-1} = \mathrm{op}[R_{\lambda,j}]|_{\mathcal{K}^r(\mathbb{R}^n,X)}, \quad \lambda \in \mathbb{C} \setminus (i\mathbb{R}) \tag{1.29}$$

with $R_{\lambda,j}(z) := (\lambda - z_j)^{-1}$, $z \in i\mathbb{R}^n$. In particular, the tuple $\boldsymbol{\mathcal{D}}$ is admissible.

(ii) *The operator tuple $\boldsymbol{\mathcal{D}}$ has an $\mathcal{R}$-bounded joint $H^\infty(\Omega_x)$-calculus. Moreover, we obtain the representation*

$$f(\boldsymbol{\mathcal{D}}) = \mathrm{op}[f]|_{\mathcal{K}^r(\mathbb{R}^n,X)} \in L(\mathcal{K}^r(\mathbb{R}^n, X)) \tag{1.30}$$

for all $f \in H^\infty(\Omega_x)$.

Proof. (i) (I) Let $j \in \{1, \dots, n\}$ be arbitrary. The case $r = 0$ and $\mathcal{K} = H_p$ (i.e., the $L_p(\mathbb{R}^n, X)$-realization) can be found in detail in [DV02b, Theorem 7.1, Remark 7.5]. The asserted representation of the resolvent can be obtained easily by standard arguments. The resolvents commute due to the representation as Fourier multipliers (cf. [DV02b, Theorem 7.2]). Therefore, the tuple $\boldsymbol{\mathcal{D}}$ is admissible in the sense of Definition 1.16.

(II) Now let $\mathcal{K} = H_p$. We will make use of the operator $\Phi := \mathrm{op}[\Lambda_r] \in L(H^r_p(\mathbb{R}^n, X), L_p(\mathbb{R}^n, X))$. In the following, we will write $\mathcal{D}_{j,r}$ for the $H^r_p(\mathbb{R}^n, X)$-realization of ∂_j. In order to apply Proposition 1.32 (i) we have to show $D(\mathcal{D}_{j,r}) = \Phi^{-1}(D(\mathcal{D}_{j,0}))$ and $\mathcal{D}_{j,r}u = \Phi^{-1}\mathcal{D}_{j,0}\Phi u$ for all $u \in H^r_p(\mathbb{R}^n, X)$.
For $u \in D(\mathcal{D}_{j,r}) \subseteq H^r_p(\mathbb{R}^n, X)$ we have $\Phi u \in L_p(\mathbb{R}^n, X)$ and $\partial_j \Phi u = \Phi \partial_j u \in L_p(\mathbb{R}^n, X)$ according to Lemma 1.58 (ii). This implies $u = \Phi^{-1}\Phi u \in \Phi^{-1}(D(\mathcal{D}_{j,0}))$.
Let $f \in D(\mathcal{D}_{j,0}) \subseteq L_p(\mathbb{R}^n, X)$. We have $\Phi^{-1} f \in H^r_p(\mathbb{R}^n, X)$ and therefore $\partial_j \Phi^{-1} f = \Phi^{-1}\partial_j f \in H^r_p(\mathbb{R}^n, X)$. This yields $\Phi^{-1} f \in D(\mathcal{D}_{j,r})$.
Hence we obtain $D(\mathcal{D}_{j,r}) = \Phi^{-1}(D(\mathcal{D}_{j,0}))$. It is obvious that $\mathcal{D}_{j,r}u = \Phi^{-1}\mathcal{D}_{j,0}\Phi u$ for all $u \in H^r_p(\mathbb{R}^n, X)$, and therefore Proposition 1.32 (i) and (I) yield the assertion for this case. The representation of the resolvent follows from Proposition 1.32 (i) in combination with Remark 1.77 (i).

(III) Now consider $\mathcal{K} = B_{pq}$, i.e., let $\mathcal{D}_j$ be the $B^r_{pq}(\mathbb{R}^n, X)$-realization of ∂_j. Theorem 1.56 (iii) yields that Besov spaces can be obtained as real interpolation spaces of Bessel potential spaces. The bisectoriality on Besov spaces $B^r_{pq}(\mathbb{R}^n, X)$ can then be derived by Theorem 1.45 (i) (respectively, Remark 1.46) and (II) as soon as

$$(D(\mathcal{D}_{j,-m}), D(\mathcal{D}_{j,m}))_{\theta,q} = D(\mathcal{D}_j)$$

has been shown, with $m := \min\{k \in \mathbb{N} \colon |r| < k\}$ and $\theta := (r+m)/(2m)$. In this case we also have

$$B^r_{pq}(\mathbb{R}^n, X) = (H^{-m}_p(\mathbb{R}^n, X), H^m_p(\mathbb{R}^n, X))_{\theta,q}.$$

According to $1 \in \rho(\mathcal{D}_{j,-m}) \cap \rho(\mathcal{D}_{j,m})$ and the representation of the resolvents in (II) we get

$$(1-\mathcal{D}_{j,-m})^{-1} \in L(H_p^{-m}(\mathbb{R}^n,X), D(\mathcal{D}_{j,-m})),$$
$$(1-\mathcal{D}_{j,-m})^{-1}|_{H_p^m(\mathbb{R}^n,X)} = (1-\mathcal{D}_{j,m})^{-1} \in L(H_p^m(\mathbb{R}^n,X), D(\mathcal{D}_{j,m})),$$

which yield

$$\begin{aligned} A &:= (1-\mathcal{D}_{j,-m})^{-1}|_{B_{pq}^r(\mathbb{R}^n,X)} \\ &\in L(B_{pq}^r(\mathbb{R}^n,X), (D(\mathcal{D}_{j,-m}), D(\mathcal{D}_{j,m}))_{\theta,q}). \end{aligned}$$

We have $\partial_j \in L(D(\mathcal{D}_{j,m}), H_p^m(\mathbb{R}^n,X)) \cap L(D(\mathcal{D}_{j,-m}), H_p^{-m}(\mathbb{R}^n,X))$ and therefore

$$\partial_j \in L((D(\mathcal{D}_{j,-m}), D(\mathcal{D}_{j,m}))_{\theta,q}, B_{pq}^r(\mathbb{R}^n,X)).$$

The definition of $D(\mathcal{D}_j)$ then implies $(D(\mathcal{D}_{j,-m}), D(\mathcal{D}_{j,m}))_{\theta,q} \subseteq D(\mathcal{D}_j)$. Furthermore, we derive

$$\begin{aligned} A(1-\mathcal{D}_j)u &= (1-\mathcal{D}_{j,-m})^{-1}(1-\mathcal{D}_j)u \\ &= (1-\mathcal{D}_{j,-m})^{-1}(1-\mathcal{D}_{j,-m})u \\ &= u \end{aligned}$$

for all $u \in D(\mathcal{D}_j) \subseteq D(D_{j,-m})$. So we get

$$D(\mathcal{D}_j) \subseteq R(A) \subseteq (D(\mathcal{D}_{j,-m}), D(\mathcal{D}_{j,m}))_{\theta,q}.$$

Altogether we have shown

$$D(\mathcal{D}_j) = (D(D_{j,-m}), D(D_{j,m}))_{\theta,q}.$$

Remark 1.46 (ii) then yields the asserted representation of the resolvents.

