Chapter 2
Linear DAEs with variable coefficients

In this chapter we provide a comprehensive analysis of linear DAEs

$$A(t)(D(t)x(t))' + B(t)x(t) = q(t), \quad t \in \mathcal{I},$$

with properly stated leading term, by taking up the ideas of the projector based decoupling described for constant coefficient DAEs in Chapter 1. To handle the time-varying case, we proceed pointwise on the given interval and generate admissible sequences of matrix functions $G_i(\cdot) = G_{i-1}(\cdot) + B_{i-1}(\cdot)Q_{i-1}(\cdot)$ associated with admissible projector functions $Q_i(\cdot)$, instead of the former admissible matrix sequences and projectors. Thereby we incorporate into the matrix function $B_i(\cdot)$ an additional term that comprises the variations in time. This term is the crucial one of the generalization. Without this term we would be back to the so-called *local matrix pencils* which are known to be essentially inappropriate to characterize time-varying DAEs (e.g., [25, 96]). Aside from the higher technical content in the proofs, the projector based decoupling applies in precisely the same way as for constant coefficient DAEs, and fortunately, most results take the same or only slightly modified form.

In contrast to Chapter 1 which is restricted to square DAE systems, that means, the number of unknowns equals the number of equations, the present chapter is basically valid for systems of k equations and m unknowns. Following the arguments e.g., in [130], so-called rectangular systems may play their role in optimization and control. However, we emphasize that our main interest is directed to regular DAEs, with $m = k$ by definition. Nonregular DAEs, possibly with $m \neq k$, are discussed in more detail in Chapter 10.

We introduce in Section 2.1 the DAEs with properly stated leading term and describe in Section 2.2 our main tools, the admissible matrix function sequences associated to admissible projector functions and characteristic values. Widely orthogonal projector functions in Subsection 2.2.3 form a practically important particular case. The analysis of invariants in Section 2.3 serves as further justification of the concept.
The main objective of this chapter is the comprehensive characterization of *regular DAEs*, in particular, in their decoupling into an *inherent regular explicit ODE* and

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a subsystem which comprises the *inherent differentiations*. We consider the constructive existence proof of *fine* and *complete* decouplings (Theorem 2.42) to be the most important special result which describes the DAE structure as the basis of our further investigations. This leads to the intrinsic DAE theory in Section 2.6 offering solvability results, flow properties, and the *T-canonical form*. The latter appears to be an appropriate generalization of the Weierstraß–Kronecker form. Several specifications for *regular standard form DAEs* are recorded in Subsection 2.7. Section 2.9 reflects aspects of the critical point discussion and emphasizes the concept of *regularity intervals*.
In Section 2.10 we explain by means of canonical forms and reduction steps how the *strangeness* and the tractability index concepts are related to each other.

2.1 Properly stated leading terms

We consider the equation

$$A(Dx)' + Bx = q, \tag{2.1}$$

with continuous coefficients

$$A \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^n, \mathbb{R}^k)), \quad D \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^m, \mathbb{R}^n)), \quad B \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^m, \mathbb{R}^k)),$$

and the excitation $q \in \mathcal{C}(\mathcal{I}, \mathbb{R}^k)$, where $\mathcal{I} \in \mathbb{R}$ is an interval. A solution of this equation is a function belonging to the function space

$$\mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m) := \{x \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m) : Dx \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n)\},$$

which satisfies the DAE in the classical sense, that is, pointwise on the given interval.

The two coefficient functions A and D are to figure out precisely all those components of the unknown function, the first derivatives of which are actually involved in equation (2.1). For this, A and D are supposed to be well matched in the sense of the following definition, which roughly speaking means that there is no gap and no overlap of the factors within the product AD and the border between A and D is smooth.

Definition 2.1. The leading term in equation (2.1) is said to be *properly stated* on the interval $\mathcal{I}$, if the transversality condition

$$\ker A(t) \oplus \operatorname{im} D(t) = \mathbb{R}^n,\ t \in \mathcal{I}, \tag{2.2}$$

is valid and the projector valued function $R : \mathcal{I} \to L(\mathbb{R}^n)$ defined by

$$\operatorname{im} R(t) = \operatorname{im} D(t),\ \ker R(t) = \ker A(t), \quad t \in \mathcal{I},$$

is continuously differentiable.

The projector function $R \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^n))$ is named the *border projector* of the leading term of the DAE.

To shorten the phrase *properly stated leading term*, sometimes we speak of *proper leading terms*.
We explicitly point out that, in a proper leading term, both involved matrix functions A and D have necessarily constant rank. This is a consequence of the smoothness of the border projector R (see Lemma A.14).

Applying the notion of $\mathcal{C}^1$-subspaces (Definition A.19, Appendix A), a proper leading term is given, exactly if $\operatorname{im} D$ and $\ker A$ are transversal $\mathcal{C}^1$-subspaces. Equivalently (see Lemma A.14), one has a proper leading term, if condition (2.2) is satisfied and there are basis functions $\vartheta_i \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n),\ \ i = 1, \dots, n$, such that

$$\operatorname{im} D(t) = \operatorname{span}\{\vartheta_1(t), \dots, \vartheta_r(t)\}, \quad \ker A(t) = \operatorname{span}\{\vartheta_{r+1}(t), \dots, \vartheta_n(t)\}, \quad t \in \mathcal{I}.$$

Having those basis functions available, the border projector R can simply be represented as

$$R := [\vartheta_1 \dots \vartheta_n] \begin{bmatrix} \underbrace{I}_{r} & \\ & 0 \end{bmatrix} [\vartheta_1 \dots \vartheta_n]^{-1}. \tag{2.3}$$

If A and D form a properly stated leading term, then the relations

$$\operatorname{im} AD = \operatorname{im} A,\ \ \ker AD = \ker D,\ \ \operatorname{rank} A = \operatorname{rank} AD = \operatorname{rank} D =: r$$

are valid (cf. Lemma A.4), and A, AD and D have *common constant rank* r on $\mathcal{I}$.

Besides the coefficients A, D and the projector R we use a pointwise generalized inverse $D^- \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^n, \mathbb{R}^m))$ of D satisfying the relations

$$DD^-D = D,\ \ D^-DD^- = D^-,\ \ DD^- = R. \tag{2.4}$$

Such a generalized inverse exists owing to the constant-rank property of D. Namely, the orthogonal projector P_D onto $\ker D^\perp$ along $\ker D$ is continuous (Lemma A.15). If we added the fourth condition $D^-D = P_D$ to (2.4), then the resulting D^- would be uniquely determined and continuous (Proposition A.17), and this ensures the existence of a continuous generalized inverses satisfying (2.4).

By fixing only the three conditions (2.4), we have in mind some more flexibility. Here $D^-D =: P_0$ is always a continuous projector function such that $\ker P_0 = \ker D = \ker AD$. On the other hand, prescribing P_0 we fix, at the same time, D^-.

Example 2.2 (Different choices of P_0 and D^-). Write the semi-explicit DAE

$$\begin{aligned} x_1' + B_{11}x_1 + B_{12}x_2 &= q_1, \\ B_{21}x_1 + B_{22}x_2 &= q_2, \end{aligned}$$

with $m_1 + m_2 = m$ equations in the form (2.1) with properly stated leading term as

$$A = \begin{bmatrix} I \\ 0 \end{bmatrix},\ D = \begin{bmatrix} I & 0 \end{bmatrix},\ B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

such that $\ker A = \{0\}$, $\operatorname{im} D = \mathbb{R}^{m_1}$ and $R = I$. Any continuous projector function P_0 along $\ker D$ and the corresponding generalized inverse D^- have the form

$$P_0 = \begin{bmatrix} I & 0 \\ \mathfrak{A} & 0 \end{bmatrix}, \quad D^- = \begin{bmatrix} I \\ \mathfrak{A} \end{bmatrix},$$

with an arbitrary continuous block $\mathfrak{A}$. The choice $\mathfrak{A} = 0$ yields the symmetric projector P_0. □

2.2 Admissible matrix function sequences

2.2.1 Basics

Now we are ready to compose the basic sequence of matrix functions and subspaces to work with. Put

$$G_0 := AD,\ \ B_0 := B,\ \ N_0 := \ker G_0 \tag{2.5}$$

and choose projector functions $P_0, Q_0, \Pi_0 \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^m))$ such that

$$\Pi_0 = P_0 = I - Q_0,\ \ \operatorname{im} Q_0 = N_0.$$

For $i \geq 0$, as long as the expressions exist, we form

$$G_{i+1} = G_i + B_i Q_i, \tag{2.6}$$

$$N_{i+1} = \ker G_{i+1}, \tag{2.7}$$

choose projector functions P_{i+1}, Q_{i+1} such that $P_{i+1} = I - Q_{i+1},\ \operatorname{im} Q_{i+1} = N_{i+1}$, and put

$$\Pi_{i+1} := \Pi_i P_{i+1},$$

$$B_{i+1} := B_i P_i - G_{i+1} D^- (D\Pi_{i+1} D^-)' D\Pi_i. \tag{2.8}$$

We emphasize that B_{i+1} contains the derivative of $D\Pi_{i+1}D^-$, that is, this term comprises the variation in time. This term disappears in the constant coefficient case, and then we are back at the formulas (1.10) in Chapter 1. The specific form of the new term is motivated in Section 2.4.1 below, where we consider similar decoupling rearrangements for the DAE (2.1) as in Chapter 1 for the constant coefficient case.
We are most interested in continuous matrix functions G_{i+1}, B_{i+1}; in particular we have to take that $D\Pi_{i+1}D^-$ is smooth enough.
Important characteristic values of the given DAE emerge from the rank functions

$$r_j := \operatorname{rank} G_j,\ \ j \geq 0.$$

Example 2.3 (Matrix functions for Hessenberg size-1 and size-2 DAEs). Write the semi-explicit DAE

$$\begin{aligned} x_1' + B_{11}x_1 + B_{12}x_2 &= q_1, \\ B_{21}x_1 + B_{22}x_2 &= q_2, \end{aligned}$$

with $m_1 + m_2 = m$ equations in the form (2.1) as

$$A = \begin{bmatrix} I \\ 0 \end{bmatrix}, \; D = \begin{bmatrix} I & 0 \end{bmatrix}, \; B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \; D^- = \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

Then we have a proper leading term and

$$G_0 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \; Q_0 = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, \; G_1 = \begin{bmatrix} I & B_{12} \\ 0 & B_{22} \end{bmatrix}.$$

Case 1:
Let B_{22} be nonsingular on the given interval. Then G_1 is also nonsingular. It follows that $Q_1 = 0$, thus $G_2 = G_1$ and so on. The sequence becomes stationary. All rank functions r_i are constant, in particular $r_0 = m_1,\ r_1 = m$.
Case 2:
Let $B_{22} = 0$, but the product $B_{21}B_{12}$ remains nonsingular. We denote by Ω a projector function onto $\operatorname{im} B_{12}$, and by B_{12}^- a reflexive generalized inverse such that $B_{12}B_{12}^- = \Omega,\ B_{12}^-B_{12} = I$. The matrix function G_1 now has rank $r_1 = m_1$, and a nontrivial nullspace. We choose the next projector functions Q_1 and the resulting $D\Pi_1 D^-$ as

$$Q_1 = \begin{bmatrix} \Omega & 0 \\ -B_{12}^- & 0 \end{bmatrix}, \quad D\Pi_1 D^- = I - \Omega.$$

This makes it clear that, for a continuously differentiable $D\Pi_1 D^-$, we have to assume the range of B_{12} to be a $\mathcal{C}^1$-subspace (cf. A.4). Then we form the matrix functions

$$B_1 = \begin{bmatrix} B_{11} & 0 \\ B_{21} & 0 \end{bmatrix} - \begin{bmatrix} -\Omega' & 0 \\ 0 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} I + (B_{11} + \Omega')\Omega & B_{12} \\ B_{21}\Omega & 0 \end{bmatrix},$$

and consider the nullspace of G_2.
$G_2 z = 0$ means

$$z_1 + (B_{11} + \Omega')\Omega z_1 + B_{12}z_2 = 0, \quad B_{21}\Omega z_1 = 0.$$

The second equation means $B_{21}B_{12}B_{12}^- z_1 = 0$, thus $B_{12}^- z_1 = 0$, and hence $\Omega z_1 = 0$. Now the first equation simplifies to $z_1 + B_{12}z_2 = 0$. Multiplication by B_{12}^- gives $z_2 = 0$, and then $z_1 = 0$. Therefore, the matrix function G_2 is nonsingular, and again

the sequence becomes stationary.
Up to now we have not completely fixed the projector function Ω onto $\operatorname{im} B_{12}$. In particular, we can take the orthoprojector function such that $\Omega = \Omega^*$ and $\ker \Omega = \ker B_{12}^* = \operatorname{im} B_{12}^{\perp}$, which corresponds to $B_{12}^- = B_{12}^+ = (B_{12}^* B_{12})^{-1} B_{12}^*$ and

$$\ker Q_1 = \{z \in \mathbb{R}^{m_1+m_2} : B_{12}^* z_1 = 0\}.$$

□

Example 2.4 (Matrix functions for a transformed regular index-3 matrix pencil). The constant coefficient DAE

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\bar{E}} \bar{x}'(t) + \bar{x}(t) = q(t), \quad t \in \mathbb{R},$$

has Weierstraß–Kronecker canonical form, and its matrix pencil $\{\bar{E}, I\}$ is regular with Kronecker index 3. By means of the simple factorization

$$\bar{E} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =: \bar{A}\bar{D}$$

we rewrite the leading term properly as

$$\bar{A}(\bar{D}\bar{x}(t))' + \bar{x}(t) = q(t), \quad t \in \mathbb{R}.$$

Then we transform $\bar{x}(t) = K(t)x(t)$ by means of the smooth matrix function K,

$$K(t) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -t & 1 \end{bmatrix}, \quad t \in \mathbb{R},$$

being everywhere nonsingular. This yields the new DAE

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\tilde{A}} (\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -t & 1 \end{bmatrix}}_{\tilde{D}(t)} x(t))' + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -t & 1 \end{bmatrix}}_{\tilde{B}(t)} x(t) = q(t), \quad t \in \mathbb{R}. \tag{2.9}$$

Next we reformulate the DAE once again by deriving

$$(\tilde{D}(t)x(t))' = (\tilde{D}(t) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t))' = \tilde{D}(t) \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t) \right)' + \tilde{D}'(t) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t),$$

which leads to the further equivalent DAE

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & -t & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{A(t)} \Bigg(\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} x(t) \Bigg)' + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -t & 1 \end{bmatrix}}_{B(t)} x(t) = q(t), \quad t \in \mathbb{R}. \tag{2.10}$$

Observe that the local matrix pencil $\{A(t)D, B(t)\}$ is singular for all $t \in \mathbb{R}$.
We construct a matrix function sequence for the DAE (2.10). The DAE is expected to be regular with index 3, as its equivalent constant coefficient counterpart. A closer look to the solutions strengthens this expectation. We have

$$A(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -t & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; D(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; B(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -t & 1 \end{bmatrix}, \; G_0(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -t & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

and $R(t) = D(t)$. Set $D(t)^- = D(t)$ and $\Pi_0(t) = P_0(t) = D(t)$. Next we compute $G_1(t) = G_0(t) + B(t)Q_0(t)$ as well as a projector $Q_1(t)$ onto $\ker G_1(t) = N_1(t)$:

$$G_1(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -t & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q_1(t) = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & t & 0 \end{bmatrix}.$$

This leads to

$$\Pi_1(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -t & 1 \end{bmatrix}, \; B_1(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -t & 1 \end{bmatrix}, \; G_2(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1-t & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

A suitable projector function Q_2 and the resulting B_2 and G_3 are:

$$Q_2(t) = \begin{bmatrix} 0 & -t & 1 \\ 0 & t & -1 \\ 0 & -t(1-t) & 1-t \end{bmatrix}, \; \Pi_2(t) = 0, \; B_2(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -t & 1 \end{bmatrix}, \; G_3(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1-t & 1 \\ 0 & -t & 1 \end{bmatrix}.$$

The matrix functions G_i, $i = 0,1,2$, are singular with constant ranks, and G_3 is the first matrix function that is nonsingular. Later on, this turns out to be typical for regular index-3 DAEs (cf. Definition 10.2), and meets our expectation in comparison with the constant coefficient case (cf. Theorem 1.31). At this place it should be mentioned that here the term $B_0P_0Q_1$ vanishes identically, which corresponds to the singular local matrix pencil. This fact makes the term $G_1D^-(D\Pi_1D)'D\Pi_0Q_1$ crucial for G_2 to incorporate a nontrivial increment with respect to G_1.
Observe that the nullspaces and projectors fulfill the relations

$$\begin{aligned} &N_0(t) \cap N_1(t) = \{0\}, \quad (N_0(t) + N_1(t)) \cap N_2(t) = \{0\}, \\ &Q_1(t)Q_0(t) = 0, \quad Q_2(t)Q_0(t) = 0, \quad Q_2(t)Q_1(t) = 0. \end{aligned}$$

The matrix functions G_i as well as the projector functions Q_i are continuous and it holds that $\operatorname{im} G_0 = \operatorname{im} G_1 = \operatorname{im} G_2 \subset \operatorname{im} G_3$. $\square$

Any matrix function sequence (2.5)–(2.8) generates subspaces

$$\operatorname{im} G_0 \subseteq \cdots \subseteq \operatorname{im} G_i \subseteq \operatorname{im} G_{i+1}$$

of nondecreasing dimensions.
To show several useful properties we introduce the additional projector functions $\mathcal{W}_j : \mathcal{I} \to L(\mathbb{R}^k)$ and generalized inverses $G_j^- : \mathcal{I} \to L(\mathbb{R}^k, \mathbb{R}^m)$ of G_j such that

$$\ker \mathcal{W}_j = \operatorname{im} G_j, \tag{2.11}$$

$$G_j G_j^- G_j = G_j,\ \ G_j^- G_j G_j^- = G_j^-,\ \ G_j^- G_j = P_j,\ \ G_j G_j^- = I - \mathcal{W}_j. \tag{2.12}$$

Proposition 2.5. *Let the DAE (2.1) have a properly stated leading term. Then, for each matrix function sequence (2.5)–(2.8) the following relations are satisfied:*

(1) $\ker \Pi_i \subseteq \ker B_{i+1}$,
(2) $\mathcal{W}_{i+1} B_{i+1} = \mathcal{W}_{i+1} B_i = \cdots = \mathcal{W}_{i+1} B_0 = \mathcal{W}_{i+1} B$,
$\mathcal{W}_{i+1} B_{i+1} = \mathcal{W}_{i+1} B_0 = \mathcal{W}_{i+1} B_0 \Pi_i$,
(3) $G_{i+1} = (G_i + \mathcal{W}_i B Q_i) F_{i+1}$ *with* $F_{i+1} = I + G_i^- B_i Q_i$ *and*
$\operatorname{im} G_{i+1} = \operatorname{im} G_i \oplus \operatorname{im} \mathcal{W}_i B Q_i$,
(4) $N_i \cap \ker B_i = N_i \cap N_{i+1} \subseteq N_{i+1} \cap \ker B_{i+1}$,
(5) $N_{i-1} \cap N_i \subseteq N_i \cap N_{i+1}$,
(6) $\operatorname{im} G_i + \operatorname{im} B_i \subseteq \operatorname{im}[AD, B] = \operatorname{im}[G_0, B_0]$.

Proof. (1) From (2.8) we successively derive an expression for B_{i+1}:

$$\begin{aligned} B_{i+1} &= \Big(B_{i-1}P_{i-1} - G_i D^-(D\Pi_i D^-)' D\Pi_{i-1}\Big)P_i - G_{i+1}D^-(D\Pi_{i+1}D^-)' D\Pi_i \\ &= B_{i-1}P_{i-1}P_i - \sum_{j=i}^{i+1} G_j D^-(D\Pi_j D^-)' D\Pi_i, \end{aligned}$$

hence

$$B_{i+1} = B_0 \Pi_i - \sum_{j=1}^{i+1} G_j D^-(D\Pi_j D^-)' D\Pi_i, \tag{2.13}$$

but this immediately verifies assertion (1).
(2) Because of $\operatorname{im} G_j \subseteq \operatorname{im} G_{i+1}$ for $j \le i+1$, we have $\mathcal{W}_{i+1} B_{i+1} = \mathcal{W}_{i+1} B_0 \Pi_i$ due to (2.13). Taking into account also the inclusion $\operatorname{im} B_j Q_j = \operatorname{im} G_{j+1} Q_j \subseteq \operatorname{im} G_{j+1} \subseteq \operatorname{im} G_{i+1}$, for $j \le i$, we obtain from (2.8) that $\mathcal{W}_{i+1} B_{i+1} = \mathcal{W}_{i+1} B_i P_i = \mathcal{W}_{i+1} B_i - \mathcal{W}_{i+1} B_i Q_i = \mathcal{W}_{i+1} B_i = \mathcal{W}_{i+1} B_{i-1} P_{i-1} = \mathcal{W}_{i+1} B_{i-1} = \cdots = \mathcal{W}_{i+1} B_0$, which proves assertion (2).
(3) We rearrange G_{i+1} as

$$G_{i+1} = G_i + G_i G_i^- B_i Q_i + (I - G_i G_i^-) B_i Q_i = G_i((I + G_i^- B_i Q_i) + \mathcal{W}_i B_i Q_i.$$

Because of $Q_i G_i^- = Q_i P_i G_i^- = 0$ the matrix function $F_{i+1} := I + G_i^- B_i Q_i$ remains nonsingular (see Lemma A.3) and the factorization

$$G_{i+1} = (G_i + \mathcal{W}_i B_i Q_i)F_{i+1} = (G_i + \mathcal{W}_i B Q_i)F_{i+1}$$

holds true. This yields assertion (3).
(4) $z \in N_i \cap \ker B_i$, i.e., $G_i z = 0$, $B_i z = 0$, leads to $z = Q_i z$ and $G_{i+1} z = B_i Q_i z = B_i z = 0$, thus $z \in N_i \cap N_{i+1}$. Conversely, $z \in N_i \cap N_{i+1}$ yields $z = Q_i z$, $B_i z = B_i Q_i z = G_{i+1} z = 0$, i.e., $z \in N_i \cap \ker B_i$ and we are done with assertion (4).
(5) From $z \in N_{i-1} \cap N_i$ it follows that $z = Q_{i-1} z$ and $B_i z = B_i Q_{i-1} z = B_i P_{i-1} Q_{i-1} z = 0$ because of $B_i = B_i P_{i-1}$ (cf. (2.13)), hence $z \in N_i \cap \ker B_i = N_i \cap N_{i+1}$.
(6) follows from $\operatorname{im} G_0 + \operatorname{im} B_0 = \operatorname{im}[G_0, B_0]$ by induction. Namely, $\operatorname{im} G_i + \operatorname{im} B_i \subseteq \operatorname{im}[G_0, B_0]$ implies $\operatorname{im} B_i Q_i \subseteq \operatorname{im}[G_0, B_0]$, hence $\operatorname{im} G_{i+1} \subseteq \operatorname{im}[G_i, B_0 Q_i] \subseteq \operatorname{im}[G_0, B_0]$, and further $\operatorname{im} B_{i+1} \subseteq \operatorname{im}[G_{i+1}, B_i] \subseteq \operatorname{im}[G_0, B_0]$. □

2.2.2 Admissible projector functions and characteristic values

In Chapter 1 on constant coefficient DAEs, useful decoupling properties are obtained by restricting the variety of possible projectors Q_i and somehow choosing smart ones, so-called *admissible* ones. Here we take up this idea again, and we incorporate conditions concerning ranks and dimensions to ensure the continuity of the matrix functions associated to the DAE. Possible rank changes will be treated as critical points discussed later on in Section 2.9. The following definition generalizes Definition 1.10.

Definition 2.6. Given the DAE (2.1) with properly stated leading term, Q_0 denotes a continuous projector function onto $\ker D$ and $P_0 = I - Q_0$. The generalized inverse D^- is given by $DD^-D = D$, $D^-DD^- = D^-$, $DD^- = R$, $D^-D = P_0$.
For a given level $\kappa \in \mathbb{N}$, we call the sequence $G_0, \dots, G_\kappa$ an *admissible matrix function sequence* associated to the DAE on the interval $\mathcal{I}$, if it is built by the rule

Set $G_0 := AD$, $B_0 := B, N_0 := \ker G_0$.
For $i \geq 1$:

$$G_i := G_{i-1} + B_{i-1} Q_{i-1},$$
$$B_i := B_{i-1} P_{i-1} - G_i D^- (D\Pi_i D^-)' D\Pi_{i-1}$$
$$N_i := \ker G_i, \quad \widehat{N_i} := (N_0 + \cdots + N_{i-1}) \cap N_i,$$

fix a complement X_i such that $N_0 + \cdots + N_{i-1} = \widehat{N_i} \oplus X_i$,
choose a projector Q_i such that $\operatorname{im} Q_i = N_i$ and $X_i \subseteq \ker Q_i$,
set $P_i := I - Q_i$, $\Pi_i := \Pi_{i-1} P_i$

and, additionally,

(a) G_i has constant rank r_i on $\mathcal{I}$, $i = 0, \dots, \kappa$,
(b) the intersection $\widehat{N_i}$ has constant dimension $u_i := \dim \widehat{N_i}$ on $\mathcal{I}$,

(c) the product function Π_i is continuous on $\mathcal{I}$ and $D\Pi_i D^-$ is there continuously differentiable, $i = 0, \dots, \kappa$.

The projector functions $Q_0, \dots, Q_\kappa$ in an admissible matrix function sequence are said to be *admissible* themselves.
An admissible matrix function sequence $G_0, \dots, G_\kappa$ is said to be *regular admissible*, if

$$\widehat{N_i} = \{0\}, \quad \forall\, i = 1, \dots, \kappa.$$

Then, also the projector functions $Q_0, \dots, Q_\kappa$ are called *regular admissible*.

Examples 2.3 and 2.4 already show regular admissible matrix function sequences.
The matrix functions $G_0, \dots, G_\kappa$ in an admissible sequence are a priori continuous on the given interval.
If $G_0, \dots, G_\kappa$ are admissible, besides the nullspaces $N_0, \dots, N_\kappa$ and the intersection spaces $\widehat{N}_1, \dots, \widehat{N}_\kappa$ also the sum spaces $N_0 + \dots + N_i$, $i = 1, \dots, \kappa$, and the complements $X_1, \dots, X_\kappa$ have constant dimension. Namely, the construction yields

$$N_0 + \dots + N_{i-1} = X_i \oplus \widehat{N_i}, \quad N_0 + \dots + N_i = X_i \oplus N_i, \quad i = 1, \dots \kappa,$$

and hence

$$\begin{aligned} \dim N_0 &= m - r_0, \\ \dim(N_0 + \dots + N_{i-1}) &= \dim X_i + u_i, \\ \dim(N_0 + \dots + N_i) &= \dim X_i + m - r_i, \quad i = 1, \dots, \kappa. \end{aligned}$$

It follows that

$$\begin{aligned} \dim(N_0 + \dots + N_i) &= \underbrace{\dim(N_0 + \dots + N_{i-1}) - u_i}_{\dim X_i} + \underbrace{m - r_i}_{\dim N_i} \\ &= \sum_{j=0}^{i-1} (m - r_j - u_{j+1}) + m - r_i = \sum_{j=0}^{i} (m - r_j) - \sum_{j=0}^{i-1} u_{j+1}. \end{aligned}$$

We are most interested in the case of trivial intersections $\widehat{N_i}$, yielding $X_i = N_0 + \dots + N_{i-1}$, and $u_i = 0$. In particular, all so-called regular DAEs in Section 2.6 belong to this latter class. Due to the trivial intersection $\widehat{N_i} = \{0\}$, the subspace $N_0 + \dots + N_i$ has dimension $\dim(N_0 + \dots + N_{i-1}) + \dim N_i$, that is, its increase is maximal at each level.

The next proposition collects benefits from admissible projector functions. Comparing with Proposition 1.13 we recognize a far-reaching conformity. The most important benefit seems to be the fact that Π_i being a product of projector functions is again a projector function, and it projects along the sum space $N_0 + \dots + N_i$ which now appears to be a $\mathcal{C}$-subspace.

We stress once more that admissible projector functions are always cross-linked with their generating admissible matrix function sequence. Nevertheless, for brevity, we simply speak of admissible projector functions or admissible projectors, dropping this natural background.

Proposition 2.7. *Given a DAE (2.1) with properly stated leading term, and an integer $\kappa \in \mathbb{N}$.*
If $Q_0,\dots,Q_\kappa$ are admissible projector functions, then the following eight relations become true for $i = 1,\dots,\kappa$.

(1) $\ker \Pi_i = N_0 + \cdots + N_i$,
(2) *the products $\Pi_i = P_0 \cdots P_i$ and $\Pi_{i-1}Q_i = P_0 \cdots P_{i-1}Q_i$, as well as $D\Pi_i D^-$ and $D\Pi_{i-1}Q_i D^-$, are projector valued functions, too,*
(3) $N_0 + \cdots + N_{i-1} \subseteq \ker \Pi_{i-1}Q_i$,
(4) $B_i = B_i \Pi_{i-1}$,
(5) $\widehat{N}_i \subseteq N_i \cap N_{i+1}$, *and hence* $\widehat{N}_i \subseteq \widehat{N}_{i+1}$,
(6) $G_{i+1}Q_j = B_jQ_j,\ 0 \le j \le i$,
(7) $D(N_0 + \cdots + N_i) = \operatorname{im} DP_0 \cdots P_{i-1}Q_i \oplus \operatorname{im} D\Pi_{i-2}Q_{i-1} \oplus \cdots \oplus \operatorname{im} DP_0Q_1$,
(8) *the products $Q_i(I - \Pi_{i-1})$ and $P_i(I - \Pi_{i-1})$ are projector functions onto $\widehat{N}_i$ and X_i, respectively.*

Additionally, the matrix functions $G_1,\dots,G_\kappa$, and $G_{\kappa+1}$ are continuous.
If $Q_0,\dots,Q_\kappa$ are regular admissible then it holds for $i = 1,\dots,\kappa$ that

$$\ker \Pi_{i-1}Q_i = \ker Q_i, \quad \textit{and} \quad Q_iQ_j = 0, \quad j = 0,\dots,i-1.$$

Proof. (1) See the proof of Proposition 1.13 (1).
(2) Due to assertion (1) it holds that $\ker \Pi_i = N_0 + \cdots + N_i$, which means $\Pi_i Q_j = 0$, $j = 0,\dots,i$. With $0 = \Pi_i Q_j = \Pi_i(I - P_j)$, we obtain $\Pi_i = \Pi_i P_j,\ j = 0,\dots,i$, which yields $\Pi_i\Pi_i = \Pi_i$. Derive further

$$\begin{aligned}(\Pi_{i-1}Q_i)^2 &= (\Pi_{i-1} - \Pi_i)(\Pi_{i-1} - \Pi_i)\\ &= \Pi_{i-1} - \underbrace{\Pi_{i-1}\Pi_i}_{=\Pi_{i-1}P_i} - \underbrace{\Pi_i\Pi_{i-1}}_{=\Pi_i} + \Pi_i = \Pi_{i-1}Q_i,\end{aligned}$$

$$(D\Pi_i D^-)^2 = D\Pi_i \underbrace{D^-D}_{=P_0} \Pi_i D^- = D\Pi_i D^-,$$

$$(D\Pi_{i-1}Q_iD^-)^2 = D\Pi_{i-1}Q_i \underbrace{D^-D}_{=P_0} \Pi_{i-1}Q_iD^- = D(\Pi_{i-1}Q_i)^2D^- = D\Pi_{i-1}Q_iD^-.$$

(3) See the proof of Proposition 1.13 (3).
(4) The detailed structure of B_i given in (2.13) and the projector property of Π_{i-1} (cf. (1)) proves the statement.
(5) $z \in N_i \cap (N_0 + \cdots + N_{i-1})$ means that $z = Q_iz,\ \Pi_{i-1}z = 0$, hence

$$G_{i+1}z = G_iz + B_iQ_iz = B_iz = B_i\Pi_{i-1}z = 0.$$

(6) For $0 \le j \le i$, it follows with (4) from

$$\begin{aligned} G_{i+1} = G_i + B_iQ_i &= G_0 + B_0Q_0 + B_1Q_1 + \cdots + B_iQ_i \\ &= G_0 + B_0Q_0 + B_1P_0Q_1 + \cdots + B_iP_0 \cdots P_{i-1}Q_i \end{aligned}$$

that

$$G_{i+1}Q_j = (G_0 + B_0Q_0 + \cdots + B_jP_0 \cdots P_{j-1}Q_j)Q_j = (G_j + B_jQ_j)Q_j = B_jQ_j.$$

(7) From $\ker \Pi_i = N_0 + \cdots + N_i$ it follows that

$$\begin{aligned} D(N_0 + \cdots + N_i) &= D\operatorname{im}(I - \Pi_i) = D\operatorname{im}(Q_0 + P_0Q_1 + \cdots + \Pi_{i-1}Q_i) \\ &= D\{\operatorname{im} Q_0 \oplus \operatorname{im} P_0Q_1 \oplus \cdots \oplus \operatorname{im} \Pi_{i-1}Q_i\} \\ &= \operatorname{im} DP_0Q_1 \oplus \cdots \oplus \operatorname{im} D\Pi_{i-1}Q_i. \end{aligned}$$

This proves assertion (7).
(8) We have (cf. (3))

$$Q_i(I - \Pi_{i-1})Q_i(I - \Pi_{i-1}) = (Q_i - Q_i \underbrace{\Pi_{i-1}Q_i)(I - \Pi_{i-1})}_{=0} = Q_i(I - \Pi_{i-1}).$$

Further, $z = Q_i(I - \Pi_{i-1})z$ implies $z \in N_i$, $\Pi_{i-1}z = \Pi_{i-1}Q_i(I - \Pi_{i-1})z = 0$, and hence $z \in \widehat{N_i}$.
Conversely, from $z \in \widehat{N_i}$ it follows that $z = Q_iz$ and $z = (I - \Pi_{i-1})z$, thus $z = Q_i(I - \Pi_{i-1})z$. Similarly, we compute

$$P_i(I - \Pi_{i-1})P_i(I - \Pi_{i-1}) = P_i(I - \Pi_{i-1}) - P_i(I - \Pi_{i-1})Q_i(I - \Pi_{i-1}) = P_i(I - \Pi_{i-1}).$$

From $z = P_i(I - \Pi_{i-1})z$ it follows that $Q_iz = 0$, $\Pi_{i-1}z = \Pi_i(I - \Pi_{i-1})z = 0$, therefore $z \in X_i$.
Conversely, $z \in X_i$ yields $z = P_iz$, $z = (I - \Pi_{i-1})z$, and hence $z = P_i(I - \Pi_{i-1})z$. This verifies (8).
Next we verify the continuity of the matrix functions G_i. Applying the representation (2.13) of the matrix function B_i we express

$$G_{i+1} = G_i + B_0\Pi_{i-1}Q_i - \sum_{j=1}^{i} G_jD^-(D\Pi_jD^-)'D\Pi_{i-1}Q_i,$$

which shows that, supposing that previous matrix functions $G_0, \ldots, G_i$ are continuous, the continuity of $\Pi_{i-1}Q_i = \Pi_{i-1} - \Pi_i$ implies G_{i+1} is also continuous.
Finally, let $Q_0, \ldots, Q_\kappa$ be regular admissible. $\Pi_{i-1}Q_iz = 0$ implies $Q_iz = (I - \Pi_{i-1})Q_iz \in N_0 + \cdots + N_{i-1}$, hence $Q_iz \in \widehat{N_i}$, therefore $Q_iz = 0$. It remains to apply (3). □

As in the constant coefficient case, there is a great variety of admissible projector functions, and the matrix functions G_i clearly depend on the special choice of the projector functions Q_j, including the way complements X_j in the decomposition of $N_0 + \cdots + N_{j-1}$ are chosen. Fortunately, there are invariants, in particular, invariant subspaces and subspace dimensions, as shown by the next assertion.

Theorem 2.8. *Let the DAE (2.1) have a properly stated leading term. Then, for a given $\kappa \in \mathbb{N}$, if admissible projector functions up to level κ do at all exist, then the subspaces*

$$\operatorname{im} G_j, \quad N_0 + \cdots + N_j, \quad S_j := \ker \mathcal{W}_j B, \quad j = 0, \ldots, \kappa + 1,$$

as well as the numbers

$$r_j := \operatorname{rank} G_j,\ j = 0, \ldots, \kappa, \quad u_j := \dim \widehat{N}_j,\ j = 1, \ldots, \kappa,$$

and the functions $r_{\kappa+1} : \mathcal{I} \to \mathbb{N} \cup \{0\}$, $\quad u_{\kappa+1} : \mathcal{I} \to \mathbb{N} \cup \{0\}$ are independent of the special choice of admissible projector functions $Q_0, \ldots, Q_\kappa$.

