Chapter 11
Target Identification and Tracking

In this chapter we first recall the notion of contracted GPTs. Then we show that the CGPTs have some nice properties, such as simple rotation and translation formulas, simple relation with shape symmetry, etc. More importantly, we derive new invariants for the CGPTs. Based on those invariants, we develop a dictionary matching algorithm. We suppose that the unknown shape of the target is an exact copy of some element from the dictionary, up to a rigid transform and dilatation. Using the invariants, we identify the target in the dictionary with a low computational cost. We also apply the Extended Kalman Filter to track both the location and the orientation of a mobile target from MSR data.

11.1 Complex CGPTs Under Rigid Motions and Scaling

As we will see later, a complex combination of CGPTs is most convenient when we consider the transforms of CGPTs under dilatation and rigid motions, i.e., shift and rotation. Therefore, for a double index mn, with $m, n = 1, 2, \ldots$, we make use of the complex combination of CGPTs given by (7.10), where the CGPTs, $M_{mn}^{cc}, M_{mn}^{ss}, M_{mn}^{cs}$, and M_{mn}^{sc}, are defined by (7.6)–(7.9).

Then, from (4.2), we observe that

$$\mathbb{N}_{mn}^{(1)}(\lambda, D) = \int_{\partial D} P_n(y)(\lambda I - \mathcal{K}_D^*)^{-1}[\langle \nu, \nabla P_m \rangle](y)\, d\sigma(y) \ ,$$

$$\mathbb{N}_{mn}^{(2)}(\lambda, D) = \int_{\partial D} P_n(y)(\lambda I - \mathcal{K}_D^*)^{-1}[\langle \nu, \nabla \overline{P_m} \rangle](y)\, d\sigma(y) \ ,$$

H. Ammari et al., *Mathematical and Statistical Methods for Multistatic Imaging*, Lecture Notes in Mathematics 2098, DOI 10.1007/978-3-319-02585-8_11,

where P_n and P_m are defined by (7.4). In order to simplify the notation, we drop λ in the following and write simply $\mathbb{N}^{(1)}_{mn}(D), \mathbb{N}^{(2)}_{mn}(D)$.

We consider the translation, the rotation and the dilatation of the domain D by introducing the following notation:

(i) Shift: $T_z D = \{x + z, x \in D\}$, for $z \in \mathbb{R}^2$;
(ii) Rotation: $R_\theta D = \{e^{i\theta} x, x \in D\}$, for $\theta \in [0, 2\pi)$;
(iii) Scaling: $sD = \{sx, x \in D\}$, for $s > 0$.

The following properties for the complex CGPTs hold. They are much simpler than those associated with the GPTs, which are derived in Sect. 4.2.

Proposition 11.1. *For all integers m, n, and geometric parameters θ, s, and z, the following holds:*

$$\mathbb{N}^{(1)}_{mn}(R_\theta D) = e^{i(m+n)\theta} \mathbb{N}^{(1)}_{mn}(D), \quad \mathbb{N}^{(2)}_{mn}(R_\theta D) = e^{i(n-m)\theta} \mathbb{N}^{(2)}_{mn}(D) , \tag{11.1}$$

$$\mathbb{N}^{(1)}_{mn}(sD) = s^{m+n} \mathbb{N}^{(1)}_{mn}(D), \quad \mathbb{N}^{(2)}_{mn}(sD) = s^{m+n} \mathbb{N}^{(2)}_{mn}(D) , \tag{11.2}$$

$$\mathbb{N}^{(1)}_{mn}(T_z D) = \sum_{l=1}^{m} \sum_{k=1}^{n} C^z_{ml} \mathbb{N}^{(1)}_{lk}(D) C^z_{nk}, \quad \mathbb{N}^{(2)}_{mn}(T_z D) = \sum_{l=1}^{m} \sum_{k=1}^{n} \overline{C^z_{ml}} \mathbb{N}^{(2)}_{lk}(D) C^z_{nk} , \tag{11.3}$$

where C^z is a lower triangle matrix with the m, n-th entry given by

$$C^z_{mn} = \binom{m}{n} z^{m-n} , \tag{11.4}$$

and $\overline{C^z}$ denotes its conjugate. Here, we identify $z = (z_1, z_2)$ with $z = z_1 + iz_2$.

An ingredient that we will need in the proof is the following chain rule between the gradient of a function and its push forward under transformation. In fact, for any diffeomorphism Ψ from $\mathbb{R}^2$ to $\mathbb{R}^2$ and any scalar-valued differentiable map f on $\mathbb{R}^2$, we have

$$\mathrm{d}(f \circ \Psi)\big|_x (h) = \left(\mathrm{d}f\big|_{\Psi(x)} \circ \mathrm{d}\Psi\big|_x \right)(h) , \tag{11.5}$$

for any tangent vector $h \in \mathbb{R}^2$, with $\mathrm{d}\Psi$ being the differential of Ψ.

Proof (of Proposition 11.1). We will follow proofs of similar relations that can be found in Chap. 4. Let us first show (11.1) for the rotated domain $D_\theta := R_\theta D$. For a function $\phi(y), y \in \partial D$, we define a function $\phi^\theta(y_\theta)$, $y_\theta := R_\theta y \in \partial D_\theta$ by

$$\phi^\theta(y_\theta) = \phi \circ R_{-\theta}(y_\theta) = \phi(y) \ .$$

It is proved in (4.10) that $\lambda I - \mathcal{K}_D^*$ is invariant under the rotation map, that is,

$$(\lambda I - \mathcal{K}_{D_\theta}^*)[\phi^\theta](y_\theta) = (\lambda I - \mathcal{K}_D^*)[\phi](y) \ . \tag{11.6}$$

We also check that $P_m(R_\theta y) = e^{im\theta} P_m(y)$.