(ii) (I) If $r=0$ and $\mathcal{K} = H_p$ (i.e., in the case of the $L_p(\mathbb{R}^n,X)$-realization), the existence of a bounded H^∞-calculus is a direct consequence of [DV02b, Theorem 7.3, Remark 7.5]. The representation of $f(\boldsymbol{\mathcal{D}})$ ($f \in H_0^\infty(\Omega_x)$) by Fourier-multipliers can be found there, too. Due to the representation of $f(\boldsymbol{\mathcal{D}})$ as a Fourier multiplier we can prove the $\mathcal{R}$-boundedness of the family

$$\{f(\boldsymbol{\mathcal{D}})\colon f \in H_0^\infty(\Omega_x),\ \|f\|_\infty \le 1\} \subseteq L_p(\mathbb{R}^n,X)$$

by Theorem 1.74, (1.19), and Theorem 1.8.
Let $f \in H^\infty(\Omega_x)$. Then Theorem 1.27 implies $f(\boldsymbol{\mathcal{D}}) \in L(L_p(\mathbb{R}^n,X))$. Using Remark 1.21 (iii), Lemma 1.19 (i), and Proposition 1.75 we obtain

$$\begin{aligned} f(\boldsymbol{\mathcal{D}})u &= \lim_{j\to\infty} (\psi_{j,n}f)(\boldsymbol{\mathcal{D}})u = \lim_{j\to\infty} \mathrm{op}[\psi_{j,n}f]u \\ &= \lim_{j\to\infty} \mathrm{op}[\psi_{j,n}]\mathrm{op}[f]u = \lim_{j\to\infty} \psi_{j,n}(\boldsymbol{\mathcal{D}})\mathrm{op}[f]u = \mathrm{op}[f]u \end{aligned}$$

for all $u \in L_p(\mathbb{R}^n, X)$. So we have proved the claimed representation (1.30) for all $f \in H^\infty(\Omega_x)$.

(II) Proposition 1.32 (iii) with $\Phi := \Lambda_k$ and the same arguments as in part (i) show that $\boldsymbol{\mathcal{D}}$ admits an $\mathcal{R}$-bounded $H^\infty(\Omega_x)$-calculus on $H^r_p(\mathbb{R}^n, X)$ for all $r \in \mathbb{R}$.

(III) Repeating the arguments from part (i), Theorem 1.45 (iii), Remark 1.46, and (II) we can show that $\boldsymbol{\mathcal{D}}$ admits an $\mathcal{R}$-bounded $H^\infty(\Omega_x)$-calculus on $B^r_{pq}(\mathbb{R}^n, X)$ for all $r \in \mathbb{R}$, and

$$f(\boldsymbol{\mathcal{D}}) = [f(\mathcal{D}_{1,-m}, \dots, \mathcal{D}_{n,-m})]\,|_{B^r_{pq}(\mathbb{R}^n,X)} = \mathrm{op}[f]|_{B^r_{pq}(\mathbb{R}^n,X)}$$

for all $f \in H^\infty(\Omega_x)$. □

Similarly to the space derivatives ∂_j, $j = 1, \dots, n$, considered above, we will now analyze the realization of the time derivative ∂_t. For this, we fix $s \geq 0$, $p, q \in (1, \infty)$ and $\mathcal{F} \in \{B_{pq}, H_p\}$ where we assume $s > 0$ if $\mathcal{F} = B_{pq}$. Then the ${}_0\mathcal{F}^s(\mathbb{R}_+, X)$-realization of ∂_t is defined as the operator $\mathcal{D}_t$ given by

$$\begin{aligned} \mathcal{D}_t \colon {}_0\mathcal{F}^s(\mathbb{R}_+, X) \supseteq D(\mathcal{D}_t) &\to {}_0\mathcal{F}^s(\mathbb{R}_+, X), \\ u &\mapsto \partial_t u \end{aligned} \tag{1.31}$$

where $D(\mathcal{D}_t) := {}_0\mathcal{F}^{s+1}(\mathbb{R}_+, X)$. Again we also use the more precise notation $\mathcal{D}_t^{\mathcal{N}}$ for the $\mathcal{N}$-realization of ∂_t. Note that for $u \in D(\mathcal{D}_t)$ we have $\partial_t u \in {}_0\mathcal{F}^s(\mathbb{R}_+, X)$ by Proposition 1.78.

We will also consider the time-derivative operator on the whole space defined by

$$\widetilde{\mathcal{D}_t} \colon \mathcal{F}^s(\mathbb{R}, X) \supseteq D(\widetilde{\mathcal{D}_t}) \to \mathcal{F}^s(\mathbb{R}, X), \quad u \mapsto \partial_t u$$

where $D(\widetilde{\mathcal{D}_t}) := \mathcal{F}^{s+1}(\mathbb{R}, X)$.

In the following we are interested in the spectral properties of $\mathcal{D}_t$.

Lemma 1.82. *For all $f \in {}_0\mathcal{F}^s(\mathbb{R}_+, X)$ we have $e_0^+ \mathcal{D}_t f = \widetilde{\mathcal{D}_t} e_0^+ f$.*

Proof. Due to Remark 1.67 (i) we have

$$A := e_0^+ \mathcal{D}_t - \widetilde{\mathcal{D}_t} e_0^+ \in L({}_0\mathcal{F}^{s+1}(\mathbb{R}_+, X), \mathcal{F}^s(\mathbb{R}, X))$$

and $A\varphi = 0$ for all $\varphi \in \mathscr{D}(\mathbb{R}_+)$. The assertion then follows from the density of $\mathscr{D}(\mathbb{R}_+)$ in ${}_0\mathcal{F}^{s+1}(\mathbb{R}_+, X)$, cf. Remark 1.67 (iv). □

In the following theorem, we use the notation σ_r, σ_p, and σ_c for the residual spectrum, the point spectrum, and the continuous spectrum of an operator, respectively.

Theorem 1.83 (Spectrum of the time derivative). *For the ${}_0\mathcal{F}^s(\mathbb{R}_+, X)$-realization $\mathcal{D}_t$ of ∂_t we have*

$$\begin{aligned}\rho(\mathcal{D}_t) &= \{z \in \mathbb{C}\colon \operatorname{Re}(z) < 0\},\\ \sigma_r(\mathcal{D}_t) &= \{z \in \mathbb{C}\colon \operatorname{Re}(z) > 0\},\\ \sigma_p(\mathcal{D}_t) &= \emptyset,\\ \sigma_c(\mathcal{D}_t) &= i\mathbb{R}.\end{aligned}$$

In particular, $\mathcal{D}_t$ is a sectorial operator with $\varphi_{\mathcal{D}_t} = \frac{\pi}{2}$ and

$$(\lambda - \mathcal{D}_t)^{-1}u = r_0^+(\lambda - \widetilde{\mathcal{D}_t})^{-1}e_0^+u, \quad u \in {}_0\mathcal{F}^s(\mathbb{R}_+, X),$$

for all λ with $\operatorname{Re}(\lambda) < 0$.

Proof. Let $u \in D(\mathcal{D}_t)$ and $\lambda \in \mathbb{C}$ with $(\lambda - \mathcal{D}_t)u = 0$. According to Sobolev's embedding theorem (cf. Remark 1.67 (v)) we have ${}_0\mathcal{F}^{s+1}(\mathbb{R}_+, X) \hookrightarrow {}_0\mathcal{F}^1(\mathbb{R}_+, X) \hookrightarrow C_b([0,\infty), X)$. Due to the characterization in Remark 1.67 (iii) and (v) we have the classical vanishing trace $u(0) = 0$. In fact, we even have $u \in C_b^1([0,\infty), X)$ since $\partial_t u = \lambda u \in C_b([0,\infty), X)$. For arbitrary $x' \in X'$ (the dual space of X) we define the complex-valued function $\widetilde{u} := x' \circ u \in C_b^1([0,\infty))$. The function $\widetilde{u}$ solves the scalar ordinary differential equation

$$\lambda\widetilde{u} - \partial_t\widetilde{u} = 0, \quad \widetilde{u}(0) = 0.$$

With the classical theory of ordinary differential equation we trivially obtain $\widetilde{u} = 0$ and therefore also $u = 0$. Hence $(\lambda - \mathcal{D}_t)$ is one-to-one for all $\lambda \in \mathbb{C}$, which yields $\sigma_p(\mathcal{D}_t) = \emptyset$.