Proof. These assertions are immediate consequences of Lemma 2.12 below at the end of the present section. □

Definition 2.9. If the DAE (2.1) with properly stated leading term has an admissible matrix functions sequence up to level κ, then the integers

$$r_j = \operatorname{rank} G_j,\ j = 0, \ldots, \kappa, \quad u_j = \dim \widehat{N}_j,\ j = 1, \ldots, \kappa,$$

are called *characteristic values* of the DAE.

The characteristic values prove to be invariant under regular transformations and refactorizations (cf. Section 2.3, Theorems 2.18 and 2.21), which justifies this notation. For constant regular matrix pairs, these characteristic values describe the infinite eigenstructure (Corollary 1.32).

The associated subspace $S_0 = \ker \mathcal{W}_0 B$ has its special meaning. At given $t \in \mathcal{I}$, the subspace

$$S_0(t) = \ker \mathcal{W}_0(t) B(t) = \{z \in \mathbb{R}^m : B(t)z \in \operatorname{im} G_0(t) = \operatorname{im} A(t)\}$$

contains all solution values $x(t)$ of the solutions of the homogeneous equation $A(Dx)' + Bx = 0$. As we will see later, for so-called regular index-1 DAEs, the subspace $S_0(t)$ consists at all of those solution values, that means, for each element of $S_0(t)$ there exists a solution passing through it. For regular DAEs with a higher index, the sets of corresponding solution values form proper subspaces of $S_0(t)$.
In general, the associated subspaces satisfy the relations

$$S_{i+1} = S_i + N_i = S_i + N_0 + \cdots + N_i = S_0 + N_0 + \cdots + N_i,\ \ i = 0, \ldots, \kappa.$$

Namely, because of $\operatorname{im} G_i \subseteq \operatorname{im} G_{i+1}$, it holds that $\mathcal{W}_{i+1} = \mathcal{W}_{i+1}\mathcal{W}_i$, hence $S_{i+1} = \ker \mathcal{W}_{i+1}B = \ker \mathcal{W}_{i+1}\mathcal{W}_i B \supseteq \ker \mathcal{W}_i B = S_i$, and Proposition 2.5 (2) yields $S_{i+1} = \ker \mathcal{W}_{i+1}B_{i+1} \supseteq \ker B_{i+1} \supseteq N_0 + \cdots + N_i$.

Summarizing, the following three sequences of subspaces are associated with each admissible matrix function sequence:

$$\operatorname{im} G_0 \subseteq \operatorname{im} G_1 \subseteq \cdots \subseteq \operatorname{im} G_i \subseteq \cdots \subseteq \operatorname{im}[AD\ B] \subseteq \mathbb{R}^k, \tag{2.14}$$

$$N_0 \subseteq N_0 + N_1 \subseteq \cdots \subseteq N_0 + \cdots + N_i \subseteq \cdots \subseteq \mathbb{R}^m, \tag{2.15}$$

and

$$S_0 \subseteq S_1 \subseteq \cdots \subseteq S_i \subseteq \cdots \subseteq \mathbb{R}^m. \tag{2.16}$$

All of these subspaces are independent of the special choice of the admissible projector functions. In all three cases, the dimension does not decrease if the index increases. We are looking for criteria indicating that a certain G_μ already has the maximal possible rank. For instance, if we meet an injective matrix G_μ as in Examples 2.3 and 2.4, then the sequence becomes stationary with $Q_\mu = 0$, $G_{\mu+1} = G_\mu$, and so on. Therefore, the smallest index μ such that the matrix function G_μ is injective, indicates at the same time that $\operatorname{im} G_\mu$ is maximal, but $\operatorname{im} G_{\mu-1}$ is a proper subspace, if $\mu \geq 1$. The general case is more subtle. It may happen that no injective G_μ exists. Eventually one reaches

$$\operatorname{im} G_\mu = \operatorname{im}[AD\ B]; \tag{2.17}$$

however, this is not necessarily the case, as the next example shows.

Example 2.10 (Admissible matrix sequence for a nonregular DAE). Set $m = k = 3$, $n = 2$, and consider the constant coefficient DAE

$$\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}\left(\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}x\right)' + \begin{bmatrix}1&0&1\\0&1&0\\0&1&0\end{bmatrix}x = q, \tag{2.18}$$

which is nonregular due to the singular matrix pencil. Here we have $\operatorname{im}[AD\ B] = \mathbb{R}^3$. Compute successively

$$G_0 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}, \quad Q_0 = \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}, \quad \mathcal{W}_0 = \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix},$$

$$G_1 = \begin{bmatrix}1&0&1\\0&1&0\\0&0&0\end{bmatrix}, \quad Q_1 = \begin{bmatrix}1&0&0\\0&0&0\\-1&0&0\end{bmatrix}, \quad \mathcal{W}_1 = \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}, \quad B_1 = \begin{bmatrix}1&0&0\\0&1&0\\0&1&0\end{bmatrix},$$

$$G_2 = \begin{bmatrix}2&0&1\\0&1&0\\0&0&0\end{bmatrix}, \quad \Pi_1 = \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}.$$

We read off $N_0 = \{z \in \mathbb{R}^3 : z_1 = z_2 = 0\}$, $N_1 = \{z \in \mathbb{R}^3 : z_2 = 0, z_1 + z_3 = 0\}$ and $N_2 = \{z \in \mathbb{R}^3 : z_2 = 0, 2z_1 + z_3 = 0\}$. The intersection $N_0 \cap N_1$ is trivial, and the condition $Q_1 Q_0 = 0$ is fulfilled. We have further

$$N_0 + N_1 = \{z \in \mathbb{R}^3 : z_2 = 0\}, \quad (N_0 + N_1) \cap N_2 = \widehat{N_2} = N_2 \subseteq N_0 + N_1,$$
$$\text{thus } N_0 + N_1 = N_0 + N_1 + N_2 \text{ and } N_0 + N_1 = N_2 \oplus N_0.$$

We can put $X_2 = N_0$, and compute

$$Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}, \text{ with } X_2 \subseteq \ker Q_2, \quad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The projectors Q_0, Q_1, Q_2 are admissible. It holds that $B_2 Q_2 = 0$, $G_3 = G_2$, $N_3 = N_2$, and $\Pi_2 = \Pi_1$, and further

$$S_0 = \{z \in \mathbb{R}^3 : z_2 = 0\}, \quad S_0 = S_1 = S_2 = S_3.$$

We continue the matrix function sequence by $Q_3 := Q_2$, $B_3 = B_2$, $B_3 Q_3 = 0$, $G_4 = G_3$, and so on. It follows that no G_i is injective, and

$$\begin{aligned} &\operatorname{im} G_0 = \cdots = \operatorname{im} G_i = \cdots = \mathbb{R}^2 \times \{0\} \subset \operatorname{im}[AD\ B] = \mathbb{R}^3, \\ &S_0 = \cdots = S_i = \cdots = \mathbb{R} \times \{0\} \times \mathbb{R}, \\ &N_0 \subset N_0 + N_1 = N_0 + N_1 + N_2 = \cdots = \mathbb{R} \times \{0\} \times \mathbb{R}, \end{aligned}$$

and the maximal range is already $\operatorname{im} G_0$. A closer look at the DAE (2.18) gives

$$\begin{aligned} x_1' + x_1 + x_3 &= q_1, \\ x_2' + x_2 &= q_2, \\ x_2 &= q_3. \end{aligned}$$

This model is somewhat dubious. It is in parts over- and underdetermined, and much room for interpretations is left (cf. Chapter 10). □

Our next example is much nicer and more important with respect to applications. It is a so-called *Hessenberg form size-3 DAE* and might be considered as the linear prototype of the system describing constrained mechanical motion (see Example 3.41 and Section 3.5).

Example 2.11 (Admissible sequence for the Hessenberg size 3 DAE). Consider the system

$$\begin{bmatrix} x_1' \\ x_2' \\ 0 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & 0 \\ 0 & B_{32} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{2.19}$$

with $m = m_1 + m_2 + m_3$ equations, $m_1 \geq m_2 \geq m_3 \geq 1$, $k = m$ components, and a nonsingular product $B_{32} B_{21} B_{13}$. Put $n = m_1 + m_2$,

$$A = \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}, \quad D^- = \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & 0 \\ 0 & B_{32} & 0 \end{bmatrix},$$

and write this DAE in the form (2.1).

Owing to the nonsingularity of the $m_3 \times m_3$ matrix function product $B_{32}B_{21}B_{13}$, the matrix functions B_{13} and $B_{21}B_{13}$ have full column rank m_3 each, and B_{32} has full row rank m_3. This yields $\operatorname{im}[AD\ B] = \mathbb{R}^m$. Further, since B_{13} and $B_{21}B_{13}$ have constant rank, there are continuous reflexive generalized inverses B_{13}^- and $(B_{21}B_{13})^-$ such that (see Proposition A.17)

$$\begin{aligned} B_{13}^- B_{13} = I, \quad & \Omega_1 := B_{13}B_{13}^- && \text{is a projector onto} \quad \operatorname{im} B_{13}, \\ (B_{21}B_{13})^- B_{21}B_{13} = I, \quad & \Omega_2 := B_{21}B_{13}(B_{21}B_{13})^- && \text{is a projector onto} \quad \operatorname{im} B_{21}B_{13}. \end{aligned}$$

Let the coefficient function B be smooth enough so that the derivatives used below do exist. In particular, Ω_1 and Ω_2 are assumed to be continuously differentiable. We start constructing the matrix function sequence by

$$G_0 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}, \quad B_0 = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & 0 \\ 0 & B_{32} & 0 \end{bmatrix}, \quad G_1 = \begin{bmatrix} I & 0 & B_{13} \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It follows that

$$\begin{aligned} N_0 &= \{z \in \mathbb{R}^m : z_1 = 0,\ z_2 = 0\}, \quad N_1 = \{z \in \mathbb{R}^m : z_1 + B_{13}z_3 = 0,\ z_2 = 0\}, \\ \widehat{N_1} &= N_0 \cap N_1 = \{0\}, \quad X_1 = N_0, \\ N_0 + N_1 &= N_0 \oplus N_1 = \{z \in \mathbb{R}^m : z_2 = 0,\ z_1 \in \operatorname{im} B_{13}\}. \end{aligned}$$

The matrix functions G_0 and G_1 have constant rank, $r_0 = r_1 = n$. Compute the projector functions

$$Q_1 = \begin{bmatrix} \Omega_1 & 0 & 0 \\ 0 & 0 & 0 \\ -B_{13}^- & 0 & 0 \end{bmatrix}, \quad D\Pi_1 D^- = \begin{bmatrix} I - \Omega_1 & 0 \\ 0 & I \end{bmatrix},$$

such that $\operatorname{im} Q_1 = N_1$ and $Q_1Q_0 = 0$, that is $\ker Q_1 \supseteq X_1$. Q_1 is continuous, and $D\Pi_1D^-$ is continuously differentiable. In consequence, Q_0, Q_1 are admissible. Next we form

$$B_1 = \begin{bmatrix} B_{11} + \Omega_1' & B_{12} & 0 \\ B_{21} & B_{22} & 0 \\ 0 & B_{32} & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} I + (B_{11} + \Omega_1')\Omega_1 & 0 & B_{13} \\ B_{21}\Omega_1 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For $z \in \mathbb{R}^{m_1+m_2+m_3}$ with $z_1 \in \ker \Omega_1$ it holds that $\operatorname{im} G_2 = \begin{bmatrix} z_1 + B_{13}z_3 \\ z_2 \\ 0 \end{bmatrix}$, since $\operatorname{im} B_{13} = \operatorname{im} \Omega_1$. This proves the inclusion

$$\operatorname{im} G_2 \subseteq \mathbb{R}^n \times \{0\} = \{G_2 z : z \in \mathbb{R}^{m_1+m_2+m_3}, z_1 \in \ker \Omega_1\} \subseteq \operatorname{im} G_2,$$

and we obtain $\operatorname{im} G_2 = \mathbb{R}^n \times \{0\}$, and $r_2 = \operatorname{rank} G_2 = m_1 + m_2 = n$. Then we investigate the nullspace of G_2. If $z \in \mathbb{R}^m$ satisfies $G_2 z = 0$, then

$$z_1 + (B_{11} + \Omega_1')\Omega_1 z_1 + B_{13}z_3 = 0, \tag{2.20}$$

$$B_{21}\Omega_1 z_1 + z_2 = 0. \tag{2.21}$$

In turn, equation (2.20) decomposes into

$$\begin{aligned} (I - \Omega_1)z_1 + (I - \Omega_1)(B_{11} + \Omega_1')\Omega_1 z_1 &= 0, \\ B_{13}^-(I + B_{13}^-(B_{11} + \Omega_1'))\Omega_1 z_1 + z_3 &= 0. \end{aligned}$$

Similarly, considering that $\operatorname{im} B_{21}B_{13} = \operatorname{im} B_{21}B_{13}B_{13}^-$ is valid, we derive from (2.21) the relations

$$z_2 = \Omega_2 z_2, \quad B_{13}^- z_1 = -(B_{21}B_{13})^- z_2.$$

Altogether this yields

$$N_2 = \{z \in \mathbb{R}^m : z_2 = \Omega_2 z_2, z_1 = \mathcal{E}_1\Omega_2 z_2, z_3 = \mathcal{E}_3\Omega_2 z_2\}, \quad \widehat{N_2} = \{0\}, \quad X_2 = N_0 + N_1,$$

with

$$\begin{aligned} \mathcal{E}_1 &:= -(I - (I - \Omega_1)(B_{11} + \Omega_1')\Omega_1)B_{13}(B_{21}B_{13})^- \\ &= -(I - (I - \Omega_1)(B_{11} + \Omega_1'))B_{13}(B_{21}B_{13})^-, \\ \mathcal{E}_3 &:= -B_{13}^-(I + (B_{11} + \Omega_1'))B_{13}(B_{21}B_{13})^-. \end{aligned}$$

Notice that $\mathcal{E}_1 = \mathcal{E}_1\Omega_2$, $\mathcal{E}_3 = \mathcal{E}_3\Omega_2$. The projector functions

$$Q_2 = \begin{bmatrix} 0 & \mathcal{E}_1 & 0 \\ 0 & \Omega_2 & 0 \\ 0 & \mathcal{E}_3 & 0 \end{bmatrix}, \quad D\Pi_2 D^- = \begin{bmatrix} I - \Omega_1 & -(I - \Omega_1)\mathcal{E}_1 \\ 0 & I - \Omega_2 \end{bmatrix},$$

fulfill the required admissibility conditions, in particular, $Q_2 Q_0 = 0$, $Q_2 Q_1 = 0$, and hence Q_0, Q_1, Q_2 are admissible. The resulting B_2, G_3 have the form:

$$B_2 = \begin{bmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} & 0 \\ \mathcal{B}_{21} & \mathcal{B}_{22} & 0 \\ 0 & B_{32} & 0 \end{bmatrix}, \quad G_3 = \begin{bmatrix} I + (B_{11} + \Omega_1')\Omega_1 & \mathcal{B}_{11}\mathcal{E}_1 + \mathcal{B}_{12}\Omega_2 & B_{13} \\ B_{21}\Omega_1 & I + \mathcal{B}_{21}\mathcal{E}_1 + \mathcal{B}_{22}\Omega_2 & 0 \\ 0 & B_{32}\Omega_2 & 0 \end{bmatrix}.$$

The detailed form of the entries $\mathcal{B}_{ij}$ does not matter in this context. We show G_3 to be nonsingular. Namely, $G_3 z = 0$ implies $B_{32}\Omega_2 z_2 = 0$, thus $\Omega_2 z_2 = 0$, and further

$B_{21}\Omega_1 z_1 + z_2 = 0$. The latter equation yields $(I - \Omega_2)z_2 = 0$ and $B_{21}\Omega_1 z_1 = 0$, and this gives $\Omega_1 z_1 = 0$, $z_2 = 0$. Now, the first line of the system $G_3 z = 0$ simplifies to $z_1 + B_{13}z_3 = 0$. In turn, $(I-\Omega_1)z_1 = 0$ follows, and hence $z_1 = 0$, $z_3 = 0$. The matrix function G_3 is nonsingular in fact, and we stop the construction.
In summary, our basic subspaces behaves as

$$\operatorname{im} G_0 = \operatorname{im} G_1 = \operatorname{im} G_2 \subset \operatorname{im} G_3 = \operatorname{im}[AD\ B] = \mathbb{R}^m,$$
$$N_0 \subset N_0 + N_1 \subset N_0 + N_1 + N_2 = N_0 + N_1 + N_2 + N_3 \subset \mathbb{R}^m.$$

The additionally associated projector functions $\mathcal{W}_i$ onto $\operatorname{im} G_i$ and the subspaces $S_i = \ker \mathcal{W}_i B$ are here:

$$\mathcal{W}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}, \quad \mathcal{W}_0 = \mathcal{W}_1 = \mathcal{W}_2, \quad \mathcal{W}_3 = 0,$$

and

$$S_0 = \{z \in \mathbb{R}^m : B_{32}z_2 = 0\}, \quad S_0 = S_1 = S_2 \subset S_3 = \mathbb{R}^m.$$

The last relation is typical for the large class of DAEs named Hessenberg form DAEs (cf. Section 3.5). While $\operatorname{im} G_3$ and S_3 reach the maximal dimension m, the dimension of the resulting maximal subspace $N_0 + N_1 + N_2$ is less than m.
Notice that the relation $\mathcal{W}_0 B Q_0 = 0$ indicates that $\operatorname{im} G_0 = \operatorname{im} G_1$ holds true, and we can recognize this fact before explicitly computing G_1 (cf. Proposition 2.5(3)). Similarly, $\mathcal{W}_1 B Q_1 = 0$ indicates that $\operatorname{im} G_1 = \operatorname{im} G_2$. Furthermore, we know that $r_3 = r_2 + \operatorname{rank}(\mathcal{W}_2 B Q_2) = n + m_3 = m$ before we compute G_3. □

Now we come to an important auxiliary result which stands behind Theorem 2.8, and which generalizes Lemma 1.18.

Lemma 2.12. *Given the DAE (2.1) with properly stated leading term, if there are two admissible projector function sequences $Q_0, \dots, Q_\kappa$ and $\bar{Q}_0, \dots, \bar{Q}_\kappa$, both admissible on $\mathcal{I}$, then the associated matrix functions and subspaces are related by the following properties:*

(1) $\ker \bar{\Pi}_j = \bar{N}_0 + \cdots + \bar{N}_j = N_0 + \cdots + N_j = \ker \Pi_j$, $j = 0, \dots, \kappa$,

(2) $\bar{G}_j = G_j Z_j$,
$\bar{B}_j = B_j - G_j Z_j \bar{D}^- (D\bar{\Pi}_j \bar{D}^-)' D\Pi_j + G_j \sum_{l=0}^{j-1} Q_l \mathfrak{A}_{jl}$, $j = 1, \dots, \kappa$,
with nonsingular matrix functions $Z_0, \dots, Z_{\kappa+1}$ given by
$Z_0 := I$, $Z_{i+1} := Y_{i+1} Z_i$, $i = 0, \dots, \kappa$,

$$Y_1 := I + Q_0(\bar{Q}_0 - Q_0) = I + Q_0 \bar{Q}_0 P_0,$$
$$Y_{i+1} := I + Q_i(\bar{\Pi}_{i-1}\bar{Q}_i - \Pi_{i-1}Q_i) + \sum_{l=0}^{i-1} Q_l \mathfrak{A}_{il} \bar{Q}_i, \ i = 1, \dots, \kappa,$$

and certain continuous coefficients $\mathfrak{A}_{il}$ that satisfy condition $\mathfrak{A}_{il} = \mathfrak{A}_{il}\bar{\Pi}_{i-1}$,

(3) $Z_i(\bar{N}_i \cap (\bar{N}_0 + \cdots + \bar{N}_{i-1})) = N_i \cap (N_0 + \cdots + N_{i-1}), \quad i = 1,\ldots,\kappa,$
(4) $\bar{G}_{\kappa+1} = G_{\kappa+1}Z_{\kappa+1}, \quad \bar{N}_0 + \cdots + \bar{N}_{\kappa+1} = N_0 + \cdots + N_{\kappa+1},$
$Z_{\kappa+1}(\bar{N}_{\kappa+1} \cap (\bar{N}_0 + \cdots + \bar{N}_\kappa)) = N_{\kappa+1} \cap (N_0 + \cdots + N_\kappa).$

Proof. We have $G_0 = AD = \bar{G}_0$, $B_0 = B = \bar{B}_0$, $\ker P_0 = N_0 = \bar{N}_0 = \ker \bar{P}_0$, and hence $P_0 = P_0\bar{P}_0$, $\bar{P}_0 = \bar{P}_0 P_0$.
The generalized inverses D^- and $\bar{D}^-$ of D satisfy the properties $DD^- = D\bar{D}^- = R$, $D^-D = P_0$, $\bar{D}^-D = \bar{P}_0$, and therefore $\bar{D}^- = \bar{D}^-D\bar{D}^- = \bar{D}^-DD^- = \bar{P}_0 D^-$, $D^- = P_0\bar{D}^-$.
Compare $G_1 = G_0 + B_0Q_0$ and

$$\begin{aligned}\bar{G}_1 &= \bar{G}_0 + \bar{B}_0\bar{Q}_0 = G_0 + B_0\bar{Q}_0 = G_0 + B_0Q_0\bar{Q}_0\\ &= (G_0 + B_0Q_0)(P_0 + \bar{Q}_0) = G_1Z_1,\end{aligned}$$

where $Z_1 := Y_1 := P_0 + \bar{Q}_0 = I + Q_0\bar{Q}_0P_0 = I + Q_0(\bar{Q}_0 - Q_0)$. Z_1 is invertible; it has the inverse $Z_1^{-1} = I - Q_0\bar{Q}_0P_0$.
The nullspaces N_1 and $\bar{N}_1$ are, due to $\bar{G}_1 = G_1Z_1$, related by $\bar{N}_1 = Z_1^{-1}N_1 \subseteq N_0 + N_1$. This implies $\bar{N}_0 + \bar{N}_1 = N_0 + (Z_1^{-1}N_1) \subseteq N_0 + N_1$. From $N_1 = Z_1\bar{N}_1 \subseteq N_0 + \bar{N}_1 = \bar{N}_0 + \bar{N}_1$, we obtain $\bar{N}_0 + \bar{N}_1 = N_0 + N_1$.

Since the projectors P_0P_1 and $\bar{P}_0\bar{P}_1$ have the common nullspace $N_0 + N_1 = \bar{N}_0 + \bar{N}_1$, we may now derive

$$\begin{aligned}D\bar{P}_0\bar{P}_1\bar{D}^- &= D\bar{P}_0\bar{P}_1 \overbrace{P_0P_1}^{=P_0P_1P_0} \bar{P}_0D^- = D\bar{P}_0\bar{P}_1P_0P_1D^- = D\bar{P}_0\bar{P}_1\bar{D}^-DP_0P_1D^-,\\ DP_0P_1D^- &= DP_0P_1D^-D\bar{P}_0\bar{P}_1\bar{D}^-.\end{aligned}$$

Taking into account the relation $0 = \bar{G}_1\bar{Q}_1 = G_1\bar{Q}_1 + G_1(Z_1 - I)\bar{Q}_1$, thus $G_1\bar{Q}_1 = -G_1(Z_1 - I)\bar{Q}_1$ we obtain (cf. Appendix B for details)

$$\bar{B}_1 = B_1 - G_1Z_1\bar{D}^-(D\bar{P}_0\bar{P}_1D^-)'D.$$

This gives the basis for proving our assertion by induction. The proof is carried out in detail in Appendix B. A technically easier version for the time-invariant case is given in Chapter 1, Lemma 1.18. □

2.2.3 *Widely orthogonal projector functions*

For each DAE with properly stated leading term, we can always start the matrix function sequence by choosing Q_0 to be the orthogonal projector onto $N_0 = \ker D$, that means, $Q_0 = Q_0^*$, $P_0 = P_0^*$. On the next level, applying the decomposition $\mathbb{R}^m = (N_0 \cap N_1)^\perp \oplus (N_0 \cap N_1)$ we determine X_1 in the decomposition $N_0 = X_1 \oplus (N_0 \cap N_1)$ by $X_1 = N_0 \cap (N_0 \cap N_1)^\perp$. This leads to $N_0 + N_1 = (X_1 \oplus (N_0 \cap N_1)) + N_1 = X_1 \oplus N_1$ and $\mathbb{R}^m = (N_0 + N_1)^\perp \oplus (N_0 + N_1) = (N_0 + N_1)^\perp \oplus X_1 \oplus N_1$. In this way Q_1 is uniquely determined as $\operatorname{im} Q_1 = N_1$, $\ker Q_1 = (N_0 + N_1)^\perp \oplus X_1$.

On the next levels, if $Q_0,\dots,Q_{i-1}$ are admissible, we first apply the decomposition $\mathbb{R}^m = (\widehat{N_i})^\perp \oplus \widehat{N_i}$, and choose

$$X_i = (N_0+\cdots+N_{i-1}) \cap (\widehat{N_i})^\perp. \tag{2.22}$$

The resulting decompositions $N_0+\cdots+N_i = X_i \oplus N_i$, and
$\mathbb{R}^m = (N_0+\cdots+N_i)^\perp \oplus (N_0+\cdots+N_i) = (N_0+\cdots+N_i)^\perp \oplus X_i \oplus N_i$ allow for the choice

$$\operatorname{im} Q_i = N_i,\ \ker Q_i = (N_0+\cdots+N_i)^\perp \oplus X_i. \tag{2.23}$$

Definition 2.13. Admissible projector functions $Q_0,\dots,Q_\kappa$ are called *widely orthogonal* if $Q_0 = Q_0^*$ and both (2.22) and (2.23) are fulfilled for $i = 1,\dots,\kappa$.

Example 2.14 (Widely orthogonal projectors). The admissible projector functions Q_0, Q_1 built for the Hessenberg size-2 DAE in Example 2.3 with $\Omega = \Omega^*$ are widely orthogonal. In particular, it holds that

$$\ker Q_1 = \{z \in \mathbb{R}^{m_1+m_2} : B_{12}^* z_1 = 0\} = (N_0 \oplus N_1)^\perp \oplus N_0.$$

□

Widely orthogonal projector functions are uniquely fixed by construction. They provide special symmetry properties. In fact, applying widely orthogonal projector functions, the decompositions

$$x(t) = \Pi_i(t)x(t) + \Pi_{i-1}(t)Q_i(t)x(t) + \cdots + \Pi_0(t)Q_1(t)x(t) + Q_0(t)x(t)$$

are orthogonal ones for all t owing to the following proposition.

Proposition 2.15. *If $Q_0,\dots,Q_\kappa$ are widely orthogonal, then Π_i, $i = 0,\dots,\kappa$, and $\Pi_{i-1}Q_i, i = 1,\dots,\kappa$, are symmetric.*

Proof. Let $Q_0,\dots,Q_\kappa$ be widely orthogonal. In particular, it holds that $\Pi_0 = \Pi_0^*$, $\ker \Pi_0 = N_0$, $\operatorname{im} \Pi_0 = N_0^\perp$.
Compute $\operatorname{im} \Pi_1 = \operatorname{im} P_0P_1 = P_0 \operatorname{im} P_1 = P_0((N_0+N_1)^\perp \oplus X_1) = P_0(N_0+N_1)^\perp$
$= P_0(N_0^\perp \cap N_1^\perp) = N_0^\perp \cap N_1^\perp = (N_0+N_1)^\perp$.
To use induction, assume that $\operatorname{im} \Pi_j = (N_0+\cdots+N_j)^\perp,\ j \le i-1$.
Due to Proposition 2.7 (1) we know that $\ker \Pi_i = N_0+\cdots+N_i$ is true; further $\Pi_{i-1}X_i = 0$. From (2.23) it follows that

$$\begin{aligned}\operatorname{im} \Pi_i &= \Pi_{i-1}\operatorname{im} P_i = \Pi_{i-1}((N_0+\cdots+N_i)^\perp \oplus X_i)\\ &= \Pi_{i-1}(N_0+\cdots+N_i)^\perp = \Pi_{i-1}((N_0+\cdots+N_{i-1})^\perp \cap N_i^\perp)\\ &= (N_0+\cdots+N_{i-1})^\perp \cap N_i^\perp = (N_0+\cdots+N_i)^\perp.\end{aligned}$$

Since Π_i is a projector, and $\ker \Pi_i = N_0+\cdots+N_i$, $\operatorname{im} \Pi_i = (N_0+\cdots+N_i)^\perp$, Π_i must be the orthoprojector.
Finally, derive $(\Pi_{i-1}Q_i)^* = (\Pi_{i-1} - \Pi_{i-1}P_i)^* = \Pi_{i-1} - \Pi_{i-1}P_i = \Pi_{i-1}Q_i$. □

Proposition 2.16. *If, for the DAE (2.1) with properly stated leading term, there exist any admissible projector functions $Q_0, \dots, Q_\kappa$, and if $DD^* \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^n))$, then also widely orthogonal projector functions can be chosen (they do exist).*

Proof. Let $Q_0, \dots, Q_\kappa$ be admissible. Then, in particular the subspaces $N_0 + \cdots + N_i$, $i = 0, \dots, \kappa$ are continuous. The subspaces $\operatorname{im} D\Pi_0 Q_1, \dots, \operatorname{im} D\Pi_{\kappa-1} Q_\kappa$ belong to the class $\mathcal{C}^1$, since the projectors $D\Pi_0 Q_1 D^-, \dots, D\Pi_{\kappa-1} Q_\kappa D^-$ do so. Taking Proposition 2.7 into account we know the subspaces $D(N_0 + \cdots + N_i)$, $i = 1, \dots, \kappa$, to be continuously differentiable.
Now we construct widely orthogonal projectors. Choose $\bar{Q}_0 = \bar{Q}_0^*$, and form $\bar{G}_1 = G_0 + B_0 \bar{Q}_0$. Due to Lemma 2.12 (d) it holds that $\bar{G}_1 = G_1 Z_1$, $\bar{N}_0 + \bar{N}_1 = N_0 + N_1$, $Z_1(\bar{N}_0 \cap \bar{N}_1) = N_0 \cap N_1$. Since Z_1 is nonsingular, $\bar{G}_1$ has constant rank r_1, and the intersection $\overline{N\mathcal{U}}_1 = \bar{N}_1 \cap \bar{N}_0$ has constant dimension u_1. Put $\bar{X}_1 = \bar{N}_0 \cap (\bar{N}_0 \cap \bar{N}_1)^\perp$ and fix the projector $\bar{Q}_1$ by means of $\operatorname{im} \bar{Q}_1 = \bar{N}_1$, $\ker \bar{Q}_1 = \bar{X}_1 \oplus (\bar{N}_0 + \bar{N}_1)^\perp$. $\bar{Q}_1$ is continuous, but for the sequence $\bar{Q}_0, \bar{Q}_1$ to be admissible, $D\bar{\Pi}_1 \bar{D}^-$ has to belong to the class $\mathcal{C}^1$. This projector has the nullspace $\ker D\bar{\Pi}_1 \bar{D}^- = D(\bar{N}_0 + \bar{N}_1) \oplus \ker R = D(N_0 + N_1) \oplus \ker R$, which is already known to belong to $\mathcal{C}^1$. If $D\bar{\Pi}_1 \bar{D}^-$ has a range that is a $\mathcal{C}^1$ subspace, then $D\bar{\Pi}_1 \bar{D}^-$ itself is continuously differentiable. Derive $\operatorname{im} D\bar{\Pi}_1 \bar{D}^- = \operatorname{im} D\bar{\Pi}_1 = D(\bar{N}_0 + \bar{N}_1)^\perp = D(N_0 + N_1)^\perp = DD^*(D(N_0 + N_1))^\perp$. Since $D(N_0 + N_1)$ belongs to the class $\mathcal{C}^1$, so does $(D(N_0 + N_1))^\perp$. It turns out that $D\bar{\Pi}_1 \bar{D}^-$ is in fact continuously differentiable, and hence, $\bar{Q}_0, \bar{Q}_1$ are admissible.
To use induction, assume that $\bar{Q}_0, \dots, \bar{Q}_{i-1}$ are admissible and widely orthogonal. Lemma 2.12 (d) yields $\bar{G}_i = G_i Z_i$, $\bar{N}_0 + \cdots + \bar{N}_{i-1} = N_0 + \cdots + N_{i-1}$, $\bar{N}_0 + \cdots + \bar{N}_i = N_0 + \cdots + N_i$, $Z_i(\bar{N}_i \cap (\bar{N}_0 + \cdots + \bar{N}_{i-1})) = N_i \cap (N_0 + \cdots + N_{i-1})$. Since Z_i is nonsingular, it follows that $\bar{G}_i$ has constant rank r_i and the intersection $\overline{N\mathcal{U}}_i = \bar{N}_i \cap (\bar{N}_0 + \cdots + \bar{N}_{i-1})$ has constant dimension u_i. The involved subspaces are continuous. Put

$$\bar{X}_i = (\bar{N}_0 + \cdots + \bar{N}_{i-1}) \cap ((\bar{N}_0 + \cdots + \bar{N}_{i-1}) \cap \bar{N}_i)^\perp$$

and choose $\bar{Q}_i$ to be the projector onto $\bar{N}_i$ along $(\bar{N}_0 + \cdots + \bar{N}_i)^\perp \oplus \bar{X}_i$.
$\bar{Q}_0, \dots, \bar{Q}_{i-1}, \bar{Q}_i$ would be admissible if $D\bar{\Pi}_i \bar{D}^-$ was continuously differentiable. We know $\ker D\bar{\Pi}_i \bar{D}^- = D(N_0 + \cdots + N_i) \oplus \ker R$ to be already continuously differentiable. On the other hand, we have $\operatorname{im} D\bar{\Pi}_i \bar{D}^- = D \operatorname{im} \bar{\Pi}_i = D(N_0 + \cdots + N_i)^\perp = DD^*(D(N_0 + \cdots + N_i))^\perp$, hence $\operatorname{im} D\bar{\Pi}_i \bar{D}^-$ belongs to the class $\mathcal{C}^1$. $\square$

The widely orthogonal projectors have the advantage that they are uniquely determined. This proves its value in theoretical investigations, for instance in verifying Theorem 3.33 on necessary and sufficient regularity conditions for nonlinear DAEs, as well as for investigating critical points. Moreover, in practical calculations, in general, there might be difficulties in ensuring the continuity of the projector functions Π_i. Fortunately, owing to their uniqueness the widely orthogonal projector functions are continuous a priori.
By Proposition 2.16, at least for all DAEs with properly stated leading term, and with a continuously differentiable coefficient D, we may access widely orthogonal

projector functions. However, if D is just continuous, and if DD^* fails to be continuously differentiable as required, then it may happen in fact that admissible projector functions exist but the special widely orthogonal projector functions do not exist for lack of smoothness. The following example shows this situation. At this point we emphasize that most DAEs are given with a smooth D, and our example is rather academic.

Example 2.17 (Lack of smoothness for widely orthogonal projectors). Given the DAE

$$\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}\left(\begin{bmatrix}1&\alpha&0\\0&1&0\end{bmatrix}x\right)' + \begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}x = q,$$

with a continuous scalar function α, the DAE has the coefficients

$$A=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix},\quad D=\begin{bmatrix}1&\alpha&0\\0&1&0\end{bmatrix},\quad B=\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix},\quad R=\begin{bmatrix}1&0\\0&1\end{bmatrix}.$$

First we construct an admissible matrix function sequence. Set and derive

$$D^-=\begin{bmatrix}1&-\alpha\\0&1\\0&0\end{bmatrix},\quad G_0=\begin{bmatrix}1&\alpha&0\\0&1&0\\0&0&0\end{bmatrix},\quad Q_0=\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix},\quad G_1=\begin{bmatrix}1&\alpha&0\\0&1&-1\\0&0&0\end{bmatrix},\tag{2.24}$$

and further

$$Q_1=\begin{bmatrix}0&-\alpha&0\\0&1&0\\0&1&0\end{bmatrix},\quad Q_1Q_0=0,\quad D\Pi_1D^-=\begin{bmatrix}1&0\\0&0\end{bmatrix},\quad G_2=\begin{bmatrix}1&\alpha&0\\0&1&-1\\0&1&0\end{bmatrix}.$$

The projector functions Q_0, Q_1 are admissible, and G_2 is nonsingular, such that $Q_2 = 0$. This sequence is admissible for each arbitrary continuous α; however it fails to be widely orthogonal. Namely, the product Π_0Q_1 is not symmetric.

