Chapter 2
OPTGAME3: A Dynamic Game Solver and an Economic Example

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Abstract In this paper we present the OPTGAME3 algorithm, which can be used to calculate equilibrium and optimum control solutions of dynamic games. The algorithm was programmed in C# and MATLAB[1] and allows the calculation of approximate cooperative Pareto-optimal solutions and non-cooperative Nash and Stackelberg equilibrium solutions. In addition we present an application of the OPTGAME3 algorithm where we use a small stylized nonlinear two-country macroeconomic model of a monetary union for analysing the interactions between fiscal (governments) and monetary (common central bank) policy makers, assuming different objective functions of these decision makers. Several dynamic game experiments are run for different information patterns and solution concepts. We show how the policy makers react optimally to demand and supply shocks. Some comments are given about possible applications to the recent sovereign debt crisis in Europe.

Keywords Numerical methods for control and dynamic games • Economic dynamics • Monetary union

[1]The source code of the OPTGAME3 algorithm is available from the authors on request.

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2.1 Introduction

When we think about economic policy making in one single country, it is preferable to consider the government controlling fiscal policy and the central bank controlling monetary policy as independent players. When considering a country inside a monetary union where monetary policy is no longer an instrument of national institutions, it is essential to look at the government and the central bank of the monetary union separately. Moreover, the interests of other countries inside the union, which primarily pursue their own national interests and do not necessarily care about the spillovers of their actions to other countries should also be taken into account by the decision makers when determining the best policy actions. Such problems can best be modelled by using the concepts and methods of dynamic game theory, which has been developed mostly by engineers and mathematicians but which has proved to be a valuable analytical tool for economists, too (see, e.g., Başar and Olsder 1999; Dockner et al. 2000; Petit 1990).

The theory of dynamic games is well developed for linear-quadratic games. It is also well known that considering linear problems alone is a very strong limitation, thus a lot of research is required to extend the theory for nonlinear games. This paper follows this line of research and presents an algorithm which is designed for the solution of nonlinear-quadratic dynamic tracking games. The algorithm is called OPTGAME3 and is programmed in C# and MATLAB. Due to their nonlinearity, the problems cannot be solved analytically but only numerically. The algorithm allows the calculation of approximate cooperative Pareto-optimal solutions and non-cooperative Nash and Stackelberg equilibrium solutions.

In addition we present an application of the OPTGAME3 algorithm for a monetary union. Dynamic games have been used by several authors (Hager et al. 2001; Pohjola 1986) for modelling conflicts between monetary and fiscal policies. There is also a large body of literature on dynamic conflicts between policy makers from different countries on issues of international stabilization (Hughes Hallett 1986; Levine and Brociner 1994; Miller and Salmon 1985). Both types of conflict are present in a monetary union, because a supranational central bank interacts strategically with sovereign governments as national fiscal policy makers in the member states. Such conflicts can be analysed using either large empirical macroeconomic models (Engwerda et al. 2012; Haber et al. 2002; Plasmans et al. 2006) or small stylized models (van Aarle et al. 2002; Neck and Behrens 2004, 2009). We follow the latter line of research and use a small stylized nonlinear two-country macroeconomic model of a monetary union for analysing the interactions between fiscal (governments) and monetary (common central bank) policy makers, assuming different objective functions of these decision makers. We show how the policy makers react optimally to demand and supply shocks. Some comments are given about possible applications to the recent sovereign debt crisis in Europe.

2.2 The Dynamic Game Problem

We consider intertemporal nonlinear game-theoretic problems which are given in tracking form. The players aim at minimizing quadratic deviations of the equilibrium values from given target (desired) values. Thus each player minimizes an objective function J^i:

$$\min_{u_1^i,\dots,u_T^i} J^i = \min_{u_1^i,\dots,u_T^i} \sum_{t=1}^{T} L_t^i(x_t, u_t^1, \dots, u_t^N), \qquad i = 1, \dots, N, \tag{2.1}$$

with

$$L_t^i(x_t, u_t^1, \dots, u_t^N) = \frac{1}{2}[X_t - \tilde{X}_t^i]' \Omega_t^i [X_t - \tilde{X}_t^i], \qquad i = 1, \dots, N, \tag{2.2}$$

The parameter N denotes the number of players (decision makers). T is the terminal period of the finite planning horizon, i.e. the duration of the game. X_t is an aggregated vector

$$X_t := [x_t \;\; u_t^1 \;\; u_t^2 \;\; \dots \;\; u_t^N]', \tag{2.3}$$

which consists of an $(n_x \times 1)$ vector of state variables

$$x_t := [x_t^1 \;\; x_t^2 \;\; \dots \;\; x_t^{n_x}]', \tag{2.4}$$

and N $(n_i \times 1)$ vectors of control variables determined by the players $i = 1, \dots, N$:

$$\begin{aligned} u_t^1 &:= [u_t^{11} \;\; u_t^{12} \;\; \dots \;\; u_t^{1n_1}]', \\ u_t^2 &:= [u_t^{21} \;\; u_t^{22} \;\; \dots \;\; u_t^{2n_2}]', \\ &\vdots \\ u_t^N &:= [u_t^{N1} \;\; u_t^{N2} \;\; \dots \;\; u_t^{Nn_N}]'. \end{aligned} \tag{2.5}$$

Thus X_t (for all $t = 1, \dots, T$) is an r-dimensional vector, where

$$r := n_x + n_1 + n_2 + \dots + n_N. \tag{2.6}$$

The desired levels of the state variables and the control variables of each player enter the quadratic objective functions (as given by (2.1) and (2.2)) via the terms

$$\tilde{X}_t^i := [\tilde{x}_t^i \;\; \tilde{u}_t^{i1} \; \tilde{u}_t^{i2} \;\; \dots \;\; \tilde{u}_t^{iN}]'. \tag{2.7}$$

It must be pointed out that each player $i = 1, \ldots, N$ may be allowed to observe and monitor the control variables of the other players, i.e. deviations of other control variables can be punished in one's own objective function.[2]

Finally, (2.2) contains an $(r \times r)$ penalty matrix Ω_t^i $(i = 1, \ldots, N)$, weighting the deviations of states and controls from their desired levels in any time period t $(t = 1, \ldots, T)$. Thus the matrices

$$\Omega_t^i = \begin{bmatrix} Q_t^i & 0 & \cdots & 0 \\ 0 & R_t^{i1} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & R_t^{iN} \end{bmatrix}, \; i = 1, \ldots, N, \; t = 1, \ldots, T, \tag{2.8}$$

are of block-diagonal form, where the blocks Q_t^i and R_t^{ij} $(i, j = 1, \ldots, N)$ are symmetric. These blocks Q_t^i and R_t^{ij} correspond to penalty matrices for the states and the controls, respectively. The matrices $Q_t^i \geq 0$ are positive semi-definite for all $i = 1, \ldots, N$; the matrices R_t^{ij} are positive semi-definite for $i \neq j$ but positive definite for $i = j$. This guarantees that the matrices $R_t^{ii} > 0$ are invertible, a necessary prerequisite for the analytical tractability of the algorithm.