Let $\operatorname{Re}\lambda < 0$. Then we can use the results about the space-derivative given above. From (1.29) we obtain

$$(\lambda - \widetilde{\mathcal{D}_t})^{-1} = \operatorname{op}[R_\lambda]|_{\mathcal{F}^s(\mathbb{R}_+,X)}, \quad R_\lambda(z) := (\lambda - z)^{-1}, \quad z \in i\mathbb{R}.$$

The function R_λ has a holomorphic extension to a bisector, i.e., $R_\lambda \in H^\infty(\Sigma_\delta)$ for small $\delta > 0$. Therefore we can apply Proposition 1.78 for $\operatorname{op}[R_\lambda]$. It can be easily seen by Lemma 1.82 and Remark 1.67 (i) that

$$\begin{aligned}&(\lambda - \mathcal{D}_t)r_0^+(\lambda - \widetilde{\mathcal{D}_t})^{-1}e_0^+f\\ &\qquad = r_0^+e_0^+(\lambda - \mathcal{D}_t)r_0^+(\lambda - \widetilde{\mathcal{D}_t})^{-1}e_0^+f\\ &\qquad = r_0^+(\lambda - \widetilde{\mathcal{D}_t})e_0^+r_0^+(\lambda - \widetilde{\mathcal{D}_t})^{-1}e_0^+f\\ &\qquad = r_0^+(\lambda - \widetilde{\mathcal{D}_t})(\lambda - \widetilde{\mathcal{D}_t})^{-1}e_0^+f = f\end{aligned}$$

for all $f \in {}_0\mathcal{F}^s(\mathbb{R}_+, X)$. In the same way we obtain

$$r_0^+(\lambda - \widetilde{\mathcal{D}_t})^{-1}e_0^+(\lambda - \mathcal{D}_t)f = f, \quad f \in {}_0\mathcal{F}^{s+1}(\mathbb{R}_+, X)$$

by Lemma 1.82. Altogether we obtain $\{z \in \mathbb{C}\colon \operatorname{Re}(z) < 0\} \subseteq \rho(\mathcal{D}_t)$ and

$$(\lambda - \mathcal{D}_t)^{-1} = r_0^+ (\lambda - \widetilde{\mathcal{D}_t})^{-1} e_0^+$$

for all λ with $\operatorname{Re}\lambda < 0$. With this representation of the resolvents and Remark 1.67 (i) we derive, for every $\theta' \in (\pi/2, \pi)$,

$$\begin{aligned} &\sup_{\lambda \in \mathbb{C}\setminus \overline{S}_{\theta'}} \|\lambda(\lambda - \mathcal{D}_t)^{-1}\|_{L({}_0\mathcal{F}^s(\mathbb{R}_+,X))} \\ &\qquad \le C \sup_{\lambda \in \mathbb{C}\setminus \overline{S}_{\theta'}} \|\lambda(\lambda - \widetilde{\mathcal{D}_t})^{-1}\|_{L(\mathcal{F}^s(\mathbb{R},X))} < \infty \end{aligned} \tag{1.32}$$

from the bisectoriality of $\widetilde{\mathcal{D}_t}$.

For λ with $\operatorname{Re}(\lambda) > 0$ we have to determine the range of $\lambda - \mathcal{D}_t$. For this, we consider the adjoint operator $(\lambda - \mathcal{D}_t)'$. Due to the embedding ${}_0\mathcal{F}^s(\mathbb{R}_+, X) \hookrightarrow L_p(\mathbb{R}_+, X)$ and Hölder's inequality we can define a functional in $({}_0\mathcal{F}^s(\mathbb{R}_+, X))'$ by

$$\begin{aligned} F_\lambda\colon {}_0\mathcal{F}^s(\mathbb{R}_+, X) &\to \mathbb{C}, \\ f &\mapsto \int_0^\infty \langle f(t), e^{-\lambda t} x' \rangle_{X \times X'} dt \end{aligned}$$

for $x' \in X'$. We easily obtain $\int_0^\infty e^{-\lambda t}\varphi'(t)dt = \int_0^\infty \lambda e^{-\lambda t}\varphi(t)dt$ for all $\varphi \in \mathscr{D}(\mathbb{R}_+, X)$. The density of the test functions and ${}_0\mathcal{F}^{s+1}(\mathbb{R}_+, X) \hookrightarrow {}_0H^1_p(\mathbb{R}_+, X)$ show that this is even true for all $\varphi \in D(\mathcal{D}_t) = {}_0\mathcal{F}^{s+1}(\mathbb{R}_+, X)$. Thus, we can conclude

$$F_\lambda((\lambda - \mathcal{D}_t)f) = x' \left(\int_0^\infty \lambda e^{-\lambda t} f(t) dt - \int_0^\infty e^{-\lambda t} f'(t) dt \right) = 0$$

for all $f \in D(\mathcal{D}_t)$. Hence we obtain $F_\lambda \in D((\lambda - \mathcal{D}_t)')$ and

$$0 \neq F_\lambda \in \ker((\lambda - \mathcal{D}_t)') = R(\lambda - \mathcal{D}_t)^\perp. \tag{1.33}$$

Here

$$R(\lambda - \mathcal{D}_t)^\perp := \left\{ F \in \left({}_0\mathcal{F}^s(\mathbb{R}_+, X)\right)' : \mathcal{F}|_{R(\lambda - \mathcal{D}_t)} = 0 \right\}$$

denotes the annihilator of $R(\lambda - \mathcal{D}_t)$ and the equality in (1.33) follows from [Kat76, p. 168]. Consequently, $R(\lambda - \mathcal{D}_t)$ cannot be dense in ${}_0\mathcal{F}^s(\mathbb{R}_+, X)$ and we conclude $\{z \in \mathbb{C}\colon \operatorname{Re}(z) > 0\} \subseteq \sigma_r(\mathcal{D}_t)$. As the spectrum is closed, this implies $i\mathbb{R} \subseteq \sigma(\mathcal{D}_t)$.

Due to $\sigma_p(\mathcal{D}_t) = \emptyset$ the operator $\mu i - \mathcal{D}_t$ is injective for all $\mu \in \mathbb{R}$. According to (1.32) and $|\lambda| \le C_{\theta'}|\lambda + i\mu|$, $\lambda \in \mathbb{C} \setminus \overline{S}_{\theta'}$, the operator $\mathcal{D}_t - i\mu$ satisfies the resolvent estimates in Lemma 1.4. Therefore we obtain the density of $R(\mathcal{D}_t - i\mu)$ in ${}_0\mathcal{F}^s(\mathbb{R}_+, X)$ by reflexivity of X (cf. Remark 1.13 (i)). Hence we have $i\mathbb{R} \subseteq \sigma_c(\mathcal{D}_t)$.

Altogether we have $\{z \in \mathbb{C}\colon \operatorname{Re}(z) < 0\} \subseteq \rho(\mathcal{D}_t)$, $\sigma_p(\mathcal{D}_t) = \emptyset$, $\{z \in \mathbb{C}\colon \operatorname{Re}(z) > 0\} \subseteq \sigma_r(\mathcal{D}_t)$, and $i\mathbb{R} \subseteq \sigma_c(\mathcal{D}_t)$. Thus we have obtained the full characterization of the spectrum. □

Thanks to our previous results we easily obtain the existence of an $\mathcal{R}$-bounded H^∞-calculus of $\mathcal{D}_t$. This was already stated in [DSS08, Proposition 2.7] but our approach is slightly different.