We will focus on the relation for $\mathbb{N}_{mn}^{(1)}$, the other one can be proved in the same way. By definition, we have

$$\begin{aligned} \mathbb{N}_{mn}^{(1)}(D) &= \int_{\partial D} P_n(y) \phi_{D,m}(y) d\sigma(y) \ , \\ \mathbb{N}_{mn}^{(1)}(D_\theta) &= \int_{\partial D_\theta} P_n(y_\theta) \phi_{D_\theta,m}(y_\theta) d\sigma(y_\theta) \ , \end{aligned} \tag{11.7}$$

where

$$\begin{aligned} \phi_{D,m}(y) &= (\lambda I - \mathcal{K}_D^*)^{-1}[\langle \nu, \nabla P_m \rangle](y) \ , \\ \phi_{D_\theta,m}(y_\theta) &= (\lambda I - \mathcal{K}_{D_\theta}^*)^{-1}[\langle \nu, \nabla P_m \rangle](y_\theta) \ . \end{aligned}$$

Note that the last function differs from $\phi_{D,m}^\theta$. By the change of variables $y_\theta = R_\theta y$ in the first expression of (11.7), we obtain

$$\begin{aligned} \mathbb{N}_{mn}^{(1)}(D) &= \int_{\partial D_\theta} P_n(R_{-\theta} y_\theta) \phi_{D,m}(R_{-\theta} y_\theta) d\sigma(y_\theta) \\ &= e^{-in\theta} \int_{\partial D_\theta} P_n(y_\theta) \phi_{D,m}^\theta(y_\theta) d\sigma(y_\theta) \ . \end{aligned}$$

From (11.6), we have

$$\begin{aligned} (\lambda I - \mathcal{K}_{D_\theta}^*)[\phi_{D,m}^\theta](y_\theta) &= (\lambda I - \mathcal{K}_D^*)[\phi_{D,m}](y) \\ &= \langle \nu_y, \nabla P_m(y) \rangle \ . \end{aligned}$$

Moreover, $P_m(y) = e^{-im\theta} P_m(y_\theta)$ so that, by applying the chain rule (11.5) with $f = P_m$, $T = R_\theta$, $x = y$ and $h = \nu_y$, we can conclude that

$$\begin{aligned} \langle \nu_y, \nabla P_m(y) \rangle &= e^{-im\theta} \langle R_\theta \nu_y, \nabla P_m(y_\theta) \rangle \\ &= e^{-im\theta} \langle \nu_{y_\theta}, \nabla P_m(y_\theta) \rangle \ . \end{aligned}$$

Therefore, $\phi^\theta_{D,m} = e^{-im\theta}\phi_{D_\theta,m}$, and we conclude that

$$\mathbb{N}^{(1)}_{mn}(D_\theta) = e^{i(m+n)\theta}\mathbb{N}^{(1)}_{mn}(D) \ .$$

The second identity in (11.1) results from the same computation as above (the minus sign comes from the conjugate in the definition of $\mathbb{N}^{(2)}$), and the two equations in (11.2) are proved in the same way, replacing the transformed function ϕ^θ by

$$\phi^s(sy) = \phi(y) \ .$$

Thus, only (11.3) remains. Since the difference between these two comes from the conjugation, we will focus only on the first identity in (11.3). The strategy will be once again the following: for a function $\phi(y), y \in \partial D$, we define a function $\phi^z(y_z), y_z = y + z \in \partial D_z$, with $D_z := T_z D$, by

$$\phi^z(y_z) = \phi \circ T_{-z}(y_z) = \phi(y) \ ,$$

which also verifies an invariance relation similar to (11.6)

$$(\lambda I - \mathcal{K}^*_{D_z})[\phi^z](y_z) = (\lambda I - \mathcal{K}^*_D)[\phi](y) \ . \tag{11.8}$$

Moreover, for every integer $q \in \mathbb{N}$ one has the following

$$P_q(y_z) = (y+z)^q = \sum_{r=0}^{q} \binom{q}{r} y^r z^{q-r} \ . \tag{11.9}$$

Equations (11.7) become

$$\mathbb{N}^{(1)}_{mn}(D) = \int_{\partial D} P_n(y)\phi_{D,m}(y)d\sigma(y) \ ,$$
$$\mathbb{N}^{(1)}_{mn}(D_z) = \int_{\partial D_z} P_n(y_z)\phi_{D_z,m}(y_z)d\sigma(y_z) \ ,$$

where

$$\phi_{D,m}(y) = (\lambda I - \mathcal{K}^*_D)^{-1}[\langle \nu, \nabla P_m\rangle](y) \ ,$$
$$\phi_{D_z,m}(y_z) = (\lambda I - \mathcal{K}^*_{D_z})^{-1}[\langle \nu, \nabla P_m\rangle](y_z) \ .$$

Thus, combining (11.8) and (11.9) leads us to

$$
\begin{aligned}
(\lambda I - \mathcal{K}_{D_z}^*)[\phi_{D_z,m}](y_z) &= \langle \nu_{y_z}, \nabla P_m(y_z) \rangle \\
&= \langle \nu_y, \sum_{l=1}^{m} \binom{m}{l} z^{m-l} \nabla P_l(y) \rangle \\
&= \sum_{l=1}^{m} \binom{m}{l} z^{m-l} (\lambda I - \mathcal{K}_D^*)[\phi_{D,l}](y) \\
&= \sum_{l=1}^{m} \binom{m}{l} z^{m-l} (\lambda I - \mathcal{K}_{D_z}^*)[\phi_{D,l}^z](y_z) \,,
\end{aligned}
$$

so that we have

$$
\phi_{D_z,m}(y) = \sum_{l=1}^{m} \binom{m}{l} z^{m-l} \phi_{D,l}^z(y_z) \,.
$$

