Chapter 2
Professor in Pisa

2.1 Just Graduated and Teaching Right Away

Young Vito, student at the Scuola Normale in Pisa, graduated in physics in 1882 with a thesis on hydrodynamics written under the advisement of Enrico Betti. Other graduates from the Scuola Normale in the 1880s included Salvatore Pincherle, Gregorio Ricci-Curbastro, Luigi Bianchi, Carlo Somigliana and Mario Pieri, and later, in the 1890s, Federigo Enriques, Gaetano Scorza and Guido Fubini. Each one of these is a 'luminary' in the history of mathematics, and they provide excellent testimony to the level of the Scuola Normale. In the previous chapter, we mentioned the teaching faculty that Volterra found when he enrolled in the Scuola Normale. Above all, it was Dini and Betti who provided the orientation for his early choices. They also contributed to making Pisa a major centre for Italian mathematics, which, at the beginning of the twentieth century, would generally be acknowledged as third in the ideal ranking of mathematics by country, after France and Germany. This is a significant achievement for a mathematics that had essentially begun from scratch, in 1860, at the time of Italian unification!

Before following the young graduate in his first steps in the world of research, it is helpful to get a perspective on the situation: we are following the tracks of one who will become a leading figure in the great *leap forward* that will bring Italian mathematics close to the traditional excellence of the French and Germans, in the attempt – and here the outcomes will reveal themselves to be much more problematic – to orient the cultural and social development of the country itself. At the turn of the twentieth century there was a boom in all of the disciples in which mathematical research was traditionally divided (and in which we find many of the 'Pisan' names just mentioned): *analysis*, with the studies of Brioschi, Dini, Casorati, Arzelà, Ascoli, Peano, Pincherle, Vitali, Tonelli, Fubini, and others; *geometry*, with the contributions of Cremona, Battaglini, Bertini, Segre, Fano, Beltrami, Castelnuovo, Enriques and Severi in particular, and all members of the Italian school of algebraic geometry in general; *mathematical physics*, with Betti, Ricci-Curbastro, Bianchi, Somigliana, Levi-Civita and others. In short, Volterra

A. Guerraggio and G. Paoloni, *Vito Volterra*,
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had the great good luck to find himself in the *right place* at the *right time*, and he also deserves credit for having made a decisive contribution to that ‘right time’. His contributions regarded analysis, mathematical physics and other fields. His research work is in any case strongly unified by several constants that are easily perceived in his thoughts and in his actions. These constants are also very significant.

It was under Betti’s advisement, as we have seen, that Volterra earned his degree in 1882, and it was also with Betti that, in the month of December, immediately after receiving his degree, he began his academic career, as his assistant: ‘Dear Volterra, yesterday the faculty resolved to propose to the Minister that he anticipate the opening of the competition for the chair in rational mechanics, and that in the meantime I teach these classes until the professor who wins the competition takes over the chair. I accepted this responsibility, but I asked that the government be requested to give me an assistant, to whom will be paid a salary of 1,250 lire; and the Faculty also presented this proposal to the Minister. I have reason to believe that the Faculty’s proposal will be accepted. As soon as approval is given, I will have to propose an assistant, and I would like it to be you. Please write me if you are willing to accept’.

Volterra’s family, his mother Angelica and his uncle Alfonso, continued to stay in close contact with the young graduate even during these first steps in his adult life, and followed his progress with loving concern. Antonio Ròiti , his ‘old’ professor of physics at the Istituto Tecnico ‘Galileo Galilei’ in Florence, by now a professor at the University of Palermo, continued to advise him – even bluntly, when necessary – so that he doesn’t let good opportunities pass him by. With his nomination as Betti’s assistant, Volterra’s career accelerated unexpectedly. The following year, in spite of the initial cautiousness of his ‘boss’, he sat for and passed the competitive examination for the chair of rational mechanics that Betti had mentioned in the letter quoted above, in the same University of Pisa where he had earned his degree only a few months earlier!

Volterra taught for 10 years in Pisa, holding the courses in rational mechanics and graphic statics (as well as mathematical physics, after Betti’s death). For a short while he would also be asked to work in the library of the Scuola Normale. These were important years for Pisa as well as its university. It is easy to see how they would be decisive for a university professor who was not yet 30 years old!