Theorem 1.84. *The vector-valued time-derivative operator $\mathcal{D}_t$ has an $\mathcal{R}$-bounded H^∞-calculus in ${}_0\mathcal{F}^s(\mathbb{R}_+, X)$ with $\varphi_{\mathcal{D}_t}^{\mathcal{R},\infty} = \frac{\pi}{2}$. For all $f \in H^\infty(S_\theta)$ with $\theta \in (\frac{\pi}{2}, \pi)$ we can represent $f(\mathcal{D}_t)$ by*

$$f(\mathcal{D}_t) = \big(r_0^+ f(\widetilde{\mathcal{D}_t}) e_0^+\big)|_{{}_0\mathcal{F}^s(\mathbb{R}_+,X)} = \big(r_0^+ \operatorname{op}[f] e_0^+\big)|_{{}_0\mathcal{F}^s(\mathbb{R}_+,X)}. \tag{1.34}$$

Proof. Due to Theorem 1.83 the resolvent of $\mathcal{D}_t$ can be written as

$$(\lambda - \mathcal{D}_t)^{-1} u = r_0^+ (\lambda - \widetilde{\mathcal{D}_t})^{-1} e_0^+ u, \quad u \in {}_0\mathcal{F}^s(\mathbb{R}_+, X), \quad \operatorname{Re}\lambda < 0.$$

Using Remark 1.21 (iii) and Definition 1.18 we infer

$$f(\mathcal{D}_t) u = r_0^+ f(\widetilde{\mathcal{D}_t}) e_0^+ u, \quad u \in {}_0\mathcal{F}^s(\mathbb{R}_+, X)$$

for all $f \in H_0^\infty(S_\theta) \subseteq H^\infty(\Sigma_{\theta-\pi/2})$ and $\theta \in (\frac{\pi}{2}, \pi)$. So we derive the representation (1.34) for all $f \in H_0^\infty(S_\theta)$ by (1.30). As in part (I) of the proof of Theorem 1.81 (ii) we can also verify this representation for all $f \in H^\infty(S_\theta)$.

According to Theorem 1.81 (ii) we have the $\mathcal{R}$-boundedness of

$$\{f(\widetilde{\mathcal{D}_t})\colon f \in H_0^\infty(S_\theta),\ \|f\|_\infty \le 1\} \subseteq L(\mathcal{F}^s(\mathbb{R}, X)).$$

The boundedness of all involved operators and Remark 1.7 (iii) then yield the $\mathcal{R}$-boundedness of

$$\{f(\mathcal{D}_t)\colon f \in H_0^\infty(S_\theta),\ \|f\|_\infty \le 1\} \subseteq L({}_0\mathcal{F}^s(\mathbb{R}_+, X))$$

for all $\theta \in (\frac{\pi}{2}, \pi)$. Thus, we even get $\varphi_{\mathcal{D}_t}^{\mathcal{R},\infty} = \frac{\pi}{2}$. □

c) Joint space-time H^∞-calculus

Now we are going to show that the tuple (∂_t, ∇_x) admits a bounded joint H^∞-calculus. We will consider the operators in the space ${}_0\mathcal{F}^s(\mathbb{R}_+, \mathcal{K}^r(\mathbb{R}^n, X))$ where $\mathcal{F}, \mathcal{K} \in \{B_{pq}, H_p\}$ as before. As $\mathcal{K}^r(\mathbb{R}^n, X)$ is a Banach space of class $\mathcal{HT}$ with property (α), too, the results from Subsection b) can be applied directly to $\mathcal{D}_t$. For the space derivatives $\boldsymbol{\mathcal{D}} = (\mathcal{D}_1, \dots, \mathcal{D}_n)$, we know the existence of an $\mathcal{R}$-bounded joint H^∞-calculus in the space $\mathcal{K}^r(\mathbb{R}^n, X)$. We now have to consider the natural extension to ${}_0\mathcal{F}^s(\mathbb{R}_+, \mathcal{K}^r(\mathbb{R}^n, X))$, i.e., now the space derivatives act on functions of time and space. Note that we identify a function $u = u(t, x)$ with the vector-valued function $t \mapsto u(t, \cdot)$, and therefore the natural extension of $\mathcal{D}_j$ can be written as $t \mapsto (D_j \circ u)(t, \cdot)$.

In the following, we fix $p_0, p_1, q_0, q_1 \in (1, \infty)$, $\mathcal{F} \in \{B_{p_0 q_0}, H_{p_0}\}$, $\mathcal{K} \in \{B_{p_1 q_1}, H_{p_1}\}$, $s \ge 0$ ($s > 0$ if $\mathcal{F} = B_{p_0 q_0}$), and $r \in \mathbb{R}$.

We start with a definition of the natural extension of an operator.

Definition and Lemma 1.85 (Natural extension). *Let X and Y be Banach spaces of class $\mathcal{HT}$. For a densely defined closed linear operator $A\colon X \supseteq D(A) \to Y$ the* natural extension *of A to ${}_0\mathcal{F}^s(\mathbb{R}_+, X)$ given by*

$$A^+\colon {}_0\mathcal{F}^s(\mathbb{R}_+, X) \supseteq D(A^+) \to {}_0\mathcal{F}^s(\mathbb{R}_+, Y),$$
$$u \mapsto A \circ u$$

is a well-defined operator with dense domain $D(A^+) := {}_0\mathcal{F}^s(\mathbb{R}_+, D(A))$.

Proof. First, we consider the case $s \in \mathbb{N}_0$ and $\mathcal{F} = H_{p_0}$. We have $\partial_t^k(A \circ u) = A \circ (\partial_t^k u)$ for all $k \le s$ and all $u \in D(A^+) = {}_0H^s_{p_0}(\mathbb{R}_+, D(A))$. Thus, we obtain

$$\partial_t^k u \in {}_0H^{s-k}_{p_0}(\mathbb{R}_+, D(A)), \quad k \le s$$

for $u \in D(A^+)$. We have $\lim_{t\to 0} \partial_t^k u(t) = 0$ in $D(A)$ for all $k \le s-1$, which leads to

$$\lim_{t\to 0} \partial_t^k (A \circ u)(t) = \lim_{t\to 0} A(\partial_t^k u(t)) = 0, \quad k \le s-1.$$

So we have proved $A \circ u \in {}_0H^s_{p_0}(\mathbb{R}_+, Y)$ for $u \in {}_0H^s_{p_0}(\mathbb{R}_+, D(A))$ and

$$A^+ \in L({}_0H^s_{p_0}(\mathbb{R}_+, D(A)), {}_0H^s_{p_0}(\mathbb{R}_+, Y)) \tag{1.35}$$

with $\|A^+\|_{L({}_0H^s_{p_0}(\mathbb{R}_+, D(A)), {}_0H^s_{p_0}(\mathbb{R}_+, Y))} \le 1$. Note that $D(A)$ is equipped with the graph norm.

Next, we consider the case $s \ge 0$ and $\mathcal{F}$ as in Definition 1.71. We directly get

$$A^+ \in L({}_0\mathcal{F}^s(\mathbb{R}_+, D(A)), {}_0\mathcal{F}^s(\mathbb{R}_+, Y)) \tag{1.36}$$

by an interpolation argument, (1.35), and Theorem 1.56. The real and complex interpolation methods are exact (cf. Theorem 1.39) and therefore we also have

$$\|A^+\|_{L({}_0\mathcal{F}^s(\mathbb{R}_+, D(A)), {}_0\mathcal{F}^s(\mathbb{R}_+, Y))} \le 1. \tag{1.37}$$

Altogether we conclude that A^+ is a well-defined operator.