Hence, returning to the definition of $\mathbb{N}_{mn}^{(1)}(D_z)$ with the substitution $y_z \leftrightarrow y$, we obtain

$$
\begin{aligned}
\mathbb{N}_{mn}^{(1)}(D_z) &= \sum_{l=1}^{m} \binom{m}{l} z^{m-l} \int_{\partial D_z} P_n(y_z) \phi_{D,l}^z(y_z) d\sigma(y_z) \,, \\
&= \sum_{l=1}^{m} \sum_{k=1}^{n} \binom{m}{l} \binom{n}{k} z^{m-l} z^{n-k} \mathbb{N}_{lk}^{(1)}(D) \,,
\end{aligned}
$$

which is the desired result. Note that the index k begins with $k = 1$ because $\int_{\partial D_z} \phi_{D,l}^z = 0$. This completes the proof.

11.1.1 Some Properties of the Complex CGPTs

We define the complex CGPT matrices by $\mathbb{N}^{(1)} := (\mathbb{N}_{mn}^{(1)})_{m,n}$ and $\mathbb{N}^{(2)} := (\mathbb{N}_{mn}^{(2)})_{m,n}$. We set $w = se^{i\theta}$ and introduce the diagonal matrix G^w with the m-th diagonal entry given by $w^m = s^m e^{im\theta}$. Proposition 11.1 implies immediately that

$$
\mathbb{N}^{(1)}(T_z s R_\theta D) = C^z G^w \mathbb{N}^{(1)}(D) G^w (C^z)^T \,, \tag{11.10}
$$

$$
\mathbb{N}^{(2)}(T_z s R_\theta D) = \overline{C^z G^w} \mathbb{N}^{(2)}(D) G^w (C^z)^T \,, \tag{11.11}
$$

where C^z is defined by (11.4). Relations (11.10) and (11.11) still hold for the truncated CGPTs of finite order, due to the triangular shape of the matrix C^z.

Using the symmetry of the CGPTs [31, Theorem 4.11] and the positivity of the GPTs as proved in [31], we easily establish the following result.

Proposition 11.2. *The complex CGPT matrix* $\mathbb{N}^{(1)}$ *is symmetric:* $(\mathbb{N}^{(1)})^T = \mathbb{N}^{(1)}$*, and* $\mathbb{N}^{(2)}$ *is Hermitian:* $(\overline{\mathbb{N}^{(2)}})^T = \mathbb{N}^{(2)}$*. Consequently, the diagonal elements of* $\mathbb{N}^{(2)}$ *are strictly positive if* $\lambda > 0$ *and strictly negative if* $\lambda < 0$*.*

Furthermore, the CGPTs of rotation invariant shapes have special structures:

Proposition 11.3. *Suppose that* D *is invariant under rotation of angle* $2\pi/p$ *for some integer* $p \geq 2$*, i.e.,* $R_{2\pi/p}D = D$*, then*

$$\mathbb{N}^{(1)}_{mn}(D) = 0 \textit{ if } p \textit{ does not divide } (m+n)\,, \tag{11.12}$$

$$\mathbb{N}^{(2)}_{mn}(D) = 0 \textit{ if } p \textit{ does not divide } (m-n)\,. \tag{11.13}$$

Proof. Suppose that p does not divide $(m+n)$, and define $r := 2\pi(n+m)/p \bmod 2\pi$. Then by the rotation symmetry of D and the symmetry property of the CGPTs, we have

$$\mathbb{N}^{(1)}_{mn}(D) = \mathbb{N}^{(1)}_{mn}(R_{2\pi/p}D) = e^{i(m+n)2\pi/p}\mathbb{N}^{(1)}_{mn}(D) = e^{ir}\mathbb{N}^{(1)}_{mn}(D)\,.$$

Since $r < 2\pi$ and $r \neq 0$, we conclude that $\mathbb{N}^{(1)}_{mn}(D) = 0$. The proof of (11.13) is similar.

11.2 Shape Identification by the CGPTs

We call a *dictionary* $\mathcal{D}$ a collection of standard shapes, which are centered at the origin and with characteristic sizes of order 1. Given the CGPTs of an unknown shape D, and assuming that D is obtained from a certain element $B \in \mathcal{D}$ by applying some unknown rotation θ, scaling s and translation z, i.e., $D = T_z s R_\theta B$, our objective is to recognize B from $\mathcal{D}$. For doing so, one may proceed by first reconstructing the shape D using its CGPTs through some optimization procedures as proposed in [37], and then match the reconstructed shape with $\mathcal{D}$. However, such a method may be time-consuming and the recognition efficiency depends on the shape reconstruction algorithm.

We propose in Sects. 11.2.1 and 11.2.2 two shape identification algorithms using the CGPTs. The first one matches the CGPTs of data with that of the dictionary element by estimating the transform parameters, while the second one is based on a transform invariant shape descriptor obtained from the CGPTs. The second approach is computationally more efficient. Both of them operate directly in the data domain which consists of CGPTs and avoid the need for reconstructing the shape D. The heart of our approach is some basic algebraic equations between the CGPTs of D and B that can be deduced easily from (11.10) and (11.11). Particularly, the first four equations read:

$$\mathbb{N}_{11}^{(1)}(D) = w^2 \mathbb{N}_{11}^{(1)}(B) \,, \tag{11.14}$$

$$\mathbb{N}_{12}^{(1)}(D) = 2\mathbb{N}_{11}^{(1)}(D)z + w^3 \mathbb{N}_{12}^{(1)}(B) \,, \tag{11.15}$$

$$\mathbb{N}_{11}^{(2)}(D) = s^2 \mathbb{N}_{11}^{(2)}(B) \,, \tag{11.16}$$

$$\mathbb{N}_{12}^{(2)}(D) = 2\mathbb{N}_{11}^{(2)}(D)z + s^2 w \mathbb{N}_{12}^{(2)}(B) \,, \tag{11.17}$$

where $w = se^{i\theta}$.