Of average height and build for that period (just under 170 cm tall), with dark brown hair, he began to grow a beard. At this point in time he didn’t express any particular political opinions, although it appears from some phrases in his letters to his family that he tended towards a moderately conservative stance, and supported the monarchy. Of course, this was the prevailing political climate in the Scuola Normale where he had been trained. When Dini told Betti about the positive outcome of the administrative elections in a Tuscan college in 1882, he wrote that ‘we have even conquered the citadel of progressivism’,[1] a joking way of summing up the political leanings held by the group of mathematicians. In his

[1]The letter is quoted in M. Berengo, *Cultura e instituzioni nell’Ottocento italiano* (Bologna: Il Mulino, 2004).

first contacts with students, *Professor* Volterra showed himself to be particularly severe and demanding. All things considered, however, he still moved with a certain reserve and uneasiness in the academic world. Starting in 1887 he lived with his mother, who left Florence definitively to come to be with him. Those who knew him best spoke of him as an extremely cordial young man who was immediately likeable. Ernesto Pascal (1865–1940), almost the same age as Volterra, and like him, destined to teach mathematics, first at the University of Pavia, and then in that of Naples, had this to say about him: 'Prof. Volterra is an angelic young man, of characteristic modesty'.[2]

The extent of his talents began to recognised beyond the confines of Pisa. Gradually the scientific and academic worlds in Italy opened their doors to him. In 1887 he was awarded the gold medal for mathematics by the Società dei XL, the forerunner of the Italian National Academy of Sciences. The following year he was nominated *corresponding member* of the Accademia dei Lincei (he would be elected as an even more prestigious *national member* in 1899). In 1891, he became a member of the Palermo's Circolo Matematico, and a Knight of the Order of the Crown of Italy. In 1892 – at the death of Betti – he was elected dean of the Faculty of Sciences of the University of Pisa. He also succeeded his mentor in the direction of the journal *Nuovo Cimento*.

2.2 Scientific Work During the Period in Pisa

The years in Pisa following his winning the chair in rational mechanics were very important for Volterra from the point of view of his research as well as the development of his scientific personality. First he had worked in analysis with Dini, and then in mathematical physics with Betti. In his case (but also, in actual fact, that of Betti) it is difficult to draw a clear line between the works in analysis and those in mathematical physics. We will attempt in any case to do this – in order to provide an orientation within a body of work that is not only vast, but also suggests right away several different ways in which it can be read – but it must be noted that this distinction has to be applied with a certain degree of caution, in part because in the articles of 'physics' the use of instruments of analysis is far from marginal, and in part because those we will call 'analytical' very often comprise points of view, examples and applications drawn from physics.

In any case, the various works – whether of analysis or of mathematical physics – are never reduced to a mere proof of a theorem. Rather, theorems, corollaries and observations are used as the means for developing a genuine scientific discourse.

[2]The quote appears in a letter from Pascal to the mathematics historian Federico Amodeo, dated 23 November 1887, published in *Dalla "Moderna Geometria" alla "Nuova Geometria Italiana" Viaggiando per Napoli, Torino e dintorni. Lettere Di: Sannia, Segre, Peano, Castelnuovo, D'Ovidio, Del Pezzo, Pascal e Altri a Federico Amodeo*, Franco and Nicla Palladino, eds. (Florence: Olshki, 2006).

Sometimes the investigations almost necessarily include long prefaces or theoretical 'parentheses' in order to construct the instruments for their formalisation; other times they are more absorbed in the possibility of solving new problems in physics or making a contribution to the clarification of older experimental observations. In presenting some of Volterra's research works, we will have several opportunities to underline their strong unitary character. For the moment we will limit ourselves to providing an example of the style he adopted, by quoting a passage from 'Sopra alcune condizioni caratteristiche delle funzioni di una variabile complessa' (On some characteristic conditions of functions of a complex variable). This long, involved paper – about 50 pages! – dated 12 May 1882 includes a note saying that the author is a student at the Scuola Normale. In the introduction to this work of – according to us – analysis, we read that 'the present paper provides a solution to the problem of the determination of functions of complex variables defined under certain conditions in finite fields. These solutions lead to the integration of the differential equation $\Delta^2 \mu = 0$ with given boundary conditions, as can be expected due to the connections between the two problems. It should be noted that the formulae found solve an equal number of questions of physics relative to the distribution of temperature and constant galvanic currents'.