In a frequent special case, a discount factor α is used to calculate the penalty matrix Ω_t^i in time period t:

$$\Omega_t^i = \alpha^{t-1} \Omega_0^i, \qquad i = 1, \ldots, N, \; t = 1, \ldots, T, \tag{2.9}$$

where the initial penalty matrix Ω_0^i of player i is given.

The dynamic system, which constrains the choices of the decision makers, is given in state-space form by a first-order system of nonlinear difference equations:

$$x_t = f(x_{t-1}, x_t, u_t^1, \ldots, u_t^N, z_t), \quad x_0 = \bar{x}_0. \tag{2.10}$$

$\bar{x}_0$ contains the initial values of the state variables. The vector z_t contains non-controlled exogenous variables. f is a vector-valued function where f^k $(k = 1, \ldots, n_x)$ denotes the kth component of f. For the algorithm, we require that the first and second derivatives of the system function f with respect to x_t, x_{t-1} and $u_t^1, \ldots, u_t^N$ exist and are continuous. The assumption of a first-order system of difference equations as stated in (2.10) is not really restrictive as higher-order difference equations can be reduced to systems of first-order difference equations by suitably redefining variables as new state variables and augmenting the state vector.

[2]For example, the central bank in a monetary union, which controls monetary policy, can also penalize "bad" fiscal policies of member countries.

Equations (2.1), (2.2), and (2.10) define a nonlinear dynamic tracking game problem to be solved. That means, we try to find N trajectories of control variables u_t^i, $i = 1, \ldots, N$ which minimize the postulated objective functions subject to the dynamic system. In the next section, the OPTGAME3 algorithm, which is designed to solve such types of problems, is presented.

2.3 OPTGAME3

This section describes the OPTGAME algorithm in its third version (OPTGAME3), which was programmed in C# and MATLAB. For a better understanding, first a very simplified structure of the OPTGAME algorithm will be presented.

Algorithm 1 Rough structure of the OPTGAME algorithm

1: iteration step $k = 0$
2: initialize input parameters $x_0, (\overset{\circ}{u}{}_t^i)_{t=1}^T, (\tilde{x}_t^i)_{t=1}^T, (\tilde{u}_t^{ij})_{t=1}^T, (z_t)_{t=1}^T$ and $f(\ldots)$
3: calculate tentative path for states $x_t = f(x_{t-1}, x_t, \overset{\circ}{u}{}_t^1, \ldots, \overset{\circ}{u}{}_t^N, z_t), t = 1, \ldots, T$
4: **while** the stopping criterion is not met (*nonlinearity loop*) **do**
5: **for** T to 1 (*backward loop*) **do**
6: linearise the system of equations: $x_t = A_t x_{t-1} + \sum_{i=1}^N B_t^i u_t^i + c_t$
7: min J^i, get feedback matrices: G_t^i and g_t^i
8: **end for**
9: **for** 1 to T (*forward loop*) **do**
10: calculate the solution: $u_t^{i*} = G_t^i x_{t-1}^* + g_t^i$ and $x_t^* = f(x_{t-1}^*, x_t, u_t^{1*}, \ldots, u_t^{N*}, z_t)$
11: **end for**
12: at the end of the forward loop the solution for the current iteration of the nonlinearity loop is calculated: $(u_t^{i*}, x_t^*)_{t=1}^T$
13: set new tentative control paths: $u_t^{i*} \to \overset{\circ}{u}{}_t^i \quad \forall\ t, i$
14: calculate new tentative path for states $x_t = f(x_{t-1}, x_t, \overset{\circ}{u}{}_t^1, \ldots, \overset{\circ}{u}{}_t^N, z_t), t = 1, \ldots, T$
15: $k \to k + 1$
16: **end while**
17: final solution is calculated: $(u_t^{i*})_{t=1}^T, (x_t^*)_{t=1}^T, J^{i*}, J^*$

The algorithm starts with the input of all required data. As indicated in step 2, for all players ($i = 1, \ldots, N$) the initial tentative paths of the control variables $(\overset{\circ}{u}{}_t^i)_{t=1}^T$ are given as inputs. In order to find an initial tentative path for the state variables we apply an appropriate system solver like Newton–Raphson, Gauss–Seidel, Levenberg–Marquardt or Trust region to $x_t - f(x_{t-1}, x_t, \overset{\circ}{u}{}_t^1, \ldots, \overset{\circ}{u}{}_t^N, z_t) = 0$ in step 3. After that the nonlinearity loop is started where we iteratively approximate the final solution of the nonlinear dynamic tracking game. To this end, following a procedure introduced by Chow (1975) for optimum control problems, we linearise the nonlinear system f along the tentative path determined in the previous iteration steps. Note that we do not linearise the system only once prior to launching the optimization procedure (cf. step 4) but repeatedly linearise the entire system during the iterative optimization process along the current tentative paths (for both controls and states). This allows for replacing the autonomous nonlinear sytem by a non-autonomous linear system evaluated along a tentative path that changes with each iteration step. Accordingly, for each time period t we compute the reduced form of the linearised structure of (2.10) and approximate the nonlinear system by a linear system with time-dependent parameters in step 6.

The dynamic tracking game can then be solved for the linearised system using known optimization techniques, which results in feedback matrices G_t^i and g_t^i (see step 7). These feedback matrices allow us to calculate in a forward loop the solutions (u_t^{i*} and x_t^*) of the current iteration of the nonlinearity loop and, at the end of the nonlinearity loop, the final solutions. If the new tentative path falls into an ϵ-tube around the old tentative path, no variable differs by more than a value of ϵ between two successive iterations, and, for ϵ small enough, the consecutive paths are (more or less) identical. In particular, then, the state path calculated according to the nonlinear system dynamics (2.10) by using $x_t - f(x_{t-1}, x_t, u_t^1, \ldots, u_t^N, z_t) = 0$ equals the state path calculated according to the linearized system representation evaluated along the current tentative path (step 5). In other words, the algorithm has converged,[3] and the paths obtained indeed solve the original problem, i.e. (2.1) and (2.2) subject to (2.10).

The core of the OPTGAME algorithm appears in step 7, where the linearised system has to be optimized by each player. The optimization technique for minimizing the objective functions depends on the type of the game or solution concept. The OPTGAME3 algorithm determines four game strategies: one cooperative (Pareto optimal) and three non-cooperative game types: the Nash game for the open-loop information pattern, the Nash game for the feedback information pattern, and the Stackelberg game for the feedback information pattern.

Generally, open-loop Nash equilibrium solutions of affine-quadratic games are determined using Pontryagin's maximum principle. Feedback Nash and Stackelberg equilibrium solutions of affine-quadratic games are calculated using the dynamic programming (Hamilton–Jacobi–Bellman) technique. How to calculate the dynamic game solutions depending on the type of the game will be discussed separately in the next subsections.[4]

2.3.1 The Pareto-Optimal Solution

To determine a cooperative solution of the dynamic game, we have to define a joint objective function of all the players. This joint objective function J corresponds to

[3]Note that if convergence has not been obtained before k has reached its terminal value, then the iteration process terminates without succeeding in finding an equilibrium feedback solution.