11.2.1 CGPTs Matching

Determination of Transform Parameters

Suppose that the complex CGPT matrices $\mathbb{N}^{(1)}(B), \mathbb{N}^{(2)}(B)$ of the true shape B are given. Then, from (11.16), we obtain that

$$s = \sqrt{\mathbb{N}_{11}^{(2)}(D) / \mathbb{N}_{11}^{(2)}(B)} \,. \tag{11.18}$$

Case 1: Rotational Symmetric Shape.

If the shape B has rotational symmetry, i.e., $R_{2\pi/p}B = B$ for some $p \geq 2$, then from Proposition 11.3 we have $\mathbb{N}_{12}^{(2)}(B) = 0$ and the translation parameter z is uniquely determined from (11.17) by

$$z = \frac{\mathbb{N}_{12}^{(2)}(D)}{2\mathbb{N}_{11}^{(2)}(D)} \,. \tag{11.19}$$

On the contrary, the rotation parameter θ (or $e^{i\theta}$) can only be determined up to a multiple of $2\pi/p$, from CGPTs of order $\lceil p/2 \rceil$ at least. Although explicit expressions of $e^{ip\theta}$ can be deduced from (11.14)–(11.17) (or higher-order equations if necessary), we propose to recover $e^{ip\theta}$ by solving the least-squares problem:

$$\min_{\theta} \left(\|\mathbb{N}^{(1)}(T_z s R_\theta B) - \mathbb{N}^{(1)}(D)\|_F^2 + \|\mathbb{N}^{(2)}(T_z s R_\theta B) - \mathbb{N}^{(2)}(D)\|_F^2 \right) \,. \tag{11.20}$$

Here, s and z are given by (11.18) and (11.19) respectively, and $\mathbb{N}^{(1)}(D)$ and $\mathbb{N}^{(2)}(D)$ are the truncated complex CGPTs matrices of dimension $\lceil p/2 \rceil \times \lceil p/2 \rceil$.

Case 2: Non Rotational Symmetric Shape.

Consider a non rotational symmetric shape B which satisfies the assumption:

$$\mathbb{N}_{11}^{(1)}(B) \neq 0 \quad \text{and} \quad \det \begin{pmatrix} \mathbb{N}_{11}^{(1)}(B) \; \mathbb{N}_{11}^{(2)}(B) \\ \mathbb{N}_{12}^{(1)}(B) \; \mathbb{N}_{12}^{(2)}(B) \end{pmatrix} \neq 0 \, . \tag{11.21}$$

From (11.15) and (11.17), it follows that we can uniquely determine the translation z and the rotation parameter $w = e^{i\theta}$ from CGPTs of orders one and two by solving the following linear system:

$$\begin{aligned} \mathbb{N}_{12}^{(1)}(D)/\mathbb{N}_{11}^{(1)}(D) &= 2z + w\mathbb{N}_{12}^{(1)}(B)/\mathbb{N}_{11}^{(1)}(B) \, , \\ \mathbb{N}_{12}^{(2)}(D)/\mathbb{N}_{11}^{(2)}(D) &= 2z + w\mathbb{N}_{12}^{(2)}(B)/\mathbb{N}_{11}^{(2)}(B) \, . \end{aligned} \tag{11.22}$$

Debiasing by Least-Squares Solutions

In practice (for both the rotational symmetric and non rotational symmetric cases), the values of the parameters z, s and θ provided by the analytical formulas and numerical procedures above may be inexact, due to the noise in the data and the ill-conditioned character of the linear system (11.22). Let z^*, s^*, θ^* be the true transform parameters, which can be considered as perturbations around the estimations z, s, θ obtained above:

$$z^* = z + \delta_z, \;\; s^* = s\delta_s, \;\text{ and } \theta^* = \theta + \delta_\theta \, , \tag{11.23}$$

for δ_z, δ_θ small and δ_s close to 1. To find these perturbations, we solve a nonlinear least-squares problem:

$$\min_{z', s', \theta'} \left(\|\mathbb{N}^{(1)}(T_{z'} s' R_{\theta'} B) - \mathbb{N}^{(1)}(D)\|_F^2 + \|\mathbb{N}^{(2)}(T_{z'} s' R_{\theta'} B) - \mathbb{N}^{(2)}(D)\|_F^2 \right) \, , \tag{11.24}$$

with (z, s, θ) as an initial guess. Here, the order of the CGPTs in (11.24) is taken to be 2 in the non rotational case and $\max(2, [p/2])$ in the rotational symmetric case. Thanks to the relations (11.10) and (11.11), one can calculate explicitly the derivatives of the objective function, therefore can solve (11.24) by means of standard gradient-based optimization methods.