For the time being, let's set aside this unifying aspect and for the sake of convenience adopt a more rigid classification. Volterra's first paper – the one that Betti would write about in his report to the Minister of Public Education, published in the *Nuovo Cimento* – dealt with the calculation of the potential of an ellipsoid. In all, during the period in Pisa and up to the beginning of the 1890s, Volterra would publish no fewer than about 20 (!) articles on mathematical physics. This output is mainly concentrated between 1882 and 1885, and in the final years of his time in Pisa. It goes from potential theory to the first observations of elasticity, passing through various questions of hydrodynamics, electrochemistry (suggested to him by Antonio Ròiti), mechanics, optics, electrostatics and electrodynamics (with particular emphasis on analytical aspects and relations to the calculus of variations).

The most important of these studies, 'Sur les vibrations dans les milieux birefringents' (On vibrations in birefringent media), was published in 1891 in *Acta Mathematica*, the journal published by the Swedish mathematician Gösta Mittag-Leffler (1846–1927), and this time it wasn't preceded by a earlier paper on the topic published with the Lincei. It was concerned with analysing the mathematical laws of the propagation of light in doubly refracting materials and substances, and their property of decomposing incident rays into two polarised rays that vibrate in planes that are perpendicular to each other.

The problem had a long history. It had already been addressed by Christiaan Huygens in his 1690 *Traité de la lumière*. Later it attracted the attention of physicists and mathematicians such as Fresnel, Hamilton, Plücker, Navier, Cauchy, Green and Stokes, with an increasing focus on the model provided by the theory of elasticity.[3]

[3]See the essay by Lars Gårding, 'History of the Mathematics of Double Refraction', *Archive for History of Exact Sciences* 40, 4 (1989): pp. 355–385.

In 1860, the French mathematician and engineer Gabriel Lamé published a collection of his lectures at the Sorbonne in a volume entitled *Théorie mathématique de l'elasticité des corps solides*. This is what Volterra had to say: 'Lamé dedicated the 22nd and 23rd of these lessons to the mathematical theory of elasticity in search of a single centre of disturbance in the propagation of light in birefringent media'. In order to study the phenomenon from a mathematical point of view, Lamé formulated a system of partial differential equations for which he found special solutions. These formed the basis for the consideration of the Russian mathematician Sophie Kowalevski, who also relied on a method used earlier by Weierstrass to solve simpler systems. The aim was to get beyond some of the physical inconsistencies in Lamé's reasoning, while at the same time obtaining the general solution to the system. Her research was published in 1885 in *Acta Mathematica*. Mittag-Leffler had been able to bring her academic wandering to an end, by finding her a teaching position (and later a chair) at the University of Stockholm. Even the Swedish intellectuals had to struggle to accept the fact that a woman could dedicate herself to scientific research. The playwright August Strindberg was outraged over her appointment, considering a woman professor of mathematics to be a 'monstrous' phenomenon, 'pernicious, useless, and unpleasant'.

Sophie Kowalevski (1850–1891) was the first woman in the world to receive a doctorate in mathematics and to enjoy significant fame in the scientific world. To find another woman mathematician of her calibre you have to go back to Maria Gaetana Agnesi (1715–1799). Of Russian nationality, in order to be able to continue her studies and be granted a passport, she 'invented' a marriage of convenience. This is how she was able to study in Germany, where in 1874 she earned her doctorate under the advisement of Karl Weierstrass. Earlier she had been in Paris, during the time of the Paris Comune. Her political ideas were not very 'orthodox', and this did not help improve her situation, nor her prospects for an academic career. Luckily, as mentioned, she found the support of mathematicians such as Weierstrass and Mittag-Leffler. In 1888, she was awarded the *Prix Bordin* of the French Academy of Science. It would not be until 1908 that another woman – Marie Curie – obtained a university chair.