Next we construct widely orthogonal projector functions. We start with the same matrix functions Q_0, D^- and G_1 (see (2.24)). Compute further

$$N_0\oplus N_1=\operatorname{span}\left\{\begin{bmatrix}0\\0\\1\end{bmatrix},\begin{bmatrix}-\alpha\\1\\1\end{bmatrix}\right\},\quad (N_0\oplus N_1)^\perp=\operatorname{span}\begin{bmatrix}1\\\alpha\\0\end{bmatrix}.$$

The required projector function onto N_1 along $N_0\oplus(N_0\oplus N_1)^\perp$ is

$$Q_1=\frac{1}{1+\alpha^2}\begin{bmatrix}\alpha^2&-\alpha&0\\-\alpha&1&0\\-\alpha&1&0\end{bmatrix},\quad\text{and it follows that}\quad D\Pi_1D^-=\begin{bmatrix}1&0\\\frac{\alpha}{1+\alpha^2}&0\end{bmatrix}.$$

We recognize that, in the given setting, $D\Pi_1 D^-$ is just continuous. If we additionally assume that $\alpha \in \mathcal{C}^1(\mathcal{I},\mathbb{R})$, then Q_0, Q_1 appear to be admissible. Notice that in this case $DD^* = \begin{bmatrix} 1+\alpha^2 & \alpha \\ \alpha & 1 \end{bmatrix}$ is continuously differentiable, which confirms Proposition 2.16 once more.

Let us stress that this special DAE is solvable for arbitrary continuous α. From this point of view there is no need to assume α to be $\mathcal{C}^1$. Namely, the detailed equations are

$$\begin{aligned} (x_1+\alpha x_2)' &= q_1, \\ x_2' - x_3 &= q_2, \\ x_2 &= q_3, \end{aligned}$$

with the solutions

$$\begin{aligned} x_1(t)+\alpha(t)x_2(t) &= x_1(0)+\alpha(0)x_2(0)+\int_0^t q_3(s)ds, \\ x_2(t) &= q_3(t), \\ x_3(t) &= q_3'(t)-q_2(t). \end{aligned}$$

It turns out that widely orthogonal projectors need some specific slightly higher smoothness which is not justified by solvability. □

2.3 Invariants under transformations and refactorizations

Given the DAE (2.1) with continuous coefficients and properly stated leading term, we premultiply this equation by a nonsingular matrix function $L \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^k))$ and transform the unknown $x = K\bar{x}$ by means of a nonsingular matrix function $K \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^m))$ such that the DAE

$$\bar{A}(\bar{D}\bar{x})' + \bar{B}\bar{x} = \bar{q} \tag{2.25}$$

results, where $\bar{q} := Lq$, and

$$\bar{A} := LA, \quad \bar{D} := DK, \quad \bar{B} := LBK. \tag{2.26}$$

These transformed coefficients are continuous as are the original ones. Moreover, $\bar{A}$ and $\bar{D}$ inherit from A and D the constant ranks, and the leading term of (2.25) is properly stated (cf. Definition 2.1) with the same border projector $\bar{R} = R$ as $\ker \bar{A} = \ker A$, $\operatorname{im} \bar{D} = \operatorname{im} D$.

Suppose that the original DAE (2.1) has admissible projectors $Q_0, \ldots, Q_\kappa$. We form a corresponding matrix function sequence for the transformed DAE (2.25) starting with

$$\bar{G}_0 = \bar{A}\bar{D} = LADK = LG_0K, \quad \bar{B}_0 = \bar{B} = LB_0K,$$
$$\bar{Q}_0 := K^{-1}Q_0K, \quad \bar{D}^- = K^{-1}D^-, \quad \bar{P}_0 = K^{-1}P_0K,$$

such that $\bar{D}\bar{D}^- = DD^- = R$, $\bar{D}^-\bar{D} = \bar{P}_0$, and

$$\bar{G}_1 = \bar{G}_0 + \bar{B}_0\bar{Q}_0 = L(G_0 + B_0Q_0)K = LG_1K.$$

This yields $\bar{N}_0 = K^{-1}N_0$, $\bar{N}_1 = K^{-1}N_1$, $\bar{N}_0 \cap \bar{N}_1 = K^{-1}(N_0 \cap N_1)$. Choose $\bar{Q}_1 := K^{-1}Q_1K$ which corresponds to $\bar{X}_1 := K^{-1}X_1$. Proceeding in this way at each level, $i = 1, \ldots, \kappa$, with

$$\bar{Q}_i := K^{-1}Q_iK$$

it follows that $\bar{\Pi}_i = K^{-1}\Pi_iK$, $\bar{D}\bar{\Pi}_i\bar{D}^- = D\Pi_iD^-$, $\bar{X}_i = K^{-1}X_i$, $\overline{N\mathcal{U}}_i = K^{-1}\widehat{N}_i$, and

$$\bar{G}_{i+1} = LG_{i+1}K, \quad \bar{B}_{i+1} = LB_{i+1}K.$$

This shows that $\bar{Q}_0, \ldots, \bar{Q}_\kappa$ are admissible for (2.25), and the following assertion becomes evident.

Theorem 2.18. *If the DAE* (2.1) *has an admissible matrix function sequence up to level* $\kappa \in \mathbb{N}$, *with characteristic values* r_i, u_i, $i = 1, \ldots, \kappa$, *then the transformed equation* (2.25) *also has an admissible matrix function sequence up to level* κ, *with the same characteristic values, i.e.,* $\bar{r}_i = r_i$, $\bar{u}_i = u_i$, $i = 1, \ldots, \kappa$.

By Theorem 2.18 the characteristic values are invariant under transformations of the unknown function as well as under premultiplications of the DAE. This feature seems to be rather trivial. The invariance with respect to refactorizations of the leading term, which we verify next, is more subtle.

First we explain what *refactorization* means. For the given DAE (2.1) with properly stated leading term, we consider the product AD to represent a *factorization of the leading term* and we ask whether we can turn to a different factorization $AD = \bar{A}\bar{D}$ such that $\ker \bar{A}$ and $\operatorname{im} \bar{D}$ are again transversal $\mathcal{C}^1$-subspaces. For instance, in Example 2.4, equation (2.10) results from equation (2.9) by taking a different factorization.

In general, we describe the change to a different factorization as follows:
Let $H \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^s, \mathbb{R}^n))$ be given together with a generalized inverse $H^- \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^n, \mathbb{R}^s))$ such that

$$H^-HH^- = H^-, \quad HH^-H = H, \quad RHH^-R = R. \tag{2.27}$$

H has constant rank greater than or equal to the rank of the border projector R. In particular, one can use any nonsingular $H \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^n))$. However, we do not restrict ourselves to square nonsingular matrix functions H.
Due to $AR = ARHH^-R$ we may write

$$\begin{aligned} A(Dx)' = ARHH^-R(Dx)' &= ARH(H^-RDx)' - ARH(H^-R)'Dx \\ &= AH(H^-Dx)' - AH(H^-R)'Dx. \end{aligned}$$

This leads to the new DAE

$$\bar{A}(\bar{D}x)' + \bar{B}x = q \tag{2.28}$$

with the continuous coefficients

$$\bar{A} := AH, \quad \bar{D} := H^- D, \quad \bar{B} := B - ARH(H^- R)' D. \tag{2.29}$$

Because of $\bar{A}\bar{D} = AD$ we call this procedure that changes (2.1) to (2.28) a *refactorization of the leading term*. It holds that

$$\ker \bar{A} = \ker AH = \ker RH, \quad \operatorname{im} \bar{D} = \operatorname{im} H^- D = \operatorname{im} H^- R;$$

further $(H^- RH)^2 = H^- RHH^- RH = H^- RH$. It becomes clear that $H^- RH \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^s))$ is actually the border projector corresponding to the new DAE (2.28), and (2.28) has a properly stated leading term.
We emphasize that the old border space $\mathbb{R}^n$ and the new one $\mathbb{R}^s$ may actually have different dimensions, and this is accompanied by different sizes of the involved matrix functions. Here, the only restriction is $n, s \geq r := \operatorname{rank} D$.

Example 2.19 (A simple refactorization changing the border space dimension). The semi-explicit DAE

$$x_1' + B_{11}x_1 + B_{12}x_2 = q_1,$$
$$B_{21}x_1 + B_{22}x_2 = q_2,$$

comprising m_1 and m_2 equations can be written with proper leading term in different ways, for instance as

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} x \right)' + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} x = q \tag{2.30}$$

as well as

$$\begin{bmatrix} I \\ 0 \end{bmatrix} \left(\begin{bmatrix} I & 0 \end{bmatrix} x \right)' + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} x = q. \tag{2.31}$$

The border projector R of the DAE (2.30) as well as H and H^-,

$$R = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad H^- = \begin{bmatrix} I & 0 \end{bmatrix},$$

satisfy condition (2.27). The DAE (2.31) results from the DAE (2.30) by refactorization of the leading term by means of H. The border projector of the DAE (2.31) is simply $\bar{R} = H^- RH = I$. The dimension of the border space is reduced from $m_1 + m_2$ in (2.30) to m_1 in (2.31). □

Example 2.20 (Nontrivial refactorization). The following two DAEs are given in Example 2.4,

$$\underbrace{\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}}_{\tilde{A}}\Bigg(\underbrace{\begin{bmatrix}0&0&0\\0&1&0\\0&-t&1\end{bmatrix}}_{\tilde{D}(t)}x(t)\Bigg)'+\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&-t&1\end{bmatrix}}_{\tilde{B}(t)}x(t)=q(t),\quad t\in\mathbb{R}$$

and

$$\underbrace{\begin{bmatrix}0&1&0\\0&-t&1\\0&0&0\end{bmatrix}}_{A(t)}\Bigg(\underbrace{\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}}_{D}x(t)\Bigg)'+\underbrace{\begin{bmatrix}1&0&0\\0&0&0\\0&-t&1\end{bmatrix}}_{B(t)}x(t)=q(t),\quad t\in\mathbb{R}.$$

The border projector of the last DAE is simply $R=D$. The nonsingular matrix function

$$H(t)=\begin{bmatrix}1&0&0\\0&1&0\\0&t&1\end{bmatrix},\quad H(t)^-=H(t)^{-1}=\begin{bmatrix}1&0&0\\0&1&0\\0&-t&1\end{bmatrix}$$

fulfills condition (2.27). Comparing the coefficients, one proves that the first DAE results from the refactorization of the second DAE with H. Conversely, one obtains the second DAE by refactorization of the first one with H^{-1}.
Observe that the matrix pencil $\{\tilde{A}\tilde{D}(t),\tilde{B}(t)\}$ is regular with Kronecker index 3, while $\{A(t)D,B(t)\}$ is a singular pencil. This confirms once more the well-known fact that local matrix pencils are inapplicable to characterize time-varying DAEs.

□

Theorem 2.21. *Let the DAE (2.1) have a properly stated leading term and an admissible matrix function sequence up to level $\kappa\in\mathbb{N}$ and characteristic values $r_0,\ldots,r_\kappa$, $u_1,\ldots,u_\kappa$.*
Let the matrix functions $H\in\mathcal{C}^1(\mathcal{I},L(\mathbb{R}^s,\mathbb{R}^n))$ and $H^-\in\mathcal{C}^1(\mathcal{I},L(\mathbb{R}^n,\mathbb{R}^s))$ satisfy condition (2.27).

(a) *Then the refactorized DAE (2.28) also has a properly stated leading term and an admissible matrix function sequence up to level κ. Its characteristic values coincide with that of (2.1).*
(b) *The subspaces* $\operatorname{im} G_i$, $N_0+\cdots+N_i$, $i=0,\ldots,\kappa$, *are invariant.*

Proof. Put $F_1:=I$.
We use induction to show that the following relations are valid:

$$\bar{G}_i=G_iF_i\cdots F_1,\tag{2.32}$$

$$\bar{Q}_i:=(F_i\cdots F_1)^{-1}Q_iF_i\cdots F_1,\quad \bar{\Pi}_{i-1}\bar{Q}_i=\Pi_{i-1}Q_i,\quad \bar{\Pi}_i=\Pi_i,\tag{2.33}$$

$$\bar{B}_i=B_i-G_iD^-H(H^-R)'D\Pi_i+G_i\sum_{j=0}^{i-1}Q_jZ_{ij}\Pi_{i-1},\tag{2.34}$$

with nonsingular

$$F_i := I + P_{i-1} \sum_{j=0}^{i-2} Q_j Z_{i-1j} \Pi_{i-2} Q_{i-1}, \quad i = 1, \ldots, \kappa.$$

The coefficients $Z_{\ell j}$ are continuous matrix functions whose special form does not matter at all.
Since $\bar{G}_0 = \bar{A}\bar{D} = AD = G_0$ we may choose $\bar{D}^- = D^- H$, $\bar{Q}_0 = Q_0$. It follows that $\bar{\Pi}_0 = \Pi_0$, $\bar{B}_0 = \bar{B} = B - ARH(H^- R)' D$ and $\bar{B}_0 \bar{Q}_0 = BQ_0 = B_0 Q_0$, hence $\bar{G}_1 = \bar{G}_0 + \bar{B}_0 \bar{Q}_0 = G_0 + B_0 Q_0 = G_1 = G_1 F_1$. Choose $\bar{Q}_1 = Q_1 = F_1^{-1} Q_1$ such that $\bar{\Pi}_1 = \Pi_1$, $\bar{\Pi}_0 \bar{Q}_1 = \Pi_0 Q_1$, $\bar{D}\bar{\Pi}_1 \bar{D}^- = H^- D \Pi_1 D^- H$, and further

$$\begin{aligned}
\bar{B}_1 &= \bar{B}_0 \bar{P}_0 - \bar{G}_1 \bar{D}^- (\bar{D}\bar{\Pi}_1 \bar{D}^-)' \bar{D}\bar{\Pi}_0 \\
&= B_0 P_0 - ARH(H^- R)' D - G_1 D^- H (H^- D \Pi_1 D^- H)' H^- D \Pi_0 \\
&= B_0 P_0 - G_1 D^- (D\Pi_1 D^-)' D\Pi_0 + G_1 D^- (D\Pi_1 D^-)' D\Pi_0 \\
&\qquad\qquad -ARH(H^- R)' D - G_1 D^- H (H^- RD\Pi_1 D^- RH)' H^- D\Pi_0 \\
&= B_1 + G_1 D^- (D\Pi_1 D^-)' D\Pi_0 - ARH(H^- R)' D - G_1 D^- H \{(H^- R)' D\Pi_1 D^- RH \\
&\qquad\qquad + H^- R (D\Pi_1 D^-)' RH + H^- RD\Pi_1 D^- (RH)'\} H^- D \\
&= B_1 - ARH(H^- R)' D - G_1 D^- H (H^- R)' D\Pi_1 - G_1 \Pi_1 D^- (RH)' H^- RD \\
&= B_1 - G_1 D^- H (H^- R)' D\Pi_1 - ARH(H^- R)' D + G_1 \Pi_1 D^- RH (H^- R)' D.
\end{aligned}$$

In the last expression we have used that

$$D^- (RHH^- R)' D = D^- R' D = 0.$$

Compute $G_1 \Pi_1 D^- RH(H^- R)' D - ARH(H^- R)' D = G_1 (\Pi_1 - I) D^- RH (H^- R)' D$ and

$$\begin{aligned}
G_1 (\Pi_1 - I) &= G_1 ((I - Q_0)(I - Q_1) - I) = G_1 (-Q_0 - Q_1 + Q_0 Q_1) \\
&= G_1 (-Q_0 + Q_0 Q_1) = -G_1 Q_0 P_1.
\end{aligned}$$

This yields the required expression

$$\bar{B}_1 = B_1 - G_1 D^- H (H^- R)' D\Pi_1 + G_1 Q_0 Z_{10} \Pi_0$$

with $Z_{10} := -Q_0 P_1 D^- RH (H^- R)' D$.
Next, supposing the relations (2.32)–(2.34) to be given up to i, we show their validity for $i+1$. Derive

$$\begin{aligned}
\bar{G}_{i+1} &= \bar{G}_i + \bar{B}_i \bar{Q}_i = \{G_i + \bar{B}_i (F_i \cdots F_1)^{-1} Q_i\} F_i \cdots F_1 \\
&= \{G_i + \bar{B}_i \Pi_{i-1} (F_i \cdots F_1)^{-1} Q_i\} F_i \cdots F_1,
\end{aligned}$$

and, because of $\Pi_{i-1} F_1^{-1} \cdots F_i^{-1} = \Pi_{i-1}$, we obtain further

$$\begin{aligned}
\bar{G}_{i+1} &= \Big\{G_i + B_iQ_i - G_iD^-H(H^-R)'D\Pi_iQ_i + G_i\sum_{j=0}^{i-1}Q_jZ_{ij}\Pi_{i-1}Q_i\Big\}F_i\cdots F_1\\
&= \Big\{G_{i+1} + G_i\sum_{j=0}^{i-1}Q_jZ_{ij}\Pi_{i-1}Q_i\Big\}F_i\cdots F_1\\
&= G_{i+1}\Big\{I + P_i\sum_{j=0}^{i-1}Q_jZ_{ij}\Pi_{i-1}Q_i\Big\}F_i\cdots F_1\\
&= G_{i+1}F_{i+1}F_i\cdots F_1,
\end{aligned}$$

with nonsingular matrix functions

$$F_{i+1} = I + P_i\sum_{j=0}^{i-1}Q_jZ_{ij}\Pi_{i-1}Q_i, \quad F_{i+1}^{-1} = I - P_i\sum_{j=0}^{i-1}Q_jZ_{ij}\Pi_{i-1}Q_i.$$

Put $\bar{Q}_{i+1} := (F_{i+1}\cdots F_1)^{-1}Q_{i+1}F_{i+1}\cdots F_1$, and compute

$$\begin{aligned}
\bar{\Pi}_i\bar{Q}_{i+1} &= \Pi_i\bar{Q}_{i+1} = \Pi_iF_1^{-1}\cdots F_{i+1}^{-1}Q_{i+1}F_{i+1}\cdots F_1\\
&= \Pi_iQ_{i+1}F_{i+1}\cdots F_1 = \Pi_iQ_{i+1}\Pi_iF_{i+1}\cdots F_1 = \Pi_iQ_{i+1}\Pi_i = \Pi_iQ_{i+1},
\end{aligned}$$

$$\bar{\Pi}_{i+1} = \bar{\Pi}_i - \bar{\Pi}_i\bar{Q}_{i+1} = \Pi_i - \Pi_iQ_{i+1} = \Pi_{i+1}.$$

It remains to verify the expression for $\bar{B}_{i+1}$. We derive

$$\begin{aligned}
\bar{B}_{i+1} &= \bar{B}_i\bar{P}_i - \bar{G}_{i+1}\bar{D}^-(\bar{D}\bar{\Pi}_{i+1}\bar{D}^-)'\bar{D}\bar{\Pi}_i\\
&= \bar{B}_i\Pi_i - G_{i+1}F_{i+1}\cdots F_1D^-H(H^-D\Pi_{i+1}D^-H)'H^-D\Pi_i,
\end{aligned}$$

and

$$\begin{aligned}
\bar{B}_{i+1} = \Big\{&B_i - G_iD^-H(H^-R)'D\Pi_i + G_i\sum_{j=0}^{i-1}Q_jZ_{ij}\Pi_{i-1}\Big\}\Pi_i\\
&-G_{i+1}(F_{i+1}\cdots F_1 - I)D^-H(H^-D\Pi_{i+1}D^-H)'H^-D\Pi_i\\
&-G_{i+1}D^-H\{(H^-R)'RD\Pi_{i+1}D^-RH + H^-R(D\Pi_{i+1}D^-)'RH\\
&+H^-RD\Pi_{i+1}D^-(RH)'\}H^-D\Pi_i,
\end{aligned}$$

and

$$\begin{aligned}
\bar{B}_{i+1} = {}&B_iP_i - G_iD^-H(H^-R)'D\Pi_i + G_i\sum_{j=0}^{i-1}Q_jZ_{ij}\Pi_i\\
&-G_{i+1}D^-H(H^-R)'D\Pi_{i+1} - G_{i+1}D^-(D\Pi_{i+1}D^-)'D\Pi_i\\
&-G_{i+1}\Pi_{i+1}D^-(RH)'H^-RD\Pi_i\\
&-G_{i+1}(F_{i+1}\cdots F_1 - I)D^-H(H^-D\Pi_{i+1}D^-H)'H^-D\Pi_i,
\end{aligned}$$

and

$$\begin{aligned}\bar{B}_{i+1} = B_{i+1} - G_{i+1}D^-H(H^-R)'D\Pi_{i+1} - G_{i+1}P_iD^-H(H^-R)'D\Pi_i \\ + G_{i+1}\Pi_{i+1}D^-H(H^-R)'D\Pi_i + G_{i+1}P_i\sum_{j=0}^{i-1}Q_jZ_{ij}\Pi_i \\ - G_{i+1}(F_{i+1}\cdots F_i - I)D^-H(H^-D\Pi_{i+1}D^-H)'H^-D\Pi_i.\end{aligned}$$

Finally, decomposing

$$P_i\sum_{j=0}^{i-1}Q_jZ_{ij}\Pi_i = \sum_{j=0}^{i-1}Q_jZ_{ij}\Pi_i - Q_i\sum_{j=0}^{i-1}Q_jZ_{ij}\Pi_i,$$

and expressing

$$F_{i+1}\cdots F_1 - I = \sum_{j=0}^{i}Q_j\mathfrak{A}_{i+1,j},$$

and taking into account that

$$G_{i+1}\{\Pi_{i+1} - P_i\}D^-H(H^-R)'D\Pi_i = G_{i+1}\sum_{j=0}^{i}Q_i\mathfrak{B}_{i+1,j}D^-H(H^-R)'D\Pi_i$$

we obtain

$$\bar{B}_{i+1} = B_{i+1} - G_{i+1}D^-H(H^-R)'D\Pi_{i+1} + \sum_{j=0}^{i}Q_jZ_{i+1,j}D\Pi_i.$$

□

By Theorem 2.21, the characteristic values and the tractability index are invariant under refactorizations of the leading term. In this way, the size of A and D may change or not (cf. Examples 2.4 and 2.19).

It is worth mentioning that also the associated function space accommodating the solutions of the DAE remains invariant under refactorizations as the next proposition shows.

Proposition 2.22. *Given the matrix function $D \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^m, \mathbb{R}^n))$ and the projector function $R \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^n))$ onto $\operatorname{im} D$, let $H \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^s, \mathbb{R}^n))$ be given together with a generalized inverse $H^- \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^n, \mathbb{R}^s))$ such that $H^-HH^- = H^-$, $HH^-H = H$, and $RHH^-R = R$. Then, for $\bar{D} = H^-D$, it holds that*

$$\mathcal{C}^1_{\bar{D}}(\mathcal{I}, \mathbb{R}^m) = \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m).$$

Proof. For any $x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ we find $\bar{D}x = H^-Dx \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^s)$, and hence $x \in \mathcal{C}^1_{\bar{D}}(\mathcal{I}, \mathbb{R}^m)$. Conversely, for $x \in \mathcal{C}^1_{\bar{D}}(\mathcal{I}, \mathbb{R}^m)$, we find $Dx = RDx = RHH^-Dx = RH\bar{D}x \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^s)$, and hence $x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$. □

2.4 Decoupling regular DAEs

The main objective of this section is the characterization of *regular DAEs* by means of admissible matrix function sequences and the projector based *structural decoupling* of each regular DAE (2.1) into an inherent regular ODE

$$u' - (D\Pi_{\mu-1}D^-)'u + D\Pi_{\mu-1}G_\mu^{-1}B_\mu D^- u = D\Pi_{\mu-1}G_\mu^{-1}q$$

and a triangular subsystem of several equations including differentiations

$$\begin{bmatrix} 0 & \mathcal{N}_{01} & \cdots & \mathcal{N}_{0,\mu-1} \\ & 0 & \ddots & \vdots \\ & & \ddots & \mathcal{N}_{\mu-2,\mu-1} \\ & & & 0 \end{bmatrix} \begin{bmatrix} 0 \\ (Dv_1)' \\ \vdots \\ (Dv_{\mu-1})' \end{bmatrix}$$
$$+ \begin{bmatrix} I & \mathcal{M}_{01} & \cdots & \mathcal{M}_{0,\mu-1} \\ & I & \ddots & \vdots \\ & & \ddots & \mathcal{M}_{\mu-2,\mu-1} \\ & & & I \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{\mu-1} \end{bmatrix} + \begin{bmatrix} \mathcal{H}_0 \\ \mathcal{H}_1 \\ \vdots \\ \mathcal{H}_{\mu-1} \end{bmatrix} D^- u = \begin{bmatrix} \mathcal{L}_0 \\ \mathcal{L}_1 \\ \vdots \\ \mathcal{L}_{\mu-1} \end{bmatrix} q.$$

This structural decoupling is associated with the decomposition (see Theorem 2.30)

$$x = D^- u + v_0 + v_1 + \cdots + v_{\mu-1}.$$

2.4.1 Preliminary decoupling rearrangements

We apply admissible projector functions $Q_0, \ldots, Q_\kappa$ to rearrange terms within the DAE (2.1) in a similar way as done in Chapter 1 on constant coefficient DAEs for obtaining decoupled systems. The objective of the rearrangements is to place a matrix function G_κ in front of the derivative component $(D\Pi_\kappa x)'$, the rank of which is as large as possible, and at the same time to separate terms living in $N_0 + \cdots + N_\kappa$. We emphasize that we do not change the given DAE at all, and we do not transform the variables. We work just with the given DAE and its unknown. What we do are *rearrangements of terms and separations or decouplings of solution components* by means of projector functions. We proceed stepwise. Within this procedure, the special form of the matrix functions B_i appears to make good sense.
This part is valid for general DAEs with proper leading term, possibly with less or more variables than equations ($m \neq k$). The rearranged DAE versions serve then as the basis for further decouplings and solutions in the present chapter and also in Chapter 10.

First rewrite (2.1) as

$$G_0 D^-(Dx)' + B_0 x = q, \tag{2.35}$$

and then as

$$G_0D^-(Dx)' + B_0(Q_0x + P_0x) = q$$

and rearrange this in order to increase the rank of the leading coefficient to

$$(G_0 + B_0Q_0)(D^-(Dx)' + Q_0x) + B_0P_0x = q,$$

or

$$G_1D^-(Dx)' + B_0P_0x + G_1Q_0x = q. \tag{2.36}$$

Compute

$$\begin{aligned} P_1D^-(Dx)' &= P_0P_1D^-(Dx)' + Q_0P_1D^-(Dx)' \\ &= D^-DP_0P_1D^-(Dx)' + Q_0P_1D^-(Dx)' \\ &= D^-(DP_0P_1x)' - D^-(DP_0P_1D^-)'Dx + Q_0P_1D^-(Dx)' \\ &= D^-(DP_0P_1x)' - D^-(DP_0P_1D^-)'Dx - (I-P_0)Q_1D^-(Dx)' \\ &= D^-(D\Pi_1x)' - D^-(D\Pi_1D^-)'Dx - (I-\Pi_0)Q_1D^-(D\Pi_0x)', \end{aligned}$$

and hence

$$G_1D^-(Dx)' = G_1D^-(D\Pi_1x)' - G_1D^-(D\Pi_1D^-)'DP_0x - G_1(I-\Pi_0)Q_1D^-(D\Pi_0x)'.$$

Inserting this into (2.36) yields

$$\begin{aligned} G_1D^-(D\Pi_1x)' \;&+\; (B_0P_0 - G_1D^-(D\Pi_1D^-)'DP_0)x \\ &+\; G_1\{Q_0x - (I-\Pi_0)Q_1D^-(Dx)'\} = q, \end{aligned}$$

and, regarding the definition of the matrix function B_1,

$$G_1D^-(D\Pi_1x)' + B_1x + G_1\{Q_0x - (I-\Pi_0)Q_1D^-(Dx)'\} = q. \tag{2.37}$$

Note that, if $N_0 \cap N_1 = 0$, then the derivative $(D\Pi_1x)'$ is no longer involved in the term

$$Q_1D^-(Dx)' = Q_1D^-DP_0Q_1D^-(Dx)' = Q_1D^-(DP_0Q_1x)' - Q_1D^-(DP_0Q_1D^-)'Dx.$$

In the next step we move a part of the term B_1x in (2.37) to the leading term, and so on. Proposition 2.23 describes the result of these systematic rearrangements.

Proposition 2.23. *Let the DAE (2.1) with properly stated leading term have the admissible projectors $Q_0, \dots, Q_\kappa$, where $\kappa \in \mathbb{N} \cup \{0\}$.*

(1) *Then this DAE can be rewritten in the form*

$$G_\kappa D^-(D\Pi_\kappa x)' + B_\kappa x + G_\kappa \sum_{l=0}^{\kappa-1} \{Q_l x + (I-\Pi_l)(P_l - Q_{l+1}P_l)D^-(D\Pi_l x)'\} = q. \tag{2.38}$$

(2) *If, additionally, all intersections $\widehat{N}_i$, $i=1,\dots,\kappa$, are trivial, then the DAE (2.1) can be rewritten as*

$$\begin{aligned} &G_\kappa D^-(D\Pi_\kappa x)' + B_\kappa x \\ &\quad + G_\kappa \sum_{l=0}^{\kappa-1}\{Q_l x - (I-\Pi_l)Q_{l+1}D^-(D\Pi_l Q_{l+1}x)' + V_l D\Pi_l x\} = q, \end{aligned} \tag{2.39}$$

with coefficients

$$V_l = (I-\Pi_l)\{P_l D^-(D\Pi_l D^-)' - Q_{l+1}D^-(D\Pi_{l+1}D^-)'\}D\Pi_l D^-,\ l=0,\dots,\kappa-1.$$

Comparing with the rearranged DAE obtained in the constant coefficient case (cf. (1.35)), now we observe the extra terms V_l caused by time-dependent movements of certain subspaces. They disappear in the time-invariant case.

Proof (of Proposition 2.23). (1) In the case of $\kappa=0$, equation (2.35) is just a trivial reformulation of (2.1). For $\kappa=1$ we are done by considering (2.37). For applying induction, we suppose for $i+1\le\kappa$, that (2.1) can be rewritten as

$$G_i D^-(D\Pi_i x)' + B_i x + G_i\sum_{l=0}^{i-1}\{Q_l x + (I-\Pi_l)(P_l - Q_{l+1}P_l)D^-(D\Pi_l x)'\} = q. \tag{2.40}$$

Represent $B_i x = B_i P_i x + B_i Q_i x = B_i P_i x + G_{i+1}Q_i x$ and derive

$$\begin{aligned} G_i D^-(D\Pi_i x)' &= G_{i+1}P_{i+1}P_i D^-(D\Pi_i x)' \\ &= G_{i+1}\{\Pi_{i+1}P_i D^-(D\Pi_i x)' + (I-\Pi_i)P_{i+1}P_i D^-(D\Pi_i x)'\} \\ &= G_{i+1}\{D^- D\Pi_{i+1}D^-(D\Pi_i x)' + (I-\Pi_i)P_{i+1}P_i D^-(D\Pi_i x)'\} \\ &= G_{i+1}D^-(D\Pi_{i+1}x)' - G_{i+1}D^-(D\Pi_{i+1}D^-)'D\Pi_i x \\ &\qquad + G_{i+1}(I-\Pi_i)(P_i - Q_{i+1}P_i)D^-(D\Pi_i x)'). \end{aligned}$$

Taking into account that $(I-\Pi_i) = Q_0P_1\cdots P_i + \cdots + Q_{i-1}P_i + Q_i$ and $G_iQ_l = G_{i+1}Q_l$, $l=0,\dots,i-1$, we realize that (2.40) can be reformulated to

$$\begin{aligned} &G_{i+1}D^-(D\Pi_{i+1}x)' + (B_iP_i - G_{i+1}D^-(D\Pi_{i+1}D^-)'D\Pi_i)x \\ &\qquad + G_{i+1}Q_i x + G_{i+1}\sum_{l=0}^{i-1}\{Q_l x + (I-\Pi_l)(P_l - Q_{l+1}P_l)D^-(D\Pi_l x)'\} \\ &\qquad\qquad + G_{i+1}(I-\Pi_i)(P_i - Q_{i+1}P_i)D^-(D\Pi_i x)' = q. \end{aligned}$$

We obtain in fact

$$G_{i+1}D^-(D\Pi_{i+1}x)' + B_{i+1}x + G_{i+1}\sum_{l=0}^{i}\{Q_l x + (I-\Pi_l)(P_l - Q_{l+1}P_l)D^-(D\Pi_l x)'\} = q$$

as we tried for.

(2) Finally assuming $\widehat{N}_i=\{0\}$, $i=1,\dots,\kappa$, and taking into account Proposition 2.7,

we compute the part in question as

$$\begin{aligned}\mathcal{F} &:= \sum_{l=0}^{k-1}(I-\Pi_l)(P_l-Q_{l+1}P_l)D^-(D\Pi_l x)' = \sum_{l=0}^{k-1}(I-\Pi_l)(P_l-Q_{l+1})D^-(D\Pi_l x)' \\ &= \sum_{l=0}^{k-1}(I-\Pi_l)\Big\{P_l D^-(D\Pi_l x)' - Q_{l+1}D^- D\Pi_l Q_{l+1}D^-(D\Pi_l x)'\Big\}.\end{aligned}$$

Applying the relations

$$\begin{aligned}(D\Pi_l x)' &= (D\Pi_l D^-)'(D\Pi_l x) + D\Pi_l D^-(D\Pi_l x)', \\ (I-\Pi_l)P_l D^- D\Pi_l D^- &= (I-\Pi_l)P_l \Pi_l D^- = 0, \\ D\Pi_l Q_{l+1}D^-(D\Pi_l x)' &= (D\Pi_l Q_{l+1}x)' - (D\Pi_l Q_{l+1}D^-)' D\Pi_l x, \\ Q_{l+1}(D\Pi_l Q_{l+1}D^-)' D\Pi_l &= Q_{l+1}(D\Pi_l D^-)' D\Pi_l - Q_{l+1}(D\Pi_{l+1}D^-)' D\Pi_l \\ &= -Q_{l+1}(D\Pi_{l+1}D^-)' D\Pi_l,\end{aligned}$$

we obtain, with the coefficients V_l described by the assertion,

$$\begin{aligned}\mathcal{F} = \sum_{l=0}^{k-1}(I-\Pi_l)\Big\{&P_l D^-(D\Pi_l D^-)' D\Pi_l x + Q_{l+1}D^-(D\Pi_l Q_{l+1}D^-)' D\Pi_l x \\ &- Q_{l+1}D^-(D\Pi_l Q_{l+1}x)'\Big\} = \sum_{l=0}^{k-1}\Big\{V_l D\Pi_l x - (I-\Pi_l)Q_{l+1}D^-(D\Pi_l Q_{l+1}x)'\Big\},\end{aligned}$$

and this completes the proof. □

How can one make use of the rearranged version of the DAE (2.1) and the structural information included in this version? We discuss this question in the next subsection for the case of regular DAEs, that is, if $m = k$ and a nonsingular G_μ exists. We study nonregular cases in Chapter 10.

For the moment, to gain a first impression, we cast a glance at the simplest situation, if G_0 already has maximal rank. Later on we assign the *tractability index 0* to each DAE whose matrix functions G_0 already have maximal rank. Then the DAE (2.35) splits into the two parts

$$G_0 D^-(Dx)' + G_0 G_0^- B_0 x = G_0 G_0^- q, \quad \mathcal{W}_0 B_0 x = \mathcal{W}_0 q. \tag{2.41}$$

Since $\operatorname{im} G_0$ is maximal, it holds that $\operatorname{im} B_0 Q_0 \subseteq \operatorname{im} G_1 = \operatorname{im} G_0$, hence $\mathcal{W}_0 B_0 = \mathcal{W}_0 B_0 P_0$. Further, since $DG_0^- G_0 = D$, we find the DAE (2.35) to be equivalent to the system

$$(Dx)' - R' Dx + DG_0^- B_0 D^- Dx + DG_0^- B_0 Q_0 x = DG_0^- q, \;\; \mathcal{W}_0 B_0 D^- Dx = \mathcal{W}_0 q, \tag{2.42}$$

the solution of which decomposes as $x = D^- Dx + Q_0 x$. It becomes clear that this DAE comprises an explicit ODE for Dx, that has an undetermined part $Q_0 x$ to be chosen arbitrarily. The ODE for Dx is accompanied by a consistency condition applied to Dx and q. If G_0 is surjective, the consistency condition disappears. If G_0 is injective, then the undetermined component Q_0x disappears. If G_0 is nonsingular, which happens just for $m = k$, then the DAE is nothing other than a regular implicit ODE with respect to Dx.