Algorithm 11.1 Shape identification based on CGPT matching

Input: the first k-th order CGPTs $\mathbb{N}^{(1)}(D), \mathbb{N}^{(2)}(D)$ of an unknown shape D
for $B_n \in \mathcal{D}$ **do**
1. Estimate z, s, and θ using the procedures described in Sect. 11.2.1;
2. $\tilde{D} \leftarrow R_{-\theta} s^{-1} T_{-z} D$, and calculate $\mathbb{N}^{(1)}(\tilde{D})$ and $\mathbb{N}^{(2)}(\tilde{D})$;
3. $E^{(1)} \leftarrow \mathbb{N}^{(1)}(B_n) - \mathbb{N}^{(1)}(\tilde{D})$, and $E^{(2)} \leftarrow \mathbb{N}^{(2)}(B_n) - \mathbb{N}^{(2)}(\tilde{D})$;
4. $e_n \leftarrow (\|E^{(1)}\|_F^2 + \|E^{(2)}\|_F^2)^{1/2} / (\|\mathbb{N}^{(1)}(B_n)\|_F^2 + \|\mathbb{N}^{(2)}(B_n)\|_F^2)^{1/2}$;
5. $n \leftarrow n + 1$;

end for
Output: the true dictionary element $n^* \leftarrow \mathrm{argmin}_n e_n$.

First Algorithm for Shape Identification

For each dictionary element, we determine the transform parameters as above, then measure the similarity of the complex CGPT matrices using the Frobenius norm, and choose the most similar element as the identified shape. Intuitively, the true dictionary element will give the correct transform parameters and hence the most similar CGPTs. This procedure is described in Algorithm 11.1.

11.2.2 Transform Invariant Shape Descriptors

From (11.16) and (11.17) we deduce the following identity:

$$\frac{\mathbb{N}_{12}^{(2)}(D)}{2\mathbb{N}_{11}^{(2)}(D)} = z + se^{i\theta} \frac{\mathbb{N}_{12}^{(2)}(B)}{2\mathbb{N}_{11}^{(2)}(B)} , \tag{11.25}$$

which is well defined since $\mathbb{N}_{11}^{(2)} \neq 0$ thanks to the Proposition 11.2. Identity (11.25) shows a very simple relationship between $\frac{\mathbb{N}_{12}^{(2)}(B)}{2\mathbb{N}_{11}^{(2)}(B)}$ and $\frac{\mathbb{N}_{12}^{(2)}(D)}{2\mathbb{N}_{11}^{(2)}(D)}$ for $D = T_z s R_\theta B$.

Let $u = \frac{\mathbb{N}_{12}^{(2)}(D)}{2\mathbb{N}_{11}^{(2)}(D)}$. We first define the following quantities which are translation invariant:

$$\mathcal{J}^{(1)}(D) = \mathbb{N}^{(1)}(T_{-u}D) = C^{-u}\mathbb{N}^{(1)}(D)(C^{-u})^T , \tag{11.26}$$

$$\mathcal{J}^{(2)}(D) = \mathbb{N}^{(2)}(T_{-u}D) = \overline{C^{-u}}\mathbb{N}^{(2)}(D)(C^{-u})^T , \tag{11.27}$$

with the matrix C^{-u} being the same as in Proposition 11.1. From $\mathcal{J}^{(1)}(D) = (\mathcal{J}_{mm}^{(1)}(D))_{m,n}$ and $\mathcal{J}^{(2)}(D) = (\mathcal{J}_{mm}^{(2)}(D))_{m,n}$, we define, for any indices m, n, the scaling invariant quantities:

Algorithm 11.2 Shape identification based on transform invariant descriptors

Input: the first k-th order shape descriptors $\mathcal{I}^{(1)}(D), \mathcal{I}^{(2)}(D)$ of an unknown shape D
for $B_n \in \mathcal{D}$ **do**
1. $e_n \leftarrow \left(\|\mathcal{I}^{(1)}(B_n) - \mathcal{I}^{(1)}(D)\|_F^2 + \|\mathcal{I}^{(2)}(B_n) - \mathcal{I}^{(2)}(D)\|_F^2\right)^{1/2}$;
2. $n \leftarrow n+1$;

end for
Output: the true dictionary element $n^* \leftarrow \mathrm{argmin}_n e_n$.

$$\mathcal{S}^{(1)}_{mn}(D) = \frac{\mathcal{J}^{(1)}_{mn}(D)}{\left(\mathcal{J}^{(2)}_{mm}(D)\mathcal{J}^{(2)}_{nn}(D)\right)^{1/2}}, \quad \mathcal{S}^{(2)}_{mn}(D) = \frac{\mathcal{J}^{(2)}_{mn}(D)}{\left(\mathcal{J}^{(2)}_{mm}(D)\mathcal{J}^{(2)}_{nn}(D)\right)^{1/2}}. \tag{11.28}$$

Finally, we introduce the CGPT-based shape descriptors $\mathcal{I}^{(1)} = (\mathcal{I}^{(1)}_{mn})_{m,n}$ and $\mathcal{I}^{(2)} = (\mathcal{I}^{(2)}_{mn})_{m,n}$:

$$\mathcal{I}^{(1)}_{mn}(D) = |\mathcal{S}^{(1)}_{mn}(D)| \quad \text{and} \quad \mathcal{I}^{(2)}_{mn}(D) = |\mathcal{S}^{(2)}_{mn}(D)|, \tag{11.29}$$

where $|\cdot|$ denotes the modulus of a complex number. Constructed in this way, $\mathcal{I}^{(1)}$ and $\mathcal{I}^{(2)}$ are clearly invariant under translation, rotation, and scaling.

It is worth emphasizing the symmetry property, $\mathcal{I}^{(1)}_{mn} = \mathcal{I}^{(1)}_{nm}, \mathcal{I}^{(2)}_{mn} = \mathcal{I}^{(2)}_{nm}$, and the fact that $\mathcal{I}^{(2)}_{mm} = 1$ for any m.