Going back to Volterra, this explains why he published his work in *Acta Mathematica*, because it is the journal that published the work of Kowalevski. Volterra's contribution grew out of a critical examination of the procedure followed by Lamè. The transformation of his solution into another form made it possible to uncover an error that even Kowalevski had missed, and which thus nullified her search for the general solution: 'This property [of being multi-valued functions] is hardly perceived at first glance, when one examines these quantities in the form that Lamé had given them. This is why he was mistaken when he believed that they could represent the luminous vibrations coming from a centre of disturbance. The same functions appear in the paper of Mme. Kovalevskii. When one sees that they are multi-valued functions, one also sees that the method discovered by Weierstrass for the integration of the partial differential equations can be used for integrating the Lamé equations by using the coordinates of Weber'. On 17 April 1892, he wrote in more explicit terms to the French physicist and philosopher of science Pierre

Duhem: 'Some time ago I began research in the electromagnetic theory of light, but I have had to interrupt my work because I recognised, to my great surprise, that the integrals relative to double refraction given by Lamé and Mme. Kowalevski were affected by the same analytical error, although the starting points for the geometers who have studied this question were different'.

As was the case in mathematical physics, in the period Volterra spent in Pisa, his publications concerning analysis numbered about 20! After his papers of 1881, which we discussed in the previous chapter, Volterra never again addressed questions related to the foundations of the discipline, such as the relationships between derivation and integration. This was either a fortunate choice, or happy intuition, given that both the interest in rigorisation of real analysis and the moment of Dini's greatest influence were already past their prime. It was Betti who, with increasing frequency, presented the articles of his former student and by then young colleague to the Academica dei Lincei.

The problems concerned regarded above all the study of functions of complex variables and differential equations (ordinary and partial). In 1890 there appeared two articles on the calculus of variations. In 'Sopra un'estensione della teoria di Jacobi-Hamilton del calcolo delle variazioni' (On an extension of the Hamilton-Jacobi theory of the calculus of variations), Volterra generalised to double integrals the (partial) differential equations of Hamilton-Jacobi, which at the time had only been used in the so-called 'simpler problem' (which deals with determining the function $f(x)$ that makes the quantity $J = \int_a^b f(y, y, y')dx$ minimum or maximum), considering J as a function of a, b, $y(a)$, $y(b)$. For Volterra, this provided another opportunity to use his new theory of functions of lines: 'If one goes from simple integrals to the case of double integrals, we have one or more lines forming the boundary of the field of integration and on this must lie the arbitrary values of the unknown function'. With the functions of lines, we arrive to the most significant research of Volterra's years in Pisa. Functional analysis was invented by Volterra – a young man just over 25 years old! – with two papers published in 1887 entitled 'Sopra le funzioni che dipendono da altre funzioni' (On functions that depend on other functions) and 'Sopre le funzioni dependenti da linee' (On functions that depend on lines), which were followed by others – very closely related – in which the focus was moved to certain questions of complex analysis. These included 'Sopra un'estensione della teoria di Riemann nelle funzioni di variabili complesse' (On an extension of Riemann's theory of functions of complex variables), 'Sur une généralisation de la théorie des functions d'une variable imaginaire' (On a generalisation of the theory of functions of an imaginary variable, published in *Acta Mathematica*, Mittag-Leffler's journal) and 'Delle variabili complesse negli iperspazi' (On complex variables in hyperspaces).

The term *functional analysis* refers to the study of functionals (and later, their spaces). At that time Volterra did not yet use term, which would be coined in 1903 by Jacques Hadamard (1865–1963), one of the most important mathematicians of the first half of the twentieth century, and another of Volterra's close French friends. He preferred to use the terms that appear in the titles of his papers: 'functions that depend on other functions' and 'functions that depend on lines'. The meaning is the same, even though the term 'functional' would be adopted by later generations, and would become the definitive one. What is a functional, or a function that depends on lines?

The generalisation of the concept of function is simple. While we use the term *real function of a real variable* to indicate a correspondence that associates a real number to another real number, term functional is used to indicate a correspondence between any given set and a real number. In Volterra's terminology, which was initially less general, it is a correspondence which associates a real number to an element of the set of continuous functions or of the curve that represent them geometrically. He thus talks about a function that depends on another function – a concept that must must not be confused with that of a composite function – or a function that depends on a line. The generalisation involves the independent variable of the correspondence, not the dependent variable. In the passage from functions to functionals, everything that has been said regarding the behaviour of the independent variable (for example, its approaching a limit value) will have to be reformulated, while those that involve the dependent variable can be 'copied' without qualms because – from this point of view – nothing has changed.