Example 2.24 (Nonregular DAE). The DAE

$$\begin{bmatrix} t \\ 1 \end{bmatrix} \left(\begin{bmatrix} -1 & t \end{bmatrix} x(t) \right)' + \begin{bmatrix} 1 & -t \\ 0 & 0 \end{bmatrix} x(t) = q(t)$$

leads to

$$G_0(t) = \begin{bmatrix} -t & t^2 \\ -1 & t \end{bmatrix}, \quad Q_0(t) = \begin{bmatrix} 0 & t \\ 0 & 1 \end{bmatrix}, \quad B_0(t) = \begin{bmatrix} 1 & -t \\ 0 & 0 \end{bmatrix}, \quad G_1 = G_0.$$

Compute further

$$D^-(t) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \qquad R = 1, \quad G_0^-(t) = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{W}_0(t) = \begin{bmatrix} 1 & -t \\ 0 & 0 \end{bmatrix},$$
$$DG_0^- B_0 D^- = 0, \qquad B_0 Q_0 = 0, \quad DG_0^- = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

For the second equation in formula (2.42) we obtain

$$\mathcal{W}_0 B_0 x = \mathcal{W}_0 q \Leftrightarrow -x_1 + t x_2 = q_1 - t q_2$$

and the inherent explicit ODE in formula (2.42) reads

$$(-x_1 + t x_2)' = q_2.$$

In this way the consistency condition $(q_1 - t q_2)' = q_2$ follows. The solution is

$$\begin{aligned} x(t) &= D^-(-x_1 + t x_2) + Q_0 x \\ &= \begin{bmatrix} x_1 - t x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} t x_2 \\ x_2 \end{bmatrix}, \end{aligned}$$

with an arbitrary continuous function x_2. □

Of course, if the tractability index is greater than 0, things become much more subtle.

2.4.2 Regularity and basic decoupling of regular DAEs

We define regularity for DAEs after the model of classical ODE theory. The system

$$A(t)x'(t)+B(t)x(t)=q(t),\ \ t\in\mathcal{I}, \tag{2.43}$$

with continuous coefficients, is named a regular implicit ODE or an ODE having solely regular line-elements, if the matrix $A(t)\in L(\mathbb{R}^m)$ remains nonsingular on the given interval. Then the homogeneous version of this ODE has a solution space of dimension m and the inhomogeneous ODE is solvable for each continuous excitation q. No question, these properties are maintained, if one turns to a subinterval. On the other hand, a point at which the full-rank condition of the matrix $A(t)$ becomes defective is a critical point, and different kinds of singularities are known to arise (e.g. [123]).
Roughly speaking, in our view, a regular DAE should have similar properties. It should be such that the homogeneous version has a finite-dimensional solution space and no consistency conditions related to the excitations q arise for inhomogeneous equations, which rules out DAEs with more or less unknowns than equations. Additionally, each restriction of a DAE to a subinterval should also inherit all characteristic values.

In the case of constant coefficients, regularity of DAEs is bound to regular pairs of square matrices. In turn, regularity of matrix pairs can be characterized by means of admissible matrix sequences and the associated characteristic values, as described in Section 1.2. A pair of $m\times m$ matrices is regular, if and only if an admissible matrix sequence shows a nonsingular matrix G_μ and the characteristic value $r_\mu=m$. Then the Kronecker index of the given matrix pair results as the smallest such index μ. The same idea applies now to DAEs with time-varying coefficients, too. However, we are now facing continuous *matrix functions* in distinction to the constant matrices in Chapter 1. While, in the case of constant coefficients, admissible projectors do always exist, their existence is now tied to several *rank conditions*. These rank conditions are indeed relevant to the problem. A point at which these rank conditions are defective is considered as a critical point.

We turn back to the DAE (2.1), i.e.,

$$A(t)(D(t)x(t))'+B(t)x(t)=q(t),\ \ t\in\mathcal{I}. \tag{2.44}$$

We are looking for solutions in the function space $\mathcal{C}_D^1(\mathcal{I},\mathbb{R}^m)$. Recall that the ranks $r_i=\operatorname{rank}G_i$ in admissible matrix function sequences (see Definitions 2.6, 2.9, Theorem 2.8) give the meaning of characteristics of the DAE on the given interval. The following regularity notion proves to meet the above expectations.

Definition 2.25. The DAE (2.44) with properly stated leading term and $m=k$ is said to be, on the given interval,

(1) *regular with tractability index 0*, if $r_0=m$,
(2) *regular with tractability index* $\mu\in\mathbb{N}$, if there is an admissible matrix function sequence with characteristic values $r_{\mu-1}<r_\mu=m$,
(3) *regular*, if the DAE is regular with any tractability index μ (i.e., case (1) or (2) apply).

This regularity notion is well-defined in the sense that it is independent of the special choice of the admissible projector functions, which is guaranteed by Theorem 2.8.

Since for a regular DAE the matrix function G_μ is nonsingular, all intersections $\widehat{N_i} = N_i \cap (N_0 + \cdots + N_{i-1})$ are trivial, as a consequence of Proposition 2.7. Then it holds that

$$X_i = (N_0 + \cdots + N_{i-1}) \ominus \widehat{N_i} = N_0 + \cdots + N_{i-1} = N_0 \oplus \cdots \oplus N_{i-1} \subseteq \ker Q_i,$$

$i = 1, \ldots, \mu - 1$, thus $Q_i(I - \Pi_{i-1}) = 0$, and, equivalently,

$$Q_i Q_j = 0, \quad 0 \leq j \leq i-1, \; i = 1, \ldots, \mu - 1. \tag{2.45}$$

Additionally, Proposition 2.7 (4) yields $G_\mu Q_j = B_j Q_j$, thus

$$Q_j = G_\mu^{-1} B_j \Pi_{j-1} Q_j, \quad j = 1, \ldots, \mu - 1. \tag{2.46}$$

While, in the general Definition 2.6, only the part $\Pi_{j-1} Q_j = \Pi_{j-1} - \Pi_j$ of an admissible projector function Q_j is required to be continues, for a regular DAE, the admissible projector functions are continuous in all their components, as follows from the representation (2.46).
We emphasize once again that, for regular DAEs, the admissible projector functions are always *regular* admissible, and they are continuous in all components. At this place, we draw the readers attention to the fact that, in papers dealing exclusively with regular DAEs, the requirements for trivial intersections $\widehat{N_i}$ and the continuity of Q_i are usually already incorporated into the admissibility notion (e.g., [170]) or into the regularity notion (e.g., [167], [137]). Then, the relations (2.46) are constituent parts of the definitions (see also the recent monograph [194]).

Here is a further special quality of regular DAEs: The associated subspaces (cf. Theorem 2.8)

$$S_i = \ker \mathcal{W}_i B = \{z \in \mathbb{R}^m : B_i z \in \operatorname{im} G_i\} = S_{i-1} + N_{i-1}$$

are now $\mathcal{C}$-subspaces, too. They have the constant dimensions r_i. This can be immediately checked. By Lemma A.9, the nonsingularity of G_μ implies the decomposition $N_{\mu-1} \oplus S_{\mu-1} = \mathbb{R}^m$, thus $\dim S_{\mu-1} = r_{\mu-1}$. Regarding the relation $\ker(G_{\mu-2} + \mathcal{W}_{\mu-2} B_{\mu-2} Q_{\mu-2}) = N_{\mu-2} \cap S_{\mu-2}$, we conclude by Proposition 2.5 (3) that $N_{\mu-2} \cap S_{\mu-2}$ has the same dimension as $N_{\mu-1}$ has. This means $\dim N_{\mu-2} \cap S_{\mu-2} = m - r_{\mu-1}$. Next, the representation $S_{\mu-1} = S_{\mu-2} + N_{\mu-2}$ leads to $r_{\mu-1} = \dim S_{\mu-2} + (m - r_{\mu-2}) - (m - r_{\mu-1})$, therefore $\dim S_{\mu-2} = r_{\mu-2}$, and so on.

We decouple the regular DAE (2.44) into its characteristic components, in a similar way as we did with constant coefficient DAEs in Subsection 1.2.2. Since G_μ is nonsingular, by introducing $Q_\mu = 0$, $P_\mu = I$, $\Pi_\mu = \Pi_{\mu-1}$, the sequence $Q_0, \ldots, Q_{\mu-1}, Q_\mu$ is admissible, and we can apply Proposition 2.23. The DAE (2.44) can be rewritten as

$$G_\mu D^-(D\Pi_{\mu-1}x)' + B_\mu x \qquad (2.47)$$
$$+ G_\mu \sum_{l=0}^{\mu-1} \{Q_l x - (I-\Pi_l)Q_{l+1}D^-(D\Pi_l Q_{l+1}x)' + V_l D\Pi_l x\} = q.$$

If the coefficients were constant, we would have $D^-(D\Pi_{\mu-1}x)' = (D^-D\Pi_{\mu-1}x)' = (\Pi_{\mu-1}x)'$, further $D^-(D\Pi_l Q_{l+1}x)' = (\Pi_l Q_{l+1}x)'$, and $V_l = 0$. This means that formula (2.47) precisely generalizes formula (1.35) obtained for constant coefficients. The new formula (2.47) contains the extra terms V_l which arise from subspaces moving with time. They disappear in the time-invariant case.

In Subsection 1.2.2, the decoupled version of the DAE is generated by the scaling with G_μ^{-1}, and then by the splitting by means of the projectors $\Pi_{\mu-1}$ and $I-\Pi_{\mu-1}$. Here we go a slightly different way and use $D\Pi_{\mu-1}$ instead of $\Pi_{\mu-1}$. Since $\Pi_{\mu-1}$ can be recovered from $D\Pi_{\mu-1}$ due to $\Pi_{\mu-1} = D^-D\Pi_{\mu-1}$, no information gets lost. Equation (2.47) scaled by G_μ^{-1} reads

$$D^-(D\Pi_{\mu-1}x)' + G_\mu^{-1}B_\mu x \qquad (2.48)$$
$$+ \sum_{l=0}^{\mu-1} \{Q_l x - (I-\Pi_l)Q_{l+1}D^-(D\Pi_l Q_{l+1}x)' + V_l D\Pi_l x\} = G_\mu^{-1}q.$$

The detailed expression for V_l (Proposition 2.23) is

$$V_l = (I-\Pi_l)\{P_l D^-(D\Pi_l D^-)' - Q_{l+1}D^-(D\Pi_{l+1}D^-)'\}D\Pi_l D^-.$$

This yields $D\Pi_{\mu-1}V_l = 0$, $l = 0,\ldots,\mu-1$, and multiplying (2.48) by $D\Pi_{\mu-1}$ results in the equation

$$D\Pi_{\mu-1}D^-(D\Pi_{\mu-1}x)' + D\Pi_{\mu-1}G_\mu^{-1}B_\mu x = D\Pi_{\mu-1}G_\mu^{-1}. \qquad (2.49)$$

Applying the $\mathcal{C}^1$-property of the projector $D\Pi_{\mu-1}D^-$, and recognizing that $B_\mu = B_\mu \Pi_{\mu-1} = B_\mu D^-D\Pi_{\mu-1}$, we get

$$(D\Pi_{\mu-1}x)' - (D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}x + D\Pi_{\mu-1}G_\mu^{-1}B_\mu D^-D\Pi_{\mu-1}x = D\Pi_{\mu-1}G_\mu^{-1}q. \qquad (2.50)$$

Equation (2.50) is an explicit ODE with respect to the component $D\Pi_{\mu-1}x$. A similar ODE is described by formula (1.37) for the time-invariant case. Our new ODE (2.50) generalizes the ODE (1.37) in the sense that, due to $D^-D\Pi_{\mu-1} = \Pi_{\mu-1}$, equation (2.50) multiplied by D^- coincides with (1.37) for constant coefficients.

Definition 2.26. For the regular DAE (2.44) with tractability index μ, and admissible projector functions $Q_0,\ldots,Q_{\mu-1}$, the resulting explicit regular ODE

$$u' - (D\Pi_{\mu-1}D^-)'u + D\Pi_{\mu-1}G_\mu^{-1}B_\mu D^-u = D\Pi_{\mu-1}G_\mu^{-1}q \qquad (2.51)$$

is called an *inherent explicit regular ODE* (IERODE) of the DAE.

It should be pointed out that there is a great variety of admissible projector functions. In consequence, there are various projector functions $\Pi_{\mu-1}$, and the IERODE (2.51) is not unique, except for the index-1 case. So far, we know the nullspace $N_0+\cdots+N_{\mu-1}$ of the projector function $\Pi_{\mu-1}$ to be independent of the choice of the admissible projector functions $Q_0,\ldots,Q_{\mu-1}$, which means the subspace $N_0+\cdots+N_{\mu-1}$ is unique; it is determined by the DAE coefficients only (Theorem 2.8). Later on we introduce advanced *fine decouplings* which make the corresponding IERODE unique.

Lemma 2.27. *If the DAE (2.44) is regular with index μ, and $Q_0,\ldots,Q_{\mu-1}$ are admissible, then the subspace $\operatorname{im} D\Pi_{\mu-1}$ is an invariant subspace for the IERODE (2.51), that is, the following assertion is valid for the solutions $u\in\mathcal{C}^1(\mathcal{I},\mathbb{R}^n)$ of the ODE (2.51):*

$$u(t_*)\in\operatorname{im}(D\Pi_{\mu-1})(t_*),\ \textit{with a certain}\ t_*\in\mathcal{I}\ \Leftrightarrow\ u(t)\in\operatorname{im}(D\Pi_{\mu-1})(t)\ \forall\, t\in\mathcal{I}.$$

Proof. Let $\bar u\in\mathcal{C}^1(\mathcal{I},\mathbb{R}^n)$ denote a solution of (2.51) with $\bar u(t_*)=(D\Pi_{\mu-1}D^-)(t_*)\bar u(t_*)$. We multiply the identity

$$\bar u' - (D\Pi_{\mu-1}D^-)'\bar u + D\Pi_{\mu-1}G_\mu^{-1}D^-\bar u = D\Pi_{\mu-1}G_\mu^{-1}q$$

by $I-D\Pi_{\mu-1}D^-$, and introduce the function $\bar v:=(I-D\Pi_{\mu-1}D^-)\bar u\in\mathcal{C}^1(\mathcal{I},\mathbb{R}^n)$. This gives

$$(I-D\Pi_{\mu-1}D^-)\bar u' - (I-D\Pi_{\mu-1}D^-)(D\Pi_{\mu-1}D^-)'\bar u = 0,$$

further,

$$\bar v' - (I-D\Pi_{\mu-1}D^-)'\bar u - (I-D\Pi_{\mu-1}D^-)(D\Pi_{\mu-1}D^-)'\,\bar u = 0,$$

and

$$\bar v' - (I-D\Pi_{\mu-1}D^-)'\bar v = 0.$$

Because of $\bar v(t_*)=0$, $\bar v$ must vanish identically, and hence $\bar u = D\Pi_{\mu-1}D^-\bar u$ holds true. $\square$

We leave the IERODE for a while, and turn back to the scaled version (2.48) of the DAE (2.44). Now we consider the other part of this equation, which results from multiplication by the projector function $I-\Pi_{\mu-1}$. First we express

$$\begin{aligned}
(I-\Pi_{\mu-1})D^-(D\Pi_{\mu-1}x)' &+ (I-\Pi_{\mu-1})G_\mu^{-1}B_\mu x\\
&= (I-\Pi_{\mu-1})G_\mu^{-1}\{G_\mu D^-(D\Pi_{\mu-1}x)' + B_{\mu-1}P_{\mu-1}x\\
&\qquad\qquad\qquad - G_\mu D^-(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}x\}\\
&= (I-\Pi_{\mu-1})G_\mu^{-1}\{B_{\mu-1}P_{\mu-1}x + G_\mu D^-D\Pi_{\mu-1}D^-(D\Pi_{\mu-1}x)'\}\\
&= (I-\Pi_{\mu-1})G_\mu^{-1}B_{\mu-1}\Pi_{\mu-1}x,
\end{aligned}$$

and then obtain the equation

$$
\begin{aligned}
(I-\Pi_{\mu-1})G_\mu^{-1}B_{\mu-1}\Pi_{\mu-1}x+\sum_{l=0}^{\mu-1}\{Q_lx+V_lD\Pi_lx\}& \\
-\sum_{l=0}^{\mu-2}(I-\Pi_l)Q_{l+1}D^-(D\Pi_lQ_{l+1}x)' &= (I-\Pi_{\mu-1})G_\mu^{-1}q,
\end{aligned}
\tag{2.52}
$$

which is the precise counterpart of equation (1.38). Again, the extra terms V_l comprise the time variation. By means of the decompositions

$$
\begin{aligned}
D\Pi_lx &= D\Pi_l(\Pi_{\mu-1}+I-\Pi_{\mu-1})x = D\Pi_{\mu-1}x+D\Pi_l(I-P_{l+1}\cdots P_{\mu-1})x\\
&= D\Pi_{\mu-1}x+D\Pi_l(Q_{l+1}+P_{l+1}Q_{l+2}+\cdots+P_{l+1}\cdots P_{\mu-2}Q_{\mu-1})x\\
&= D\Pi_{\mu-1}x+D\Pi_l(Q_{l+1}+\cdots+D\Pi_{\mu-2}Q_{\mu-1})x,
\end{aligned}
$$

we rearrange the terms in (2.52) once more as

$$
\begin{aligned}
\sum_{l=0}^{\mu-1}Q_lx-\sum_{l=0}^{\mu-2}(I-\Pi_l)Q_{l+1}D^-(D\Pi_lQ_{l+1}x)'+\sum_{l=0}^{\mu-2}\mathcal{M}_{l+1}D\Pi_lQ_{l+1}x& \\
+\mathcal{K}\Pi_{\mu-1}x &= (I-\Pi_{\mu-1})G_\mu^{-1}q,
\end{aligned}
\tag{2.53}
$$

with the continuous coefficients

$$
\begin{aligned}
\mathcal{K}:&=(I-\Pi_{\mu-1})G_\mu^{-1}B_{\mu-1}\Pi_{\mu-1}+\sum_{l=0}^{\mu-1}V_lD\Pi_{\mu-1}\\
&=(I-\Pi_{\mu-1})G_\mu^{-1}B_{\mu-1}\Pi_{\mu-1}+\sum_{l=0}^{\mu-1}(I-\Pi_l)\Big\{P_lD^-(D\Pi_lD^-)'\\
&\qquad -Q_{l+1}D^-(D\Pi_{l+1}D^-)'\Big\}D\Pi_{\mu-1}\\
&=(I-\Pi_{\mu-1})G_\mu^{-1}B_{\mu-1}\Pi_{\mu-1}+\sum_{l=1}^{\mu-1}(I-\Pi_{l-1})(P_l-Q_l)D^-(D\Pi_lD^-)'D\Pi_{\mu-1}
\end{aligned}
\tag{2.54}
$$

and

$$
\begin{aligned}
\mathcal{M}_{l+1}:&=\sum_{j=0}^{l}V_jD\Pi_lQ_{l+1}D^-\\
&=\sum_{j=0}^{l}(I-\Pi_j)\{P_jD^-(D\Pi_jD^-)'-Q_{j+1}D^-(D\Pi_{j+1}D^-)'\}D\Pi_lQ_{l+1}D^-,\\
&\qquad\qquad l=0,\ldots,\mu-2.
\end{aligned}
\tag{2.55}
$$

The coefficients $\mathcal{M}_{l+1}$ vanish together with the V_j in the constant coefficient case.

Next we provide a further splitting of the subsystem (2.53) according to the decomposition

$$I-\Pi_{\mu-1}=Q_0P_1\cdots P_{\mu-1}+\cdots+Q_{\mu-2}P_{\mu-1}+Q_{\mu-1}$$

into μ parts. Notice that the products $Q_iP_{i+1}\cdots P_{\mu-1}$ are also continuous projectors. To prepare the further decoupling we provide some useful properties of our projectors and coefficients.

Lemma 2.28. *For the regular DAE (2.44) with tractability index μ, and admissible projector functions $Q_0,\ldots,Q_{\mu-1}$, the following relations become true:*

$$\begin{array}{llr}
(1) & Q_iP_{i+1}\cdots P_{\mu-1}(I-\Pi_l)=0, & l=0,\ldots,i-1,\\
 & & i=1,\ldots,\mu-2,\\
 & Q_{\mu-1}(I-\Pi_l)=0, & l=0,\ldots,\mu-2,\\
(2) & Q_iP_{i+1}\cdots P_{\mu-1}(I-\Pi_i)=Q_i, & i=0,\ldots,\mu-2,\\
 & Q_{\mu-1}(I-\Pi_{\mu-1})=Q_{\mu-1}, & \\
(3) & Q_iP_{i+1}\cdots P_{\mu-1}(I-\Pi_{i+s})=Q_iP_{i+1}\cdots P_{i+s}, & s=1,\ldots,\mu-1-i,\\
 & & i=0,\ldots,\mu-2,\\
(4) & Q_iP_{i+1}\cdots P_{\mu-1}\mathcal{M}_{l+1}=0, & l=0,\ldots,i-1,\\
 & & i=0,\ldots,\mu-2,\\
 & Q_{\mu-1}\mathcal{M}_{l+1}=0, & l=0,\ldots,\mu-2,\\
(5) & Q_iP_{i+1}\cdots P_{\mu-1}Q_s=0 \textit{ if } s\neq i, & s=0,\ldots,\mu-1,\\
 & Q_iP_{i+1}\cdots P_{\mu-1}Q_i=Q_i, & i=0,\ldots,\mu-2,
\end{array}$$

$$(6)\qquad \mathcal{M}_j=\sum_{l=1}^{j-1}(I-\Pi_{l-1})(P_l-Q_l)D^-(D\Pi_{j-1}Q_jD^-)'D\Pi_{j-1}Q_jD^-,\qquad j=1,\ldots,\mu-1,$$

$$(7)\qquad \Pi_{\mu-1}G_\mu^{-1}B_\mu=\Pi_{\mu-1}G_\mu^{-1}B_0\Pi_{\mu-1},\ \textit{and hence}\quad D\Pi_{\mu-1}G_\mu^{-1}B_\mu D^-=D\Pi_{\mu-1}G_\mu^{-1}BD^-.$$

Proof. (1) The first part of the assertion results from the relation $Q_iP_{i+1}\cdots P_{\mu-1}=Q_iP_{i+1}\cdots P_{\mu-1}\Pi_{i-1}$, and the inclusion $\operatorname{im}(I-\Pi_l)\subseteq\ker\Pi_{i-1}$, $l\leq i-1$. The second part is a consequence of the inclusion $\operatorname{im}(I-\Pi_l)\subseteq\ker Q_{\mu-1}$, $l\leq\mu-2$.
(2) This is a consequence of the relations $P_{i+1}\cdots P_{\mu-1}(I-\Pi_i)=(I-\Pi_i)$ and $Q_i(I-\Pi_i)=Q_i$.
(3) We have

$$Q_iP_{i+1}\cdots P_{\mu-1}\Pi_{\mu-1}=0,\quad \text{thus}\quad Q_iP_{i+1}\cdots P_{\mu-1}(I-\Pi_{\mu-1})=Q_iP_{i+1}\cdots P_{\mu-1}.$$

Taking into account that $Q_j(I-\Pi_{i+s})=0$ for $j>i+s$, we find

$$\begin{aligned}Q_iP_{i+1}\cdots P_{\mu-1}(I-\Pi_{i+s}) &= Q_iP_{i+1}\cdots P_{i+s}P_{i+s+1}\cdots P_{\mu-1}(I-\Pi_{i+s})\\ &= Q_iP_{i+1}\cdots P_{i+s}P_{i+s+1}\cdots P_{\mu-1}(I-\Pi_{i+s})\\ &= Q_iP_{i+1}\cdots P_{i+s}(I-\Pi_{i+s}) = Q_iP_{i+1}\cdots P_{i+s}.\end{aligned}$$

(4) This is a consequence of (1).
(5) This is evident.
(6) We derive

$$\begin{aligned}\mathcal{M}_j &= \sum_{l=1}^{j-1}(I-\Pi_l)P_lD^-(D\Pi_lD^-)'D\Pi_{j-1}Q_jD^-\\ &\qquad -\sum_{l=0}^{j-2}(I-\Pi_l)Q_{l+1}D^-(D\Pi_{l+1}D^-)'D\Pi_{j-1}Q_jD^-\\ &= \sum_{l=1}^{j-1}(I-\Pi_l)P_lD^-\Big\{(D\Pi_{j-1}Q_jD^-)'-D\Pi_lD^-(D\Pi_{j-1}Q_jD^-)'\Big\}D\Pi_{j-1}Q_jD^-\\ &\qquad -\sum_{l=0}^{j-2}(I-\Pi_l)Q_{l+1}D^-\Big\{(D\Pi_{j-1}Q_jD^-)'\\ &\qquad\qquad -D\Pi_{l+1}D^-(D\Pi_{j-1}Q_jD^-)'\Big\}D\Pi_{j-1}Q_jD^-\\ &= \sum_{l=1}^{j-1}(I-\Pi_l)P_lD^-(D\Pi_{j-1}Q_jD^-)'D\Pi_{j-1}Q_jD^-\\ &\qquad -\sum_{l=0}^{j-2}(I-\Pi_l)Q_{l+1}D^-(D\Pi_{j-1}Q_jD^-)'D\Pi_{j-1}Q_jD^-\\ &= \sum_{l=1}^{j-1}(I-\Pi_{l-1})P_lD^-(D\Pi_{j-1}Q_jD^-)'D\Pi_{j-1}Q_jD^-\\ &\qquad -\sum_{l=1}^{j-1}(I-\Pi_{l-1})Q_lD^-(D\Pi_{j-1}Q_jD^-)'D\Pi_{j-1}Q_jD^-.\end{aligned}$$

(7) Owing to $P_\mu = I$, it holds that

$$\begin{aligned}B_\mu &= B_{\mu-1}P_{\mu-1} - G_\mu D^-(D\Pi_\mu D^-)'D\Pi_{\mu-1}\\ &= B_{\mu-1}P_{\mu-1} - G_\mu D^-(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}.\end{aligned}$$

We compute

$$\begin{aligned}\Pi_{\mu-1}G_\mu^{-1}B_\mu &= \Pi_{\mu-1}G_\mu^{-1}\{B_{\mu-1}P_{\mu-1} - G_\mu D^-(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}\}\\ &= \Pi_{\mu-1}G_\mu^{-1}B_{\mu-1}\Pi_{\mu-1} - \underbrace{\Pi_{\mu-1}D^-(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}}_{=0}.\end{aligned}$$

The next step is

$$\begin{aligned}\Pi_{\mu-1}G_\mu^{-1}B_{\mu-1}\Pi_{\mu-1} &= \Pi_{\mu-1}G_\mu^{-1}\{B_{\mu-2}P_{\mu-2} - G_{\mu-1}D^-(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-2}\}\Pi_{\mu-1}\\ &= \Pi_{\mu-1}G_\mu^{-1}B_{\mu-2}\Pi_{\mu-1} - \underbrace{\Pi_{\mu-1}P_{\mu-1}D^-(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}}_{=0},\end{aligned}$$

and so on. □

As announced before we split the subsystem (2.53) into μ parts. Multiplying by the projector functions $Q_iP_{i+1}\cdots P_{\mu-1}$, $i=0,\dots,\mu-2$, and $Q_{\mu-1}$, and regarding Lemma 2.28 one attains the system

$$\begin{aligned}Q_ix - Q_iQ_{i+1}D^-(D\Pi_iQ_{i+1}x)' - \sum_{l=i+1}^{\mu-2} Q_iP_{i+1}\cdots P_lQ_{l+1}D^-(D\Pi_lQ_{l+1}x)' \qquad (2.56)\\ + \sum_{l=i}^{\mu-2} Q_iP_{i+1}\cdots P_{\mu-1}\mathcal{M}_{l+1}D\Pi_lQ_{l+1}x\\ = -Q_iP_{i+1}\cdots P_{\mu-1}\mathcal{K}\Pi_{\mu-1}x + Q_iP_{i+1}\cdots P_{\mu-1}G_\mu^{-1}q, \qquad i=0,\dots,\mu-2,\end{aligned}$$

as well as

$$Q_{\mu-1}x = -Q_{\mu-1}\mathcal{K}\Pi_{\mu-1}x + Q_{\mu-1}G_\mu^{-1}q. \tag{2.57}$$

Equation (2.57) determines $Q_{\mu-1}x$ in terms of q and $\Pi_{\mu-1}x$. The i-th equation in (2.56) determines Q_ix in terms of q, $\Pi_{\mu-1}x$, $Q_{\mu-1}x,\dots,Q_{i+1}x$, and so on, that is, the system (2.56), (2.57) successively determines all components of $I-\Pi_{\mu-1} = Q_0 + \Pi_0Q_1 + \cdots + \Pi_{\mu-2}Q_{\mu-1}$ in a unique way. Comparing with the constant coefficient case, we recognize that, the system (2.56), (2.57) generalizes the system (1.40), (1.41).

So far, the regular DAE (2.44) decouples into the IERODE (2.51) and the subsystem (2.56), (2.57) by means of each arbitrary admissible matrix function sequence. The solutions of the DAE can be expressed as

$$x = \Pi_{\mu-1}x + (I-\Pi_{\mu-1}x) = D^-u + (I-\Pi_{\mu-1})x,$$

whereby $(I-\Pi_{\mu-1})x$ is determined by the subsystem (2.56), (2.57), and $u = D\Pi_{\mu-1}D^-u$ is a solution of the IERODE, which belongs to its invariant subspace.

The property

$$\ker Q_i = \ker \Pi_{i-1}Q_i, \;\; i=1,\dots,\mu-1, \tag{2.58}$$

is valid, since we may represent $Q_i = (I+(I-\Pi_{i-1})Q_i)\Pi_{i-1}Q_i$ with the nonsingular factor $I+(I-\Pi_{i-1})Q_i$, $i=1,\dots,\mu-1$. This allows us to compute Q_ix from $\Pi_{i-1}Q_ix$ and vice versa. We take advantage of this in the following rather cosmetic changes.

Denote (cf. (1.45))

$$v_0 := Q_0x, \quad v_i := \Pi_{i-1}Q_ix, \quad i=1,\dots,\mu-1, \tag{2.59}$$

$$u := D\Pi_{\mu-1}x, \tag{2.60}$$

such that we have the solution expression

$$x = v_0 + v_1 + \cdots + v_{\mu-1} + D^- u. \tag{2.61}$$

Multiply equation (2.57) by $\Pi_{\mu-2}$, and, if $i \geq 1$, the i-th equation in (2.56) by Π_{i-1}. This yields the following system which determines the functions $v_{\mu-1}, \ldots, v_0$ in terms of q and u:

$$\begin{bmatrix} 0 & \mathcal{N}_{01} & \cdots & \mathcal{N}_{0,\mu-1} \\ & 0 & \ddots & \vdots \\ & & \ddots & \mathcal{N}_{\mu-2,\mu-1} \\ & & & 0 \end{bmatrix} \left(\mathcal{D} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{\mu-1} \end{bmatrix} \right)' \tag{2.62}$$
$$+ \begin{bmatrix} I & \mathcal{M}_{01} & \cdots & \mathcal{M}_{0,\mu-1} \\ & I & \ddots & \vdots \\ & & \ddots & \mathcal{M}_{\mu-2,\mu-1} \\ & & & I \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{\mu-1} \end{bmatrix} + \begin{bmatrix} \mathcal{H}_0 \\ \mathcal{H}_1 \\ \vdots \\ \mathcal{H}_{\mu-1} \end{bmatrix} D^- u = \begin{bmatrix} \mathcal{L}_0 \\ \mathcal{L}_1 \\ \vdots \\ \mathcal{L}_{\mu-1} \end{bmatrix} q.$$

The matrix function $\mathcal{D} := (\mathcal{D}_{ij})_{i,j=0}^{\mu-1}$ has as entries the blocks $\mathcal{D}_{ii} = D\Pi_{i-1}Q_i$, $i = 1, \ldots, \mu-1$, $\mathcal{D}_{00} = 0$, and $\mathcal{D}_{ij} = 0$, if $i \neq j$. This matrix function is block-diagonal if $n = m$. The further coefficients in (2.62) are also continuous, and their detailed form is

$$\begin{aligned}
\mathcal{N}_{01} &:= -Q_0Q_1D^-, && \\
\mathcal{N}_{0j} &:= -Q_0P_1 \cdots P_{j-1}Q_jD^-, && j = 2, \ldots, \mu-1, \\
\mathcal{N}_{i,i+1} &:= -\Pi_{i-1}Q_iQ_{i+1}D^-, && \\
\mathcal{N}_{ij} &:= -\Pi_{i-1}Q_iP_{i+1} \cdots P_{j-1}Q_jD^-, && j = i+2, \ldots, \mu-1,\ i = 1, \ldots, \mu-2, \\
\mathcal{M}_{0j} &:= Q_0P_1 \cdots P_{\mu-1}\mathcal{M}_jD\Pi_{j-1}Q_j, && j = 1, \ldots, \mu-1, \\
\mathcal{M}_{ij} &:= \Pi_{i-1}Q_iP_{i+1} \cdots P_{\mu-1}\mathcal{M}_jD\Pi_{j-1}Q_j, && j = i+1, \ldots, \mu-1,\ i = 1, \ldots, \mu-2, \\
\mathcal{L}_0 &:= Q_0P_1 \cdots P_{\mu-1}G_\mu^{-1}, && \\
\mathcal{L}_i &:= \Pi_{i-1}Q_iP_{i+1} \cdots P_{\mu-1}G_\mu^{-1}, && i = 1, \ldots, \mu-2, \\
\mathcal{L}_{\mu-1} &:= \Pi_{\mu-2}Q_{\mu-1}G_\mu^{-1}, &&
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}_0 &:= Q_0P_1 \cdots P_{\mu-1}\mathcal{K}\Pi_{\mu-1}, && \\
\mathcal{H}_i &:= \Pi_{i-1}Q_iP_{i+1} \cdots P_{\mu-1}\mathcal{K}\Pi_{\mu-1}, && i = 1, \ldots, \mu-2, \\
\mathcal{H}_{\mu-1} &:= \Pi_{\mu-2}Q_{\mu-1}\mathcal{K}\Pi_{\mu-1}, &&
\end{aligned}$$

with $\mathcal{K}$ and $\mathcal{M}_j$ defined by formulas (2.54), (2.55). Introducing the matrix functions $\mathcal{N}$, $\mathcal{M}$, $\mathcal{H}$, $\mathcal{L}$ of appropriate sizes according to (2.62), we write this subsystem as

$$\mathcal{N}(\mathcal{D}v)' + \mathcal{M}v + \mathcal{H}D^- u = \mathcal{L}q, \tag{2.63}$$

whereby the vector function v contains the entries $v_0, \ldots, v_{\mu-1}$.

Again, we draw the reader's attention to the great consistency with (1.46). The difficulties caused by the time-variations are now hidden in the coefficients $\mathcal{M}_{ij}$ which disappear for constant coefficients.
We emphasize that the system (2.62) is nothing other than a more transparent reformulation of the former subsystem (2.56), (2.57). The next proposition records important properties.

Proposition 2.29. *Let the DAE (2.44) be regular with tractability index μ, and let $Q_0, \ldots, Q_{\mu-1}$ be admissible projector functions. Then the coefficient functions in (2.62) have the further properties:*

(1) $\mathcal{N}_{ij} = \mathcal{N}_{ij} D\Pi_{j-1} Q_j D^-$ *and* $\mathcal{N}_{ij} D = \mathcal{N}_{ij} D\Pi_{j-1} Q_j$*, for* $j = 1, \ldots, \mu-1$, $i = 0, \ldots, \mu-2$.
(2) $\operatorname{rank} \mathcal{N}_{i,i+1} = \operatorname{rank} \mathcal{N}_{i,i+1} D = m - r_{i+1}$*, for* $i = 0, \ldots, \mu-2$.
(3) $\ker \mathcal{N}_{i,i+1} = \ker D\Pi_i Q_{i+1} D^-$*, and* $\ker \mathcal{N}_{i,i+1} D = \ker \Pi_i Q_{i+1}$*, for* $i = 0, \ldots, \mu-2$.
(4) *The subsystem (2.62) is a DAE with properly stated leading term.*
(5) *The square matrix function $\mathcal{N}\mathcal{D}$ is pointwise nilpotent with index μ, more precisely,* $(\mathcal{N}\mathcal{D})^\mu = 0$ *and* $\operatorname{rank}(\mathcal{N}\mathcal{D})^{\mu-1} = m - r_{\mu-1} > 0$.
(6) $\mathcal{M}_{i,i+1} = 0, \quad i = 0, \ldots, \mu-2$.