Second Algorithm for Shape Identification

Thanks to the transform invariance of the new shape descriptors, there is no need now for calculating the transform parameters, and the similarity between a dictionary element and the unknown shape can be directly measured from $\mathcal{I}^{(1)}$ and $\mathcal{I}^{(2)}$. As in Algorithm 11.1, we use the Frobenius norm as the distance between two shape descriptors and compare with all the elements of the dictionary. We propose a simplified method for shape identification, as described in Algorithm 11.2.

11.3 Target Tracking

In this section we apply an Extended Kalman Filter to track both the location and the orientation of a mobile target from multistatic response measurements. As shown in Sect. 1.10.2, the Extended Kalman Filter (EKF) is a generalization of the Kalman Filter (KF) to nonlinear dynamical systems.

It is robust with respect to noise and computationally inexpensive, therefore is well suited for real-time applications such as tracking. One should have in mind that, in real applications, one would like to localize the target and reconstruct its orientation directly from the MSR data without reconstructing the GPTs.

11.3.1 Location and Orientation Tracking of a Mobile Target

We denote by $z_t = (x_t, y_t)^T \in \mathbb{R}^2$ the position and $\theta_t \in [0, 2\pi)$ the orientation of a target D_t at the instant t, such that the shape of the target D_t is given by:

$$D_t = z_t + R_{\theta_t} B \ , \tag{11.30}$$

where R_{θ_t} is the rotation by θ_t. We assume that the CGPTs of B have been reconstructed and the shape has been correctly identified from a dictionary, so that the CGPT matrix $\mathbb{M} := \mathbb{M}(B)$ of order $K \geq 2$ is available. We use the same notation as in the previous chapter. Then we have the MSR matrix:

$$A_t = L(\mathbb{M}_t) + E_t + W_t \ , \tag{11.31}$$

where $\mathbb{M}_t$ is the CGPT of D_t, E_t is the truncation error, and W_t the measurement noise of time t. In the case of circular configuration with coincident arrays of sources and receivers, the linear operator L takes the form:

$$L(\mathbb{M}_t) = CD\mathbb{M}_t DC^T \ . \tag{11.32}$$

The objective of *tracking* is to estimate the target location z_t and orientation θ_t from the MSR data stream A_t. Before developing a CGPT-based tracking algorithm, we establish a simple relation between $\mathbb{M}_t$ and $\mathbb{M}$.

Time Relationship Between CGPTs

Let $u = (1, i)^T$. The complex CGPT $\mathbb{N}^{(1)}, \mathbb{N}^{(2)}$ are defined by

$$\mathbb{N}^{(1)}_{mn} = (M^{cc}_{mn} - M^{ss}_{mn}) + i(M^{cs}_{mn} + M^{sc}_{mn}) = u^T M_{mn} u \ ,$$
$$\mathbb{N}^{(2)}_{mn} = (M^{cc}_{mn} + M^{ss}_{mn}) + i(M^{cs}_{mn} - M^{sc}_{mn}) = u^* M_{mn} u \ .$$

Therefore, we have

$$\mathbb{N}^{(1)} = U^T \mathbb{M} U, \quad \text{and} \quad \mathbb{N}^{(2)} = U^* \mathbb{M} U \; , \tag{11.33}$$

where the matrix U of dimension $2K \times K$ is defined by

$$U = \begin{pmatrix} u & 0 & \dots & 0 \\ 0 & u & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & u \end{pmatrix} \; . \tag{11.34}$$

To recover the CGPT $\mathbb{M}_{mn}$ from the complex CGPTs $\mathbb{N}^{(1)}, \mathbb{N}^{(2)}$, we simply use the relations

$$\begin{aligned} M^{cc}_{mn} &= \frac{1}{2}\Re e(\mathbb{N}^{(1)}_{mn} + \mathbb{N}^{(2)}_{mn}), \; M^{cs}_{mn} = \frac{1}{2}\Im m(\mathbb{N}^{(1)}_{mn} + \mathbb{N}^{(2)}_{mn}) \; , \\ M^{sc}_{mn} &= \frac{1}{2}\Im m(\mathbb{N}^{(1)}_{mn} - \mathbb{N}^{(2)}_{mn}), \; M^{ss}_{mn} = \frac{1}{2}\Re e(\mathbb{N}^{(2)}_{mn} - \mathbb{N}^{(1)}_{mn}) \; . \end{aligned} \tag{11.35}$$

For two targets D_t and B satisfying (11.30), the following relationships between their complex CGPTs hold:

$$\mathbb{N}^{(1)}(D_t) = F_t^T \mathbb{N}^{(1)}(B) F_t \; , \tag{11.36a}$$

$$\mathbb{N}^{(2)}(D_t) = F_t^* \mathbb{N}^{(2)}(B) F_t \; , \tag{11.36b}$$

where F_t is a upper triangle matrix with the (m, n)-th entry given by

$$(F_t)_{mn} = \binom{n}{m} (x_t + iy_t)^{n-m} e^{im\theta_t} \; . \tag{11.37}$$

Linear Operator T_t

Now one can find explicitly a linear operator T_t which depends only on z_t, θ_t, such that $\mathbb{M}_t = T_t(\mathbb{M})$, and the equation (11.31) becomes:

$$A_t = L(T_t(\mathbb{M})) + E_t + W_t \; . \tag{11.38}$$

For doing so, we set $J_t := UF_t$, where U is given by (11.34). Then, a straightforward computation using (11.33), (11.35), and (11.36) shows that

$$\begin{aligned} M^{cc}(D_t) &= \Re e J_t^T \mathbb{M} \Re e J_t, \; M^{cs}(D_t) = \Re e J_t^T \mathbb{M} \Im m J_t \; , \\ M^{sc}(D_t) &= \Im m J_t^T \mathbb{M} \Re e J_t, \; M^{ss}(D_t) = \Im m J_t^T \mathbb{M} \Im m J_t \; , \end{aligned} \tag{11.39}$$

where $M^{cc}(D_t), M^{cs}(D_t), M^{sc}(D_t), M^{ss}(D_t)$ are the CGPTs. By interlacing all these four terms we get the operator T_t:

$$T_t(\mathbb{M}) = \Re eU(\Re eJ_t^T \mathbb{M} \Re eJ_t)\Re eU^T + \Re eU(\Re eJ_t^T \mathbb{M} \Im mJ_t)\Im mU^T + \Im mU(\Im mJ_t^T \mathbb{M} \Re eJ_t)\Re eU^T + \Im mU(\Im mJ_t^T \mathbb{M} \Im mJ_t)\Im mU^T = \mathbb{M}_t \,. \tag{11.40}$$