For mathematicians, ideas for new research can come from previous studies – these are the so-called *internal* motivations – where further reflection is aimed at making concepts more precise and extending them. Ideas for research, however, can also come from observing nature in the real world – the so-called *external* motivations. Of course, as far as mathematics is concerned, especially modern mathematics, the idea of 'real world' is understood in an indirect sense. There might be cases where a mathematician studies planetary motion or a particular upheaval in the stock market and is able right away to formulate a new theorem, but more often it happens that the echoes of the real world filter in from others – astronomers (for planetary motion) or economists (for the stock market) – and that the mathematician actually works with a world that is already on paper, that is, the world of astronomers or economists. Both internal and external motivations were at play in the generalisation performed by Volterra with functionals, in a way that was something of a paradigm. In the paper published in *Acta Mathematica* cited above, he wrote: 'These functions arise in a number of questions in physics. They can also be related to analytical questions'. Later he would reaffirm this: 'I realised the necessity of considering the functions of lines because many natural phenomena lead to the study of quantities that depend on an infinite number of variables. Many problems of analysis also lead to the same quantities'.[4]

Volterra was driven to invent the theory of functionals by the ascertainment that expressions that depend on other functions are present in many analytical developments, such as, for example, the soluion of partial differential equations (where an integral comes to depend on one or more abitrary functions) or in complex analysis. In fact, one of the initial motivations was that he expected to be able to apply the new theory successfully to some studies of complex analysis:

[4]Vito Volterra, 'Funzioni di linee, equazioni integrali e integro-differenziabilii', *Anales de la Sociadad Cientifica Argentina*, 1921.

'I allow myself to mention in this paper some of the considerations that serve to clarify the concepts I believe it necessary to introduce in order to extend Riemann's theory on functions of complex variables, which I think can be used to good advantage in various other research projects'.[5] In the same year, 1877, Poincaré proved that the integral of a function of two complex variables, extended to a surface contained in the domain set of its boundary, turns out to depend only on this. Volterra observed: 'Poincaré, in generalizing Cauchy's theorem, has shown that the integral of a uniform function of two imaginary variables taken over a closed surface is zero. It can be inferred from this that, when the surface of integration is not closed, the integral depends on the lines that form the contour of the surface. So we see that the integration of functions of two variables leads to functions of the lines'.[6]

These are internal motivations. The external references are to the 'many experiments in physics and mechanics' where likewise, spontaneously, the concept of function can be glimpsed: 'for example, the temperature at a point in a thin plate conductor depends on all the values of the temperature at the edges; the infinitesimal displacement of a point on a non-extendable flexible surface depends on all of the components of displacement of the points on the edges, parallel to a given direction'. Even the geometric version of functionals, considered in the work on the set of closed curves in a three-dimensional space, is familiar to the physicist, given that 'it appears spontaneously when one thinks of certain electrical phenomena. Think of an electric current that flows in a closed linear circuit with an intensity equal to I and that is located in a magnetic field. The potential energy of the current will depend only on the shape, the position of the circuit and on the direction in which the current flows; thus to each closed line that is traced in the magnetic field followed in a given direction, there will correspond a value of potential energy'.

The originality of the internal motivations did not escape Hadamard, who underlined the almost aesthetic value of the theory formulated by Volterra: 'much more surprising is the fate of the extension given to that initial conception in the last part of the nineteenth century, chiefly under the powerful impulse of Volterra. Why was the great Italian geometer lead to work on functions in the same way in which infinitesimal calculus had worked on numbers, that is, to consider a function as a continuously variable element? Only because he realised that it this was a harmonious way of completing the architecture of the mathematical building, just as the architect sees that the building would be better poised by the addition of a new wing'. The French mathematician did not mention what we have called external motivations, but he cannot help but underline the unexpected applicability of the new concepts: 'that 'functionals', as we called the new conception, could be in direct relation with reality, could not be thought of otherwise as mere absurdity.

[5]Vito Volterra, 'Sopra le funzioni che dipendono da altre funzioni', *Rendiconti Accademia dei Lincei* 4, 3 (1887): pp. 97–105, 141–146 and 153–158.

[6]Vito Volterra, 'Sur une généralisation de la théorie des fonctions d'une variable imaginaire', *Acta Mathematica* 12 (1889): pp. 233–286.