Proof. (1) This is given by the construction.
(2) Because of $\mathcal{N}_{i,i+1} = \mathcal{N}_{i,i+1} DD^-$, the matrix functions $\mathcal{N}_{i,i+1}$ and $\mathcal{N}_{i,i+1} D$ have equal rank. To show that this is precisely $m - r_{i+1}$ we apply the same arguments as for Lemma 1.27. First we validate the relation

$$\operatorname{im} Q_i Q_{i+1} = N_i \cap S_i.$$

Namely, $z \in N_i \cap S_i$ implies $z = Q_i z$ and $B_i z = G_i w$, therefore, $(G_i + B_i Q_i)(P_i w + Q_i z) = 0$, further $(P_i w + Q_i z) = Q_{i+1}(P_i w + Q_i z) = Q_{i+1} w$, $Q_i z = Q_i Q_{i+1} w$, and hence $z = Q_i z = Q_i Q_{i+1} w$.
Conversely, $z \in \operatorname{im} Q_i Q_{i+1}$ yields $z = Q_i z$, $z = Q_i Q_{i+1} w$. Then the identity $(G_i + B_i Q_i) Q_{i+1} = 0$ leads to $B_i z = B_i Q_i Q_{i+1} w = -G_i Q_{i+1} w$, thus $z \in N_i \cap S_i$.
The intersection $N_i \cap S_i$ has the same dimension as N_{i+1}, so that we attain $\dim \operatorname{im} Q_i Q_{i+1} = \dim N_{i+1} = m - r_{i+1}$.
(3) From (1) we derive the inclusions

$$\ker D\Pi_i Q_{i+1} D^- \subseteq \ker \mathcal{N}_{i,i+1}, \quad \ker \Pi_i Q_{i+1} \subseteq \ker \mathcal{N}_{i,i+1} D.$$

Because of $\Pi_i Q_{i+1} = D^-(D\Pi_i Q_{i+1} D^-)D$, and $\ker \Pi_i Q_{i+1} = \ker Q_{i+1}$, the assertion becomes true for reasons of dimensions.
(4) We provide the subspaces

$$\ker\mathcal{N} = \left\{z = \begin{bmatrix} z_0 \\ \vdots \\ z_{\mu-1} \end{bmatrix} \in \mathbb{R}^{n\mu} : z_i \in \ker\Pi_{i-1}Q_i,\ i = 1,\ldots,\mu-1\right\}$$

and

$$\operatorname{im}\mathcal{D} = \left\{z = \begin{bmatrix} z_0 \\ \vdots \\ z_{\mu-1} \end{bmatrix} \in \mathbb{R}^{n\mu} : z_i \in \operatorname{im}\Pi_{i-1}Q_i,\ i = 1,\ldots,\mu-1\right\}$$

which obviously fulfill the condition $\ker\mathcal{N} \oplus \operatorname{im}\mathcal{D} = \mathbb{R}^{n\mu}$. The border projector is $\mathcal{R} = \operatorname{diag}(0, D\Pi_0Q_1D^-,\ldots,D\Pi_{\mu-2}Q_{\mu-1}D^-)$, and it is continuously differentiable.
(5) The matrix function $\mathcal{N}\mathcal{D}$ is by nature strictly block upper triangular, and its main entries $(\mathcal{N}\mathcal{D})_{i,i+1} = \mathcal{N}_{i,i+1}D$ have constant rank $m - r_{i+1}$, for $i = 0,\ldots,\mu-2$.
The matrix function $(\mathcal{N}\mathcal{D})^2$ has zero-entries on the block positions $(i,i+1)$, and the dominating entries are

$$((\mathcal{N}\mathcal{D})^2)_{i,i+2} = \mathcal{N}_{i,i+1}D\mathcal{N}_{i+1,i+2}D = \Pi_{i-1}Q_iQ_{i+1}\Pi_iQ_{i+1}Q_{i+2} = \Pi_{i-1}Q_iQ_{i+1}Q_{i+2},$$

which have rank $m - r_{i+2}$, and so on.
In $(\mathcal{N}\mathcal{D})^{\mu-1}$ there remains exactly one nontrivial block in the upper right corner, $((\mathcal{N}\mathcal{D})^{\mu-1})_{0,\mu-1} = (-1)^{\mu-1}Q_0Q_1\cdots Q_{\mu-1}$, and it has rank $m - r_{\mu-1}$.
(6) This property is a direct consequence of the representation of $\mathcal{M}_{i+1}$ in Lemma 2.28 (6) and Lemma 2.28 (1). □

By this proposition, the subsystem (2.62) is in turn a regular DAE with tractability index μ and transparent structure. Property (6) slightly eases the structure of (2.62). We emphasize that the DAE (2.62) lives in $\mathbb{R}^{m\mu}$. The solutions belong to the function space $\mathcal{C}^1_{\mathcal{D}}(\mathcal{I},\mathbb{R}^{m\mu})$. Owing to the special form of the matrix function $\mathcal{L}$ on the right-hand side, each solution of (2.62) satisfies the conditions $v_0 = Q_0v_0$ and $v_i = \Pi_{i-1}Q_iv_i$, for $i = 1,\ldots,\mu-1$.

We now formulate the main result concerning the basic decoupling:

Theorem 2.30. *Let the DAE (2.44) be regular with tractability index μ, and let $Q_0,\ldots,Q_{\mu-1}$ be admissible projector functions. Then the DAE is equivalent via (2.59)–(2.61) to the system consisting of the IERODE (2.51) related to its invariant subspace* $\operatorname{im}D\Pi_{\mu-1}$, *and the subsystem (2.62).*

Proof. If $x \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$ is a solution of the DAE, then the component $u := D\Pi_{\mu-1}x \in \mathcal{C}^1(\mathcal{I},\mathbb{R}^m)$ satisfies the IERODE (2.51) and belongs to the invariant subspace $\operatorname{im}\Pi_{\mu-1}$. The functions $v_0 := Q_0x \in \mathcal{C}(\mathcal{I},\mathbb{R}^m)$, $v_i := \Pi_{i-1}Q_ix \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$, $i = 1,\ldots,\mu-1$, form the unique solution of the system (2.62) corresponding to u. Thereby, we recognize that $D\Pi_{\mu-1}x = D\Pi_{\mu-1}D^-Dx$, $Dv_i := D\Pi_{i-1}Q_ix = D\Pi_{i-1}Q_iD^-Dx$, $i = 1,\ldots,\mu-1$, are continuously differentiable functions since Dx and the used projectors are so.
Conversely, let $u = D\Pi_{\mu-1}x$ denote a solution of the IERODE, and let $v_0,\ldots,v_{\mu-1}$ form a solution of the subsystem (2.62). Then, it holds that $v_i = \Pi_{i-1}Q_iv_i$, for $i = 1,\ldots,\mu-1$, and $v_0 = Q_0v_0$. The functions u and $Dv_i = D\Pi_{i-1}Q_iv_i$, $i = 1,\ldots,$

$\mu-1$, are continuously differentiable. The composed function $x := D^- u + v_0 + v_1 + \cdots + v_{\mu-1}$ is continuous and has a continuous part Dx. It remains to insert x into the DAE, and to recognize that x fulfills the DAE. □

The coefficients of the IERODE and the system (2.62) are determined in terms of the DAE coefficients and the admissible matrix function sequence resulting from these coefficients. We can make use of these equations unless we suppose that there is a solution of the DAE. Considering the IERODE (2.51) and the system (2.62) as equations with unknown functions $u \in \mathcal{C}^1(\mathcal{I},\mathbb{R}^n)$, $v_0 \in \mathcal{C}(\mathcal{I},\mathbb{R}^m)$, $v_i \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$, $i = 1,\ldots,\mu-1$, we may solve these equations and construct continuous functions $x := D^- u + v_0 + v_1 + \cdots + v_{\mu-1}$ with $Dx = DD^- u + Dv_1 + \cdots + Dv_{\mu-1}$ being continuously differentiable, such that x satisfies the DAE. In this way we restrict our interest to those solutions u of the IERODE that have the property $u = D\Pi_{\mu-1}D^- u$. In this way one can prove the existence of DAE solutions, supposing the excitation and the coefficients to be sufficiently smooth.

The following additional description of the coupling coefficients $\mathcal{H}_0,\ldots,\mathcal{H}_{\mu-1}$ in the subsystem (2.62), which tie the solution u of the IERODE into this subsystem, supports the idea of an advanced decoupling. We draw the reader's attention to the consistency with Theorem 1.22 which provides the easier time-invariant counterpart of a complete decoupling. This lemma plays its role when constructing fine decouplings. Further, we make use of the given special representation of the coefficient $\mathcal{H}_0$ when describing the canonical projector function associated to the space of consistent values for the homogeneous DAE in the next subsection.

Lemma 2.31. *Let the DAE (2.44) be regular with tractability index μ. Let $Q_0,\ldots,Q_{\mu-1}$ be admissible projector functions, and*

$$\begin{aligned} Q_{0*} &:= Q_0 P_1 \cdots P_{\mu-1} G_\mu^{-1}\{B_0 + G_0 D^-(D\Pi_{\mu-1}D^-)'D\},\\ Q_{k*} &:= Q_k P_{k+1} \cdots P_{\mu-1} G_\mu^{-1}\{B_k + G_k D^-(D\Pi_{\mu-1}D^-)'D\Pi_{k-1}\},\ k = 1,\ldots,\mu-2,\\ Q_{\mu-1*} &:= Q_{\mu-1} G_\mu^{-1} B_{\mu-1}. \end{aligned}$$

(1) *Then the coupling coefficients of the subsystem (2.62) have the representations*

$$\begin{aligned} \mathcal{H}_0 &= Q_{0*}\Pi_{\mu-1},\\ \mathcal{H}_k &= \Pi_{k-1} Q_{k*}\Pi_{\mu-1},\ k = 1,\ldots,\mu-2,\\ \mathcal{H}_{\mu-1} &= \Pi_{\mu-2} Q_{\mu-1*}\Pi_{\mu-1}. \end{aligned}$$

(2) *The $Q_{0*},\ldots,Q_{\mu-1*}$ are also continuous projector functions onto the subspaces $N_0,\ldots,N_{\mu-1}$, and it holds that $Q_{k*} = Q_{k*}\Pi_{k-1}$ for $k = 1,\ldots,\mu-1$.*

Proof. (1) For $k = 0,\ldots,\mu-2$, we express

$$\begin{aligned} \mathcal{A}_k :&= Q_k P_{k+1} \cdots P_{\mu-1} \mathcal{K}\Pi_{\mu-1} \quad \text{(cf. (2.54) for } \mathcal{K} \text{ and Prop. 2.23 for } V_l)\\ &= Q_k P_{k+1} \cdots P_{\mu-1} G_\mu^{-1} B_{\mu-1}\Pi_{\mu-1} + Q_k P_{k+1} \cdots P_{\mu-1} \sum_{l=0}^{\mu-1} V_l D\Pi_{\mu-1}. \end{aligned}$$

Regarding the identity $\Pi_l D^-(D\Pi_l D^-)'D\Pi_l = 0$ we derive first

$$\begin{aligned}
&\Pi_{k-1}\sum_{l=0}^{\mu-1} V_l D\Pi_{\mu-1} = \Pi_{k-1}\sum_{l=k}^{\mu-1} V_l D\Pi_{\mu-1}\\
&= \Pi_{k-1}\sum_{l=k}^{\mu-1}\{\underbrace{(I-\Pi_l)P_l}_{P_l-\Pi_l} D^-(D\Pi_l D^-)'D\Pi_{\mu-1}\\
&\qquad\qquad\qquad\qquad - (I-\Pi_l)Q_{l+1}D^-(D\Pi_{l+1}D^-)'D\Pi_{\mu-1}\}\\
&= \Pi_{k-1}\sum_{l=k}^{\mu-1}\{P_l D^-(D\Pi_l D^-)' - (I-\Pi_l)Q_{l+1}D^-(D\Pi_{l+1}D^-)'D\Pi_{\mu-1}D^-\}D\Pi_{\mu-1}\\
&= \Pi_{k-1}\sum_{l=k}^{\mu-1}\{P_l D^-(D\Pi_l D^-)' - (I-\Pi_l)Q_{l+1}D^-(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}D^-\}D\Pi_{\mu-1}.
\end{aligned}$$

Then, taking into account that $Q_\mu = 0$, as well as the properties

$$Q_k P_{k+1}\cdots P_{\mu-1} = Q_k P_{k+1}\cdots P_{\mu-1}\Pi_{k-1},\quad Q_k P_{k+1}\cdots P_{\mu-1}P_k = Q_k P_{k+1}\cdots P_{\mu-1}\Pi_k,$$
$$Q_k P_{k+1}\cdots P_{\mu-1}Q_l = 0,\ \text{ if }\ l \ge k+1,$$

we compute

$$\begin{aligned}
&Q_k P_{k+1}\cdots P_{\mu-1}\sum_{l=0}^{\mu-1} V_l D\Pi_{\mu-1}\\
&= Q_k P_{k+1}\cdots P_{\mu-1}\sum_{l=k+1}^{\mu-1} D^-(D\Pi_l D^-)'D\Pi_{\mu-1}\\
&\qquad\qquad + Q_k P_{k+1}\cdots P_{\mu-1}\underbrace{\sum_{l=k}^{\mu-1}\Pi_l Q_{l+1}}_{\Pi_k-\Pi_{\mu-1}} D^-(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}\\
&= Q_k P_{k+1}\cdots P_{\mu-1}\sum_{l=k+1}^{\mu-1} D^-(D\Pi_l D^-)'D\Pi_{\mu-1}\\
&\qquad\qquad + Q_k P_{k+1}\cdots P_{\mu-1}P_k(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}.
\end{aligned}$$

This leads to

$$\begin{aligned}
\mathcal{A}_k = Q_k P_{k+1}\cdots P_{\mu-1}G_\mu^{-1}&\Big\{B_k\Pi_{\mu-1} - \sum_{j=k+1}^{\mu-1} G_j D^-(D\Pi_j D^-)'D\Pi_{\mu-1}\Big\}\\
&+ Q_k P_{k+1}\cdots P_{\mu-1}\sum_{l=k+1}^{\mu-1} D^-(D\Pi_l D^-)'D\Pi_{\mu-1}\\
&+ Q_k P_{k+1}\cdots P_{\mu-1}P_k(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}.
\end{aligned}$$

Due to $Q_kP_{k+1}\cdots P_{\mu-1}G_\mu^{-1}G_j = Q_kP_{k+1}\cdots P_{\mu-1}$, for $j \geq k+1$, it follows that

$$\begin{aligned}
\mathcal{A}_k = Q_kP_{k+1}\cdots P_{\mu-1}G_\mu^{-1}B_k\Pi_{\mu-1} - Q_kP_{k+1}\cdots P_{\mu-1}\sum_{j=k+1}^{\mu-1} D^-(D\Pi_jD^-)'D\Pi_{\mu-1}\\
+ Q_kP_{k+1}\cdots P_{\mu-1}\sum_{l=k+1}^{\mu-1} D^-(D\Pi_lD^-)'D\Pi_{\mu-1}\\
+ Q_kP_{k+1}\cdots P_{\mu-1}P_k(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}\\
= Q_kP_{k+1}\cdots P_{\mu-1}G_\mu^{-1}B_k\Pi_{\mu-1} + Q_kP_{k+1}\cdots P_{\mu-1}P_kD^-(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}\\
= Q_{k*}\Pi_{\mu-1},
\end{aligned}$$

which proves the relations $\mathcal{H}_0 = Q_0P_1\cdots P_{\mu-1}\mathcal{K}\Pi_{\mu-1} = Q_{0*}\Pi_{\mu-1}$, and $\mathcal{H}_k = \Pi_{k-1}Q_kP_{k+1}\cdots P_{\mu-1}\mathcal{K}\Pi_{\mu-1} = \Pi_{k-1}\mathcal{A}_k\Pi_{\mu-1} = Q_{k*}\Pi_{\mu-1}$, $k = 1,\ldots,\mu-2$. Moreover, it holds that $\mathcal{H}_{\mu-1} = \Pi_{\mu-2}Q_{\mu-1}\mathcal{K} = Q_{\mu-1}G_\mu^{-1}B_{\mu-1}\Pi_{\mu-1} = \Pi_{\mu-2}Q_{\mu-1*}\Pi_{\mu-1}$.

(2) Derive

$$\begin{aligned}
Q_{k*}Q_k &= Q_kP_{k+1}\cdots P_{\mu-1}G_\mu^{-1}\{B_k + G_kD^-(D\Pi_{\mu-1}D^-)'D\Pi_{k-1}\}Q_k\\
&= Q_kP_{k+1}\cdots P_{\mu-1}G_\mu^{-1}B_kQ_k + Q_kP_{k+1}\cdots P_{\mu-1}P_kD^-(D\Pi_{\mu-1}D^-)'D\Pi_{k-1}Q_k\\
&= \underbrace{Q_kP_{k+1}\cdots P_{\mu-1}Q_k}_{=Q_k} - \underbrace{Q_kP_{k+1}\cdots P_{\mu-1}P_kD^-(D\Pi_{\mu-1}}_{=0}D^-)(D\Pi_{k-1}Q_kD^-)'D.
\end{aligned}$$

Then, $Q_{k*}Q_{k*} = Q_{k*}$ follows. The remaining part is evident. □

2.4.3 *Fine and complete decouplings*

Now we advance the decoupling of the subsystem (2.62) of the regular DAE (2.44). As benefits of such a refined decoupling we get further natural information on the DAE being independent of the choice of projectors in the given context. In particular, we fix a unique natural IERODE.

2.4.3.1 Index-1 case

Take a closer look at the special case of regular index-1 DAEs. Let the DAE (2.44) be regular with tractability index 1. The matrix function $G_0 = AD$ is singular with constant rank. We take an arbitrary continuous projector function Q_0. The resulting matrix function $G_1 = G_0 + BQ_0$ is nonsingular. It follows that $Q_1 = 0$, $\Pi_1 = \Pi_0$ and $V_0 = 0$ (cf. Proposition 2.23), further $B_1 = BP_0 - G_1D^-(D\Pi_0D^-)'D\Pi_0 = BP_0$. The DAE scaled by G_1^{-1} is (cf. (2.48)) now

$$D^-(D\Pi_0 x)' + G_1^{-1}BP_0x + Q_0x = G_1^{-1}q.$$

Multiplication by $D\Pi_0 = D$ and $I - \Pi_0 = Q_0$ leads to the system

$$(Dx)' - R'Dx + DG_1^{-1}BD^-Dx = DG_1^{-1}q, \tag{2.64}$$

$$Q_0x + Q_0G_1^{-1}BD^-Dx = Q_0G_1^{-1}q, \tag{2.65}$$

and the solution expression $x = D^-Dx + Q_0x$. Equation (2.65) stands for the subsystem (2.62), i.e., for

$$Q_0x + \mathcal{H}_0D^-Dx = \mathcal{L}_0q,$$

with $\mathcal{H}_0 = Q_0\mathcal{K}\Pi_0 = Q_0G_1^{-1}B\Pi_0 = Q_0G_1^{-1}BP_0,\ \mathcal{L}_0 = Q_0G_1^{-1}$.
The nonsingularity of G_1 implies the decomposition $S_0 \oplus N_0 = \mathbb{R}^m$ (cf. Lemma A.9), and the matrix function $Q_0G_1^{-1}B$ is a representation of the projector function onto N_0 along S_0.
We can choose Q_0 to be the special projector function onto N_0 along S_0 from the beginning. The benefit of this choice consists in the property $\mathcal{H}_0 = Q_0G_1^{-1}BP_0 = 0$, that is, the subsystems (2.65) uncouples from (2.64).

Example 2.32 (Decoupling of a semi-explicit index-1 DAE). We reconsider the semi-explicit DAE from Example 2.3

$$\begin{bmatrix} I \\ 0 \end{bmatrix} \left(\begin{bmatrix} I & 0 \end{bmatrix} x\right)' + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} x = q$$

with nonsingular B_{22}. Here we have the subspaces

$$N_0 = \{z \in \mathbb{R}^{m_1+m_2} : z_1 = 0\} \quad \text{and} \quad S_0 = \{z \in \mathbb{R}^{m_1+m_2} : B_{21}z_1 + B_{22}z_2 = 0\},$$

and the projector function onto N_0 along S_0 is given by

$$Q_0 = \begin{bmatrix} 0 & 0 \\ B_{22}^{-1}B_{21} & I \end{bmatrix}.$$

This projector is reasonable owing to the property $\mathcal{H}_0 = 0$, although it is far from being orthogonal. It yields

$$D^- = \begin{bmatrix} I \\ -B_{22}^{-1}B_{21} \end{bmatrix}, G_1 = \begin{bmatrix} I + B_{12}B_{22}^{-1}B_{21} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, G_1^{-1} = \begin{bmatrix} I & -B_{12}B_{22}^{-1} \\ -B_{22}^{-1}B_{21} & (I + B_{22}^{-1}B_{21}B_{12}) \end{bmatrix},$$

and the IERODE

$$x_1' + (B_{11} - B_{12}B_{22}^{-1}B_{21})x_1 = q_1 - B_{12}B_{22}^{-1}q_2.$$

Notice that in Example 2.3, Q_0 is chosen to be the orthoprojector, but precisely the same IERODE results for this choice. □

The last observation reflects a general property of regular index-1 DAEs as the following proposition states.

Proposition 2.33. *Let the DAE (2.44) be regular with index 1. Then its IERODE*

$$u' - R'u + DG_1^{-1}BD^-u = DG_1^{-1}q$$

is actually independent of the special choice of the continuous projector function Q_0.

Proof. We compare the IERODEs built for two different projector functions Q_0 and $\bar{Q}_0$. It holds that $\bar{G}_1 = G_0 + B\bar{Q}_0 = G_0 + BQ_0\bar{Q}_0 = G_1(P_0 + \bar{Q}_0) = G_1(I + Q_0\bar{Q}_0P_0)$ and $\bar{D}^- = \bar{D}^-D\bar{D}^- = \bar{D}^-R = \bar{D}^-DD^- = \bar{P}_0D^-$, therefore $D\bar{G}_1^{-1} = DG_1^{-1}$, $D\bar{G}_1^{-1}B\bar{D}^- = DG_1^{-1}B(I - \bar{Q}_0)D^- = DG_1^{-1}B(I - Q_0\bar{Q}_0)D^- = DG_1^{-1}BD^-$. □

Regular index-1 DAEs are transparent and simple, and the coefficients of their IERODEs are *always* independent of the projector choice. However, higher index DAEs are different.

2.4.3.2 Index-2 case

We take a closer look at the simplest class among regular higher index DAEs, the DAEs with tractability index $\mu = 2$.

Let the DAE (2.44) be regular with tractability index $\mu = 2$. Then the IERODE (2.51) and the subsystem (2.62) reduce to

$$u' - (D\Pi_1D^-)'u + D\Pi_1G_2^{-1}B_1D^-u = D\Pi_1G_2^{-1}q,$$

and

$$\begin{bmatrix} 0 & -Q_0Q_1D^- \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & 0 \\ 0 & D\Pi_0Q_1 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \right)' + \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} + \begin{bmatrix} \mathcal{H}_0 \\ \mathcal{H}_1 \end{bmatrix} D^-u = \begin{bmatrix} Q_0P_1G_2^{-1} \\ \Pi_0Q_1G_2^{-1} \end{bmatrix} q,$$

with

$$\begin{aligned}
\mathcal{H}_0 &= Q_0P_1\mathcal{K}\Pi_1 = Q_0P_1G_2^{-1}B_1\Pi_1 + Q_0(P_1 - Q_1)D^-(D\Pi_1D^-)'D\Pi_1 \\
&= Q_0P_1G_2^{-1}B_0\Pi_1 + Q_0P_1D^-(D\Pi_1D^-)'D\Pi_1 \\
\mathcal{H}_1 &= \Pi_0Q_1\mathcal{K}\Pi_1 = \Pi_0Q_1G_2^{-1}B_1\Pi_1.
\end{aligned}$$

Owing to the nonsingularity of G_2, the decomposition (cf. Lemma A.9)

$$N_1 \oplus S_1 = \mathbb{R}^m$$

is given, and the expression $Q_1G_2^{-1}B_1$ appearing in $\mathcal{H}_1$ reminds us of the representation of the special projector function onto N_1 along S_1 (cf. Lemma A.10) which is uniquely determined. In fact, $Q_1G_2^{-1}B_1$ is this projector function. The subspaces N_1 and S_1 are given before one has to choose the projector function Q_1, and hence one can settle on the projector function Q_1 onto N_1 along S_1 at the beginning.

Thereby, the necessary admissibility condition $N_0 \subseteq \ker Q_1$ is fulfilled because of $N_0 \subseteq S_1 = \ker Q_1$. It follows that

$$Q_1 G_2^{-1} B_1 \Pi_1 = Q_1 G_2^{-1} B_1 P_1 = Q_1 P_1 = 0, \quad \mathcal{H}_1 = \Pi_0 Q_1 G_2^{-1} B_1 \Pi_1 = 0.$$

Example 2.34 (Advanced decoupling of Hessenberg size-2 DAEs). Consider once again the so-called Hessenberg size-2 DAE

$$\begin{bmatrix} I \\ 0 \end{bmatrix} \left(\begin{bmatrix} I & 0 \end{bmatrix} x \right)' + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & 0 \end{bmatrix} x = q, \tag{2.66}$$

with the nonsingular product $B_{21}B_{12}$. Suppose the subspaces $\operatorname{im} B_{12}$ and $\ker B_{21}$ to be $\mathcal{C}^1$-subspaces. In Example 2.3, admissible matrix functions are built. This DAE is regular with index 2, and the projector functions

$$Q_0 = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, \quad Q_1 = \begin{bmatrix} \Omega & 0 \\ -B_{12}^- & 0 \end{bmatrix}, \quad \Omega := B_{12} B_{12}^-, \tag{2.67}$$

are admissible, for each arbitrary reflexive inverse B_{12}^- such that Ω is continuously differentiable. We have further $D\Pi_1 D^- = I - \Omega$ and

$$S_0 = S_1 = \{z \in \mathbb{R}^{m_1+m_2} : B_{21} z_1 = 0\}.$$

In contrast to Example 2.3, where widely orthogonal projectors are chosen and

$$\ker Q_1 = \{z \in \mathbb{R}^{m_1+m_2} : B_{12}^* z_1 = 0\} = (N_0 \oplus N_1)^\perp \oplus N_0,$$

now we set $B_{12}^- := (B_{21}B_{12})^{-1} B_{21}$ such that Ω projects $\mathbb{R}^{m_1}$ onto $\operatorname{im} B_{12}$ along $\ker B_{21}$, and Q_1 projects $\mathbb{R}^m$ onto N_1 along

$$\ker Q_1 = \{z \in \mathbb{R}^{m_1+m_2} : B_{21} z_1 = 0\} = S_1.$$

Except for the very special case, if $\ker B_{12}^* = \ker B_{21}$, a nonsymmetric projector function $D\Pi_1 D^- = I - \Omega = I - B_{12}(B_{21}B_{12})^{-1} B_{21}$ results. However, as we already know, this choice has the advantage of a vanishing coupling coefficient $\mathcal{H}_1$.
In contrast to the admissible projector functions (2.67), the projector functions

$$Q_0 = \begin{bmatrix} 0 & 0 \\ B_{12}^-(B_{11} - \Omega')(I - \Omega) & I \end{bmatrix}, \quad Q_1 = \begin{bmatrix} \Omega & 0 \\ -B_{12}^- & 0 \end{bmatrix}, \quad \Omega := B_{12} B_{12}^-, \tag{2.68}$$

form a further pair of admissible projector functions again yielding $D\Pi_1 D^- = I - \Omega$. With $B_{12}^- := (B_{21}B_{12})^{-1} B_{21}$, this choice forces both coefficients $\mathcal{H}_1$ and $\mathcal{H}_0$ to disappear, and the subsystem (2.62) uncouples from the IERODE. One can check that the resulting IERODE coincides with that from (2.67). □

As mentioned before, the index-2 case has the simplest higher index structure. The higher the index, the greater the variety of admissible projector functions. We recall Example 1.26 which shows several completely decoupling projectors for a time-invariant regular matrix pair with Kronecker index 2.

2.4.3.3 General benefits from fine decouplings

Definition 2.35. Let the DAE (2.44) be regular with tractability index μ, and let $Q_0, \dots, Q_{\mu-1}$ be admissible projector functions.

(1) If the coupling coefficients $\mathcal{H}_1, \dots, \mathcal{H}_{\mu-1}$ of the subsystem (2.62) vanish, then we speak of *fine decoupling projector functions* $Q_0, \dots, Q_{\mu-1}$, and of a *fine decoupling*.

(2) If all the coupling coefficients $\mathcal{H}_0, \dots, \mathcal{H}_{\mu-1}$ of the subsystem (2.62) vanish, then we speak of *complete decoupling projector functions* $Q_0, \dots, Q_{\mu-1}$, and of a *complete decoupling*.

Special fine and complete decoupling projector functions Q_0, Q_1 are built in Examples 2.34 and (2.32).

Owing to the linearity of the DAE (2.44) its homogeneous version

$$A(t)(D(t)x(t))' + B(t)x(t) = 0, \quad t \in \mathcal{J}, \tag{2.69}$$

plays its role, and in particular the subspace

$$S_{can}(t) := \{z \in \mathbb{R}^m : \exists x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m),\, A(Dx)' + Bx = 0,\, x(t) = z\},\ t \in \mathcal{I}.$$

The subspace $S_{can}(t)$ represents the geometric locus of all solution values of the homogeneous DAE (2.69) at time t. In other words, $S_{can}(t)$ is the linear *space of consistent initial values* at time t for the homogeneous DAE.

For implicit regular ODEs (2.43), $S_{can}(t) = \mathbb{R}^m$ is simply the entire time-invariant state space $\mathbb{R}^m$. In contrast, for intrinsic DAEs, the proper inclusion

$$S_{can}(t) \subseteq S_0(t)$$

is valid. While $S_0(t)$ represents the so-called *obvious constraint* associated with the DAE (2.69), the subspace $S_{can}(t)$ serves, so to say, as the complete final constraint which also incorporates all hidden constraints.

In particular, for the semi-explicit DAE in Example 2.3, we find the obvious constraint

$$S_0(t) = \{z \in \mathbb{R}^{m_1+m_2} : z_2 = -B_{22}(t)^{-1}B_{21}(t)z_1\},\ \dim S_0(t) = m_1,$$

and further

$$S_{can}(t) = \{z \in \mathbb{R}^{m_1+m_2} : z_2 = -B_{22}(t)^{-1}B_{21}(t)z_1\} = S_0(t),$$

supposing $B_{22}(t)$ remains nonsingular. However, if $B_{22}(t) \equiv 0$, but $B_{21}(t)B_{12}(t)$ remains nonsingular, then

$$\begin{aligned} S_{can}(t) = \{z \in \mathbb{R}^{m_1+m_2} :& B_{21}(t)z_1 = 0, \\ & z_2 = -[(B_{21}B_{21})^{-1}B_{21}(B_{11} - (B_{12}((B_{21}B_{21})^{-1}B_{21})')](t)z_1\} \end{aligned}$$

is merely a proper subspace of the obvious constraint

$$S_0(t) = \{z \in \mathbb{R}^{m_1+m_2} : B_{21}(t)z_1 = 0\}.$$

Example 2.4 confronts us even with a zero-dimensional subspace $S_{can}(t) = \{0\}$.

Except for those simpler cases, the canonical subspace S_{can} is not easy to access. It coincides with the finite eigenspace of the matrix pencil for regular linear time-invariant DAEs. Theorem 2.39 below provides a description by means of fine decoupling projector functions.

Definition 2.36. For the regular DAE (2.44) the time-varying subspaces $S_{can}(t)$, $t \in \mathcal{I}$, and $N_{can}(t) := N_0(t) + \cdots + N_{\mu-1}(t)$, $t \in \mathcal{I}$, are said to be the *canonical subspaces* of the DAE.

By Theorem 2.8, N_{can} is known to be independent of the special choice of admissible projectors, which justifies the notion. The canonical subspaces of the linear DAE generalize the finite and infinite eigenspaces of matrix pencils.

Applying fine decoupling projector functions $Q_0, \ldots, Q_{\mu-1}$, the subsystem (2.62) corresponding to the homogeneous DAE simplifies to

$$\begin{bmatrix} 0 & \mathcal{N}_{01} & \cdots & \mathcal{N}_{0,\mu-1} \\ & 0 & \ddots & \vdots \\ & & \ddots & \mathcal{N}_{\mu-2,\mu-1} \\ & & & 0 \end{bmatrix} \left(\mathcal{D} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{\mu-1} \end{bmatrix} \right)' + \begin{bmatrix} I & \mathcal{M}_{01} & \cdots & \mathcal{M}_{0,\mu-1} \\ & I & \ddots & \vdots \\ & & \ddots & \mathcal{M}_{\mu-2,\mu-1} \\ & & & I \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{\mu-1} \end{bmatrix}$$
$$+ \begin{bmatrix} \mathcal{H}_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} D^- u = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{2.70}$$

For given u, its solution components are determined successively as

$$v_{\mu-1} = 0, \ \ldots, \ v_1 = 0, \ v_0 = -\mathcal{H}_0 D^- u,$$

and hence each solution $x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ of the homogeneous DAE possesses the representation

$$x = D^- u + v_0 = (I - \mathcal{H}_0)D^- u = (I - Q_{0*}\Pi_{\mu-1})D^- D\Pi_{\mu-1}D^- u = (I - Q_{0*})\Pi_{\mu-1}D^- u,$$

whereby $u = D\Pi_{\mu-1}D^- u$ is a solution of the homogeneous IERODE

$$u' - (D\Pi_{\mu-1}D^-)'u + D\Pi_{\mu-1}G_\mu^{-1}BD^-u = 0,$$

and Q_{0*} is defined in Lemma 2.31. Owing to the relations $P_0Q_{0*} = 0$, the continuous matrix function $(I - Q_{0*})\Pi_{\mu-1}$ is also a projector function, and the nullspace is easily checked to be

$$\ker(I - Q_{0*})\Pi_{\mu-1} = N_{can}.$$

Since each solution of the homogeneous DAE can be represented in this way, the inclusion

$$S_{can} \subseteq \operatorname{im}(I - Q_{0*})\Pi_{\mu-1}$$

is valid. On the other hand, through each element of $\operatorname{im}((I - Q_{0*}(t))\Pi_{\mu-1}(t))$, at time t, there passes a DAE solution, and we obtain

$$\operatorname{im}(I - Q_{0*})\Pi_{\mu-1} = S_{can}.$$

In fact, fixing an arbitrary pair $x_0 \in \operatorname{im}((I - Q_{0*}(t_0))\Pi_{\mu-1}(t_0))$, $t_0 \in \mathcal{I}$, we determine the unique solution u of the standard IVP

$$u' - (D\Pi_{\mu-1}D^-)'u + D\Pi_{\mu-1}G_\mu^{-1}BD^-u = 0, \quad u(t_0) = D(t_0)\Pi_{\mu-1}(t_0)x_0,$$

and then the DAE solution $x := (I - Q_{0*})\Pi_{\mu-1}D^-u$. It follows that $x(t_0) = (I - Q_{0*}(t_0))\Pi_{\mu-1}(t_0)x_0 = x_0$. In consequence, the DAE solution passes through $x_0 \in \operatorname{im}((I - Q_{0*}(t_0))\Pi_{\mu-1}(t_0))$.
Owing to the projector properties, the decomposition

$$N_{can}(t) \oplus S_{can}(t) = \mathbb{R}^m, \quad t \in \mathcal{I}, \tag{2.71}$$

becomes valid. Moreover, now we see that S_{can} is a $\mathcal{C}$-subspace of dimension $d = m - \sum_{i=0}^{\mu-1}(m - r_i)$.

Definition 2.37. For a regular DAE (2.44) with tractability index μ, which has a fine decoupling, the projector function $\Pi_{can} \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^m))$ being uniquely determined by

$$\operatorname{im}\Pi_{can} = S_{can}, \quad \ker\Pi_{can} = N_{can}$$

is named *the canonical projector function* of the DAE.

We emphasize that both canonical subspaces S_{can} and N_{can}, and the canonical projector function Π_{can}, depend on the index μ. Sometimes it is reasonable to indicate this by writing $S_{can\ \mu}$, $N_{can\ \mu}$ and $\Pi_{can\ \mu}$.
The canonical projector plays the same role as the spectral projector does in the time-invariant case.

Remark 2.38. In earlier papers also the subspaces S_i (e.g., [159]) and the single projector functions $Q_0, \ldots, Q_{\mu-1}$ forming a fine decoupling (e.g., [157], [164]) are named canonical. This applies, in particular, to the projector function $Q_{\mu-1}$ onto $N_{\mu-1}$ along $S_{\mu-1}$. We do not use this notation. We know the canonical projector function Π_{can} in Definition 2.37 to be unique, however, for higher index cases, the single factors P_i in the given representation by means of fine decoupling projectors are not uniquely determined as is demonstrated by Example 1.26.