We have to prove the density of the domain. For this we need the concept of tensor products. For a Banach space Z we define

$$\mathscr{D}(\mathbb{R}_+) \otimes Z := \operatorname{span}\{\varphi \cdot z\colon \varphi \in \mathscr{D}(\mathbb{R}_+),\, z \in Z\} \subseteq \mathscr{D}(\mathbb{R}_+, Z)$$

endowed with the subspace topology of the locally convex space $\mathscr{D}(\mathbb{R}_+, Z)$. One can easily show

$$\mathscr{D}(\mathbb{R}_+) \otimes D(A) \overset{d}{\hookrightarrow} \mathscr{D}(\mathbb{R}_+) \otimes X, \tag{1.38}$$

and in [Ama03, Theorem 1.3.6 (i)] one can find the dense embedding

$$\mathscr{D}(\mathbb{R}_+) \otimes X \overset{d}{\hookrightarrow} \mathscr{D}(\mathbb{R}_+, X). \tag{1.39}$$

Altogether we obtain $\mathscr{D}(\mathbb{R}_+) \otimes D(A) \overset{d}{\hookrightarrow} {}_0\mathcal{F}^s(\mathbb{R}_+, X)$ due to (1.38), (1.39), and Remark 1.67 (iv). So we have proved

$$\mathscr{D}(\mathbb{R}_+) \otimes D(A) \overset{d}{\hookrightarrow} {}_0\mathcal{F}^s(\mathbb{R}_+, X),$$

which yields the density of ${}_0\mathcal{F}^s(\mathbb{R}_+, D(A))$ in ${}_0\mathcal{F}^s(\mathbb{R}_+, X)$ because of $\mathscr{D}(\mathbb{R}_+) \otimes D(A) \subseteq {}_0\mathcal{F}^s(\mathbb{R}_+, D(A))$. □

Lemma 1.86. (i) *In the same situation as in Definition and Lemma 1.85 we have*

$$A^+ \in L({}_0\mathcal{F}^s(\mathbb{R}_+, X), {}_0\mathcal{F}^s(\mathbb{R}_+, Y))$$

if $A \in L(X, Y)$. In particular, we have

$$A^+ \in L_{\mathrm{Isom}}({}_0\mathcal{F}^s(\mathbb{R}_+, X), {}_0\mathcal{F}^s(\mathbb{R}_+, Y))$$

if and only if $A \in L_{\mathrm{Isom}}(X, Y)$. In both cases we also have

$$\|A^+\|_{L({}_0\mathcal{F}^s(\mathbb{R}_+,X),{}_0\mathcal{F}^s(\mathbb{R}_+,Y))} \leq \|A\|_{L(X,Y)}.$$

(ii) *Let $\mathcal{T} \subseteq L(X, Y)$ be $\mathcal{R}$-bounded. Then the family of extended operators*

$$\mathcal{T}^+ := \{T^+ : T \in \mathcal{T}\} \subseteq L({}_0\mathcal{F}^s(\mathbb{R}_+, X), {}_0\mathcal{F}^s(\mathbb{R}_+, Y))$$

is also $\mathcal{R}$-bounded.

Proof. (i) The boundedness of A^+ follows from (1.37) and the equivalence of the norms $\|\cdot\|_{D(A)}$ and $\|\cdot\|_X$. The estimate can be obtained by the same arguments as in the proof of Definition and Lemma 1.85.

If A is bijective, it is obvious that $(A^{-1})^+$ is the inverse of A^+ and therefore A^+ is also bijective. Let A^+ be bijective and let $x \in X$ with $Ax = 0$. For $\varphi \in \mathscr{D}(\mathbb{R}_+) \setminus \{0\}$ we then derive $A^+(\varphi \cdot x) = 0$, which yields $\varphi \cdot x = 0$. Thus we get $x = 0$. Let $y \in Y$ and $\varphi \in \mathscr{D}(\mathbb{R}_+) \setminus \{0\}$. Then there exists $f \in {}_0\mathcal{F}^s(\mathbb{R}_+, X)$ with $A^+ f = \varphi \cdot y$. For $t_0 \in \mathbb{R}_+$ with $\varphi(t_0) \neq 0$ we then derive $A(f(t_0)/\varphi(t_0)) = y$. Hence A is also bijective.

(ii) First, we consider the case $s \in \mathbb{N}_0$ and $\mathcal{F} = H_{p_0}$. Let $m \in \mathbb{N}$, $(T_k)_{k=1,\dots,m} \subseteq$

$\mathcal{T}$, and $(f_k)_{k=1,\dots,m} \subseteq {}_0H^s_{p_0}(\mathbb{R}_+, X)$. Then we have

$$\begin{aligned}
&\Big\| \sum_{k=1}^m r_k T_k^+ f_k \Big\|_{L_{p_0}([0,1],{}_0H^s_{p_0}(\mathbb{R}_+,Y))} \\
&\qquad = \Big(\int_0^1 \sum_{j=1}^s \Big\| \sum_{k=1}^m r_k(\omega) T_k^+ \Big(\Big(\frac{d}{dt} \Big)^j f_k \Big) \Big\|^{p_0}_{L_{p_0}(\mathbb{R}_+,Y)} d\omega \Big)^{1/p_0} \\
&\qquad = \Big(\sum_{j=1}^s \int_{\mathbb{R}_+} \int_0^1 \Big\| \sum_{k=1}^m r_k(\omega) T_k \Big[\Big(\Big(\frac{d}{dt} \Big)^j f_k \Big)(t) \Big] \Big\|^{p_0}_Y d\omega dt \Big)^{1/p_0} \\
&\qquad \le \mathcal{R}_{p_0}(\mathcal{T}) \Big(\sum_{j=1}^s \int_{\mathbb{R}_+} \int_0^1 \Big\| \sum_{k=1}^m r_k(\omega) \Big(\Big(\frac{d}{dt} \Big)^j f_k \Big)(t) \Big\|^{p_0}_X d\omega dt \Big)^{1/p_0} \\
&\qquad = \Big\| \sum_{k=1}^m r_k f_k \Big\|_{L_{p_0}([0,1],{}_0H^s_{p_0}(\mathbb{R}_+,X))}
\end{aligned}$$

by Fubini's theorem and the $\mathcal{R}$-boundedness of $\mathcal{T}$. An interpolation argument based on Theorem 1.44 and Remark 1.67 (ii) then yields the assertion for the whole fractional Bessel potential and Besov scale. □

The following result shows that the natural extension is compatible with the H^∞-calculus.

Lemma 1.87. *Let X be a Banach space of class $\mathcal{HT}$ and let $\mathbf{T} = (T_1, \dots, T_N)$ be an admissible tuple on X with $T_k \colon X \supseteq D(T_k) \to X$, $k = 1, \dots, N$, and let $\Omega \subseteq \mathbb{C}^n$ be chosen in accordance to $\mathbf{T}$ as in the text after Definition 1.16. The following permanence properties for the natural extension $\mathbf{T}^+ := (T_1^+, \dots, T_N^+)$ on ${}_0\mathcal{F}^s(\mathbb{R}_+, X)$ hold.*

(i) *If T_k is sectorial, then T_k^+ is also sectorial with $\varphi_{T_k^+} = \varphi_{T_k}$ and*

$$(\lambda - T_k^+)^{-1} = \big[(\lambda - T_k)^{-1} \big]^+$$

for all $\lambda \in \rho(T_k)$. The analog result holds for bisectorial operators.

(ii) *The natural extension $\mathbf{T}^+$ is an admissible tuple on ${}_0\mathcal{F}^s(\mathbb{R}_+, X)$.*

(iii) *We have*

$$f(\mathbf{T}^+) = f(\mathbf{T})^+$$

for all $f \in H_0^\infty(\Omega)$.

(iv) *If $\mathbf{T}$ admits a bounded $H^\infty(\Omega)$-calculus, then $\mathbf{T}^+$ also admits a bounded $H^\infty(\Omega)$-calculus with*

$$f(\mathbf{T}^+) = f(\mathbf{T})^+$$

for all $f \in H^\infty(\Omega)$.

(v) *If* $\mathbf{T}$ *admits an* $\mathcal{R}$*-bounded* $H^\infty(\Omega)$*-calculus, then* $\mathbf{T}^+$ *also admits an* $\mathcal{R}$*-bounded* $H^\infty(\Omega)$*-calculus.*

Proof. (i) Let T_k be sectorial or bisectorial. Then T_k^+ is also a densely defined operator by Lemma and Definition 1.85. Lemma 1.86 directly yields $\rho(T_k) = \rho(T_k^+)$ and $(\lambda - T_k^+)^{-1} = \big[(\lambda - T_k)^{-1}\big]^+$ for $\lambda \in \rho(T_k)$. Now Lemma 1.86 yields the resolvent estimate for T_k^+ and thus $\varphi_{T_k^+} = \varphi_{T_k}$ (respectively, $\varphi^{(\mathrm{bi})}_{T_k^+} = \varphi^{(\mathrm{bi})}_{T_k}$).