Tracking by CGPTs

A naive way to track the location z_t and the orientation θ_t is as follows. At each time t we first reconstruct $\mathbb{M}_t$ to get the complex CGPTs

$$\mathbb{N}_{1,1}^{(1)}(D_t), \mathbb{N}_{1,2}^{(1)}(D_t), \mathbb{N}_{1,1}^{(2)}(D_t), \mathbb{N}_{1,2}^{(2)}(D_t) \,.$$

Then we find the relative movement $\varDelta z_t = z_t - z_{t-1}$ and $\varDelta\theta_t = \theta_t - \theta_{t-1}$ by solving a linear system:

$$\begin{aligned} \mathbb{N}_{12}^{(1)}(D_t)/\mathbb{N}_{11}^{(1)}(D_t) &= 2(\Re\varDelta z_t + i\Im\varDelta z_t) + e^{i\varDelta\theta_t}\mathbb{N}_{12}^{(1)}(D_{t-1})/\mathbb{N}_{11}^{(1)}(D_{t-1}), \\ \mathbb{N}_{12}^{(2)}(D_t)/\mathbb{N}_{11}^{(2)}(D_t) &= 2(\Re\varDelta z_t + i\Im\varDelta z_t) + e^{i\varDelta\theta_t}\mathbb{N}_{12}^{(2)}(D_{t-1})/\mathbb{N}_{11}^{(2)}(D_{t-1}) \,. \end{aligned} \tag{11.41}$$

The estimated path is then $z_t = \sum_{s=1}^t \varDelta z_s + z_0$, and $\theta_t = \sum_{s=1}^t \varDelta\theta_s + \theta_0$. However, such an algorithm has no practical interest. In fact, the error in the estimated path (z_t, θ_t) will propagate over time, since the noise presented in data is not properly taken into account here. In the following we apply the Extended Kalman Filter to the system (11.38) which takes advantage of the operator T_t and handles correctly the noise.

11.3.2 Tracking by the Extended Kalman Filter

In the next we establish first the *system state* and the *observation* equations, then linearize the observation equation and apply the EKF algorithm.

System State Observation Equations

We assume that the position of the target is subject to an external driving force that has the form of a white noise. In other words the velocity $(V(\tau))_{\tau\in\mathbb{R}^+}$ of the target is given in terms of a two-dimensional Brownian motion $(W_a(\tau))_{\tau\in\mathbb{R}^+}$ and its position $(Z(\tau))_{\tau\in\mathbb{R}^+}$ is given in terms of the integral of this Brownian motion:

$$V(\tau) = V_0 + \sigma_a W_a(\tau), \qquad Z(\tau) = Z_0 + \int_0^\tau V(s)ds \,.$$

The orientation $(\Theta(\tau))_{\tau\in\mathbb{R}^+}$ of the target is subject to random fluctuations and its angular velocity is given in terms of an independent white noise, so that the orientation is given in terms of a one-dimensional Brownian motion $(W_\theta(\tau))_{\tau\in\mathbb{R}^+}$:

$$\Theta(\tau) = \Theta_0 + \sigma_\theta W_\theta(\tau) \,.$$

We observe the target at discrete times $t\Delta\tau$, $t \in \mathbb{N}$, with time step $\Delta\tau$. We denote $z_t = Z(t\Delta\tau)$, $v_t = V(t\Delta\tau)$, and $\theta_t = \Theta(t\Delta\tau)$. These functions obey the recursive relations

$$\begin{aligned} v_t &= v_{t-1} + a_t, & a_t &= \sigma_a\big(W_a(t\Delta\tau) - W_a((t-1)\Delta\tau)\big)\,,\\ z_t &= z_{t-1} + v_{t-1}\Delta\tau + b_t\,, & b_t &= \sigma_a \int_{(t-1)\Delta\tau}^{t\Delta\tau} W_a(s) - W_a((t-1)\Delta\tau)ds\,,\\ \theta_t &= \theta_{t-1} + c_t, & c_t &= \sigma_\theta\big(W_\theta(t\Delta\tau) - W_\theta((t-1)\Delta\tau)\big)\,. \end{aligned} \tag{11.42}$$

Since the increments of the Brownian motions are independent from each other, the vectors $(U_t)_{t\ge1}$ given by

$$U_t = \begin{pmatrix} a_t \\ b_t \\ c_t \end{pmatrix}$$

are independent and identically distributed with the multivariate normal distribution with mean zero and covariance matrix Σ given by

$$\Sigma = \Delta\tau \begin{pmatrix} \sigma_a^2 I_2 & \frac{\sigma_a^2}{2}\Delta\tau I_2 & 0 \\ \frac{\sigma_a^2}{2}\Delta\tau I_2 & \frac{\sigma_a^2}{3}\Delta\tau^2 I_2 & 0 \\ 0 & 0 & \sigma_\theta^2 \end{pmatrix} \,. \tag{11.43}$$