Now we are in a position to gather the fruit of the construction.

Theorem 2.39. *Let the regular index-μ DAE (2.44) have a fine decoupling.*

(1) *Then the canonical subspaces S_{can} and N_{can} are $\mathcal{C}$-subspaces of dimensions $d = m - \sum_{i=0}^{\mu-1}(m - r_i)$ and $m - d$.*
(2) *The decomposition (2.71) is valid, and the canonical projector function has the representation*
$$\Pi_{can} = (I - Q_{0*})\Pi_{\mu-1},$$
with fine decoupling projector functions $Q_0, \dots, Q_{\mu-1}$.
(3) *The coefficients of the IERODE (2.51) are independent of the special choice of the fine decoupling projector functions.*

Proof. It remains to verify (3). Let two sequences of fine decoupling projector functions $Q_0, \dots, Q_{\mu-1}$ and $\bar{Q}_0, \dots, \bar{Q}_{\mu-1}$ be given. Then the canonical projector function has the representations $\Pi_{can} = (I - Q_{0*})\Pi_{\mu-1}$ and $\Pi_{can} = (I - \bar{Q}_{0*})\bar{\Pi}_{\mu-1}$. Taking into account that $\bar{D}^- = \bar{P}_0 D^-$ we derive

$$D\Pi_{\mu-1}D^- = D\Pi_{can}D^- = D\bar{\Pi}_{\mu-1}D^- = D\bar{\Pi}_{\mu-1}\bar{D}^-.$$

Then, with the help of Lemma 2.12 yielding the relation $\bar{G}_\mu = G_\mu Z_\mu$, we arrive at

$$D\bar{\Pi}_{\mu-1}\bar{G}_\mu^{-1} = D\Pi_{\mu-1}D^- DZ_\mu^{-1}G_\mu^{-1} = D\Pi_{\mu-1}G_\mu^{-1},$$

$$D\bar{\Pi}_{\mu-1}\bar{G}_\mu^{-1}B\bar{D}^- = D\Pi_{\mu-1}G_\mu^{-1}B\bar{D}^- = D\Pi_{\mu-1}G_\mu^{-1}B(I - \bar{Q}_0)D^- = D\Pi_{\mu-1}G_\mu^{-1}BD^-,$$

and this proves the assertion. □

For regular index-1 DAEs, each continuous projector function Q_0 already generates a fine decoupling. Therefore, Proposition 2.33 is now a special case of Theorem 2.39 (3).

DAEs with fine decouplings, later on named *fine* DAEs, allow an intrinsic DAE theory in Section 2.6 addressing solvability, qualitative flow behavior and the characterization of admissible excitations.

2.4.3.4 Existence of fine and complete decouplings

For regular index-2 DAEs, the admissible pair Q_0, Q_1 provides a fine decoupling, if Q_1 is chosen such that $\ker Q_1 = S_1$. This is accompanied by the requirement that $\operatorname{im} D\Pi_1 D^- = DS_1$ is a $\mathcal{C}^1$-subspace. We point out that, for fine decouplings, we need some additional smoothness with respect to the regularity notion. While regularity with index 2 comprises the *existence* of an arbitrary $\mathcal{C}^1$ decomposition (i.e., the existence of a continuously differentiable projector function $D\Pi_1 D^-$)

$$\operatorname{im} D\Pi_1 D^- \oplus \underbrace{\operatorname{im} D\Pi_0 Q_1 D^-}_{=DN_1} \oplus \ker A = \mathbb{R}^n,$$

one needs for fine decouplings that the *special* decomposition

$$DS_1 \oplus DN_1 \oplus \ker A = \mathbb{R}^n,$$

consists of $\mathcal{C}^1$-subspaces. For instance, the semi-explicit DAE in Example 2.34 possesses fine decoupling projector functions, if both subspaces $\operatorname{im} B_{12}$ and $\ker B_{21}$ are continuously differentiable. However, for regularity, it is enough if $\operatorname{im} B_{12}$ is a $\mathcal{C}^1$-subspace, as demonstrated in Example 2.3.

Assuming the coefficients A, D, B to be $\mathcal{C}^1$, and choosing a continuously differentiable projector function Q_0, the resulting DN_1 and DS_1 are always $\mathcal{C}^1$-subspaces. However, we do not feel comfortable with such a generous sufficient smoothness assumption, though it is less demanding than that in derivative array approaches, where one naturally has to require $A, D, B \in \mathcal{C}^2$ for the treatment of an index-2 problem.
We emphasize that only certain continuous subspaces are additionally assumed to belong to the class $\mathcal{C}^1$. Since the precise description of these subspaces is somewhat cumbersome, we use instead the wording *the coefficients of the DAE are sufficiently smooth* just to indicate the smoothness problem.
In essence, the additional smoothness requirements are related to the coupling coefficients $\mathcal{H}_1, \ldots, \mathcal{H}_{\mu-1}$ in the subsystem (2.62), and in particular to the special projectors introduced in Lemma 2.31. It turns out that, for a fine decoupling of a regular index-μ DAE, certain parts of the coefficients A, D, B have to be continuously differentiable up to degree $\mu - 1$. This meets the common understanding of index μ DAEs, and it is closely related to solvability conditions. We present an example for more clarity.

Example 2.40 (Smoothness for a fine decoupling). Consider the DAE

$$\underbrace{\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0 \end{bmatrix}}_{A} \Bigl(\underbrace{\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0 \end{bmatrix}}_{D} x \Bigr)' + \underbrace{\begin{bmatrix} 0&0&0&0\\0&0&0&-1\\0&-1&0&0\\\alpha&0&-1&0 \end{bmatrix}}_{B} x = 0,$$

on the interval $\mathcal{I} = [0,1]$. According to the basic continuity assumption, B is continuous, that is, $\alpha \in \mathcal{C}([0,1])$. Taking a look at the solution satisfying the initial condition $x_1(0) = 1$, that is

$$x_1(t) = 1,\ x_3(t) = \alpha(t),\ x_2(t) = x_3'(t) = \alpha'(t),\ x_4(t) = x_3''(t) = \alpha''(t)$$

we recognize that we must more reasonably assume $\alpha \in \mathcal{C}^2([0,1])$. We demonstrate by constructing a fine decoupling sequence that this is precisely the smoothness needed.

The first elements of the matrix function sequence can be chosen, respectively, computed as

$$Q_0 = \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1 \end{bmatrix},\; G_1 = \begin{bmatrix} 1&0&0&0\\ 0&1&0&-1\\ 0&0&1&0\\ 0&0&0&0 \end{bmatrix},\; Q_1 = \begin{bmatrix} 0&0&0&0\\ 0&1&0&0\\ 0&0&0&0\\ 0&1&0&0 \end{bmatrix},\; G_2 = \begin{bmatrix} 1&0&0&0\\ 0&1&0&-1\\ 0&-1&1&0\\ 0&0&0&0 \end{bmatrix}.$$

We could continue with

$$Q_2 = \begin{bmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&0&1&0\\ 0&0&1&0 \end{bmatrix},\; G_3 = \begin{bmatrix} 1&0&0&0\\ 0&1&0&-1\\ 0&-1&1&0\\ 0&0&-1&0 \end{bmatrix},$$

which shows the DAE to be regular with tractability index 3, and Q_0, Q_1, Q_2 to be admissible, if $\alpha \in \mathcal{C}([0,1])$. However, we dismiss this choice of Q_2 and compute it instead corresponding to the decomposition

$$N_2 \oplus S_2 = \{z \in \mathbb{R}^4 : z_1 = 0, z_2 = z_3 = z_4\} \oplus \{z \in \mathbb{R}^4 : \alpha z_1 = z_3\} = \mathbb{R}^4.$$

This leads to

$$Q_2 = \begin{bmatrix} 0&0&0&0\\ \alpha&0&1&0\\ \alpha&0&1&0\\ \alpha&0&1&0 \end{bmatrix},\; D\Pi_2 D^- = \Pi_2 = \begin{bmatrix} 1&0&0&0\\ 0&0&0&0\\ -\alpha&0&0&0\\ 0&0&1&0 \end{bmatrix},$$

and hence, for these Q_0, Q_1, Q_2 to be admissible, the function α is required to be continuously differentiable. The coupling coefficients related to the present projector functions are

$$\mathcal{H}_1 = \begin{bmatrix} 0&0&0&0\\ \alpha'&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}, \quad \mathcal{H}_2 = 0.$$

If α' does not vanish identically, we have not yet reached a fine decoupling. In the next round we set $\bar{Q}_0 = Q_0$ such that $\bar{G}_1 = G_1$, but then we put

$$\bar{Q}_1 := Q_{1*} := Q_1 P_2 G_3^{-1}\{B_1 + G_1 D^-(D\Pi_2 D^-)' D\Pi_0\} = \begin{bmatrix} 0&0&0&0\\ \alpha'&1&0&0\\ 0&0&0&0\\ \alpha'&1&0&0 \end{bmatrix},$$

in accordance with Lemma 2.31 (see also Lemma 2.41 below). It follows that

$$D\bar{\Pi}_1 D^- = \bar{\Pi}_1 = \begin{bmatrix} 1 & 0\,0\,0 \\ -\alpha' & 0\,0\,0 \\ 0 & 0\,0\,0 \\ 0 & 0\,1\,0 \end{bmatrix}, \; \bar{G}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ -\alpha' & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and we see that, to ensure that $D\bar{\Pi}_1 D^-$ becomes continuously differentiable, and $\bar{Q}_0, \bar{Q}_1$ admissible, we need a two times continuously differentiable function α. Then we have $\bar{N}_2 = N_2$, which allows for the choice $\bar{Q}_2 = Q_2$. The resulting $\bar{Q}_0, \bar{Q}_1, \bar{Q}_2$ are fine decoupling projector functions. □

In general, if the DAE (2.44) is regular with tractability index μ, and $Q_0, \ldots, Q_{\mu-1}$ are admissible projector functions, then the decomposition

$$N_{\mu-1} \oplus S_{\mu-1} = \mathbb{R}^m$$

holds true (cf. Lemma A.9). If the last projector function $Q_{\mu-1}$ is chosen such that the associated subspace $S_{\mu-1} \supseteq N_0 \oplus \cdots \oplus N_{\mu-2}$ becomes its nullspace, that is $\ker Q_{\mu-1} = S_{\mu-1}$, $\operatorname{im} Q_{\mu-1} = N_{\mu-1}$, then it follows (cf. Lemma A.10) that $Q_{\mu-1} = Q_{\mu-1} G_{\mu-1}^{-1} B_{\mu-1}$, and hence (cf. (2.54))

$$\begin{aligned} \mathcal{H}_{\mu-1} &:= \Pi_{\mu-2} Q_{\mu-1} \mathcal{K} \Pi_{\mu-1} = \Pi_{\mu-2} Q_{\mu-1} \mathcal{K} \\ &= \Pi_{\mu-2} \underbrace{Q_{\mu-1}(I - \Pi_{\mu-1})}_{=Q_{\mu-1}} G_\mu^{-1} B_{\mu-1} \Pi_{\mu-1} \\ &\qquad + \sum_{l=0}^{\mu-1} \Pi_{\mu-2} \underbrace{Q_{\mu-1}(I - \Pi_l)}_{=0} (P_l - Q_l)(D\Pi_l D^-)' D\Pi_{\mu-1} \\ &= \Pi_{\mu-2} Q_{\mu-1} G_\mu^{-1} B_{\mu-1} \Pi_{\mu-1} = \Pi_{\mu-2} Q_{\mu-1} \Pi_{\mu-1} = 0. \end{aligned}$$

So far one can prevail on the coefficients $\mathcal{H}_{\mu-1}$ to vanish by determining $\ker Q_{\mu-1} = S_{\mu-1}$. This confirms the existence of complete decoupling projector functions for regular index-1 DAEs, and the existence of fine decoupling projector functions for regular index-2 DAEs.

Remember that, for regular constant coefficient DAEs with arbitrary index, complete decoupling projectors are provided by Theorem 1.22. We follow the lines of [169] to prove a similar result for general regular DAEs (2.44).

Having Lemma 2.31 we are well prepared to construct fine decoupling projector functions for the general regular DAE (2.44). As in Example 2.40, we successively improve the decoupling with the help of Lemma 2.31 in several rounds. We begin by forming arbitrary admissible projector functions $Q_0, \ldots, Q_{\mu-2}$ and $G_{\mu-1}$. Then we determine $Q_{\mu-1}$ by $\ker Q_{\mu-1} = S_{\mu-1}$ and $\operatorname{im} Q_{\mu-1} = N_{\mu-1}$. This yields $G_\mu = G_{\mu-1} + B_{\mu-1} Q_{\mu-1}$ as well as

$$\begin{aligned} &Q_{\mu-1} = Q_{\mu-1} G_\mu^{-1} B_{\mu-1} = Q_{\mu-1*}, \quad \text{and} \\ &\mathcal{H}_{\mu-1} = \Pi_{\mu-2} Q_{\mu-1*} \Pi_{\mu-1} = \Pi_{\mu-2} Q_{\mu-1} \Pi_{\mu-1} = 0. \end{aligned}$$

If $\mu = 2$ we already have a fine decoupling. If $\mu \geq 3$, we assume $D\Pi_{\mu-3}Q_{\mu-2*}D^-$, which is a priori continuous, to be even continuously differentiable, and compose a new sequence from the previous one. We set

$$\bar{Q}_0 := Q_0, \ldots, \bar{Q}_{\mu-3} = Q_{\mu-3}, \quad \text{and} \quad \bar{Q}_{\mu-2} = Q_{\mu-2*}.$$

$D\bar{\Pi}_{\mu-2}D^- = D\Pi_{\mu-3}D^- - D\Pi_{\mu-3}Q_{\mu-2*}D^-$ is continuously differentiable, and the projector functions $\bar{Q}_0, \ldots, \bar{Q}_{\mu-2}$ are admissible. Further, some technical calculations yield

$$\bar{G}_{\mu-1} = G_{\mu-1} \underbrace{\{I + \bar{Q}_{\mu-2}P_{\mu-2} + (I - \Pi_{\mu-3})Q_{\mu-2}D^-(D\bar{\Pi}_{\mu-2}D^-)'D\Pi_{\mu-3}\bar{Q}_{\mu-2}\}}_{Z_{\mu-1}}.$$

The matrix function $Z_{\mu-1}$ remains nonsingular; it has the pointwise inverse

$$Z_{\mu-1}^{-1} = I - \bar{Q}_{\mu-2}P_{\mu-2} - (I - \Pi_{\mu-3})Q_{\mu-2}D^-(D\bar{\Pi}_{\mu-2}D^-)'D\Pi_{\mu-3}Q_{\mu-2}.$$

We complete the current sequence by

$$\bar{Q}_{\mu-1} := Z_{\mu-1}^{-1}Q_{\mu-1}Z_{\mu-1} = Z_{\mu-1}^{-1}Q_{\mu-1}.$$

It follows that $\bar{Q}_{\mu-1}\bar{Q}_{\mu-2} = Z_{\mu-1}^{-1}Q_{\mu-1}Q_{\mu-2*} = 0$ and $\bar{Q}_{\mu-1}\bar{Q}_i = Z_{\mu-1}^{-1}Q_{\mu-1}Q_i = 0$ for $i = 0, \ldots, \mu-3$. Applying several basic properties (e.g., $\bar{\Pi}_{\mu-2} = \bar{\Pi}_{\mu-2}\Pi_{\mu-2}$) we find the representation $D\bar{\Pi}_{\mu-1}D^- = (D\bar{\Pi}_{\mu-2}D^-)(D\Pi_{\mu-1}D^-)$ which shows the continuous differentiability of $D\bar{\Pi}_{\mu-1}D^-$. Our new sequence $\bar{Q}_0, \ldots, \bar{Q}_{\mu-1}$ is admissible. We have further $\operatorname{im}\bar{G}_{\mu-1} = \operatorname{im}G_{\mu-1}$, thus

$$\bar{S}_{\mu-1} = S_{\mu-1} = \ker \mathcal{W}_{\mu-1}B = \ker \mathcal{W}_{\mu-1}BZ_{\mu-1} = Z_{\mu-1}^{-1}S_{\mu-1}.$$

This makes it clear that, $\bar{Q}_{\mu-1} = Z_{\mu-1}^{-1}Q_{\mu-1}$ projects onto $\bar{N}_{\mu-1} = Z_{\mu-1}^{-1}N_{\mu-1}$ along $\bar{S}_{\mu-1} = Z_{\mu-1}^{-1}S_{\mu-1}$, and therefore the new coupling coefficient satisfies $\bar{\mathcal{H}}_{\mu-1} = 0$. Additionally, making further technical efforts one attains $\bar{\mathcal{H}}_{\mu-2} = 0$.
If $\mu = 3$, a fine decoupling is reached. If $\mu \geq 4$, we build the next sequence analogously as

$$\begin{gathered}\bar{\bar{Q}}_0 := \bar{Q}_0, \ldots, \bar{\bar{Q}}_{\mu-4} := \bar{Q}_{\mu-4},\ \bar{\bar{Q}}_{\mu-3} := \bar{Q}_{\mu-3*},\\ \bar{\bar{Q}}_{\mu-2} := \bar{Z}_{\mu-2}^{-1}\bar{Q}_{\mu-2}\bar{Z}_{\mu-2},\ \bar{\bar{Q}}_{\mu-1} := \bar{Z}_{\mu-1}^{-1}\bar{Q}_{\mu-1}\bar{Z}_{\mu-1}.\end{gathered}$$

Supposing $D\bar{\Pi}_{\mu-4}\bar{Q}_{\mu-3*}D^-$ to be continuously differentiable, we prove the new sequence to be admissible, and to generate the coupling coefficients

$$\bar{\bar{\mathcal{H}}}_{\mu-1} = 0, \quad \bar{\bar{\mathcal{H}}}_{\mu-2} = 0, \quad \bar{\bar{\mathcal{H}}}_{\mu-3} = 0.$$

And so on. Lemma 2.41 below guarantees the procedure reaches its goal.

Lemma 2.41. *Let the DAE (2.44) with sufficiently smooth coefficients be regular with tractability index $\mu \geq 3$, and let $Q_0, \dots, Q_{\mu-1}$ be admissible projector functions.*
Let $k \in \{1, \dots, \mu-2\}$ be fixed, and let $\bar{Q}_k$ be an additional continuous projector function onto $N_k = \ker G_k$ such that $D\Pi_{k-1}\bar{Q}_k D^-$ is continuously differentiable and the inclusion $N_0 + \cdots + N_{k-1} \subseteq \ker \bar{Q}_k$ is valid. Then the following becomes true:

(1) *The projector function sequence*

$$\begin{aligned}
&\bar{Q}_0 := Q_0, \dots, \bar{Q}_{k-1} := Q_{k-1}, \\
&\bar{Q}_k, \\
&\bar{Q}_{k+1} := Z_{k+1}^{-1} Q_{k+1} Z_{k+1}, \dots, \bar{Q}_{\mu-1} := Z_{\mu-1}^{-1} Q_{\mu-1} Z_{\mu-1},
\end{aligned}$$

with the continuous nonsingular matrix functions $Z_{k+1}, \dots, Z_{\mu-1}$ determined below, is also admissible.

(2) *If, additionally, the projector functions $Q_0, \dots, Q_{\mu-1}$ provide an advanced decoupling in the sense that the conditions (cf. Lemma 2.31)*

$$Q_{\mu-1*}\Pi_{\mu-1} = 0, \dots, \; Q_{k+1*}\Pi_{\mu-1} = 0$$

are given, then also the relations

$$\bar{Q}_{\mu-1*}\bar{\Pi}_{\mu-1} = 0, \dots, \; \bar{Q}_{k+1*}\bar{\Pi}_{\mu-1} = 0, \tag{2.72}$$

are valid, and further

$$\bar{Q}_{k*}\bar{\Pi}_{\mu-1} = (Q_{k*} - \bar{Q}_k)\Pi_{\mu-1}. \tag{2.73}$$

The matrix functions Z_i are consistent with those given in Lemma 2.12; however, for easier reading we do not access this general lemma in the proof below. In the special case given here, Lemma 2.12 yields simply $Z_0 = I, Y_1 = Z_1 = I, \dots,$ $Y_k = Z_k = I$, and further

$$\begin{aligned}
Y_{k+1} &= I + Q_k(\bar{Q}_k - Q_k) + \sum_{l=0}^{k-1} Q_l \mathfrak{A}_{kl} \bar{Q}_k = \Big(I + \sum_{l=0}^{k-1} Q_l \mathfrak{A}_{kl} Q_k\Big)\Big(I + Q_k(\bar{Q}_k - Q_k)\Big), \\
Z_{k+1} &= Y_{k+1}, \\
Y_j &= I + \sum_{l=0}^{j-2} Q_l \mathfrak{A}_{j-1l} Q_{j-1}, \quad Z_j = Y_j Z_{j-1}, \quad j = k+2, \dots, \mu.
\end{aligned}$$

Besides the general property $\ker \bar{\Pi}_j = \ker \Pi_j$, $j = 0, \dots, \mu-1$, which follows from Lemma 2.12, now it additionally holds that

$$\operatorname{im} \bar{Q}_k = \operatorname{im} Q_k, \quad \text{but} \quad \ker \bar{Q}_j = \ker Q_j, \quad j = k+1, \dots, \mu-1.$$

We refer to Appendix B for the extensive calculations proving this lemma.

Lemma 2.41 guarantees the existence of fine decoupling projector functions, and it confirms the procedure sketched above to be reasonable.
The following theorem is the time-varying counterpart of Theorem 1.22 on constant coefficient DAEs.

Theorem 2.42. *Let the DAE (2.44) be regular with tractability index μ.*

(1) *If the coefficients of the DAE are sufficiently smooth, then a fine decoupling exists.*
(2) *If there is a fine decoupling, then there is also a complete decoupling.*

Proof. (1) The first assertion is a consequence of Lemma 2.41 and the procedure described above.
(2) Let fine decoupling projectors $Q_0,\dots,Q_{\mu-1}$ be given. We form the new sequence

$$\bar{Q}_0 := Q_{0*},\ \bar{Q}_1 := Z_1^{-1}Q_1Z_1,\dots,\ \bar{Q}_{\mu-1} := Z_{\mu-1}^{-1}Q_{\mu-1}Z_{\mu-1},$$

with the matrix functions Z_j from Lemma 2.12, in particular $Z_1 = I+\bar{Q}_0P_0$. It holds that $\bar{D}^- = \bar{P}_0D^-$. Owing to the special form of Z_j, the relations $\Pi_{j-1}Z_j = \Pi_{j-1}$, $\Pi_{j-1}Z_j^{-1} = \Pi_{j-1}$ are given for $j \le i-1$. This yields $\bar{Q}_i\bar{Q}_j = \bar{Q}_iZ_j^{-1}Q_jZ_j = \bar{Q}_i\underbrace{\Pi_{i-1}Z_j^{-1}Q_j}_{=0}Z_j = 0$.
Expressing $D\bar{\Pi}_1\bar{D}^- = D\bar{P}_0Z_1^{-1}P_1Z_1\bar{P}_0D^- = D\underbrace{P_0Z_1^{-1}P_1}_{\Pi_1}Z_1\bar{P}_0D^- = D\Pi_1D^-$, and successively,

$$\begin{aligned} D\bar{\Pi}_i\bar{D}^- &= D\bar{\Pi}_{i-1}Z_i^{-1}P_iZ_i\bar{P}_D^- \\ &= D\bar{\Pi}_{i-1}\bar{D}^-DZ_i^{-1}P_iZ_i\bar{P}_D^- = D\underbrace{\Pi_{i-1}D^-DZ_i^{-1}P_i}_{\Pi_i}Z_i\bar{P}_D^- = D\Pi_iD^-, \end{aligned}$$

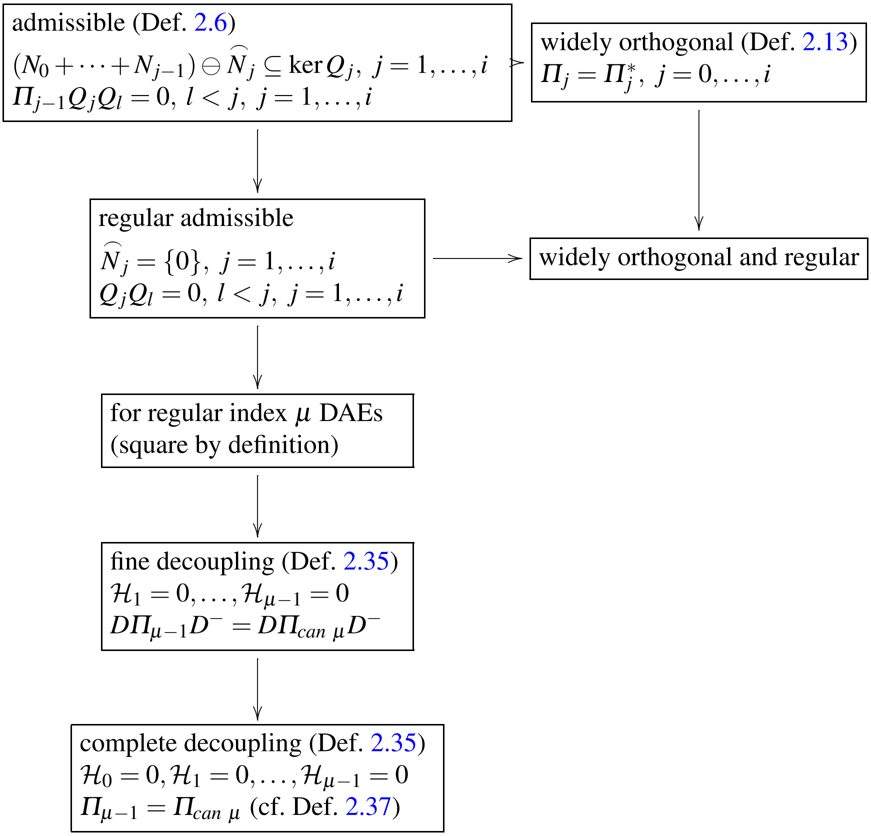
we see that the new sequence of projector functions $\bar{Q}_0,\dots,\bar{Q}_{\mu-1}$ is admissible, too. Analogously to Lemma 2.41, one shows

$$\bar{\mathcal{H}}_{\mu-1} = 0,\dots,\ \bar{\mathcal{H}}_1 = 0,\quad \bar{\mathcal{H}}_0 = (Q_{0*}-\bar{Q}_0)\Pi_{\mu-1},$$

and this completes the proof. □

2.5 Hierarchy of admissible projector function sequences for linear DAEs

The matrices $Q_0,\dots,Q_i$ are admissible projectors, where Q_j projects onto $N_j = \ker G_j$, $j=0,\dots,i$, with $P_0 := I-Q_0$, $\Pi_0 := P_0$ and $P_j := I-Q_j$, $\Pi_j := \Pi_{j-1}P_j$, $\widehat{N}_j := (N_0+\cdots+N_{j-1})\cap N_j$, $j=1,\dots,i$.



2.6 Fine regular DAEs

Here we continue to investigate regular DAEs (2.44) which have tractability index μ and fine decoupling projector functions $Q_0,\ldots,Q_{\mu-1}$. It is worth emphasizing once more that Theorem 2.42 guarantees the existence of a fine decoupling for all regular DAEs with sufficiently smooth coefficients.

Definition 2.43. Equation (2.44) is said to be a *fine DAE* on the interval $\mathcal{I}$, if it is regular there and possesses a fine decoupling.

By Theorem 2.39 and Lemma 2.31,

$$\Pi_{can}=(I-Q_{0*})\Pi_{\mu-1}=(I-\mathcal{H}_0)\Pi_{\mu-1}$$

is the canonical projector function onto S_{can} along N_{can}, and hence $D\Pi_{can}=D\Pi_{\mu-1}$, and therefore $D\Pi_{can}D^-=D\Pi_{\mu-1}D^-$, and $\operatorname{im} D\Pi_{\mu-1}=\operatorname{im} D\Pi_{can}=DS_{can}$. Taking into account also Lemma 2.28 (7), the IERODE can now be written as

$$u' - (D\Pi_{can}D^-)'u + D\Pi_{can}G_\mu^{-1}BD^-u = D\Pi_{can}G_\mu^{-1}q, \tag{2.74}$$

and, by Lemma 2.27, the subspace DS_{can} is a time-varying invariant subspace for its solutions, which means $u(t_0) \in D(t_0)S_{can}(t_0)$ implies $u(t) \in D(t)S_{can}(t)$ for all $t \in \mathcal{I}$. This invariant subspace also applies to the homogeneous version of the IERODE. Here, the IERODE is unique, its coefficients are independent of the special choice of the fine decoupling projector functions, as pointed out in the previous subsection. With regard to the fine decoupling, Proposition 2.29 (6), and the fact that $v_i = \Pi_{i-1}Q_i v_i$ holds true for $i = 1, \dots, \mu - 1$, the subsystem (2.62) simplifies slightly to

$$v_0 = -\sum_{l=1}^{\mu-1} \mathcal{N}_{0l}(Dv_l)' - \sum_{l=2}^{\mu-1} \mathcal{M}_{0l}\, v_l - \mathcal{H}_0 D^- u + \mathcal{L}_0 q, \tag{2.75}$$

$$v_i = -\sum_{l=i+1}^{\mu-1} \mathcal{N}_{il}(Dv_l)' - \sum_{l=i+2}^{\mu-1} \mathcal{M}_{il}\, v_l + \mathcal{L}_i q, \quad i = 1, \dots, \mu - 3, \tag{2.76}$$

$$v_{\mu-2} = -\mathcal{N}_{\mu-2,\mu-1}(Dv_{\mu-1})' + \mathcal{L}_{\mu-2} q, \tag{2.77}$$

$$v_{\mu-1} = \mathcal{L}_{\mu-1} q. \tag{2.78}$$

By Theorem 2.30, the DAE (2.44) is equivalent to the system consisting of the IERODE and the subsystem (2.75)–(2.78).

2.6.1 Fundamental solution matrices

The following solvability assertion is a simple consequence of the above.

Theorem 2.44. *If the homogeneous DAE is fine, then,*

(1) *for each arbitrary $x^0 \in \mathbb{R}^m$, the IVP*

$$A(Dx)' + Bx = 0, \quad x(t_0) - x^0 \in N_{can}(t_0), \tag{2.79}$$

is uniquely solvable in $C_D^1(\mathcal{I}, \mathbb{R}^m)$,

(2) *the homogeneous IVP*

$$A(Dx)' + Bx = 0, \quad x(t_0) \in N_{can}(t_0),$$

has the trivial solution only, and

(3) *through each $x_0 \in S_{can}(t_0)$ there passes exactly one solution.*

Remark 2.45. Sometimes is seems to be more comfortable to describe the initial condition in (2.79) by an equation, for instance, as

$$\Pi_{can}(t_0)(x(t_0) - x^0) = 0, \tag{2.80}$$

and as

$$C(x(t_0)-x^0)=0, \tag{2.81}$$

by any matrix C such that $\ker C=\ker \Pi_{can}(t_0)=N_{can}(t_0)$. For instance, taking arbitrary admissible projector functions $\tilde{Q}_0,\dots,\tilde{Q}_{\mu-1}$, one can choose C such that $C=C\tilde{\Pi}_{can}(t_0)$ (cf. Theorem 3.66).

Proof. (2) The initial condition yields $u(t_0)=D(t_0)\Pi_{can}(t_0)x(t_0)=0$. Then, the resulting homogeneous IVP for the IERODE admits the trivial solution $u=0$ only. Therefore, the DAE solution $x=\Pi_{can}D^-u$ vanishes identically, too.
(1) We provide the solution u of the homogeneous IERODE which satisfies the initial condition $u(t_0)=D(t_0)\Pi_{can}(t_0)x^0$. Then we form the DAE solution $x=\Pi_{can}D^-u$, and check that the initial condition is met:

$$\begin{aligned}x(t_0)-x^0&=\Pi_{can}(t_0)D(t_0)^-u(t_0)-x^0=\Pi_{can}(t_0)D(t_0)^-D(t_0)\Pi_{can}(t_0)x^0-x^0\\&=-(I-\Pi_{can}(t_0))x^0\in N_{can}(t_0).\end{aligned}$$

Owing to (2) this is the only solution of the IVP.
(3) We provide the IVP solution as in (1), with x^0 replaced by x_0. This leads to

$$x(t_0)=\Pi_{can}(t_0)D(t_0)^-u(t_0)=\Pi_{can}(t_0)D(t_0)^-D(t_0)\Pi_{can}(t_0)x_0=\Pi_{can}(t_0)x_0=0.$$

The uniqueness is ensured by (2). □

By Theorem 2.44, regular homogeneous DAEs are close to regular homogeneous ODEs. This applies also to their fundamental solution matrices.
Denote by $U(t,t_0)$ the classical fundamental solution matrix of the IERODE, that is, of the explicit ODE (2.74), which is normalized at $t_0\in\mathcal{I}$, i.e., $U(t_0,t_0)=I$.
For each arbitrary initial value $u_0\in D(t_0)S_{can}(t_0)$, the solution of the homogeneous IERODE passing through remains for ever in this invariant subspace, which means $U(t,t_0)u_0\in D(t)S_{can}(t)$ for all $t\in\mathcal{I}$, and hence

$$U(t,t_0)D(t_0)\Pi_{can}(t_0)=D(t)\Pi_{can}(t)D(t)^-U(t,t_0)D(t_0)\Pi_{can}(t_0),\quad t\in\mathcal{I}. \tag{2.82}$$

Each solution of the homogeneous DAE can now be expressed as

$$\begin{aligned}x(t)=&(I-\mathcal{H}_0(t))D(t)^-U(t,t_0)u_0=\Pi_{can}(t)D(t)^-U(t,t_0)u_0,\\&t\in\mathcal{I},\ u_0\in D(t_0)S_{can}(t_0),\end{aligned} \tag{2.83}$$

and also as

$$x(t)=\underbrace{\Pi_{can}(t)D(t)^-U(t,t_0)D(t_0)\Pi_{can}(t_0)}_{X(t,t_0)}x^0,\quad t\in\mathcal{I},\quad \text{with}\quad x^0\in\mathbb{R}^m. \tag{2.84}$$

If $x\in\mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$ satisfies the homogeneous DAE, then there is exactly one $u_0\in D(t_0)S_{can}(t_0)$ such that the expression (2.83) is valid, and there are elements $x^0\in\mathbb{R}^m$ such that (2.84) applies. Except for the index-0 case, x^0 is not unique.

Conversely, for each arbitrary $x^0 \in \mathbb{R}^m$, formula (2.84) provides a solution of the homogeneous DAE. We know that the solution values of the homogeneous DAE lie in the d-dimensional canonical subspace S_{can}, in particular $x(t_0) \in S_{can}(t_0)$. Therefore, starting from an arbitrary $x^0 \in \mathbb{R}^m$, the consistency of $x(t_0)$ with x^0 cannot be expected. What we always attain is the relation

$$x(t_0) = \Pi_{can}(t_0)x^0,$$

but the condition $x(t_0) = x_0$ is exclusively reserved for x_0 belonging to $S_{can}(t_0)$.