Functionals seemed to be an essentially and completely abstract concept of mathematicians. Now, precisely the absurd has happened'.[7]

Once the concept of functionals had been introduced, in order to work with this new mathematical reality it was necessary to construct a *calculus* that was similar to the classic one known to analysis for functions of a real variable: the notion of limit, the definition and the calculation of derivatives, and so forth. The functional analysis of the twentieth century, beginning with Fréchet's doctoral thesis of 1906,[8] studied functionals as definite correspondences on a given set and tended to privilege the study of minimal structures that had to be introduced into that set in order to develop the new calculus. One of the first definitions regarded the *metric structure*, which made it possible to specify the concept of *neighbourhood* and consider the operations of limit (and thus of derivative as well).

None of this is yet present in Volterra. He had no need for an abstraction 'pushed' this far, and for him the motivations behind his studies remained essential for determining the level of abstraction. Volterra's functions of lines – it is no coincidence that the terminology is also different – are definite correspondences to a 'concrete' set and are in any case specific, such as that constituted of functions that are continuous in an interval. There is no general definition of metric space. The definition of derivative itself still refers to the limit of a real parameter (and to a procedure that is familiar to all those who work with the calculus of variations): considering an initial function f_0 and the corresponding value of the functional U (f_0), the incremented function is written as $f_0 + tg$ where t is a real parameter and g is a pre-established function that continues to give, as t tends to *0*, the 'direction' of the increment. The limit of $[\mathrm{U}(f_0 + tg) - \mathrm{U}(f_0)]/t$ for t that tends to *0*, is the first derivative of the functional with respect to direction g.

The introduction to his 1887 paper is not written in the terse, unadorned style that we are accustomed to today, especially when the definitions are given. At the end of the nineteenth century, the habit of 'hiding' the motivations and the mental structures that led to the introduction of a certain definition was not widespread among mathematicians. Nor does Volterra spend much time examining the primary properties of the derivative of a functional. Instead, he is concerned with calculating the expressions of the different variations of a functional (first, second, …nth) because their consequential development with *Taylor's formula* serves him for the analysis that will follow. It is the applications – including applications in mathematics, to be sure – that dictate the course for the procedure. The point of reference is research as a whole, not passion for in-depth study as an end in itself.

We spoke earlier of the paradigms that are constantly present in Volterra's work. The first of these that we have the chance to underline is that of the juxtaposition and interweaving of 'pure' research and research that is in some way applied, of

[7] Jacques Hadamard, *The Mathematician's Mind: Psychology of Invention in the Mathematical Field* (Princeton: Princeton University Press, 1996), p. 130 (1st ed., 1945).

[8] Maurice Fréchet, 'Sur quelques points du calcul fonctionnel', *Rendiconti del Circolo Matematico di Palermo* 22, 1 (1906): pp. 1–72.

analysis and mathematical physics, of internal and external motivations. This leads us naturally to acknowledge the particular importance of applications, and the role that the instance of application plays in identifying the most suitable level of abstractions. We will see some aspects of this come into play a few decades later in a controversy that involved Volterra and the French mathematician Fréchet (1878–1973) regarding precisely the definition of derivative of a functional. Fréchet, author of important contributions to analysis and probability, is considered one of the founders of functional analysis. We have already mentioned his doctoral thesis of 1906, which dealt with the study of metric spaces.

Volterra's definition of the derivative of a functional – later referred to in the literature as the Gâteaux derivative[9] – generalises the usual concept of directional derivative in the choice of the increment *tg* and its progressive reduction only by means of the scalar *t*, which, tending towards *0*, reduces the incremental but leaves its form in some ways unaltered. Later, Fréchet introduced a definition of the differential of a funtional that generalises the usual concept of total differential for functions of *n* variables. The difference between the two procedures was immediately evident when Fréchet wrote the incremented function as $f_0 + g$ and evaluated the validity of the linear approximation of $[\mathrm{U}(f_0 + g) - \mathrm{U}(f_0)]$ in the whole neighbourhood of f_0 by means of the limit for g tending to 0. It is easy to compare the two definitions: that of Fréchet considers a generic increment g, making it go to 0 regardless of how it does so, and provides a more precise identity for the class of functionals – those that are 'Fréchet differentiable' – destined to play a fundamental role in *calculus*. A functional that is Fréchet differentiable is also Gâteaux differentiable, while the inverse is not true (Fig. 2.1).