[4]The mathematical details are based on Behrens and Neck (2007) and reflect the calculations and proofs in Başar and Olsder (1999).

the solution concept of Pareto-optimality and is given by a convex combination of the individual cost functions,

$$J = \sum_{t=1}^{T} \sum_{i=1}^{N} \mu^i L_t^i(x_t, u_t^1, \ldots, u_t^N), \quad \sum_{i=1}^{N} \mu^i = 1. \tag{2.11}$$

L_t^i is defined by (2.2). The parameters μ^i ($i = 1, \ldots, N$) reflect player i's "power" or "importance" in the joint objective function. Therefore the solution of the cooperative Pareto-optimal game with N players can be determined by solving a classical optimum control problem. First, define the following matrices:

$$Q_t := \sum_{i=1}^{N} \mu^i Q_t^i, \tag{2.12}$$

$$q_t := \sum_{i=1}^{N} \mu^i Q_t^i \tilde{x}_t^i, \tag{2.13}$$

$$R_t^j := \sum_{i=1}^{N} \mu^i R_t^{ij}, \quad j = 1, \ldots, N, \tag{2.14}$$

$$r_t^j := \sum_{i=1}^{N} \mu^i R_t^{ij} \tilde{u}_t^{ij}, \quad j = 1, \ldots, N. \tag{2.15}$$

The Riccati matrices H_t and h_t for all players and for all time periods $t \in 1, \ldots, T$ are derived by backward iteration according to the following system of recursive matrix equations:

$$H_{t-1} = Q_{t-1} + K_t' H_t K_t + \sum_{j=1}^{N} G_t^{j'} R_t^j G_t^j, \quad H_T = Q_T, \tag{2.16}$$

$$h_{t-1} = q_{t-1} - K_t'[H_t k_t - h_t] + \sum_{j=1}^{N} G_t^{j'} [r_t^j - R_t^j g_t^j], \quad h_T = q_T, \tag{2.17}$$

where

$$K_t := A_t + \sum_{j=1}^{N} B_t^j G_t^j, \tag{2.18}$$

$$k_t := s_t + \sum_{j=1}^{N} B_t^j g_t^j. \tag{2.19}$$

The feedback matrices G_t^i and g_t^i for $i = 1, \ldots, N$ are determined as solutions of the following set of linear matrix equations:

$$B_t^{j'} H_t A_t + [R_t^j + B_t^{j'} H_t B_t^j] G_t^j + B_t^{j'} H_t \sum_{\substack{k=1,\\ k\neq j}}^{N} B_t^k G_t^k = 0, \tag{2.20}$$

$$[R_t^j + B_t^{j'} H_t B_t^j] g_t^j + B_t^{j'} H_t \sum_{\substack{k=1,\\ k\neq j}}^{N} B_t^k g_t^k + B_t^{j'} H_t s_t - B_t^{j'} h_t - r_t^j = 0. \tag{2.21}$$

In each time period t the auxiliary matrices are determined in the following order:

1. H_t^i, h_t^i: according to (2.16) and (2.17),
2. G_t^i, g_t^i: according to (2.20) and (2.21),[5]
3. K_t, k_t: according to (2.18) and (2.19).

Using the Riccati matrices H_t^i and h_t^i for $i = 1, \ldots, N$ and the feedback matrices G_t^i and g_t^i for $i = 1, \ldots, N$ for all time periods, we can compute the matrices K_t and k_t as defined by (2.18) and (2.19). Then the states and controls forming a cooperative Pareto-optimal solution of the game can be determined by forward iteration according to:

$$x_t^* = K_t x_{t-1}^* + k_t, \quad x_0^* = x_0, \tag{2.22}$$

$$u_t^{i*} = G_t^i x_{t-1}^* + g_t^i, \quad i = 1, \ldots, N. \tag{2.23}$$

2.3.2 The Feedback Nash Equilibrium Solution

To approximate the feedback Nash equilibrium solution of the game, the algorithm proceeds as follows: Riccati matrices H_t^i and h_t^i for all players $i = 1, \ldots, N$ and for all time periods $t \in \{1, \ldots, T\}$ are derived by backward iteration according to the following system of recursive matrix equations:

$$H_{t-1}^i = Q_{t-1}^i + K_t' H_t^i K_t + \sum_{j=1}^{N} G_t^{j'} R_t^{ij} G_t^j, \quad H_T^i = Q_T^i, \tag{2.24}$$

$$h_{t-1}^i = Q_{t-1}^i \tilde{x}_{t-1}^i - K_t' [H_t^i k_t - h_t^i] + \sum_{j=1}^{N} G_t^{j'} R_t^{ij} [\tilde{u}_t^{ij} - g_t^j], \quad h_T^i = Q_T^i \tilde{x}_T^i, \tag{2.25}$$

[5]It is important to mention that we have a system with two simultaneous equations and two unknown parameters in step 2 ((2.20) and (2.21)). Therefore a system solver must be applied. In OPTGAME3, the Gauss–Seidel method is applied for this purpose.

where K_t is defined by (2.18) and k_t is given by (2.19). The feedback matrices G_t^i and g_t^i for $i = 1, \ldots, N$ are determined as solutions of the following set of linear matrix equations:

$$D_t^i G_t^i + B_t^{i'} H_t^i \sum_{\substack{j=1,\\ j\neq i}}^{N} B_t^j G_t^j + B_t^{i'} H_t^i A_t = 0, \tag{2.26}$$

$$D_t^i g_t^i + B_t^{i'} H_t^i \sum_{\substack{j=1,\\ j\neq i}}^{N} B_t^j g_t^j + v_t^i = 0, \tag{2.27}$$

where

$$D_t^i := R_t^{ii} + B_t^{i'} H_t^i B_t^i, \tag{2.28}$$

$$v_t^i := B_t^{i'} [H_t^i s_t - h_t^i] - R_t^{ii} \tilde{u}_t^{ii}. \tag{2.29}$$

In each time period t the auxiliary matrices are determined in the following order:

1. H_t^i, h_t^i: according to (2.16) and (2.17)
2. G_t^i, g_t^i: according to (2.26) and (2.27)
3. K_t, k_t: according to (2.18) and (2.19).