The composed matrix function

$$X(t,t_0) := \Pi_{can}(t)D(t)^- U(t,t_0)D(t_0)\Pi_{can}(t_0), \quad t \in \mathcal{I}, \tag{2.85}$$

arising in the solution expression (2.84) plays the role of a fundamental solution matrix of the DAE (2.44). In comparison with the (regular) ODE theory, there are several differences to be considered. By construction, it holds that $X(t_0,t_0) = \Pi_{can}(t_0)$ and

$$\operatorname{im} X(t,t_0) \subseteq S_{can}(t), \quad N_{can}(t_0) \subseteq \ker X(t,t_0), \quad t \in \mathcal{I}, \tag{2.86}$$

so that $X(t,t_0)$ is a *singular* matrix, except for the case $\mu = 0$. $X(.,t_0)$ is continuous, and $DX(.,t_0) = D\Pi_{can}D^- U(.,t_0)D(t_0)\Pi_{can}(t_0)$ is continuously differentiable, thus the columns of $X(.,t_0)$ are functions belonging to $\mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$.
We show that $X(t,t_0)$ has constant rank d. Fix an arbitrary $t \neq t_0$ and investigate the nullspace of $X(t,t_0)$. $X(t,t_0)z = 0$ means $U(t,t_0)D(t_0)\Pi_{can}(t_0)z \in \ker \Pi_{can}(t)D(t)^-$, and with regard to (2.82) this yields $U(t,t_0)D(t_0)\Pi_{can}(t_0)z = 0$, thus $D(t_0)\Pi_{can}(t_0)z = 0$, and further $\Pi_{can}(t_0)z = 0$. Owing to (2.86), and for reasons of dimensions, it follows that

$$\operatorname{im} X(t,t_0) = S_{can}(t), \quad \ker X(t,t_0) = N_{can}(t_0), \quad \operatorname{rank} X(t,t_0) = d, \quad t \in \mathcal{I}. \tag{2.87}$$

Lemma 2.46. *The matrix function*

$$X(t,t_0)^- = \Pi_{can}(t_0)D(t_0)^- U(t,t_0)^{-1} D(t)\Pi_{can}(t), \quad t \in \mathcal{I},$$

is the reflexive generalized inverse of $X(t,t_0)$ *determined by*

$$XX^-X = X, \quad X^-XX^- = X^-, \quad X^-X = \Pi_{can}(t_0), \quad XX^- = \Pi_{can}.$$

Proof. Applying the invariance (2.82), we derive

$$\begin{aligned} X^-X &= \Pi_{can}(t_0)D(t_0)^- U^{-1} D\Pi_{can}\Pi_{can}D^- UD(t_0)\Pi_{can}(t_0) \\ &= \Pi_{can}(t_0)D(t_0)^- U^{-1} \underbrace{D\Pi_{can}D^- UD(t_0)\Pi_{can}(t_0)}_{UD(t_0)\Pi_{can}(t_0)} = \Pi_{can}(t_0), \end{aligned}$$

and $X^-XX^- = (X^-X)X^- = X^-, \quad XX^-X = X(X^-X) = X.$
Next we verify the relation

$$U^{-1}D\Pi_{can} = D(t_0)\Pi_{can}(t_0)D(t_0)^- U^{-1}D\Pi_{can}, \tag{2.88}$$

which in turn implies

$$\begin{aligned} XX^- &= \Pi_{can}D^- UD(t_0)\Pi_{can}(t_0)\Pi_{can}(t_0)D(t_0)^- U^{-1}D\Pi_{can} \\ &= \Pi_{can}D^- U\underbrace{D(t_0)\Pi_{can}(t_0)D(t_0)^- U^{-1}D\Pi_{can}}_{U^{-1}D\Pi_{can}} = \Pi_{can}. \end{aligned}$$

From

$$U' - (D\Pi_{can}D^-)'U + D\Pi_{can}G_\mu^{-1}BD^- U = 0, \quad U(t_0) = 0,$$

it follows that

$$U^{-1\prime} + U^{-1}(D\Pi_{can}D^-)' - U^{-1}D\Pi_{can}G_\mu^{-1}BD^- = 0.$$

Multiplication by $D\Pi_{can}D^-$ on the right results in the explicit ODE

$$V' = V(D\Pi_{can}D^-)' + VD\Pi_{can}G_\mu^{-1}BD^-$$

for the matrix function $V = U^{-1}D\Pi_{can}D^-$. Then, the matrix function $\tilde{V} := (I - D(t_0)\Pi_{can}(t_0)D(t_0)^-)V$ vanishes identically as the solution of the classical homogeneous IVP

$$\tilde{V}' = \tilde{V}(D\Pi_{can}D^-)' + \tilde{V}D\Pi_{can}G_\mu^{-1}BD^-, \quad \tilde{V}(t_0) = 0,$$

and this proves (2.88). □

The columns of $X(.,t_0)$ are solutions of the homogeneous DAE, and the matrix function $X(.,t_0)$ itself satisfies the equation

$$A(DX)' + BX = 0, \tag{2.89}$$

as well as the initial condition

$$X(t_0,t_0) = \Pi_{can}(t_0), \tag{2.90}$$

or, equivalently,

$$\Pi_{can}(t_0)(X(t_0,t_0) - I) = 0. \tag{2.91}$$

Definition 2.47. Let the DAE (2.44) be fine. Each matrix function $Y \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^s, \mathbb{R}^m))$, $d \le s \le m$, is said to be a *fundamental solution matrix* of the DAE, if its columns belong to $\mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m)$, the equation

$$A(DY)' + BY = 0$$

is satisfied, and the condition $\operatorname{im} Y = S_{can}$ is valid.
A fundamental solution matrix is named *of minimal size*, if $s = d$, and *of maximal size*, if $s = m$.

A maximal size fundamental solution matrix Y is said to be *normalized at* t_0, if $\Pi_{can}(t_0)(Y(t_0)-I)=0$.

In this sense, the above matrix function $X(.,t_0)$ (cf. (2.85)) is a maximal size fundamental solution normalized at t_0.

Remark 2.48. Concerning fundamental solution matrices of DAEs, there is no common agreement in the literature. Minimal and maximal size fundamental solution matrices, as well as relations among them, were first described in [9] for standard form index-1 DAEs. A comprehensive analysis for regular lower index DAEs, both in standard form and with properly stated leading term, is given in [7]. This analysis applies analogously to regular DAEs with arbitrary index.
Roughly speaking, minimal size fundamental solution matrices have a certain advantage in view of computational aspects, since they have full column rank. For instance, the Moore–Penrose inverse can be easily computed. In contrast, the benefits from maximal size fundamental solution matrices are a natural normalization and useful group properties as pointed out, e.g., in [11], [7].
If $X(t,t_0)$ is the maximal size fundamental solution matrix normalized at $t_0\in\mathcal{I}$, and $X(t,t_0)^-$ is the generalized inverse described by Lemma 2.46, then it holds for all $t,t_0,t_1\in\mathcal{I}$ that

$$X(t,t_1)X(t_1,t_0)=X(t,t_0),\quad \text{and}\quad X(t,t_0)^-=X(t_0,t),$$

as immediate consequences of the construction, and Lemma 2.46.

2.6.2 Consistent initial values and flow structure

Turning to inhomogeneous DAEs, first suppose the excitation to be such that a solution exists. Before long, we shall characterize the classes of admissible functions in detail.

Definition 2.49. The function $q\in\mathcal{C}(\mathcal{I},\mathbb{R}^m)$ is named an *admissible excitation* for the DAE (2.44), if the DAE is solvable for this q, i.e., if a solution $x_q\in\mathcal{C}_D^1(\mathcal{I},\mathbb{R}^m)$ exists such that $A(Dx_q)'+Bx_q=q$.

Proposition 2.50. *Let the DAE (2.44) be fine with tractability index μ.*

(1) *Then, $q\in\mathcal{C}(\mathcal{I},\mathbb{R}^m)$ is an admissible excitation, if and only if the IVP*

$$A(Dx)'+Bx=q,\quad x(t_0)\in N_{can}(t_0), \tag{2.92}$$

admits a unique solution.

(2) *Each $q\in\mathcal{C}(\mathcal{I},\mathbb{R}^m)$, which for $\mu\geq 2$ fulfills the condition $q=G_\mu P_1\cdots P_{\mu-1}G_\mu^{-1}q$, is an admissible excitation.*

Proof. (1) Let q be admissible and x_q the associated solution. Then the function $\tilde{x}(t) := x_q(t) - X(t,t_0)x_q(t_0)$, $t \in \mathcal{I}$, satisfies the IVP (2.92). The uniqueness results from Theorem 2.44 (2). The reverse is trivial.
(2) From the condition $q = G_\mu P_1 \cdots P_{\mu-1} G_\mu^{-1} q$ it follows that

$$\begin{aligned}\mathcal{L}_i q &= \Pi_{i-1} Q_i P_{i+1} \cdots P_{\mu-1} G_\mu^{-1} q \\ &= \Pi_{i-1} Q_i P_{i+1} \cdots P_{\mu-1} P_1 \cdots P_{\mu-1} G_\mu^{-1} q = 0, \quad i = 1, \dots \mu - 2, \\ \mathcal{L}_{\mu-1} q &= \Pi_{\mu-2} Q_{\mu-1} G_\mu^{-1} q = \Pi_{\mu-2} Q_{\mu-1} P_1 \cdots P_{\mu-1} G_\mu^{-1} q = 0.\end{aligned}$$

In consequence, the subsystem (2.76)–(2.78) yields successively $v_{\mu-1}, \dots, v_1 = 0$. The IERODE (2.74) is solvable for each arbitrary continuous excitation. Denote by u_* an arbitrary solution corresponding to q. Then, the function

$$v_0 = -\mathcal{H}_0 D^- u_* + \mathcal{L}_0 q = -\mathcal{H}_0 D^- u_* + Q_0 G_\mu^{-1} q$$

results from equation (2.75), and

$$x := D^- u_* + v_0 = \Pi_{can} D^- u_* + Q_0 G_\mu^{-1} q$$

is a solution of the DAE (2.44) corresponding to this excitation q. □

For a fine index-1 DAE, all continuous functions q are admissible. For fine higher index DAEs, the additional projector function $G_\mu P_1 \cdots P_{\mu-1} G_\mu^{-1}$ cuts away the "dangerous" parts of a function, and ensures that only the zero function is differentiated within the subsystem (2.75)–(2.78). For higher index DAEs, general admissible excitations have certain smoother components. We turn back to this problem later on.

Example 2.51 (A fine index-2 DAE). Consider the DAE

$$\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix} \left(\begin{bmatrix}1 & \alpha & 0\\ 0 & 1 & 0\end{bmatrix} x\right)' + \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix} x = q.$$

Here, α is a continuous scalar function. Set and derive

$$D^- = \begin{bmatrix}1 & -\alpha\\ 0 & 1\\ 0 & 0\end{bmatrix}, \; G_0 = \begin{bmatrix}1 & \alpha & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}, \; Q_0 = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}, \; G_1 = \begin{bmatrix}1 & \alpha & 0\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix},$$

and further

$$Q_1 = \begin{bmatrix}0 & -\alpha & 0\\ 0 & 1 & 0\\ 0 & 1 & 0\end{bmatrix}, \; Q_1 Q_0 = 0, \; D\Pi_1 D^- = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \; G_2 = \begin{bmatrix}1 & \alpha & 0\\ 0 & 1 & -1\\ 0 & 1 & 0\end{bmatrix}.$$

The projector functions Q_0, Q_1 are admissible, G_2 is nonsingular, and hence the DAE is regular with tractability index 2. The given property $\ker Q_1 = S_1 = \{z \in \mathbb{R}^3 :$

$z_2 = 0\}$ indicates that Q_0, Q_1 already provide a fine decoupling. The DAE is fine. Compute additionally

$$\Pi_{can} = \Pi_1 = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_2^{-1} = \begin{bmatrix} 1 & 0 & -\alpha \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \quad G_2 P_1 G_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

A closer look at the detailed equations makes it clear that each admissible excitation q must have a continuously differentiable component q_3. By the condition $q = G_2 P_1 G_2^{-1} q$, the third component of q is put to be zero. □

Theorem 2.52. *Let the DAE (2.44) be fine. Let $q \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m)$ be an admissible excitation, and let the matrix $C \in L(\mathbb{R}^m, \mathbb{R}^s)$ have the nullspace* $\ker C = N_{can}(t_0)$.

(1) *Then, for each $x^0 \in \mathbb{R}^m$, the IVP*

$$A(Dx)' + Bx = q, \quad C(x(t_0) - x^0) = 0, \tag{2.93}$$

admits exactly one solution.

(2) *The solution of the IVP (2.93) can be expressed as*

$$x(t, t_0, x^0) = X(t, t_0)x^0 + x_q(t),$$

whereby $x_q \in \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m)$ is the unique solution of the IVP

$$A(Dx)' + Bx = q, \quad Cx(t_0) = 0. \tag{2.94}$$

Proof. (1) It holds that $C = C\Pi_{can}(t_0)$. Since q is admissible, by Proposition 2.50(1), the solution x_q exists and is unique. Then the function $x_* := X(., t_0)x^0 + x_q$ belongs to the function space $\mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m)$ and satisfies the DAE. Further, x_* meets the initial condition

$$C(x_*(t_0) - x^0) = C\Pi_{can}(t_0)(x_*(t_0) - x^0) = C\Pi_{can}(t_0)(\Pi_{can}(t_0)x^0 + x_q(t_0) - x^0) = 0,$$

and hence, x_* satisfies the IVP (2.93). By Theorem 2.44, x_* is the only IVP solution. This proves at the same time (2). □

We take a further look at the structure of the DAE solutions x_q and $x(., t_0, x^0)$. For the given admissible excitation q, we denote

$$v := v_1 + \cdots + v_{\mu-1} + \mathcal{L}_0 q - \sum_{l=1}^{\mu-1} \mathcal{N}_{0l}(Dv_l)' - \sum_{l=2}^{\mu-1} \mathcal{M}_{0l} v_l, \tag{2.95}$$

whereby $v_1, \ldots, v_{\mu-1} \in \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m)$ are determined by equations (2.76)–(2.78), depending on q. All the required derivatives exist due to the admissibility of q. If q vanishes identically, so does v. By construction, $v(t) \in N_{can}(t)$, $t \in \mathcal{I}$, and $Dv = Dv_1 + \cdots + Dv_{\mu-1}$, thus $v \in \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m)$. The function v is fully determined by q and the coefficients of the subsystem (2.75)–(2.78). It does not depend either on the initial condition nor the IERODE solution.

Introduce further the continuously differentiable function u_q as

$$u_q(t) := \int_{t_0}^{t} U(t,t_0)U(s,t_0)^{-1}D(s)\Pi_{can}(s)G_\mu^{-1}(s)q(s)ds$$
$$= U(t,t_0)\int_{t_0}^{t} X(s,t_0)^{-}G_\mu^{-1}(s)q(s)ds, \quad t \in \mathcal{I},$$

that is, as the solution of the inhomogeneous IERODE completed by the homogeneous initial condition $u(t_0) = 0$. Now the solution x_q and, in particular, its value at t_0, can be expressed as

$$x_q(t) = D(t)^{-}u_q(t) - \mathcal{H}_0(t)D(t)^{-}u_q(t) + v(t) = \Pi_{can}(t)D(t)^{-}u_q(t) + v(t),$$
$$x_q(t_0) = v(t_0) \in N_{can}(t_0).$$

The solution of the IVP (2.93) and its value at t_0 can be written in the form

$$x(t,t_0,x^0) = X(t,t_0)x^0 + \Pi_{can}(t)D(t)^{-}u_q(t) + v(t), \tag{2.96}$$
$$x(t_0,t_0,x^0) = \Pi_{can}(t_0)x^0 + v(t_0), \tag{2.97}$$

but also as

$$x(t,t_0,x^0) = \Pi_{can}(t)D(t)^{-}U(t,t_0)D(t_0)\Pi_{can}(t_0)x^0 + \Pi_{can}(t)D(t)^{-}u_q(t) + v(t)$$
$$= \Pi_{can}(t)D(t)^{-}\underbrace{\{U(t,t_0)D(t_0)\Pi_{can}(t_0)x^0 + u_q(t)\}}_{u(t,t_0,D(t_0)\Pi_{can}(t_0)x^0)} + v(t).$$

The last representation

$$x(t,t_0,x^0) = \underbrace{\Pi_{can}(t)D(t)^{-}}_{\substack{\Uparrow \\ \text{wrapping}}}\ \underbrace{u(t,t_0,D(t_0)\Pi_{can}(t_0)x^0)}_{\substack{\Uparrow \\ \text{inherent flow}}} + \underbrace{v(t)}_{\substack{\Uparrow \\ \text{perturbation}}}$$

unveils the general solution structure of fine DAEs to be the perturbed and wrapped flow of the IERODE along the invariant subspace DS_{can}. If the wrapping is thin (bounded) and the perturbation disappears, then the situation is close to regular ODEs. However, it may well happen that wrapping and perturbation dominate (cf. Example 2.57 below). In extreme cases, it holds that $S_{can} = \{0\}$, thus the inherent flow vanishes, and only the perturbation term remains (cf. Example 2.4).

From Theorem 2.52, and the representation (2.96), it follows that, for each given admissible excitation, the set

$$\mathcal{M}_{can,q}(t) := \{z + v(t) : z \in S_{can}(t)\}, \quad t \in \mathcal{I}, \tag{2.98}$$

is occupied with solution values at time t, and all solution values at time t belong to this set. In particular, for $x_0 \in \mathcal{M}_{can,q}(t_0)$ it follows that $x_0 = z_0 + v(t_0)$, $z_0 \in S_{can}(t_0)$; further $\Pi_{can}(t_0)x_0 = z_0$ and

$$x(t_0,t_0,x_0) = \Pi_{can}(t_0)x_0 + v(t_0) = z_0 + v(t_0) = x_0.$$

By construction, the inclusions

$$\begin{aligned} S_{can}(t) &\subseteq S_0(t) = \{z \in \mathbb{R}^m : B(t)z \in \operatorname{im} A(t)\} = \ker \mathcal{W}_0(t)B(t), \\ \mathcal{M}_{can,q}(t) &\subseteq \mathcal{M}_0(t) = \{x \in \mathbb{R}^m : B(t)x - q(t) \in \operatorname{im} A(t)\} \end{aligned}$$

are valid, whereby $\mathcal{W}_0(t)$ is again a projector along $\operatorname{im} A(t) = \operatorname{im} G_0(t)$. Recall that $S_{can}(t)$ and $S_0(t)$ have the dimensions $d = m - \sum_{j=0}^{\mu-1}(m - r_j) = r_0 - \sum_{j=1}^{\mu-1}(m - r_j)$ and r_0, respectively. Representing the obvious constraint set as

$$\begin{aligned} \mathcal{M}_0(t) &= \{x \in \mathbb{R}^m : \mathcal{W}_0(t)B(t)x = \mathcal{W}_0(t)q(t)\} \\ &= \{z + (\mathcal{W}_0(t)B(t))^- \mathcal{W}_0(t)q(t) : z \in S_0(t)\} \end{aligned}$$

we know that $\mathcal{M}_0(t)$, as an affine space, inherits its dimension from $S_0(t)$, while $\mathcal{M}_{can,q}(t)$ has the same dimension d as $S_{can}(t)$.

Since $d = r_0$ if $\mu = 1$, and $d < r_0$ if $\mu > 1$, $\mathcal{M}_{can,q}(t)$ coincides with $\mathcal{M}_0(t)$ for index-1 DAEs, however, for higher index DAEs, $\mathcal{M}_{can,q}(t)$ is merely a proper subset of $\mathcal{M}_0(t)$. $\mathcal{M}_{can,q}(t)$ is the set of consistent values at time t. Knowledge of this set gives rise to an adequate stability notion for DAEs. As pointed out in [7] for lower index cases, in general, $\mathcal{M}_{can,q}$ is a time-varying affine linear subspace of dimension d.

2.6.3 Stability issues

As for regular time-varying ODEs (e.g., [80]), we may consider the qualitative behavior of solutions of DAEs.

Definition 2.53. Let the fine DAE (2.44) with an admissible excitation q be given on the infinite interval $\mathcal{I} = [0, \infty)$. The DAE is said to be

(1) *stable*, if for every $\varepsilon > 0$, $t_0 \in \mathcal{I}$, a value $\delta(\varepsilon, t_0) > 0$ exists, such that the conditions $x_0, \bar{x}_0 \in \mathcal{M}_{can,q}(t_0)$, $|x_0 - \bar{x}_0| < \delta(\varepsilon, t_0)$ imply the existence of solutions $x(.,t_0,x_0)$, $x(.,t_0,\bar{x}_0) \in \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m)$ as well as the inequality

$$|x(t,t_0,x_0) - x(t,t_0,\bar{x}_0)| < \varepsilon, \quad t_0 \le t,$$

(2) *uniformly stable*, if $\delta(\varepsilon, t_0)$ in (1) is independent of t_0,
(3) *asymptotically stable*, if (1) holds true, and

$$|x(t,t_0,x_0)-x(t,t_0,\bar{x}_0)| \xrightarrow[t\to\infty]{} 0 \quad \text{for all} \quad x_0,\bar{x}_0 \in \mathcal{M}_{can,q}(t_0),\ t_0 \in \mathcal{I},$$

(4) *uniformly asymptotically stable*, if the limit in (3) is uniform with respect to t_0.

Remark 2.54. We can dispense with the explicit use of the set $\mathcal{M}_{can,q}(t_0)$ within the stability notion by turning to appropriate IVPs (cf. Theorem 2.52). This might be more comfortable from the practical point of view.
Let $C \in L(\mathbb{R}^m,\mathbb{R}^s)$ denote a matrix that has precisely $N_{can}(t_0)$ as nullspace, for instance $C = \Pi_{\mu-1}(t_0)$ or $C = \Pi_{can}(t_0)$.
The DAE (2.44) is stable, if for every $\varepsilon > 0$, $t_0 \in \mathcal{I}$, there exists a value $\delta_C(\varepsilon,t_0) > 0$ such that the IVPs

$$\begin{aligned} A(Dx)' + Bx = q, &\quad C(x(t_0) - x^0) = 0,\\ A(Dx)' + Bx = q, &\quad C(x(t_0) - \bar{x}^0) = 0, \end{aligned}$$

with x^0, $\bar{x}^0 \in \mathbb{R}^m$, $|C(x^0 - \bar{x}^0)| < \delta_C(\varepsilon,t_0))$, have solutions $x(.,t_0,x^0)$, $x(.,t_0,\bar{x}^0) \in C^1_D(\mathcal{I},\mathbb{R}^m)$, and it holds that $|x(.,t_0,x^0) - x(.,t_0,\bar{x}^0)| < \varepsilon$, for $t \geq t_0$.
This notion is equivalent to the previous one. Namely, denoting by C^- a generalized reflexive inverse of C such that $C^-C = \Pi_{can}(t_0)$, and considering the relation

$$\begin{aligned} C^-C(x^0 - \bar{x}^0) &= \Pi_{can}(t_0)x^0 - \Pi_{can}(t_0)\bar{x}^0\\ &= \underbrace{\Pi_{can}(t_0)x^0 + v(t_0)}_{=x_0\in\mathcal{M}_0(t_0)} - \underbrace{(\Pi_{can}(t_0)\bar{x}^0 + v(t_0))}_{=\bar{x}_0\in\mathcal{M}_0(t_0)} = x_0 - \bar{x}_0, \end{aligned}$$

we know that the existence of $\delta(\varepsilon,t_0)$ in Definition 2.53 implies the existence of $\delta_C(\varepsilon,t_0) = |C|\delta(\varepsilon,t_0)$. Conversely, having $\delta_C(\varepsilon,t_0)$ we may put $\delta(\varepsilon,t_0) = |C^-|\delta_C(\varepsilon,t_0)$.

Making use of the linearity,

$$x(t,t_0,x_0) - x(t,t_0,\bar{x}_0) = X(t,t_0)(x_0 - \bar{x}_0) \tag{2.99}$$

we trace back the stability questions to the growth behavior of the fundamental solution matrices. Applying normalized maximal size fundamental solution matrices we modify well-known results on flow properties of explicit ODEs (e.g., [80]) so that they can be considered for DAEs.

Theorem 2.55. *Let the DAE (2.44) be fine and the excitation q be admissible. Then the following assertions hold true, with positive constants K_{t_0}, K and α:*

(1) *If $|X(t,t_0)| \leq K_{t_0}$, $t \geq t_0$, then the DAE is stable.*
(2) *If $|X(t,t_0)| \xrightarrow[t\to\infty]{} 0$, then the DAE is asymptotically stable.*
(3) *If $|X(t,t_0)X(s,t_0)^-| \leq K$, $t_0 \leq s \leq t$, then the DAE is uniformly stable.*
(4) *If $|X(t,t_0)X(s,t_0)^-| \leq K\mathrm{e}^{-\alpha(t-s)}$, $t_0 \leq s \leq t$, then the DAE is uniformly asymptotically stable.*

Proof. (1) It suffices to put $\delta(t_0, \varepsilon) = \varepsilon / K_{t_0}$.
(2) This is now obvious.
(4) Take $x_0, \bar{x}_0 \in \mathcal{M}_{can,q}(t_0)$, $z_0 := x_0 - \bar{x}_0 \neq 0$ such that $z_0 \in S_{can}$ and $X(t,t_0)z_0$ has no zeros. For $t \geq s$, we compute

$$\frac{|X(t,t_0)z_0|}{|X(s,t_0)z_0|} = \frac{|X(t,t_0)\Pi_{can}z_0|}{|X(s,t_0)z_0|} = \frac{|X(t,t_0)X(s,t_0)^- X(s,t_0)z_0|}{|X(s,t_0)z_0|}$$
$$\leq |X(t,t_0)X(s,t_0)^-| \leq K\mathrm{e}^{-\alpha(t-s)}.$$

This implies

$$|x(t,t_0,x_0) - x(t,t_0,\bar{x}_0)| = |X(t,t_0)z_0| \leq K\mathrm{e}^{-\alpha(t-s)}|x(s,t_0,x_0) - x(s,t_0,\bar{x}_0)|.$$

(3) This is proved as (4) by letting $\alpha = 0$. □

In the theory of explicit ODEs, for instance, in the context of boundary value problems, the notion of dichotomy plays its role. The flow of a dichotomic ODE accommodates both decreasing and increasing modes. The same can happen for DAEs. As for explicit ODEs, we relate dichotomy of DAEs to the flow of homogeneous equations. More precisely, we apply maximal size fundamental solution matrices $X(t,t_0)$ normalized at a reference point t_0. The following definition resembles that for ODEs.

Definition 2.56. The fine DAE (2.44) is said to be *dichotomic*, if there are constants $K, \alpha, \beta \geq 0$, and a nontrivial projector (not equal to the zero or identity matrix) $P_{dich} \in L(\mathbb{R}^m)$ such that $P_{dich} = \Pi_{can}(t_0)P_{dich} = P_{dich}\Pi_{can}(t_0)$, and the following inequalities apply for all $t, s \in \mathcal{I}$:

$$|X(t,t_0)P_{dich}X(s,t_0)^-| \leq Ke^{-\alpha(t-s)}, \quad t \geq s,$$
$$|X(t,t_0)(I - P_{dich})X(s,t_0)^-| \leq Ke^{-\beta(s-t)}, \quad t \leq s.$$

If α, $\beta > 0$, then one speaks of an *exponential dichotomy*.

Sometimes it is reasonable to write the last inequality in the form

$$|X(t,t_0)(\Pi_{can}(t_0) - P_{dich})X(s,t_0)^-| \leq Ke^{-\beta(s-t)}, \quad t \leq s.$$

It should be pointed out that dichotomy is actually independent of the reference point t_0. Namely, for $t_1 \neq t_0$, with $P_{dich,t_1} := X(t_1,t_0)P_{dich}X(t_1,t_0)^-$ we have a projector such that $P_{dich,t_1} = \Pi_{can}(t_1)P_{dich,t_1} = P_{dich,t_1}\Pi_{can}(t_1)$ and

$$|X(t,t_1)P_{dich,t_1}X(s,t_1)^-| \leq Ke^{-\alpha(t-s)}, \quad t \geq s,$$
$$|X(t,t_1)(\Pi_{can}(t_1) - P_{dich,t_1})X(s,t_1)^-| \leq Ke^{-\beta(s-t)}, \quad t \leq s.$$

Analogously to the ODE case, the flow of a dichotomic homogeneous DAE is divided into two parts, one containing in a certain sense a nonincreasing solution, the other with nondecreasing ones. More precisely, for a nontrivial $x_0 \in \operatorname{im} P_{dich} \subseteq$

$S_{can}(t_0)$, the DAE solution $x(t,t_0,x_0) = X(t,t_0)x_0$ has no zeros, and it satisfies for $t \geq s$ the inequalities

$$\begin{aligned}\frac{|x(t,t_0,x_0)|}{|x(s,t_0,x_0)|} = \frac{|X(t,t_0)x_0|}{|X(s,t_0)x_0|} &= \frac{|X(t,t_0)P_{dich}\Pi_{can}(t_0)x_0|}{|X(s,t_0)x_0|}\\ &= \frac{|X(t,t_0)P_{dich}X(s,t_0)^{-}X(s,t_0)x_0|}{|X(s,t_0)x_0|}\\ &\leq |X(t,t_0)P_{dich}X(s,t_0)^{-}| \leq Ke^{-\alpha(t-s)}.\end{aligned}$$

For solutions $x(t,t_0,x_0) = X(t,t_0)x_0$ with $x_0 \in \operatorname{im}(I-P_{dich})\Pi_{can} \subseteq S_{can}(t_0)$ we show analogously, for $t \leq s$,

$$\begin{aligned}\frac{|x(t,t_0,x_0)|}{|x(s,t_0,x_0)|} = \frac{|X(t,t_0)x_0|}{|X(s,t_0)x_0|} &= \frac{|X(t,t_0)(I-P_{dich})\Pi_{can}(t_0)x_0|}{|X(s,t_0)x_0|}\\ &= \frac{|X(t,t_0)(I-P_{dich})X(s,t_0)^{-}X(s,t_0)x_0|}{|X(s,t_0)x_0|}\\ &\leq |X(t,t_0)(I-P_{dich})X(s,t_0)^{-}| \leq Ke^{-\beta(s-t)}.\end{aligned}$$

The canonical subspace of the dichotomic DAE decomposes into

$$S_{can}(t) = \operatorname{im} X(t,t_0) = \operatorname{im} X(t,t_0)P_{dich} \oplus \operatorname{im} X(t,t_0)(I-P_{dich}) =: S^{-}_{can}(t) \oplus S^{+}_{can}(t).$$

The following two inequalities result for $t \geq s$, and they characterize the subspaces S^{-}_{can} and S^{+}_{can} as those containing nonincreasing and nondecreasing solutions, respectively:

$$\begin{aligned}|x(t,t_0,x_0)| \leq Ke^{-\alpha(t-s)}|x(s,t_0,x_0)|, &\quad \text{if} \quad x_0 \in S^{-}_{can},\\ \frac{1}{K}e^{\beta(t-s)}|x(s,t_0,x_0)| \leq |x(t,t_0,x_0)|, &\quad \text{if} \quad x_0 \in S^{+}_{can}.\end{aligned}$$

In particular, for $s = t_0$ it follows that

$$\begin{aligned}|x(t,t_0,x_0)| \leq Ke^{-\alpha(t-t_0)}|x_0|, &\quad \text{if} \quad x_0 \in S^{-}_{can},\\ \frac{1}{K}e^{\beta(t-t_0)}|x_0| \leq |x(t,t_0,x_0)|, &\quad \text{if} \quad x_0 \in S^{+}_{can}.\end{aligned}$$

If $\alpha > 0$, and $\mathcal{I} = [t_0,\infty)$, then $|x(t,t_0,x_0)|$ tends to zero for t tending to ∞, if x_0 belongs to $S^{-}_{can}(t_0)$. If $\beta > 0$ and $x_0 \in S^{+}_{can}(t_0)$, then $x(t,t_0,x_0)$ growths unboundedly with increasing t.

As for explicit ODEs, dichotomy makes good sense on infinite intervals I. The growth behavior of fundamental solutions is also important for the condition of boundary value problems stated on compact intervals (e.g., [2] for explicit ODEs, also [146] for index-1 DAEs). Dealing with compact intervals one supposes a constant K of *moderate size*.

Example 2.57 (Dichotomic IERODE and dichotomic DAE). Consider the semi-explicit DAE

$$\begin{bmatrix} I \\ 0 \end{bmatrix} \left(\begin{bmatrix} I & 0 \end{bmatrix} x \right)' + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} x = 0,$$

consisting of three equations, $m_1 = 2, m_2 = 1, n = 2$. Let B_{22} have no zeros, and let the coefficients be such that

$$B_{11} + B_{12} \begin{bmatrix} \gamma_1 & \gamma_2 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & -\beta \end{bmatrix}, \quad \begin{bmatrix} \gamma_1 & \gamma_2 \end{bmatrix} := -B_{22}^{-1} B_{21},$$

with constants $\alpha, \beta \geq 0$. Then, the canonical projector function and the IERODE have the form (cf. Example 2.32)

$$\Pi_{can} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \gamma_1 & \gamma_2 & 0 \end{bmatrix}, \quad \text{and} \quad u' + \begin{bmatrix} \alpha & 0 \\ 0 & -\beta \end{bmatrix} u = 0.$$

The IERODE is obviously dichotomic. Compute the fundamental solution matrix of the DAE and its generalized inverse:

$$X(t,t_0) = \begin{bmatrix} e^{-\alpha(t-t_0)} & 0 & 0 \\ 0 & e^{\beta(t-t_0)} & 0 \\ \gamma_1(t) e^{-\alpha(t-t_0)} & \gamma_2(t) e^{\beta(t-t_0)} & 0 \end{bmatrix},$$

$$X(t,t_0)^- = \begin{bmatrix} e^{\alpha(t-t_0)} & 0 & 0 \\ 0 & e^{-\beta(t-t_0)} & 0 \\ \gamma_1(t_0) e^{\alpha(t-t_0)} & \gamma_2(t_0) e^{-\beta(t-t_0)} & 0 \end{bmatrix}.$$

The projector

$$P_{dich} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma_1(t_0) & 0 & 0 \end{bmatrix}, \quad \Pi_{can}(t_0) - P_{dich} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \gamma_2(t_0) & 0 \end{bmatrix},$$

meets the condition of Definition 2.56, and it follows that

$$X(t,t_0) P_{dich} X(t,t_0)^- = \mathrm{e}^{-\alpha(t-t_0)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma_1(t) & 0 & 0 \end{bmatrix}, \text{ and } \quad S_{can}^-(t) = \operatorname{span} \begin{bmatrix} 1 \\ 0 \\ \gamma_1(t) \end{bmatrix},$$

$$X(t,t_0)(\Pi_{can}(t_0) - P_{dich}) X(t,t_0)^- = \mathrm{e}^{\beta(t-t_0)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \gamma_2(t) & 0 \end{bmatrix}, \text{ and}$$

$$S_{can}^+(t) = \operatorname{span} \begin{bmatrix} 0 \\ 1 \\ \gamma_2(t) \end{bmatrix}.$$

If both γ_1 and γ_2 are bounded functions, then this DAE is dichotomic. If, additionally, α and β are positive, the DAE has an exponential dichotomy. We see that if the entries of the canonical projector remain bounded, then the dichotomy of the IERODE is passed over to the DAE. In contrast, if the functions γ_1, γ_2 grow unboundedly, the situation within the DAE may change. For instance, if $\alpha = 0$ and $\beta > 0$, then the fundamental solution

$$X(t,t_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{\beta(t-t_0)} & 0 \\ \gamma_1(t) & \gamma_2(t)e^{\beta(t-t_0)} & 0 \end{bmatrix}$$

indicates that each nontrivial solution will grow unboundedly though the IERODE is dichotomic. □

The last example is too simple in the sense that $DS_{can} = \operatorname{im} D = \mathbb{R}^n$ is valid, which happens only for regular index-1 DAEs, if A has full column rank, and D has full row rank. In general, DS_{can} is a time-varying subspace of $\operatorname{im} D$, and the IERODE at the whole does not comprise an exponential dichotomy. Here the question is whether the IERODE shows dichotomic behavior along its (time-varying) invariant subspace DS_{can}. We do not go into more details in this direction.

2.6.4 Characterizing admissible excitations and perturbation index

The fine decoupling of a regular DAE into the IERODE (2.74) and the subsystem (2.75)–(2.78) allows a precise and detailed description of admissible excitations. Remember that the equations (2.75)–(2.78), which means

$$v_0 = -\sum_{l=1}^{\mu-1} \mathcal{N}_{0l}(Dv_l)' - \sum_{l=2}^{\mu-1} \mathcal{M}_{0l}\, v_l - \mathcal{H}_0 D^- u + \mathcal{L}_0 q, \tag{2.100}$$

$$v_i = -\sum_{l=i+1}^{\mu-1} \mathcal{N}_{il}(Dv_l)' - \sum_{l=i+2}^{\mu-1} \mathcal{M}_{il}\, v_l + \mathcal{L}_i q, \quad i = 1,\ldots,\mu-3, \tag{2.101}$$

$$v_{\mu-2} = -\mathcal{N}_{\mu-2,\mu-1}(Dv_{\mu-1})' + \mathcal{L}_{\mu-2} q, \tag{2.102}$$

$$v_{\mu-1} = \mathcal{L}_{\mu-1} q, \tag{2.103}$$

constitute the subsystem (2.62) specified for fine decouplings. We quote once again the coefficients

$$
\begin{aligned}
\mathcal{N}_{01} &:= -Q_0 Q_1 D^-, && \\
\mathcal{N}_{0j} &:= -Q_0 P_1 \cdots P_{j-1} Q_j D^-, && j = 2, \dots, \mu - 1, \\
\mathcal{N}_{i,i+1} &:= -\Pi_{i-1} Q_i Q_{i+1} D^-, && \\
\mathcal{N}_{ij} &:= -\Pi_{i-1} Q_i P_{i+1} \cdots P_{j-1} Q_j D^-, && j = i+2, \dots, \mu - 1,\ i = 1, \dots, \mu - 2, \\
\mathcal{M}_{0j} &:= Q_0 P_1 \cdots P_{\mu-1} \mathcal{M}_j D \Pi_{j-1} Q_j, && j = 1, \dots, \mu - 1, \\
\mathcal{M}_{ij} &:= \Pi_{i-1} Q_i P_{i+1} \cdots P_{\mu-1} \mathcal{M}_j D \Pi_{j-1} Q_j, && j = i+1, \dots, \mu - 1,\ i = 1, \dots, \mu - 2, \\
\mathcal{L}_0 &:= Q_0 P_1 \cdots P_{\mu-1} G_\mu^{-1}, && \\
\mathcal{L}_i &:= \Pi_{i-1} Q_i P_{i+1} \cdots P_{\mu-1} G_\mu^{-1}, && i = 1, \dots, \mu - 2, \\
\mathcal{L}_{\mu-1} &:= \Pi_{\mu-2} Q_{\mu-1} G_\mu^{-1}, && \\
\mathcal{H}_0 &:= Q_0 P_1 \cdots P_{\mu-1} \mathcal{K} \Pi_{\mu-1}. &&
\end{aligned}
$$

For the detailed form of $\mathcal{K}$ and $\mathcal{M}_j$ we refer to (2.54) and (2.55), respectively. All these coefficients are continuous by construction.