Behind the two different definitions there are also a different ways of thinking about the concept of generalisation and the appropriate level of abstraction. In 1965, Fréchet wrote to another famous French mathematician, Paul Lévy: 'While I think that Volterra achieved a great step forward in giving at least one definition of the differential of a function whose argument is a function, on the other hand, I think that the definition is bad. I have not applied his definition . . . it is a definition entirely different from that of Volterra'. Earlier, in somewhat more diplomatic terms, he had written that 'the traditional method, due to Lagrange, is not to treat the functions of lines similarly to the functions of numbers, but to pass through functions calculated on a family of lines depending on a parameter and which become none other than functions of the numeric parameter. This is also the method adopted by Volterra for more general functions (of lines) studied by functional analysis. We believe that it is better to get to the bottom of things and avoid problems by abandoning the intermediate parameter and taking the line directly as an absolutely independent variable'.

[9]On the reasons that led – at least on the level of terminology – to the collocation of Volterra's contribution in second place, see one of Angelo Guerraggio's two 'updates' of Carl Boyers classic text, *Storia del Calcolo* (Milan: Bruno Mondadori, 2007).

Fig 2.1 Maurice Fréchet in 1910

Volterra did not discuss the terms of the disagreement, but he never missed an opportunity to reiterate that the development of mathematical thought cannot be guided only by aesthetic or formal criteria, by the elegance of a construction or its greatest generality. Addressing Fréchet directly in a letter of 17 November 1913, he wrote that ‘naturally I had before me at that time, in 1887, a number of problems (integral equations, equations with functional derivatives, etc.) so that I could not stop for what from my point of view were issues secondary to the general applications of concepts I had laid out’.

2.3 The First Trips Abroad

The results we have discussed were soon well known even beyond Italy’s borders. We have already mentioned the first honours Volterra received in Italy. He also received – as early as the second half of the 1880s – similar honours in the international arena, beginning with France, which it is no exaggeration to say was always thought of by Volterra as a second homeland.

In 1888 Volterra became a member of the *Société Mathématique de France*. This happened in the same year as his first trip abroad. In Pisa, while still a student, he had met the Swedish mathematician Gösta Mittag-Leffler, whom we have already come across in relation to Sophie Kowalevski. That meeting gave rise to reciprocal esteem – even on the part of the already established Swedish professor for the young Italian student – which over the course of the years turned into a lifelong friendship. Mittag-Leffler died in 1927. He was known for his work on

Fig. 2.2 Henri Poincaré

differential equations and on functions of a complex variable, and for the very active role he played in the international mathematics community, especially as a citizen of a neutral country in the aftermath of dramatic conflicts, such as the Franco-Prussian war of 1870 and World War I. In 1882, at the suggestion of the Norwegian mathematician Sophus Lie, he had founded the international journal entitled *Acta Mathematica*. With Mittag-Leffler, Volterra visited Switzerland, and had the chance to meet important, esteemed mathematicians like Karl Weierstrass, Georg Cantor, Sophie Kowalevski, Hermann Schwarz and Adolf Hurwitz. Later, during the Easter holidays of that same year, 1888, he travelled to Paris, where he met Poincaré. Poincaré invited him to come again the following year to the French capital to take part in the *Congrés international de bibliographie des sciences mathématique*, organised in concomitance with the Exposition Universelle of 1889. This was the World's Fair that celebrated the centenary of the French Revolution, and was highlighted by the inauguration of the Eiffel Tower! Where once there had been few opportunities for leading figures of the various national communities to meet, the idea for the first international congresses (Zurich 1897 and Paris 1900) was 'in the air'. Volterra attended the International Congress for Bibliography of Mathematical Science together with Giovan Battista Guccia, who had shortly before, in 1884, founded the Circolo Matematico of Palermo, which would soon enjoy significant international renown and esteem (Fig. 2.2).