Using the Riccati matrices H_t^i and h_t^i for $i = 1, \ldots, N$ and the feedback matrices G_t^i and g_t^i for $i = 1, \ldots, N$ for all time periods, we can compute the matrices K_t and k_t as defined by (2.18) and (2.19). Then approximate feedback Nash equilibrium values of the states and controls forming a solution of the game can be determined by a forward loop in accordance with:

$$x_t^* = K_t x_{t-1}^* + k_t, \quad x_0^* = x_0, \tag{2.30}$$

$$u_t^{i*} = G_t^i x_{t-1}^* + g_t^i, \quad i = 1, \ldots, N. \tag{2.31}$$

2.3.3 The Open-Loop Nash Equilibrium Solution

At the beginning of an open-loop Nash game, each of the N simultaneously acting players makes a binding commitment to stick to a chosen policy rule for the entire time horizon $t = 1, \ldots, T$. As long as these commitments hold, the solution is an equilibrium in the sense that none of the players can improve their individual welfare by one-sided deviations from the open-loop Nash equilibrium path. Although the open-loop Nash equilibrium solution is not time consistent, for certain situations this solution concept could be the right choice. Furthermore, even if this kind of policy is not very realistic, its analysis can help compare the quality of other solutions.

In the following, the procedure of finding the open-loop Nash equilibrium solution is described. For an invertible matrix

$$\Lambda_t := I + \sum_{j=1}^{N} B_t^j [R_t^{jj}]^{-1} B_t^{j'} H_t^j, \tag{2.32}$$

the Riccati matrices for player i ($i = 1, \dots, N$) are determined by backward iteration according to the following recursive system of matrix equations:

$$H_{t-1}^i = Q_{t-1}^i + A_t' H_t^i [\Lambda_t]^{-1} A_t, \;\; H_T^i = Q_T^i \tag{2.33}$$

$$h_{t-1}^i = -Q_{t-1}^i \tilde{x}_{t-1}^i + A_t' [H_t^i [\Lambda_t]^{-1} \eta_t + h_t^i], \;\; h_T^i = -Q_T^i \tilde{x}_T^i, \tag{2.34}$$

where

$$\eta_t := s_t + \sum_{j=1}^{N} B_t^j [\tilde{u}_t^{jj} - [R_t^{jj}]^{-1} B_t^{j'} h_t^j]. \tag{2.35}$$

In each time period t the auxiliary matrices are determined as follows:

1. H_t^i, h_t^i: according to (2.33) and (2.34)
2. Λ_t: according to (2.32).

With the Riccati matrices H_t^i and h_t^i, stored for all time periods $t \in \{1, \dots, T\}$, the approximate open-loop Nash equilibrium values of the state and the control variables for all players ($i = 1, \dots, N$) are determined by forward loop according to

$$x_t^* = [\Lambda_t]^{-1} [A_t x_{t-1}^* + \eta_t] \tag{2.36}$$

and

$$u_t^{i*} = \tilde{u}_t^{ii} - [R_t^{ii}]^{-1} B_t^{i'} [H_t^i x_t^* + h_t^i], \tag{2.37}$$

starting with the initial condition $x_0^* = x_0$.

2.3.4 The Feedback Stackelberg Equilibrium Solution

The feedback Stackelberg equilibrium solution is asymmetric: The Stackelberg leader (player 1) announces his decision rule, $u_t^1 = \varphi^1(x_{t-1})$, to all other players while the actions of the other players (players $i = 2, \dots, N$, the Stackelberg followers) are based on the current state x_{t-1} and on the decision of the leader according to the reaction function $u_t^i = \varphi^i(x_{t-1}, u_t^1)$. At the time of optimizing his performance, the leader considers the reaction coefficients,

$$\Psi_t^i := \frac{\partial u_t^i}{\partial u_t^1}, \quad i = 2, \ldots, N, \tag{2.38}$$

as rational reactions of the followers $i = 2, \ldots, N$. These reaction coefficients Ψ_t^i ($i = 2, \ldots, N$) are determined as solutions of the following set of $N - 1$ linear matrix equations:

$$B_t^{i'} H_t^i B_t^1 + D_t^i \Psi_t^i + B_t^{i'} H_t^i \sum_{\substack{j=2,\\ j \neq i}}^{N} B_t^j \Psi_t^j = 0, \quad i = 2, \ldots, N, \tag{2.39}$$

where H_t^i denotes the Riccati matrices of the feedback Stackelberg game calculated as

$$H_{t-1}^i = Q_{t-1}^i + K_t' H_t^i K_t + \sum_{j=1}^{N} G_t^{j'} R_t^{ij} G_t^j, \quad H_T^i = Q_T^i, \ i = 1, \ldots, N, \tag{2.40}$$

and the matrix D_t^i is given by

$$D_t^i := R_t^{ii} + B_t^{i'} H_t^i B_t^i. \tag{2.41}$$

The matrices W_t^i and w_t^i (for $i = 2, \ldots, N$), which are required for determining the feedback matrices G_t^i and g_t^i (for $i = 1, \ldots, N$), are calculated as solutions of the following set of $N - 1$ linear matrix equations:

$$D_t^i W_t^i + B_t^{i'} H_t^i \left[A_t + \sum_{\substack{j=2,\\ j \neq i}}^{N} B_t^j W_t^j \right] = 0, \tag{2.42}$$

$$D_t^i w_t^i + B_t^{i'} (H_t^i s_t - h_t^i) - R_t^{ii} \tilde{u}_t^{ii} + B_t^{i'} H_t^i \sum_{\substack{j=2,\\ j \neq i}}^{N} B_t^j w_t^j = 0. \tag{2.43}$$

Using

$$h_{t-1}^i = Q_{t-1}^i \tilde{x}_{t-1}^i - K_t' [H_t^i k_t - h_t^i] + \sum_{j=1}^{N} G_t^{j'} R_t^{ij} [\tilde{u}_t^{ij} - g_t^j], \quad h_T^i = Q_T^i \tilde{x}_T^i, \tag{2.44}$$

and given that the matrix

$$\bar{\Lambda}_t := R_t^{11} + \bar{B}_t' H_t^1 \bar{B}_t + \sum_{j=2}^{N} \Psi_t^{j'} R_t^{1j} \Psi_t^j \tag{2.45}$$

is invertible, for

$$\bar{B}_t := B_t^1 + \sum_{j=2}^{N} B_t^j \Psi_t^j \tag{2.46}$$

we can derive the Riccati matrices, H_{t-1}^i and h_{t-1}^i for $i = 1, \ldots, N$, by backward iteration according to the Riccati equations (2.40) and (2.44). The feedback matrices are determined by

$$G_t^1 := -\left[\bar{\Lambda}_t\right]^{-1} \left[\bar{B}_t' H_t^1 A_t + \sum_{j=2}^{N} \bar{D}_t^j W_t^j \right], \tag{2.47}$$

$$g_t^1 := -\left[\bar{\Lambda}_t\right]^{-1} \left[v_t^1 + \bar{v}_t + \sum_{j=2}^{N} \bar{D}_t^j w_t^j \right], \tag{2.48}$$

$$G_t^i := W_t^i + \Psi_t^i G_t^1, \quad i = 2, \ldots, N, \tag{2.49}$$

$$g_t^i := w_t^i + \Psi_t^i g_t^1, \quad i = 2, \ldots, N, \tag{2.50}$$

where

$$\bar{D}_t^i := \Psi_t^{i\prime} R_t^{1i} + \bar{B}_t' H_t^1 B_t^i, \quad i = 2, \ldots, N, \tag{2.51}$$

$$\bar{v}_t := \sum_{j=2}^{N} \Psi_t^{j\prime} \left[B_t^{j\prime} H_t^1 s_t - B_t^{j\prime} h_t^1 - R_t^{1j} \tilde{u}_t^{1j} \right]. \tag{2.52}$$