The evolution of the state vector

$$X_t = \begin{pmatrix} v_t \\ z_t \\ \theta_t \end{pmatrix}$$

takes the form

$$X_t = FX_{t-1} + U_t, \qquad F = \begin{pmatrix} I_2 & 0 & 0 \\ \Delta\tau I_2 & I_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{11.44}$$

The observation made at time t is the MSR matrix given by (11.38), where the system state X_t is implicitly included in the operator T_t. For the sake of simplicity, we suppose that the truncation error E_t is small compared to the measurement noise so that it can be dropped in (11.38), and that the Gaussian white noise W_t of different time are mutually independent. We emphasize that the velocity vector v_t of the target does not contribute to (11.38), which can be seen from (11.30). To highlight the dependence on z_t, θ_t, we introduce a function h which is nonlinear in z_t, θ_t, and takes $\mathbb{M}$ as a parameter, such that

$$h(X_t; \mathbb{M}) = h(z_t, \theta_t; \mathbb{M}) = L(T_t(\mathbb{M})) . \tag{11.45}$$

Then together with (11.44) we get the following *system state* and *observation* equations:

$$X_t = FX_{t-1} + U_t , \tag{11.46a}$$

$$V_t = h(X_t; \mathbb{M}) + W_t . \tag{11.46b}$$

Note that (11.46a) is linear, so in order to apply EKF on (11.46), we only need to linearize (11.46b), or in other words, to calculate the partial derivatives of h with respect to x_t, y_t, θ_t.

Linearization of the Observation Equation

Clearly, the operator L contains only the information concerning the acquisition system and does not depend on x_t, y_t, θ_t. So, by (11.45), we have

$$\partial_{x_t} h = L(\partial_{x_t} T_t(\mathbb{M})) . \tag{11.47}$$

Moreover, the calculation for $\partial_{x_t} T_t$ is straightforward using (11.40). We have

$$\begin{aligned}\partial_{x_t} T_t(\mathbb{M}) =& \Re e U \partial_{x_t}(\Re e J_t^T \mathbb{M} \Re e J_t) \Re e U^T + \Re e U \partial_{x_t}(\Re e J_t^T \mathbb{M} \Im m J_t) \Im m U^T + \\ & \Im m U \partial_{x_t}(\Im m J_t^T \mathbb{M} \Re e J_t) \Re e U^T + \Im m U \partial_{x_t}(\Im m J_t^T \mathbb{M} \Im m J_t) \Im m U^T ,\end{aligned} \tag{11.48}$$

where the derivatives are found by the product rule:

$$
\begin{aligned}
\partial_{x_t}(\Re e J_t^T \mathbb{M} \Re e J_t) &= \Re e(\partial_{x_t} J_t^T)\mathbb{M}\Re e J_t + \Re e J_t^T \mathbb{M} \Re e(\partial_{x_t} J_t) \,, \\
\partial_{x_t}(\Re e J_t^T \mathbb{M} \Im m J_t) &= \Re e(\partial_{x_t} J_t^T)\mathbb{M}\Im m J_t + \Re e J_t^T \mathbb{M} \Im m(\partial_{x_t} J_t) \,, \\
\partial_{x_t}(\Im m J_t^T \mathbb{M} \Re e J_t) &= \Im m(\partial_{x_t} J_t^T)\mathbb{M}\Re e J_t + \Im m J_t^T \mathbb{M} \Re e(\partial_{x_t} J_t) \,, \\
\partial_{x_t}(\Im m J_t^T \mathbb{M} \Im m J_t) &= \Im m(\partial_{x_t} J_t^T)\mathbb{M}\Im m J_t + \Im m J_t^T \mathbb{M} \Im m(\partial_{x_t} J_t) \,,
\end{aligned}
$$

and $\partial_{x_t} J_t = U \partial_{x_t} F_t$. The (m, n)-th entry of the matrix $\partial_{x_t} F_t$ is given by

$$
(\partial_{x_t} F_t)_{m,n} = \binom{n}{m} (n-m) z_t^{n-m-1} e^{im\theta_t} \,. \tag{11.49}
$$

The derivatives $\partial_{y_t} T_t(\mathbb{M})$ and $\partial_{\theta_t} T_t(\mathbb{M})$ are calculated in the same way.

Bibliography and Discussion

The results of this chapter on target identification are from [8]. They provide an efficient approach for real-time target identification using dictionary matching. They show that GPT-based representations are appropriate and natural tools for multistatic imaging. They can be generalized to electromagnetic wave propagation as well. As shown in this chapter, they can be used for tracking a mobile target from multistatic data. The results of this chapter on the location and orientation tracking are from [9]. An analysis of the ill-posed character of both the location and orientation tracking in the case of limited-view data was carried out in [9]. In [11], transformation formulas for the GPTs under rigid motions and scaling in three dimensions are given. Moreover, invariants under those transformations, which can be used as shape descriptors for dictionary matching in three dimensions, are constructed.

In [41] a shape identification and classification algorithm in echolocation is proposed. The approach is based on first extracting scattering coefficients from the reflected waves and then matching them with precomputed ones associated with a dictionary of targets. The construction of such frequency-dependent shape descriptors is based on the properties of the scattering coefficients described in Chap. 5 and some new invariants.

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http://www.springer.com/978-3-319-02584-1

Mathematical and Statistical Methods for Multistatic Imaging
Ammari, H.; Garnier, J.; Jing, W.; Kang, H.; Lim, M.; Solna, K.; Wang, H.
2013, XVII, 361 p. 61 illus., 47 illus. in color., Softcover
ISBN: 978-3-319-02584-1