The operator T_k is injective according to Remark 1.4 (ii) and therefore it is obvious that T_k^+ is also injective. Hence $R(T_k^+)$ is dense in ${}_0\mathcal{F}^s(\mathbb{R}_+, X)$ due to Remark 1.4 (ii) again.

(ii) All T_k^+ are sectorial or bisectorial so we only have to prove that the resolvents commute. Using (i) we obtain

$$\begin{aligned}(\lambda - T_k^+)^{-1}(\mu - T_j^+)^{-1} &= [(\lambda - T_k)^{-1}]^+[(\mu - T_j)^{-1}]^+ \\ &= [(\lambda - T_k)^{-1}(\mu - T_j)^{-1}]^+ = [(\mu - T_j)^{-1}(\lambda - T_k)^{-1}]^+ \\ &= (\mu - T_j^+)^{-1}(\lambda - T_k^+)^{-1}\end{aligned}$$

for all $\lambda \in \rho(T_k)$, $\mu \in \rho(T_j)$ and $j, k = 1, \dots, N$.

(iii) Let $f \in H_0^\infty(\Omega)$. By approximating the integrand by step functions, it is easy to see that the following equalities hold (see also [Kai12, Lemma 2.44] for details):

$$\begin{aligned}(f(\mathbf{T}^+)u)(t) &= (2\pi i)^{-N}\Big(\Big[\int_\Gamma f(z)\prod_{k=1}^N [(z_k - T_k)^{-1}]^+ dz\Big]u\Big)(t) \\ &= (2\pi i)^{-N}\Big(\int_\Gamma f(z)\prod_{k=1}^N [(z_k - T_k)^{-1}]^+ u\, dz\Big)(t) \\ &= (2\pi i)^{-N}\int_\Gamma f(z)\prod_{k=1}^N (z_k - T_k)^{-1}u(t)\, dz \\ &= (2\pi i)^{-N}\int_\Gamma f(z)\prod_{k=1}^N (z_k - T_k)^{-1} dz\, u(t) = (f(\mathbf{T})^+u)(t)\end{aligned}$$

for $u \in {}_0\mathcal{F}^s(\mathbb{R}_+, X)$ and almost all $t \in \mathbb{R}_+$. Hence, we obtain $f(\mathbf{T}^+)u = f(\mathbf{T})^+u$.

(iv) Lemma 1.86 and (iii) directly yield that $\mathbf{T}^+$ admits a bounded $H^\infty(\Omega)$-calculus. Remark 1.21 (iii) and Theorem 1.27 imply $f(\mathbf{T}^+) = f(\mathbf{T})^+$ for all $f \in H^\infty(\Omega)$.

(v) This easily follows from (iii) and Lemma 1.86 (ii). □

For the remainder of this subsection, let X be a Banach space of class $\mathcal{HT}$ with property (α).

Corollary 1.88. *Let $\mathcal{D}_t$ be the ${}_0\mathcal{F}^s(\mathbb{R}_+, \mathcal{K}^r(\mathbb{R}^n, X))$-realization of ∂_t and $\boldsymbol{\mathcal{D}}^+$ be the natural extension of the $\mathcal{K}^r(\mathbb{R}^n, X)$-realization $\boldsymbol{\mathcal{D}}$ to ${}_0\mathcal{F}^s(\mathbb{R}_+, \mathcal{K}^r(\mathbb{R}^n, X))$. Then we have the following results:*

(i) *The sectorial operator $\mathcal{D}_t$ admits an $\mathcal{R}$-bounded $H^\infty(S_\theta)$-calculus for all $\theta \in (\pi/2, \pi)$.*

(ii) *The operator tuple $\boldsymbol{\mathcal{D}}^+$ admits a bounded joint $H^\infty(\Omega_x)$-calculus where $\Omega_x := \prod_{j=1}^n \Sigma_{\delta_j}$ and $\delta_j \in (0, \pi/2)$, $j = 1, \dots, n$.*

(iii) *The tuple $(\mathcal{D}_t, \boldsymbol{\mathcal{D}}^+)$ is admissible.*

Proof. In parts these results have already been stated in Theorem 1.84 and Theorem 1.81 (ii). Using Lemma 1.87, the assertions in (i) and (ii) follow. The resolvents commute due to the representation as Fourier multipliers, thus the admissibility follows. □

For simplicity of notation, we will from now on again write $\boldsymbol{\mathcal{D}}$ instead of $\boldsymbol{\mathcal{D}}^+$, so all operators will be defined on ${}_0\mathcal{F}^s(\mathbb{R}_+, \mathcal{K}^r(\mathbb{R}^n, X))$. The resulting tuple will be denoted by

$$\boldsymbol{\mathcal{D}}_+ := (\mathcal{D}_t, \boldsymbol{\mathcal{D}}) = (\mathcal{D}_t, \mathcal{D}_1, \dots, \mathcal{D}_n).$$

This is the ${}_0\mathcal{F}^s(\mathbb{R}_+, \mathcal{K}^r(\mathbb{R}^n, X))$-realization of

$$\nabla_+ := (\partial_t, \nabla_x).$$

Again we will also use the more precise notation $\boldsymbol{\mathcal{D}}_+^{\mathcal{N}}$ for the $\mathcal{N}$-realization of ∇_+.

One of the main results of this chapter states that $\boldsymbol{\mathcal{D}}_+$ admits a bounded joint H^∞-calculus:

Theorem 1.89 (Joint time-space H^∞-calculus). *The tuple $\boldsymbol{\mathcal{D}}_+$ admits a bounded joint $H^\infty(\Omega)$-calculus on ${}_0\mathcal{F}^s(\mathbb{R}_+, \mathcal{K}^r(\mathbb{R}^n, X))$ where $\Omega := S_\theta \times \prod_{k=1}^n \Sigma_{\delta_k}$ with $\theta \in (\pi/2, \pi)$, $\delta_k \in (0, \pi/2)$, $k = 1, \dots, n$.*

Proof. Let $\Omega_t := S_\theta$, $\Omega_x := \prod_{j=1}^n \Sigma_{\delta_j}$, and let $h \in H_0^\infty(\Omega)$. Then we define

$$\begin{aligned} g\colon \Omega_x &\to L\big({}_0\mathcal{F}^s(\mathbb{R}_+, \mathcal{K}^r(\mathbb{R}^n, X))\big), \\ z &\mapsto \|h\|_\infty^{-1} \cdot h(\mathcal{D}_t, z), \end{aligned}$$

which is meaningful because of $h(\cdot, z) \in H_0^\infty(\Omega_t)$, $z \in \Omega_x$. Due to Lemma 1.29 we have $g \in H_0^\infty(\Omega_x, \mathcal{B}_{\boldsymbol{\mathcal{D}}_+})$ and

$$\|h\|_\infty^{-1} \cdot \|h(\cdot, z)\|_\infty \le 1, \quad z \in \Omega_x.$$

Hence, Corollary 1.88 (i) yields the $\mathcal{R}$-boundedness of $g(\Omega_x)$ and therefore $g \in H_{\mathcal{R}}^\infty(\Omega_x, \mathcal{B}_{\boldsymbol{\mathcal{D}}_+})$. Note that $\mathcal{R}_2(g(\Omega_x))$ can be estimated from above by a constant which is independent of h.