Using the Riccati matrices H_t^i and h_t^i for $i = 1, \ldots, N$ and the feedback matrices G_t^i and g_t^i for $i = 1, \ldots, N$ for all time periods, we can compute the matrices K_t and k_t by:

$$K_t := A_t + \sum_{j=1}^{N} B_t^j G_t^j, \tag{2.53}$$

$$k_t := s_t + \sum_{j=1}^{N} B_t^j g_t^j. \tag{2.54}$$

In each time period t the auxiliary matrices are determined in the following order:

1. H_t^i, h_t^i: according to (2.40) and (2.44)
2. Ψ_t^i: according to (2.39)
3. W_t^i, w_t^i: according to (2.42) and (2.43)

4. G_t^i, g_t^i: according to (2.47), (2.48), (2.49), and (2.50)
5. K_t, k_t: according to (2.53) and (2.54).

Finally, the approximate feedback Stackelberg equilibrium values of the states and controls for the game for all players ($i = 1, \dots, N$) are determined by forward iteration according to the following functional relationships:

$$x_t^* = K_t x_{t-1}^* + k_t, \quad x_0^* = x_0, \tag{2.55}$$

$$u_t^{i*} = G_t^i x_{t-1}^* + g_t^i, \quad i = 1, \dots, N. \tag{2.56}$$

2.4 An Application

2.4.1 The MUMOD1 Model

In order to show the applicability of the OPTGAME3 algorithm we use a simplified macroeconomic model of a monetary union consisting of two countries (or two blocs of countries) with a common central bank. This model is called MUMOD1 and slightly improves on the one introduced in Blueschke and Neck (2011). For a similar framework in continuous time, see van Aarle et al. (2002). The model is calibrated so as to deal with the problem of public debt targeting in a situation that resembles the one currently prevailing in the European Union, but no attempt is made to describe a monetary union in general or the EMU in every detail.

In the following, capital letters indicate nominal values, while lowercase letters correspond to real values. Variables are denoted by Roman letters and model parameters are denoted by Greek letters. Three active policy makers are considered: the governments of the two countries responsible for decisions about fiscal policy and the common central bank of the monetary union controlling monetary policy. The two countries are labelled 1 and 2 or core and periphery, respectively. The idea is to create a stylized model of a monetary union consisting of two homogeneous blocs of countries, which in the current European context might be identified with the stability-oriented bloc (core) and the PIIGS bloc (countries with problems due to high public debt).

The model is formulated in terms of deviations from a long-run growth path. The goods markets are modelled for each country by a short-run income-expenditure equilibrium relation (IS curve). The two countries under consideration are linked through their goods markets, namely exports and imports of goods and services. The common central bank decides on the prime rate, that is, a nominal rate of interest under its direct control (for instance, the rate at which it lends money to private banks).

Real output (or the deviation of short-run output from a long-run growth path) in country i $(i = 1, 2)$ at time t $(t = 1, \dots, T)$ is determined by a reduced form demand-side equilibrium equation:

$$y_{it} = \delta_i(\pi_{jt} - \pi_{it}) - \gamma_i(r_{it} - \theta) + \rho_i y_{jt} - \beta_i \pi_{it} + \kappa_i y_{i(t-1)} - \eta_i g_{it} + zd_{it}, \quad (2.57)$$

for $i \neq j$ $(i, j = 1, 2)$. The variable π_{it} denotes the rate of inflation in country i, r_{it} represents country i's real rate of interest and g_{it} denotes country i's real fiscal surplus (or, if negative, its fiscal deficit), measured in relation to real GDP. g_{it} in (2.57) is assumed to be country i's fiscal policy instrument or control variable. The natural real rate of output growth, $\theta \in [0, 1]$, is assumed to be equal to the natural real rate of interest. The parameters $\delta_i, \gamma_i, \rho_i, \beta_i, \kappa_i, \eta_i$, in (2.57) are assumed to be positive. The variables zd_{1t} and zd_{2t} are non-controlled exogenous variables and represent exogenous demand-side shocks in the goods market.

For $t = 1, \ldots, T$, the current real rate of interest for country i $(i = 1, 2)$ is given by:

$$r_{it} = I_{it} - \pi^e_{it}, \quad (2.58)$$

where π^e_{it} denotes the expected rate of inflation in country i and I_{it} denotes the nominal interest rate for country i, which is given by:

$$I_{it} = R_{Et} - \lambda_i g_{it} + \chi_i D_{it} + zhp_{it}, \quad (2.59)$$

where R_{Et} denotes the prime rate determined by the central bank of the monetary union (its control variable); $-\lambda_i$ and χ_i (λ_i and χ_i are assumed to be positive) are risk premiums for country i's fiscal deficit and public debt level. This allows for different nominal (and a fortiori also real) rates of interest in the union in spite of a common monetary policy due to the possibility of default or similar risk of a country (a bloc of countries) with high government deficit and debt. zhp_{it} allows for exogenous shocks on the nominal rate of interest, e.g. negative after-effects of a haircut or a default.

The inflation rates for each country $i = 1, 2$ and $t = 1, \ldots, T$ are determined according to an expectations-augmented Phillips curve, i.e. the actual rate of inflation depends positively on the expected rate of inflation and on the goods market excess demand (a demand-pull relation):

$$\pi_{it} = \pi^e_{it} + \xi_i y_{it} + zs_{it}, \quad (2.60)$$

where ξ_1 and ξ_2 are positive parameters; zs_{1t} and zs_{2t} denote non-controlled exogenous variables and represent exogenous supply-side shocks, such as oil price increases, introducing the possibility of cost-push inflation; π^e_{it} denotes the rate of inflation in country i expected to prevail during time period t, which is formed at (the end of) time period $t - 1$. Inflationary expectations are formed according to the hypothesis of adaptive expectations:

$$\pi^e_{it} = \varepsilon_i \pi_{i(t-1)} + (1 - \varepsilon_i)\pi^e_{i(t-1)}, \quad (2.61)$$

where $\varepsilon_i \in [0, 1]$ are positive parameters determining the speed of adjustment of expected to actual inflation.

The average values of output and inflation in the monetary union are given by:

$$y_{Et} = \omega y_{1t} + (1-\omega)y_{2t}, \quad \omega \in [0,1], \tag{2.62}$$

$$\pi_{Et} = \omega \pi_{1t} + (1-\omega)\pi_{2t}, \quad \omega \in [0,1]. \tag{2.63}$$

The parameter ω expresses the weight of country 1 in the economy of the whole monetary union as defined by its output level. The same weight ω is used for calculating union-wide inflation in (2.63).