Letters from these years show how Volterra had by that time entered into a 'circle' of correspondence and acquaintances with figures who were particularly esteemed and knowledgeable. It was, however, the meeting with Poincaré that would be decisive, much more than a mere acquaintance with a foreign colleague that made it possible to intensify the network of scientific contacts. First of all, there was the stature of the correspondent, whose family was destined to leave their mark on the history of France (and beyond): Raymond Poincaré, several times prime minister and president of the Republic during the years of World War I, was his cousin. Apart from this, in the decades at the turn of the century, Henri Poincaré (1854–1912) – mathematician, physicist, philosopher of science – was undoubtedly one of the leading figures of world science. In the year of the World's Fair and the *Congrés international*, his reputation took a great leap forward. In 1889 he won the competition established by Oscar II, King of Sweden and Norway, in honour of both his sixtieth birthday and of his royal highness's interest in mathematical progress. His paper, 'Sur le probléme de trois corps et les équations de la dynamique' (On the three-body problem and the equations of dynamics) was soon followed by his three-volume *Les methodes nouvelle de la mécanique celeste* (*New Methods of Celestial Mechanics*). It was precisely the study of dynamic systems in the context of celestial mechanics and their stability that gave rise to what we now call the theory of *deterministic chaos*. The germ was provided by an observation on the sensitivity of the evolution of the dynamic system with respect to a slight perturbation of the initial conditions: 'it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible'.[10] Poincaré created new mathematics without worrying about disciplinary confines or traditional distinctions between pure and applied mathematics, driven as he was by thirst for scientific comprehension of natural phenomena. *Pure* and *applied* were categories that were too narrow for him. He was perhaps the last 'universal' mathematician in history. His contributions ranged from mathematical physics (celestial mechanics, fluid mechanics, relativity, cosmology, optics, electricity, etc.) to the study of functions of complex variables, the qualitative theory of differential equations, non-Euclidean geometries, algebraic topology and number theory.

Volterra was particularly struck by this broadly encompassing way of conceiving and practising science. He was fascinated by the constant concern for uniting the rigour of calculus with the necessity of understanding physical reality. It was no coincidence that he repeatedly proposed Poincaré for the Nobel Prize in physics. Instead, he was called to commemorate him at Rice Institute in Houston on 10 October 1912, a few months after his death. On that occasion he recalled that for him Poincaré had been both a *maestro* and a constant source of inspiration. Volterra's sincere appreciation for Poincaré's genius and the contributions he made are evident throughout the discourse, as are the fact that he shared many of the choices that accompanied the investigations of the French mathematician:

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[10] Henri Poincaré, *Science and Method*, Francis Maitland, trans. (London: Thomas Nelson and Son, 1914), p. 68.

a mathematics at once both pure and applied, not one domesticated within rigid, predetermined classifications, one deeply and formally involved, never in any case reduced to a simple logical development of formal rules, and in which intuition and experience continued to play essential roles. Volterra wrote: 'During these last 30 years there has been no new question, connected even remotely with mathematics, which he did not subject to his deep and delicate analysis, and enrich with some discovery or fruitful point of view. I believe that no scientist so much as he lived in constant and intimate relation with the scientific world that surrounded him. … That is why, if we were to characterize the recent period of the history of mathematics by a single name, we should all give that of Poincaré, for he has been without doubt the most widely known and celebrated mathematician of recent years. … There is certainly a philosophy that is Poincaré's, and an analysis, a mathematical physics and a mechanics that are Poincaré's, which science can never forget. His renown during his life was great. Few scientists and a very few mathematicians have had celebrity equal to his. A physicist would find the reason for this in what I have just been saying, remarking that his spirit and the spirit of his time vibrated in unison, and that he was in phase with the universal vibration'.[11]

To return to our discussion of trips abroad, because in addition to French, Volterra knew German (though little English), he took advantage of that to plan a month of study in Germany. His destination was the University of Göttingen, where mathematicians of the calibre of Dirichlet and Riemann had taught, and which was to become – thanks to the impulse of Felix Klein – the liveliest centre for research in mathematics. In Göttingen Volterra had the opportunity to see Schwarz, whom he had met in Switzerland. He then went for a few days to Berlin, where he met Leopold Kronecker, one of the most important German mathematicians of the day. This was the summer of 1891. Volterra returned to Pisa in September, but his academic life would soon undergo a change.

[11] Vito Volterra, 'Henri Poincaré', Griffith Conrad Evans, trans. *The Rice Institute Pamphlet Volume Four* (Houston: Rice Institute, 1915), pp. 133–162; quotes from pp. 134, 138.

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