From Corollary 1.88 (ii) we get that $\boldsymbol{\mathcal{D}}$ has a bounded joint $H^\infty(\Omega'_x)$-calculus for $\Omega'_x := \prod_{j=1}^n \Sigma_{\delta'_j}$ with $\delta'_j \in (0, \delta_j)$, $j = 1, \dots, n$. Therefore, we can conclude with Theorem 1.28 that

$$\|g(\boldsymbol{\mathcal{D}})\|_{L({}_0\mathcal{F}^s(\mathbb{R}_+, \mathcal{K}^r(\mathbb{R}^n, X)))} \leq C_1 \cdot \mathcal{R}_2(g(\Omega_x)) \leq C_2 \tag{1.40}$$

with $C_1 = C_1(\boldsymbol{\mathcal{D}}) > 0$, $C_2 = C_2(\boldsymbol{\mathcal{D}}) > 0$. Due to Lemma 1.29 we obtain

$$g(\boldsymbol{\mathcal{D}}) \quad = \quad \|h\|_\infty^{-1} \cdot h(\boldsymbol{\mathcal{D}}_+).$$

So we get

$$\|h(\boldsymbol{\mathcal{D}}_+)\|_{L({}_0\mathcal{F}^s(\mathbb{R}_+, \mathcal{K}^r(\mathbb{R}^n, X)))} \leq C_2 \cdot \|h\|_\infty, \quad h \in H_0^\infty(\Omega)$$

by (1.40). □

Remark 1.90. Let $p_0, p_1 \in (1, \infty)$. Similarly to Proposition 1.35, for $f \in H_P(\Omega)$ we can define the maximal realization

$$f(\boldsymbol{\mathcal{D}}_+) \colon \bigcup_{s \in \mathbb{R}} D(f(\boldsymbol{\mathcal{D}}_+^{(s)})) \to \bigcup_{s \in \mathbb{R}} L_{p_0}(\mathbb{R}_+, H^s_{p_1}(\mathbb{R}^n, X))$$

where $f(\boldsymbol{\mathcal{D}}_+^{(s)})$ is defined by the joint H^∞-calculus of the $L_{p_0}(\mathbb{R}_+, H^s_{p_1}(\mathbb{R}^n, X))$-realization of ∇_+. In fact, for $u \in D(f(\boldsymbol{\mathcal{D}}_+^{(s)}))$ the assignment

$$f(\boldsymbol{\mathcal{D}}_+)u := f(\boldsymbol{\mathcal{D}}_+^{(s)})u$$

is well-defined since

$$f(\boldsymbol{\mathcal{D}}_+^{(s)})u = f(\boldsymbol{\mathcal{D}}_+^{(t)})u$$

holds for all $s \leq t$ and $u \in D(f(\boldsymbol{\mathcal{D}}_+^{(t)}))$.

Remark 1.91. It is easily seen that the analog of Theorem 1.89 also holds in the more general situation where we include a shift in the operator and exponential weights in the spaces. To include a shift in the operator (see Lemma 1.34), we replace $\mathcal{D}_t$ by $\sigma + \mathcal{D}_t$ with $\sigma \geq 0$. Finally, we can consider exponentially weighted spaces as in Remark 1.70. With the help of the isomorphism $\mathscr{M}_\varrho$ defined in (1.16), we can easily extend the results of Theorem 1.89 to the spaces ${}_0\mathcal{F}^s_\varrho(\mathbb{R}_+, \mathcal{K}^r(\mathbb{R}^n, X))$.

These straightforward generalizations yield that for every $\varrho \geq 0$ and $\sigma \geq 0$, the ${}_0\mathcal{F}^s_\varrho(\mathbb{R}_+, \mathcal{K}^r(\mathbb{R}^n, X))$-realization $\boldsymbol{\mathcal{D}}^{(\varrho)}_{+,\sigma}$ of $(\sigma + \partial_t, \nabla_x)$ admits a bounded joint H^∞-calculus in ${}_0\mathcal{F}^s_\varrho(\mathbb{R}_+, \mathcal{K}^r(\mathbb{R}^n, X))$. For $f \in H_P(\Omega)$, we also have

$$f(\boldsymbol{\mathcal{D}}^{(\varrho)}_{+,\sigma}) = f_\sigma(\boldsymbol{\mathcal{D}}^{(\varrho)}_{+,0})$$

with f_σ being defined in Lemma 1.34. Similarly, the equality

$$f(\boldsymbol{\mathcal{D}}^{(\varrho)}_{+,0})u = \mathscr{M}_\varrho^{-1} f(\boldsymbol{\mathcal{D}}^{(0)}_{+,\varrho}) \mathscr{M}_\varrho u$$

holds for all $u \in {}_0\mathcal{F}^s(\mathbb{R}_+, \mathcal{K}^r(\mathbb{R}^n, X))$. For details, we refer to [Kai12].

Having established the joint H^∞-calculus for $\boldsymbol{\mathcal{D}}_+ = (\mathcal{D}_t, \mathcal{D}_1, \dots, \mathcal{D}_n)$, we can now replace $f(\lambda, z)$ by $f(\mathcal{D}_t, \mathcal{D}_1, \dots, \mathcal{D}_n)$ for suitable functions f to obtain well-defined operators in all scales of Sobolev spaces considered above. In particular, the symbols $\Lambda_r(z) = \big(1 - \sum_{k=1}^n z_k^2\big)^{1/2}$ and $\Psi_s(\lambda) = (1+\lambda)^s$ belong to $H_P(\Omega)$. The following proposition states that the resulting operators are even isomorphisms in the corresponding spaces. This fact which is based on the representation as Fourier multipliers in Theorem 1.81 and Theorem 1.84 will be a basis for the Newton polygon approach in Chapter 2.

Proposition 1.92. *For arbitrary $r' \in \mathbb{R}$, $s' > 0$, and $s, r \geq 0$, the symbols*

$$\Lambda_r(z) := \Big(1 - \sum_{k=1}^n z_k^2\Big)^{r/2}, \quad z \in \Sigma_\delta^n, \quad \delta > 0,$$
$$\Psi_s(\lambda) := (1+\lambda)^s, \quad \lambda \in S_\theta, \quad \theta > \pi/2$$

give rise to isomorphisms

$$\begin{aligned}\Lambda_r(\boldsymbol{\mathcal{D}}_+) &= \mathrm{op}[\Lambda_r]\big|_{{}_0\mathcal{F}^{s'}(\mathbb{R}_+, \mathcal{K}^{r'+r}(\mathbb{R}^n, X))} \\ &\in L_{\mathrm{Isom}}\big({}_0\mathcal{F}^{s'}(\mathbb{R}_+, \mathcal{K}^{r'+r}(\mathbb{R}^n, X)), {}_0\mathcal{F}^{s'}(\mathbb{R}_+, \mathcal{K}^{r'}(\mathbb{R}^n, X))\big), \\ \Psi_s(\boldsymbol{\mathcal{D}}_+) &= \mathrm{op}_+[\Psi_s]\big|_{{}_0\mathcal{F}^{s+s'}(\mathbb{R}_+, \mathcal{K}^{r'}(\mathbb{R}^n, X))} \\ &\in L_{\mathrm{Isom}}\big({}_0\mathcal{F}^{s+s'}(\mathbb{R}_+, \mathcal{K}^{r'}(\mathbb{R}^n, X)), {}_0\mathcal{F}^{s'}(\mathbb{R}_+, \mathcal{K}^{r'}(\mathbb{R}^n, X))\big)\end{aligned}$$

where the domains are given by

$$D(\Lambda_r(\boldsymbol{\mathcal{D}}_+)) = {}_0\mathcal{F}^{s'}(\mathbb{R}_+, \mathcal{K}^{r'+r}(\mathbb{R}^n, X)),$$
$$D(\Psi_s(\boldsymbol{\mathcal{D}}_+)) = {}_0\mathcal{F}^{s+s'}(\mathbb{R}_+, \mathcal{K}^{r'}(\mathbb{R}^n, X)).$$

We also have

$$(\Lambda_r(\boldsymbol{\mathcal{D}}_+))^{-1} = (\Lambda_r^{-1})(\boldsymbol{\mathcal{D}}_+), \quad (\Psi_s(\boldsymbol{\mathcal{D}}))^{-1} = (\Psi_s^{-1})(\boldsymbol{\mathcal{D}}).$$