The government budget constraint is given as an equation for government debt of country i $(i = 1, 2)$:

$$D_{it} = (1 + r_{i(t-1)})D_{i(t-1)} - g_{it} + zh_{it}, \tag{2.64}$$

where D_i denotes real public debt of country i measured in relation to (real) GDP. No seigniorage effects on governments' debt are assumed to be present. zh_{it} allows us to model an exogenous shock on public debt; for instance, if negative it may express default or debt relief (a haircut).

Both national fiscal authorities are assumed to care about stabilizing inflation (π), output (y), debt (D), and fiscal deficits of their own countries (g) at each time t. This is a policy setting which seems plausible for the real EMU as well, with full employment (output at its potential level) and price level stability relating to country (or bloc) i's primary domestic goals, and government debt and deficit relating to its obligations according to the Maastricht Treaty of the European Union. The common central bank is interested in stabilizing inflation and output in the entire monetary union, also taking into account a goal of low and stable interest rates in the union.

Equations (2.57)–(2.64) constitute a dynamic game with three players, each of them having one control variable. The model contains 14 endogenous variables and four exogenous variables and is assumed to be played over a finite time horizon. The objective functions are quadratic in the paths of deviations of state and control variables from their desired values. The game is nonlinear-quadratic and hence cannot be solved analytically but only numerically. To this end, we have to specify the parameters of the model.

The parameters of the model are specified for a slightly asymmetric monetary union; see Table 2.1. Here an attempt has been made to calibrate the model parameters so as to fit for the EMU. The data used for calibration include average economic indicators for the 16 EMU countries from EUROSTAT up to the year 2007. Mainly based on the public finance situation, the EMU is divided into two blocs: a core (country or bloc 1) and a periphery (country or bloc 2). The first bloc has a weight of 60% in the entire economy of the monetary union (i.e. the parameter ω is equal to 0.6). The second bloc has a weight of 40% in the economy of the union; it consists of countries with higher public debt and deficits and higher interest and inflation rates on average. The weights correspond to the respective shares in EMU real GDP. For the other parameters of the model, we use values in accordance with econometric studies and plausibility considerations.

Table 2.1 Parameter values for an asymmetric monetary union, $i = 1, 2$

T	θ	ω	$\delta_i, \beta_i, \eta_i, \varepsilon_i$	$\gamma_i, \rho_i, \kappa_i, \xi_i, \lambda_i$	χ_i
30	3	0.6	0.5	0.25	0.0125

Table 2.2 Initial values of the two-country monetary union

$y_{i,0}$	$\pi_{i,0}$	$\pi^e_{i,0}$	$D_{1,0}$	$D_{2,0}$	$R_{E,0}$	$g_{1,0}$	$g_{2,0}$
0	2	2	60	80	3	0	0

Table 2.3 Target values for an asymmetric monetary union

$\tilde{y}_{it}$	$\tilde{D}_{1t}$	$\tilde{D}_{2t}$	$\tilde{\pi}_{it}$	$\tilde{\pi}_{Et}$	$\tilde{y}_{Et}$	$\tilde{g}_{it}$	$\tilde{R}_{Et}$
0	60↘50	80↘60	1.8	1.8	0	0	3

The initial values of the macroeconomic variables, which are the state variables of the dynamic game model, are presented in Table 2.2. The desired or ideal values assumed for the objective variables of the players are given in Table 2.3. Country 1 (the core bloc) has an initial debt level of 60% of GDP and aims to decrease this level in a linear way over time to arrive at a public debt of 50% at the end of the planning horizon. Country 2 (the periphery bloc) has an initial debt level of 80% of GDP and aims to decrease its level to 60% at the end of the planning horizon, which means that it is going to fulfil the Maastricht criterion for this economic indicator. The ideal rate of inflation is calibrated at 1.8%, which corresponds to the Eurosystem's aim of keeping inflation below, but close to, 2%. The initial values of the two blocs' government debts correspond to those at the beginning of the Great Recession, the recent financial and economic crisis. Otherwise, the initial situation is assumed to be close to equilibrium, with parameter values calibrated accordingly.

2.4.2 Equilibrium Fiscal and Monetary Policies

The MUMOD1 model can be used to simulate the effects of different shocks acting on the monetary union, which are reflected in the paths of the exogenous non-controlled variables, and of policy reactions towards these shocks. In this paper we show the applicability of the OPTGAME3 algorithm. To this end we assume a mixed asymmetric shock which occurs both on demand (zd_i) and supply side (zs_i) as given in Table 2.4.

In the first three periods, both countries experience the same negative demand shock (zd_i) which reflects a financial and economic crisis like the Great Recession. After three periods the economic environment of country 1 stabilizes again, but for country 2 the crisis continues for two more periods.

Starting with time period 3 both countries also experience adverse supply side shocks, which lead to increases in inflation rates. These shocks last three periods for both countries (or blocs) but vary in their strength. The core bloc experiences

Table 2.4 Negative asymmetric shock on demand and supply side

t	1	2	3	4	5	6	$\cdots$	30
zd_1	−2	−4	−2	0	0	0	$\cdots$	0
zd_2	−2	−4	−2	−2	−1	0	$\cdots$	0
zs_1	0	0	2	2	2	0	$\cdots$	0
zs_2	0	0	4	4	4	0	$\cdots$	0

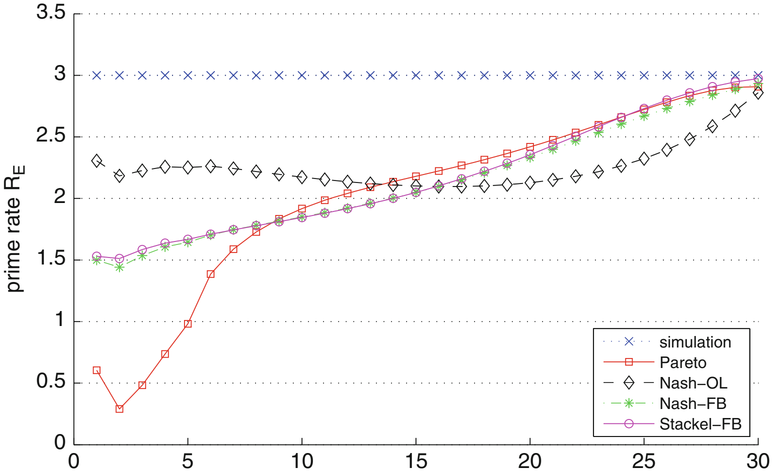

Fig. 2.1 Prime rate R_{Et} controlled by the central bank

an increase in inflation of 2 percentage points, the periphery bloc an increase of 4 percentage points.

In this section, we investigate how the dynamics of the model and the results of the policy game (2.57)–(2.64) depend on the strategy choice of the decision makers. For this game, we calculate five different solutions: a baseline solution with the shock but with policy instruments held at pre-shock levels (zero for the fiscal balance, 3 for the central bank's interest rate), three non-cooperative game solutions and one cooperative game solution. The baseline solution does not include any policy intervention and describes a simple simulation of the dynamic system. It can be interpreted as resulting from a policy ideology of market fundamentalism prescribing non-intervention in the case of a recession.

Figures 2.1–2.5 show the simulation and optimization results of our experiment. Figures 2.1 and 2.2 show the results for the control variables of the players and Figs. 2.3–2.5 show the results of selected state variables: output, inflation and public debt.

Without policy intervention (baseline scenario, denoted by "simulation"), both countries suffer dramatically from the economic downturn modelled by the demand side shock in the first periods. The output of country 1 drops by 6% and that of country 2 by more than 7%, which for several European countries is a fairly good approximation of what happened in reality. This economic crisis lowers the inflation

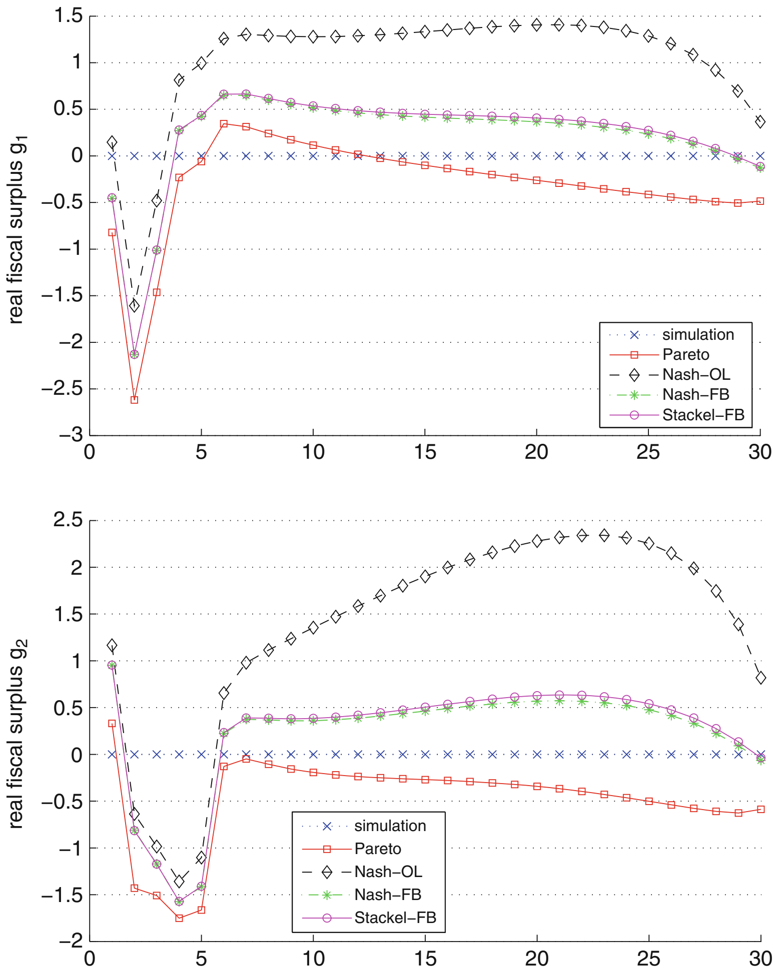

Fig. 2.2 Country i's fiscal surplus g_{it} (control variable) for $i = 1$ (core; *top*) and $i = 2$ (periphery; *bottom*)

rates to values very close to zero, but with the appearance of the supply side shock, inflation rates go up and reach 3% for country 1 and nearly 6% for country 2 in the non-controlled baseline scenario. Even more dramatic is the development of public debt. Without policy intervention it increases during the whole planning horizon and arrives at levels of 145% of GDP for country 1 (or core bloc) and 180% for country 2 (or periphery bloc), which shows a need for policy actions to stabilize the economies of the monetary union.

If the players want to react optimally to the demand and supply side shocks, their actions and their intensity depend on the presence or absence of cooperation. For example, optimal monetary policy has to be expansionary in all strategies, but

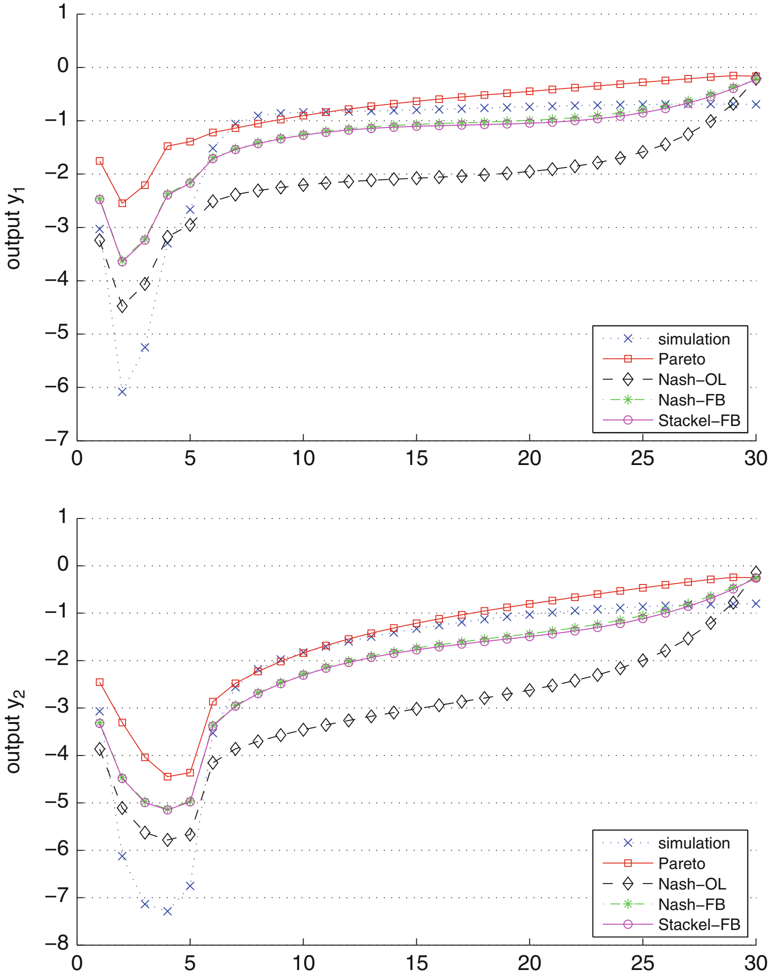

Fig. 2.3 Country i's output y_{it} for $i = 1$ (core; *top*) and $i = 2$ (periphery; *bottom*)

in the cooperative Pareto solution it is more active during the first eight periods. The open-loop Nash equilibrium solution, in contrast, is more or less constant during the whole optimization period, which causes the central bank to be less active at the beginning and relatively more active at the end of the optimization horizon.