Proof. (i) We will use the abbreviation ${}_0\mathcal{F}^{s'}(\mathcal{K}^{r'}) := {}_0\mathcal{F}^{s'}(\mathbb{R}_+, \mathcal{K}^{r'}(\mathbb{R}^n, X))$ from Definition 1.71. It is obvious that $\Lambda_r^{-1} \in H^\infty(\Sigma_\delta^n)$ and $\Lambda_r \in H_P(\Sigma_\delta^n)$. From Lemma 1.29, Lemma 1.87 (iv), and (1.30) we see that $\Lambda_r^{-1}(\boldsymbol{\mathcal{D}}_+)$ is the natural extension of $\Lambda_r^{-1}(\boldsymbol{\mathcal{D}})$ and therefore

$$\Lambda_r^{-1}(\boldsymbol{\mathcal{D}}_+) = \mathrm{op}[\Lambda_r^{-1}]\big|_{{}_0\mathcal{F}^{s'}(\mathcal{K}^{r'})} \in L\big({}_0\mathcal{F}^{s'}(\mathcal{K}^{r'})\big).$$

We even get

$$(\Lambda_r^{-1})(\boldsymbol{\mathcal{D}}_+) \in L_{\mathrm{Isom}}\big({}_0\mathcal{F}^{s'}(\mathcal{K}^{r'}), {}_0\mathcal{F}^{s'}(\mathcal{K}^{r+r'})\big)$$

by Lemma 1.58 and Lemma 1.86. Theorem 1.26 implies that $\Lambda_r(\boldsymbol{\mathcal{D}}_+) \in L({}_0\mathcal{F}^{s'}(\mathcal{K}^{r'}))$ is invertible with $[\Lambda_r(\boldsymbol{\mathcal{D}}_+)]^{-1} = (\Lambda_r^{-1})(\boldsymbol{\mathcal{D}}_+)$ and therefore

$$D(\Lambda_r(\boldsymbol{\mathcal{D}}_+)) = R(\Lambda_r^{-1}(\boldsymbol{\mathcal{D}}_+)) = {}_0\mathcal{F}^{s'}(\mathcal{K}^{r+r'})$$

and $\Lambda_r(\boldsymbol{\mathcal{D}}_+) = \mathrm{op}[\Lambda_r]\big|_{{}_0\mathcal{F}^{s'}(\mathcal{K}^{r+r'})} \in L_{\mathrm{Isom}}\big({}_0\mathcal{F}^{s'}(\mathcal{K}^{r+r'}), {}_0\mathcal{F}^{s'}(\mathcal{K}^{r'})\big)$.

(ii) We have $\Psi_s \in H_P(S_\theta)$ and $\Psi_s^{-1} \in H^\infty(S_\theta)$. In the same way as in part (i) we get

$$\Psi_s^{-1}(\boldsymbol{\mathcal{D}}_+) = r_0^+ \mathrm{op}[\Psi_s^{-1}] e_0^+ \big|_{{}_0\mathcal{F}^{s'}(\mathcal{K}^{r'})} = \mathrm{op}_+[\Psi_s^{-1}] = (\mathrm{op}_+[\Psi_s])^{-1} \in L\big({}_0\mathcal{F}^{s'}(\mathcal{K}^{r'})\big)$$

by Lemma 1.29 and Theorem 1.84. From Proposition 1.79 we get

$$\Psi_s^{-1}(\boldsymbol{\mathcal{D}}_+) \in L_{\mathrm{Isom}}\big({}_0\mathcal{F}^{s'}(\mathcal{K}^{r'}), {}_0\mathcal{F}^{s+s'}(\mathcal{K}^{r'})\big).$$

We conclude $[\Psi_s(\boldsymbol{\mathcal{D}}_+)]^{-1} = \Psi_s^{-1}(\boldsymbol{\mathcal{D}}_+)$ and

$$D(\Psi_s(\boldsymbol{\mathcal{D}}_+)) = R(\Psi_s^{-1}(\boldsymbol{\mathcal{D}}_+)) = {}_0\mathcal{F}^{s+s'}(\mathcal{K}^{r'})$$

as in part (i). □

Example 1.93 (Laplace operator). Let $r \in \mathbb{R}$ and $s > 0$. Consider the symbol $\lambda(z) := \sum_{j=1}^n z_j^2$ for $z \in \Sigma_\delta^n$. Then $\lambda \in H_P(\Sigma_\delta^n)$, and the ${}_0\mathcal{F}^s(\mathbb{R}_+, \mathcal{K}^r(\mathbb{R}^n, X))$-realization of $\lambda(\boldsymbol{\mathcal{D}}_+)$ coincides with the realization of the Laplacian Δ in this space which is given as

$$\begin{aligned} \Delta \colon {}_0\mathcal{F}^s(\mathbb{R}_+, \mathcal{K}^r(\mathbb{R}^n, X)) \supseteq D(\Delta) &\to {}_0\mathcal{F}^s(\mathbb{R}_+, \mathcal{K}^r(\mathbb{R}^n, X)), \\ u &\mapsto \Delta_x u \end{aligned}$$

where $D(\Delta) := {}_0\mathcal{F}^s(\mathbb{R}_+, \mathcal{K}^{r+2}(\mathbb{R}^n, X))$.

Proof. Note that we have $\lambda(z) = 1 - \Lambda_2(z)$ for $z \in \Omega_x$. Theorem 1.26 (iii) directly yields

$$\lambda(\boldsymbol{\mathcal{D}}_+) = 1 - \Lambda_2(\boldsymbol{\mathcal{D}}_+)$$

and therefore $D(\lambda(\boldsymbol{\mathcal{D}}_+)) = D(\Lambda_2(\boldsymbol{\mathcal{D}}_+)) = \mathcal{F}^s(\mathcal{K}^{r+2})$ due to Proposition 1.92. In particular, we see that

$$\begin{aligned} -\lambda(\boldsymbol{\mathcal{D}}_+)u &= \Lambda_2(\boldsymbol{\mathcal{D}}_+)u - u = \mathrm{op}[\Lambda_2] \circ u - u = (1 - \Delta) \circ u - u \\ &= -\Delta \circ u = -\Delta_x u \end{aligned}$$

for all $u \in \mathcal{F}^s(\mathcal{K}^{r+2})$. □

Example 1.94 (Hilbert transform). The function

$$f \colon \Sigma_\delta \to \mathbb{C}, \quad z \mapsto \frac{z}{\sqrt{-z^2}}$$

is an element of $H^\infty(\Sigma_\delta)$. Note that we have $f(z) = i \cdot \mathrm{sign}\,(\mathrm{Im}\, z)$ for $z \in \Sigma_\delta$ due to the definition of the square root $\sqrt{\zeta} = \sqrt{|\zeta|}\exp(i(\arg\zeta)/2)$ with $\arg\zeta \in (-\pi, \pi]$. For the $L_p(\mathbb{R}, X)$-realization of $f(\boldsymbol{\mathcal{D}})$ we obtain

$$\begin{aligned} f(\boldsymbol{\mathcal{D}}) &= \mathrm{op}[f]|_{L_p(\mathbb{R},X)} = \mathscr{F}^{-1} m \mathscr{F} \in L(L_p(\mathbb{R}, X)), \\ m(\xi) &:= i\xi/|\xi| = i \cdot \mathrm{sign}\,\xi, \quad \xi \in \mathbb{R} \setminus \{0\} \end{aligned}$$

due to the representation (1.30). It is shown in [Ama95, Section 4.3] that $\mathscr{F}^{-1} m \mathscr{F}$ coincides with the Hilbert transform which was defined in Definition 1.11. This implies that $f(\boldsymbol{\mathcal{D}})$ is the Hilbert transform on $L_p(\mathbb{R}, X)$. Note that we generally assumed that X is of class $\mathcal{HT}$.

Remark 1.95. Analog representations hold for Riesz transforms and the Helmholtz projection in the whole space. For the definition and further results on Riesz transforms we refer to Definition 4.36 and [Ste70, Ch. III].

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