With respect to fiscal policy, both countries are required to set expansionary actions and to create deficits in the first four periods in order to absorb the demand side shock. After that a trade-off occurs and the governments need to take care of the financial situation and to produce primary surpluses. The only exception is the cooperative Pareto solution: cooperation between the countries and the central bank

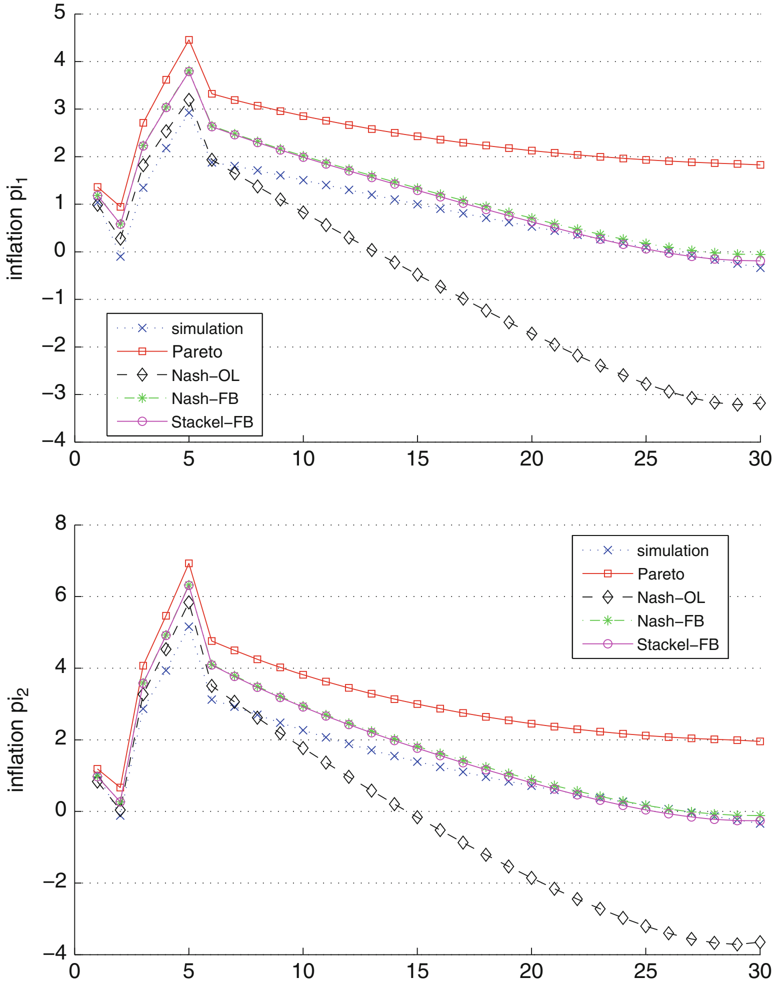

Fig. 2.4 Country i's inflation rate π_{it} for $i = 1$ (core; *top*) and $i = 2$ (periphery; *bottom*)

(which runs an expansionary monetary policy) and the high inflation means that the balance of public finances can be held close below zero. Even so the countries are able to stabilize and to bring down their public debts to the targeted values.

The non-cooperative Nash feedback and Stackelberg feedback solutions give very similar results. In comparison with a Pareto-optimal solution, the central bank acts less actively and the countries run more active fiscal policies (except during the negative demand shock). As a result, output and inflation are slightly below the values achieved in a cooperative solution, and public debt is slightly above.

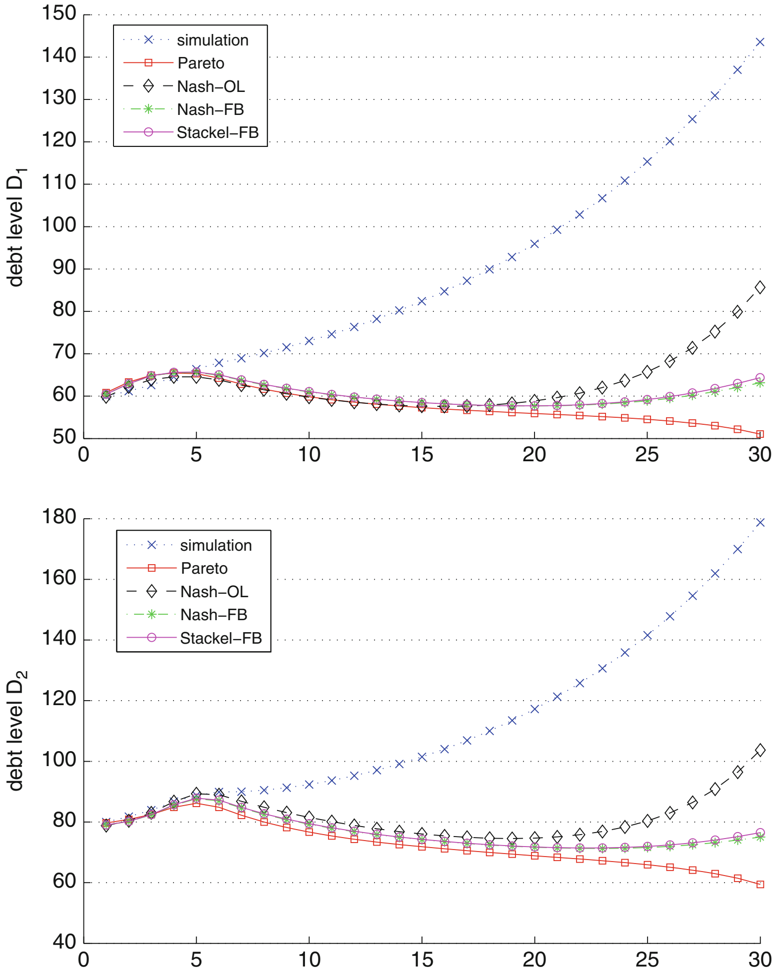

Fig. 2.5 Country i's debt level D_{it} for $i = 1$ (core; *top*) and $i = 2$ (periphery; *bottom*)

2.5 Concluding Remarks

In this paper we show the framework of the OPTGAME3 algorithm which allows us to find approximate solutions for nonlinear-quadratic dynamic tracking games. The algorithm was programmed in C# and MATLAB and includes the cooperative Pareto-optimal solution and non-cooperative Nash and Stackelberg equilibrium solutions. The applicability of the algorithm was shown using the MUMOD1 model, a small stylized nonlinear two-country macroeconomic model of a monetary union. We analyse the interaction between fiscal (governments) and monetary (common central bank) policy makers. By applying a dynamic game approach to a simple macroeconomic model of a two-country monetary union, we obtain some insights into the design of economic policies facing negative asymmetric shocks on the demand and the supply side. The monetary union is assumed to be asymmetric in the sense of consisting of a core with less initial public debt and a periphery with higher initial public debt, which is meant to reflect the situation in the EMU.

Our model implies that optimal and equilibrium policies of both the governments and the common central bank are counter-cyclical during the immediate influence of the demand shock but not afterwards. The later occurrence of a negative supply side shock increases the inflation rates and supports the countries in reducing their public debts. In the case of the Pareto solution, this leads to the situation that the countries can reduce their debts even without strongly restrictive fiscal policies. Taken together, the two negative shocks worsen the economic situation in the monetary union and produce growth rates of real output below the natural or long-run growth rate. We also show that the cooperative Pareto solution gives the best response to these shocks especially regarding output and public debt results.

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