The IERODE is solvable for each arbitrary continuous inhomogeneity, therefore, additional smoothness requirements may occur only from the subsystem equations (2.100)–(2.102).

This causes us to introduce the following function space, if $\mu \geq 2$:

$$
\begin{aligned}
\mathcal{C}^{ind\,\mu}(\mathcal{I}, \mathbb{R}^m) := \Big\{ q \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m) : & \\
v_{\mu-1} := \mathcal{L}_{\mu-1} q, \quad & D v_{\mu-1} \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n), \\
v_{\mu-2} := -\mathcal{N}_{\mu-2,\mu-1} (D v_{\mu-1})' + \mathcal{L}_{\mu-2} q, \quad & D v_{\mu-2} \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n), \\
v_i := -\sum_{l=i+1}^{\mu-1} \mathcal{N}_{il} (D v_l)' - \sum_{l=i+2}^{\mu-1} \mathcal{M}_{il}\, v_l + \mathcal{L}_i q, \quad & D v_i \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n), \\
& i = 1, \dots, \mu - 3 \Big\}.
\end{aligned}
\tag{2.104}
$$

Additionally we set for $\mu = 1 : \mathcal{C}^{ind\,1}(\mathcal{I}, \mathbb{R}^m) := \mathcal{C}(\mathcal{I}, \mathbb{R}^m)$.
The function space $\mathcal{C}^{ind\,\mu}(\mathcal{I}, \mathbb{R}^m)$ makes sense unless there are further smoothness assumptions concerning the coefficients. It contains, in particular, all continuous functions q that satisfy the condition $q = G_\mu P_1 \cdots P_{\mu-1} G_\mu^{-1} q$ (cf. Proposition 2.50), which implies $v_1 = 0, \dots, v_{\mu-1} = 0$.
The function space $\mathcal{C}^{ind\,\mu}(\mathcal{I}, \mathbb{R}^m)$ is always a proper subset of the continuous function space $\mathcal{C}(\mathcal{I}, \mathbb{R}^m)$. The particular cases $\mu = 2$ and $\mu = 3$ are described in detail as

$$
\begin{aligned}
\mathcal{C}^{ind\,2}(\mathcal{I}, \mathbb{R}^m) &:= \Big\{ q \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m) : v_1 := \mathcal{L}_1 q,\ D v_1 \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n) \Big\} \\
&= \Big\{ q \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m) : D \Pi_0 Q_1 G_2^{-1} q \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^m) \Big\} = \mathcal{C}^1_{D \Pi_0 Q_1 G_2^{-1}}(\mathcal{I}, \mathbb{R}^m),
\end{aligned}
\tag{2.105}
$$

and

$$\begin{aligned}
\mathcal{C}^{ind\,3}(\mathcal{I},\mathbb{R}^m) := \Big\{ & q \in \mathcal{C}(\mathcal{I},\mathbb{R}^m) : v_2 := \mathcal{L}_2 q,\, Dv_2 \in \mathcal{C}^1(\mathcal{I},\mathbb{R}^n), \\
& v_1 := -\mathcal{N}_{12}(Dv_2)' + \mathcal{L}_1 q,\, Dv_1 \in \mathcal{C}^1(\mathcal{I},\mathbb{R}^n) \Big\} \qquad (2.106) \\
= \Big\{ & q \in \mathcal{C}(\mathcal{I},\mathbb{R}^m) : v_2 := \Pi_1 Q_2 G_3^{-1} q,\, Dv_2 \in \mathcal{C}^1(\mathcal{I},\mathbb{R}^n), \\
& v_1 := \Pi_0 Q_1 Q_2 D^-(Dv_2)' + \Pi_0 Q_1 P_2 G_3^{-1} q,\, Dv_1 \in \mathcal{C}^1(\mathcal{I},\mathbb{R}^n) \Big\}.
\end{aligned}$$

We now introduce the linear operator $L : \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m) \to \mathcal{C}(\mathcal{I},\mathbb{R}^m)$ by means of

$$Lx := A(Dx)' + Bx, \quad x \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m), \qquad (2.107)$$

so that the DAE (2.44) is represented by the operator equation $Lx = q$, and an excitation q is admissible, exactly if it belongs to the range $\operatorname{im} L$ of the operator L.

Proposition 2.58. *If the DAE (2.44) is fine with tractability index $\mu \in \mathbb{N}$, then the linear operator L has the range*

$$\begin{aligned}
&\operatorname{im} L = \mathcal{C}(\mathcal{I},\mathbb{R}^m), && \text{if} \quad \mu = 1, \\
&\operatorname{im} L = \mathcal{C}^{ind\,\mu}(\mathcal{I},\mathbb{R}^m) \subset \mathcal{C}(\mathcal{I},\mathbb{R}^m), && \text{if} \quad \mu \geq 2.
\end{aligned}$$

Proof. The index-1 case is already known from Proposition 2.50 and the definition of L. Turn to the case $\mu \geq 2$. By means of the decoupled version, to each excitation $q \in \mathcal{C}^{ind\,\mu}(\mathcal{I},\mathbb{R}^m)$, we find a solution $x \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$ of the DAE, so that the inclusion $\mathcal{C}^{ind\,\mu}(\mathcal{I},\mathbb{R}^m) \subseteq \operatorname{im} L$ follows. Namely, owing to the properties of q (cf. (2.104)), there is a solution $v_{\mu-1} \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$ of equation (2.103), then a solution $v_{\mu-2} \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$ of (2.102), and solutions $v_i \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$ of (2.101), successively for $i = \mu-3,\ldots,1$. Furthermore, compute a solution u of the IERODE, and v_0 from equation (2.100). Finally put $x := D^- u + v_0 + \cdots + v_{\mu-1}$.
To show the reverse inclusion $\mathcal{C}^{ind\,\mu}(\mathcal{I},\mathbb{R}^m) \supseteq \operatorname{im} L$ we fix an arbitrary $x \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$ and investigate the resulting $q := A(Dx)' + Bx$. We again apply the decoupling. Denote $v_0 := Q_0 x$, and $v_i := \Pi_{i-1} Q_i x$, for $i = 1,\ldots,\mu-1$. Since the projector functions $D\Pi_{i-1} Q_i D^-$, $i = 1,\ldots,\mu-1$, and the function Dx are continuously differentiable, so are the functions $Dv_i = D\Pi_{i-1} Q_i D^- Dx$, $i = 1,\ldots,\mu-1$. Now equation (2.103) yields $v_{\mu-1} := \mathcal{L}_{\mu-1} q \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$, equation (2.102) gives $v_{\mu-2} := -\mathcal{N}_{\mu-2\,\mu-1}(Dv_{\mu-1})' + \mathcal{L}_{\mu-2} q \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$, and so on. □

At this point, the reader's attention should be directed to the fact that the linear function space $\mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$ does not necessarily contain all continuously differentiable functions. For instance, if D is continuous, but fails to be continuously differentiable, then there are constant functions x_{const} such that Dx_{const} fails to be continuously differentiable, and hence x_{const} does not belong to $\mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$. In contrast, if D is continuously differentiable and its nullspace is nontrivial, then the proper inclusion

$$\mathcal{C}^1(\mathcal{I},\mathbb{R}^m) \subset \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$$

is valid. Similar aspects are to be considered if one deals with the space $\mathcal{C}^{ind\,\mu}(\mathcal{I},\mathbb{R}^m)$ comprising the admissible excitations. For $\mu \geq 2$, only if the involved coefficients $\mathcal{L}_i$, $\mathcal{N}_{ij}$ and $\mathcal{M}_{ij}$ are sufficiently smooth, does the inclusion

$$\mathcal{C}^{\mu-1}(\mathcal{I},\mathbb{R}^m) \subset \mathcal{C}^{ind\,\mu}(\mathcal{I},\mathbb{R}^m),$$

hold true. Of course, the index-1 case is simple with

$$\mathcal{C}(\mathcal{I},\mathbb{R}^m) = \mathcal{C}^{ind\,1}(\mathcal{I},\mathbb{R}^m).$$

To achieve more transparent estimates we introduce, for each function w being continuous on $\mathcal{I}$ and $t_0, t_1 \in \mathcal{I}$, $t_0 < t_1$, the expression

$$\|w\|_\infty^{[t_0,t_1]} := \max_{t_0 \leq \tau \leq t_1} |w(\tau)|,$$

which is the maximum-norm related to the compact interval $[t_0,t_1]$. Moreover, for $q \in \mathcal{C}^{ind\,\mu}(\mathcal{I},\mathbb{R}^m)$ and $t_0, t_1 \in \mathcal{I}$, $t_0 < t_1$, we introduce

$$\|q\|_{ind\,\mu}^{[t_0,t_1]} := \|q\|_\infty^{[t_0,t_1]} + \|(Dv_{\mu-1})'\|_\infty^{[t_0,t_1]} + \cdots + \|(Dv_1)'\|_\infty^{[t_0,t_1]},$$

which means for the special cases $\mu = 2$ and $\mu = 3$:

$$\begin{aligned}
\|q\|_{ind\,2}^{[t_0,t_1]} &:= \|q\|_\infty^{[t_0,t_1]} + \|(Dv_1)'\|_\infty^{[t_0,t_1]} = \|q\|_\infty^{[t_0,t_1]} + \|(D\Pi_0 Q_1 G_2^{-1} q)'\|_\infty^{[t_0,t_1]},\\
\|q\|_{ind\,3}^{[t_0,t_1]} &:= \|q\|_\infty^{[t_0,t_1]} + \|(Dv_2)'\|_\infty^{[t_0,t_1]} + \|(Dv_1)'\|_\infty^{[t_0,t_1]}\\
&= \|q\|_\infty^{[t_0,t_1]} + \|(D\Pi_1 Q_2 G_3^{-1} q)'\|_\infty^{[t_0,t_1]}\\
&\qquad + \|(D\Pi_0 Q_1 Q_2 D^-(D\Pi_1 Q_2 G_3^{-1} q)' + D\Pi_0 Q_1 P_2 G_3^{-1} q)'\|_\infty^{[t_0,t_1]}.
\end{aligned}$$

Theorem 2.59. *Let the DAE (2.44) be fine with tractability index $\mu \in \mathbb{N}$. Let $t_0 \in \mathcal{I}$ and let C be a matrix such that* $\ker C = N_{can}(t_0)$. *Let the compact interval $[t_0,\bar{t}] \subseteq \mathcal{I}$ be fixed. Then the following assertions are true:*

(1) *The excitation q is admissible, if and only if it belongs to $\mathcal{C}^{ind\,\mu}(\mathcal{I},\mathbb{R}^m)$.*

(2) *For each pair $q \in \mathcal{C}^{ind\,\mu}(\mathcal{I},\mathbb{R}^m)$, $x^0 \in \mathbb{R}^m$, the solution $x \in \mathcal{C}_D^1(\mathcal{I},\mathbb{R}^m)$ of the IVP*

$$A(Dx)' + Bx = q, \quad C(x(t_0) - x^0) = 0, \tag{2.108}$$

satisfies the inequality

$$|x(t)| \leq \|x\|_\infty^{[t_0,t]} \leq c\left\{|\Pi_{can}(t_0)x^0| + \|q\|_{ind\,\mu}^{[t_0,t]}\right\}, \quad t_0 \leq t \leq \bar{t}, \tag{2.109}$$

whereby the constant c depends only on the interval.

(3) *If the DAE coefficients are so smooth that $\mathcal{C}^{\mu-1}(\mathcal{I},\mathbb{R}^m) \subset \mathcal{C}^{ind\,\mu}(\mathcal{I},\mathbb{R}^m)$, and*

$$\|q\|_{ind\ \mu}^{[t_0,t]} \le c_0\Big\{\|q\|_\infty^{[t_0,t]} + \sum_{l=1}^{\mu-1}\|q^{(l)}\|_\infty^{[t_0,t]}\Big\},\ \textit{for}\ q\in\mathcal{C}^{\mu-1}(\mathcal{I},\mathbb{R}^m),$$

then, for each pair $q\in\mathcal{C}^{\mu-1}(\mathcal{I},\mathbb{R}^m)$, $x^0\in\mathbb{R}^m$, *it holds that*

$$\|x\|_\infty^{[t_0,t]} \le K\Big\{|\Pi_{can}(t_0)x^0| + \|q\|_\infty^{[t_0,t]} + \sum_{l=1}^{\mu-1}\|q^{(l)}\|_\infty^{[t_0,t]}\Big\}. \tag{2.110}$$

Proof. (1) is a consequence of Proposition 2.58, and (3) results from (2). It remains to verify (2). We apply the solution representation (2.96). First we consider the function v defined by (2.95), for a given $q\in\mathcal{C}^{ind\ \mu}(\mathcal{I},\mathbb{R}^m)$. One has in detail

$$v_{\mu-1} = \mathcal{L}_{\mu-1}q, \quad \text{thus} \quad \|v_{\mu-1}\|_\infty^{[t_0,t]} \le \bar c_{\mu-1}\|q\|_{ind\ \mu}^{[t_0,t]},$$
$$v_{\mu-2} = \mathcal{L}_{\mu-2}q - \mathcal{N}_{\mu-2\,\mu-1}(Dv_{\mu-1})', \quad \text{thus} \quad \|v_{\mu-2}\|_\infty^{[t_0,t]} \le \bar c_{\mu-2}\|q\|_{ind\ \mu}^{[t_0,t]},$$

and so on, such that

$$\|v_i\|_\infty^{[t_0,t]} \le \bar c_i\|q\|_{ind\ \mu}^{[t_0,t]}, \quad i=\mu-3,\dots,1,$$

with certain constants $\bar c_i$. Then, with a suitable constant $\bar c$, it follows that

$$\|v\|_\infty^{[t_0,t]} \le \bar c\|q\|_{ind\ \mu}^{[t_0,t]}.$$

Now the representation (2.96) leads to the inequality

$$|x(t)| \le \|x\|_\infty^{[t_0,t]} \le c_1\ |\Pi_{can}(t_0)x^0| + c_2\ \|q\|_\infty^{[t_0,t]} + \|q\|_{ind\ \mu}^{[t_0,t]}, \quad t_0\le t\le \bar t,$$

with c_1 being a bound of the fundamental solution matrix $X(t,t_0)$, $c_3 := \bar c$ and c_2 resulting as a bound of the term $X(t,t_0)X(s,t_0)^- G_\mu^{-1}(s)$, whereby s varies between t_0 and t. We finish the proof by letting $c := \max\{c_1, c_2+c_3\}$. □

The inequality (2.110) suggests that the DAE has so-called *perturbation index* μ (cf. [103, 105]). The concept of perturbation index interprets the index as a measure of sensitivity of the solution with respect to perturbations of the given problem. Applied to our DAE (2.44), the definition ([105, page 478]) becomes:

Definition 2.60. Equation (2.44) has *perturbation index* μ_p along a solution x_* on the interval $[t_0,\bar t]$, if μ_p is the smallest integer such that, for all functions $\tilde x$ having a defect

$$A(D\tilde x)' + B\tilde x - q = \delta$$

there exists on $[t_0,\bar t]$ an estimate

$$|\tilde x(t) - x_*(t)| \le C\{|\tilde x(t_0) - x_*(t_0)| + \|\delta\|_\infty^{[t_0,t]} + \cdots + \|\delta^{(\mu_p-1)}\|_\infty^{[t_0,t]}\},$$

whenever the expression on the right-hand side is sufficiently small.

Owing to the linearity, the DAE (2.44) has perturbation index μ_p (along each solution) on the interval $[t_0,\bar{t}]$, if for all functions $x = \tilde{x} - x_*$ having a defect $A(Dx)' + Bx = \delta$ an estimate

$$|x(t)| \le C\{|x(t_0)| + \|\delta\|_\infty^{[t_0,t]} + \cdots + \|\delta^{(\mu_p-1)}\|_\infty^{[t_0,t]}\}, \tag{2.111}$$

is valid.
The definition of the perturbation index does not specify function classes meant for the solutions and defects, but obviously one has to suppose $\delta \in \mathcal{C}^{\mu_p-1}$, such that the notion applies to sufficiently smooth problems only. In fact, the required estimate (2.111) corresponds to the inequality (2.110), which is available for smooth problems only. Therefore, we observe that a fine DAE with tractability index μ and sufficiently smooth coefficients has at the same time perturbation index μ.

All in all, the solution $x = x(x^0,q)$ of the IVP (2.108) depends on the value x^0 as well as on the function q. It is shown that x varies smoothly with x^0 such that, concerning this aspect, the DAE solutions are close to the ODE solutions. However, solutions of higher index DAEs show an ambivalent character. With respect to their variable q they are essentially ill-posed. More precisely, the linear operator $L : \mathcal{C}_D^1(\mathcal{I},\mathbb{R}^m) \to \mathcal{C}(\mathcal{I},\mathbb{R}^m)$ described in (2.107) has the range $\operatorname{im} L = \mathcal{C}^{ind\,\mu}(\mathcal{I},\mathbb{R}^m)$ which is a proper nonclosed subset in $\mathcal{C}(\mathcal{I},\mathbb{R}^m)$, if $\mu \ge 2$. This makes the IVP (2.108) essentially ill-posed with respect to the excitations q. We recall of Example 1.5 which clearly shows this ill-posed character.

2.7 Specifications for regular standard form DAEs

At present, most of the literature on DAEs is devoted to standard form DAEs

$$E(t)x'(t) + F(t)x(t) = q(t), \quad t \in \mathcal{I}, \tag{2.112}$$

where E and F are smooth square matrix functions. Here we assume $E(t)$ to have constant rank on the given interval whereas points at which $E(t)$ change its rank are considered to be critical.
As proposed in [96], one can treat (2.112) as

$$E(t)(P(t)x(t))' + (F(t) - E(t)P'(t))x(t) = q(t), \quad t \in \mathcal{I}, \tag{2.113}$$

by means of a continuously differentiable projector function P such that $\ker P = \ker E$. The DAE (2.113) has a properly stated leading term, and all results of the previous sections apply. In particular, we build the matrix function sequence beginning with

$$A := E,\ D := P,\ R = P,\ B := F - EP',\ G_0 = E,\ B_0 := B,$$

develop decouplings, etc. However, now the new question arises: which effects are caused by a change from one projector function P to another one? Clearly, the matrix function sequence depends on the projector function P.
Suppose P and $\tilde{P}$ to be two continuously differentiable projector functions such that

$$\ker E = \ker P = \ker \tilde{P}.$$

Besides (2.113) we consider

$$E(t)(\tilde{P}(t)x(t))' + (F(t) - E(t)\tilde{P}'(t))x(t) = q(t), \quad t \in \mathcal{I}. \tag{2.114}$$

Proposition 2.22 guarantees that the function spaces $\mathcal{C}^1_P(\mathcal{I}, \mathbb{R}^m)$ and $\mathcal{C}^1_{\tilde{P}}(\mathcal{I}, \mathbb{R}^m)$ coincide. Furthermore, the DAE (2.114) results from the DAE (2.113) by a *refactorization of the leading term*. Namely, set

$$A := E,\ D := P,\ R := P,\ B := F - EP', \quad \text{and} \quad H := \tilde{P},\ H^- := \tilde{P}.$$

Then, condition (2.27) is satisfied with $RHH^-R = P\tilde{P}P = P = R$, and the refactorized DAE (2.28) coincides with (2.114) because of (cf. (2.29))

$$\begin{aligned} \bar{A} &= AH = E\tilde{P} = E, & \bar{D} &= H^-D = \tilde{P}P = \tilde{P}, \\ \bar{B} &= B - ARH(H^-R)'D = F - EP' - E\tilde{P}'P \\ &= F - E\tilde{P}P' - E\tilde{P}'P = F - E(\tilde{P}P)' \\ &= F - E\tilde{P}'. \end{aligned}$$

In consequence, by Theorem 2.21 on refactorizations, the subspaces $\operatorname{im} G_i$, S_i, and $N_0 + \cdots + N_i$, as well as the characteristic values r_i, are independent of the special choice of P. This justifies the following regularity notion for standard form DAEs which traces the problem back to Definition 2.25 for DAEs with properly stated leading terms.

Definition 2.61. The standard form DAE (2.112) is *regular with tractability index* μ, if the properly stated version (2.113) is so for one (or, equivalently, for each) continuously differentiable projector function P with $\ker P = \ker E$.
The characteristic values of (2.113) are named *characteristic values* of (2.112).
The canonical subspaces S_{can} and N_{can} of (2.113) are called *canonical subspaces* of (2.112).

While the canonical subspaces S_{can} and N_{can} are independent of the special choice of P, the IERODE resulting from (2.113) obviously depends on P:

$$u' - (P\Pi_{\mu-1})'u + P\Pi_{\mu-1}G_\mu^{-1}Bu = P\Pi_{\mu-1}G_\mu^{-1}q, \quad u \in \operatorname{im} P\Pi_{\mu-1}. \tag{2.115}$$

This is a natural consequence of the standard formulation.

When dealing with standard form DAEs, the choice $P_0 := P$, $D^- = P$ suggests itself to begin the matrix function sequence with. In fact, this is done in the related previous work. Then the accordingly specialized sequence is

$$\begin{aligned} G_0 &= E, & B_0 &= F - EP_0' = F - G_0\Pi_0', \\ G_{i+1} &= G_i + B_iQ_i, & B_{i+1} &= B_iP_i - G_{i+1}P_0\Pi_{i+1}'\Pi_i, \quad i \geq 0. \end{aligned} \tag{2.116}$$

In this context, the projector functions $Q_0, \dots, Q_\kappa$ are *regular admissible*, if

(a) the projector functions $G_0, \dots, G_\kappa$ have constant ranks,
(b) the relations $Q_iQ_j = 0$ are valid for $j = 0, \dots, i-1$, $i = 1, \dots, \kappa$,
(c) and $\Pi_0, \dots, \Pi_\kappa$ are continuously differentiable.

Then, it holds that $P\Pi_i = \Pi_i$, and the IERODE of a regular DAE (2.112) is

$$u' - \Pi_{\mu-1}'u + \Pi_{\mu-1}G_\mu^{-1}Bu = \Pi_{\mu-1}G_\mu^{-1}q, \quad u \in \operatorname{im}\Pi_{\mu-1}. \tag{2.117}$$

In previous papers exclusively devoted to regular DAEs, some higher smoothness is supposed for Q_i, and these projector functions are simply called admissible, without the addendum *regular*. A detailed description of the decoupling supported by the specialized matrix function (2.116) can be found in [194].

Remark 2.62. In earlier papers (e.g., [157], [159], [111], [160]) the matrix function sequence

$$G_{i+1} = G_i + B_iQ_i, \quad B_{i+1} = B_iP_i - G_{i+1}\Pi_{i+1}'\Pi_i, \quad i \geq 0, \tag{2.118}$$

is used, which is slightly different from (2.116). While [157], [159] provide solvability results and decouplings for regular index-2 and index-3 DAEs, [111] deserves attention in proving the invariance of the tractability index $\mu \in \mathbb{N}$ with respect to transformations (see also [160], but notice that, unfortunately, there is a misleading misprint in the sequence on page 158). In these earlier papers the famous role of the sum spaces $N_0 + \cdots + N_i$ was not yet discovered, so that the reasoning is less transparent and needs patient readers.
In [167, Remark 2.6] it is thought that the sequence (2.116) coincides with the sequence (2.118); however this is not fully correct. Because of

$$\begin{aligned} B_{i+1} &= B_iP_i - G_{i+1}P_0\Pi_{i+1}'\Pi_i = B_iP_i - G_{i+1}\Pi_{i+1}'\Pi_i + G_{i+1}Q_0\underbrace{\Pi_{i+1}'}_{(P_0\Pi_{i+1})'}\Pi_i \\ &= B_iP_i - G_{i+1}\Pi_{i+1}'\Pi_i + G_{i+1}Q_0P_0'\Pi_{i+1}, \end{aligned}$$

both matrix function sequences in fact coincide, if $Q_0P_0' = 0$. One can always arrange that $Q_0P_0' = 0$ is locally valid. Namely, for each fixed $t_* \in \mathcal{I}$, we find a neighborhood $\mathcal{N}_{t_*}$ such that $\ker E(t) \oplus \ker E(t_*)^\perp = \mathbb{R}^m$ holds true for all $t \in \mathcal{N}_{t_*}$. The projector function Q_0 onto $\ker E(t)$ along $\ker E(t_*)^\perp$ has the required property

$$Q_0P_0' = Q_0(P_0(t_*)P_0)' = Q_0P_0(t_*)P_0' = 0.$$

Owing to the independence of the choice of the projector function $P_0 = P$, the regularity notions for (2.112), defined by means of (2.116) or by (2.118), are actually

consistent, and the sum subspaces, the canonical subspaces, and the characteristic values are precisely the same.
Several papers on lower index DAEs use subspace properties rather than rank conditions for the index definition. For instance, in [163], an index-2 tractable DAE is characterized by a constant-dimensional nontrivial nullspace N_1, together with the transversality condition $N_1 \oplus S_1 = \mathbb{R}^m$. Owing to Lemma A.9, this is equivalent to the condition for G_1 to have constant rank lower than m, and the requirement for G_2 to remain nonsingular.

Theorem 2.63. *Let the DAE (2.112) be regular with tractability index μ and fine. Let the matrix $C \in L(\mathbb{R}^m, \mathbb{R}^s)$ be such that* $\ker C = N_{can}(t_0)$.

(1) *Then, the IVP*

$$Ex' + Fx = 0, \quad Cx(t_0) = 0,$$

has the zero solution only.

(2) *For each admissible excitation q, and each $x^0 \in \mathbb{R}^m$, the IVP*

$$Ex' + Fx = q, \quad C(x(t_0) - x^0) = 0,$$

has exactly one solution in $\mathcal{C}^1_P(\mathcal{I}.\mathbb{R}^m)$.

(3) *For each given admissible excitation q, the set of consistent initial values at time t_0 is*

$$\mathcal{M}_{can,q}(t_0) = \{z + v(t_0) : z \in S_{can}(t_0)\},$$

whereby v is constructed as in (2.95) by means of fine decoupling projector functions.

(4) *If the coefficients of the DAE are sufficiently smooth, then each $q \in \mathcal{C}^{\mu-1}(\mathcal{I}, \mathbb{R}^m)$ is admissible. If the interval $\mathcal{I}$ is compact, then for the IVP solution from (2), the inequality*

$$\|x\| \le K\Big(|\Pi_{can}(t_0)x^0| + \|q\|_\infty + \sum_{l=1}^{\mu-1} \|q^{(l)}\|_\infty\Big) \tag{2.119}$$

is valid with a constant K independent of q and x^0.

Proof. (1) and (2) are consequences of Theorem 2.44(2) and Theorem 2.52(1), respectively. Assertion (4) follows from Theorem 2.59(3). Assertion (3) results from the representations (2.95) and (2.98), with $D = D^- = P$. □

The inequality (2.119) indicates that the DAE has *perturbation index* μ (cf. Definition 2.60).

2.8 The T-canonical form

Definition 2.64. The structured continuous coefficient DAE with properly stated leading term

$$\left[\begin{array}{c|cccc} I_d & & & & \\ \hline & 0 & \tilde{\mathcal{N}}_{0,1} & \cdots & \tilde{\mathcal{N}}_{0,\mu-1} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \tilde{\mathcal{N}}_{\mu-2,\mu-1} \\ & & & & 0 \end{array}\right] \left(\left[\begin{array}{c|cccc} I_d & & & & \\ \hline & 0 & & & \\ & & I_{m-r_1} & & \\ & & & \ddots & \\ & & & & \ddots \\ & & & & I_{m-r_{\mu-1}} \end{array}\right] \tilde{x} \right)' \tag{2.120}$$

$$+ \left[\begin{array}{c|cccc} \tilde{\mathcal{W}} & & & & \\ \hline \tilde{\mathcal{H}}_0 & I_{m-r_0} & & & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ \tilde{\mathcal{H}}_{\mu-1} & & & & I_{m-r_{\mu-1}} \end{array}\right] \tilde{x} = \tilde{q},$$

$m = d + \sum_{j=0}^{\mu-1}(m-r_j)$, as well as its counterpart in standard form

$$\begin{bmatrix} I_d & 0 \\ 0 & \tilde{\mathcal{N}} \end{bmatrix} \tilde{x}' + \begin{bmatrix} \tilde{\mathcal{W}} & 0 \\ \tilde{\mathcal{H}} & I_{m-d} \end{bmatrix} \tilde{x} = \tilde{q}, \tag{2.121}$$

with

$$\tilde{\mathcal{N}} = \begin{bmatrix} 0 & \tilde{\mathcal{N}}_{0,1} & \cdots & \tilde{\mathcal{N}}_{0,\mu-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \tilde{\mathcal{N}}_{\mu-2,\mu-1} \\ & & & 0 \end{bmatrix},$$

are said to be in *T(ractability)-canonical form*, if the entries $\tilde{\mathcal{N}}_{0,1}, \ldots, \tilde{\mathcal{N}}_{\mu-2,\mu-1}$ are full column rank matrix functions, that is $\operatorname{rank} \mathcal{N}_{i-1,i} = m - r_i$, for $i = 1, \ldots, \mu-1$.

The subscript μ indicates the tractability index μ, and at the same time the uniform nilpotency index of the upper block triangular matrix function $\tilde{\mathcal{N}}$. $\tilde{\mathcal{N}}^{\mu}$ vanishes identically, and $\tilde{\mathcal{N}}^{\mu-1}$ has the only nontrivial entry $\tilde{\mathcal{N}}_{0,1}\tilde{\mathcal{N}}_{1,2}\cdots\tilde{\mathcal{N}}_{\mu-2,\mu-1}$ of rank $m - r_{\mu-1}$ in the upper right corner. If the coefficients $\tilde{\mathcal{H}}_0, \ldots, \tilde{\mathcal{H}}_{\mu-1}$ vanish, the T-canonical form (2.121) looks precisely like the Weierstraß–Kronecker canonical form for constant matrix pencils.

Generalizing Proposition 1.28, we show that a DAE (2.44) is regular with tractability index μ if and only if it can be brought into T-canonical form by a regular multiplication, a regular transformations of the unknown function, and a refactorization of the leading term as described in Section 2.3. This justifies the attribute *canonical*. The structural sizes $r_0, \ldots, r_{\mu-1}$ coincide with the characteristic values from the tractability index framework.

Theorem 2.65. (1) *The DAE (2.44) is regular with tractability index μ and characteristic values $r_0 \le \cdots \le r_{\mu-1} < r_\mu = m$, if and only if there are pointwise* *regular matrix functions $L,K \in \mathcal{C}(\mathcal{I},L(\mathbb{R}^m))$, and a constant-rank refactorization matrix function $H \in \mathcal{C}^1(\mathcal{I},L(\mathbb{R}^s,\mathbb{R}^n))$, $RHH^-R = R$, such that premultiplication by L, the transformation $x = K\tilde{x}$, and the refactorization of the leading term by H yield a DAE in T-canonical form, whereby the entry $\tilde{\mathcal{N}}_{i-1,i}$ has size $(m-r_{i-1}) \times (m-r_i)$ and*

$$\operatorname{rank} \tilde{\mathcal{N}}_{i-1,i} = m - r_i, \quad \textit{for} \quad i = 1,\ldots,\mu-1.$$

(2) *If the DAE (2.44) is regular with tractability index μ, and its coefficients are smooth enough for the existence of completely decoupling projector functions, then the DAE is equivalent to a T-canonical form with zero coupling coefficients $\tilde{\mathcal{H}}_0,\ldots,\tilde{\mathcal{H}}_{\mu-1}$.*

Proof. (1) If the DAE has T-canonical form, one can construct a matrix function sequence and admissible projector functions in the same way as described in Subsection 1.2.6 for constant matrix pencils, and this shows regularity and confirms the characteristic values.

The reverse implication is more difficult. Let the DAE (2.44) be regular with tractability index μ and characteristic values $r_0 \le \cdots \le r_{\mu-1} < r_\mu = m$. Let $Q_0,\ldots,Q_{\mu-1}$ be admissible projector functions. As explained in Subsection 2.4.2, the DAE decomposes into equation (2.49) being a pre-version of the IERODE and subsystem (2.63) together

$$\underbrace{\begin{bmatrix} D\Pi_{\mu-1}D^- & 0 \\ 0 & \mathcal{N} \end{bmatrix}}_{\mathfrak{A}} \Bigg(\underbrace{\begin{bmatrix} D\Pi_{\mu-1}D^- & 0 \\ 0 & \mathcal{D} \end{bmatrix}}_{\mathfrak{D}} \begin{bmatrix} u \\ v \end{bmatrix} \Bigg)' + \underbrace{\begin{bmatrix} \mathcal{W} & 0 \\ \mathcal{H}D^- & \mathcal{M} \end{bmatrix}}_{\mathfrak{B}} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \mathcal{L}_d \\ \mathcal{L} \end{bmatrix} q. \tag{2.122}$$

This is an inflated system in $\mathbb{R}^{m(\mu+1)}$, with $\mathcal{W} := D\Pi_{\mu-1}G_\mu^{-1}BD^-$, further coefficients given in Subsection 2.4.2, and the unknown functions

$$\begin{bmatrix} u \\ v \end{bmatrix} := \begin{bmatrix} u \\ v_0 \\ \vdots \\ v_{\mu-1} \end{bmatrix} := \begin{bmatrix} D\Pi_{\mu-1} \\ Q_0 \\ \Pi_0 Q_1 \\ \vdots \\ \Pi_{\mu-2}Q_{\mu-1} \end{bmatrix} x.$$

We condense this inflated system back to $\mathbb{R}^m$ in a similar way as in Proposition 1.28. The projector functions $D\Pi_{\mu-1}D^-$ and $D\Pi_{i-1}Q_iD^-$ are continuously differentiable, and so are their ranges and nullspaces. The $\mathcal{C}^1$-subspace $\operatorname{im}(D\Pi_{\mu-1}D^-)^*$ has dimension $d = m - \sum_{i=0}^{\mu-1}(m-r_i)$, and it is spanned by continuously differentiable basis functions, which means that there is a matrix function $\Gamma_d \in \mathcal{C}^1(\mathcal{I},L(\mathbb{R}^n,\mathbb{R}^d))$ such that

$$\operatorname{im}(D\Pi_{\mu-1}D^-)^* = \operatorname{im}\Gamma_d^*, \quad \ker\Gamma_d^* = \{0\},$$

and hence

$$\operatorname{im}\Gamma_d = \mathbb{R}^d, \quad \ker\Gamma_d = (\operatorname{im}(D\Pi_{\mu-1}D^-)^*)^\perp = \ker D\Pi_{\mu-1}D^-.$$

By Proposition A.17, there is a pointwise reflexive generalized inverse $\Gamma_d^- \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^d, \mathbb{R}^n))$ such that $\Gamma_d\Gamma_d^- = I_d$ and $\Gamma_d^-\Gamma_d = D\Pi_{\mu-1}D^-$. Analogously we find $\Gamma_i \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^n, \mathbb{R}^{m-r_i}))$ and $\Gamma_i^- \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^{m-r_i}, \mathbb{R}^n))$ such that for $i = 1, \ldots, \mu-1$

$$\operatorname{im}\Gamma_i = \mathbb{R}^{m-r_i}, \quad \ker\Gamma_i = \ker D\Pi_{i-1}Q_iD^-, \quad \Gamma_i\Gamma_i^- = I_{m-r_i}, \quad \Gamma_i^-\Gamma_i = D\Pi_{i-1}Q_iD^-.$$

This implies

$$\Gamma_iD = \Gamma_iD\Pi_{i-1}Q_i, \quad D^-\Gamma_i^- = \Pi_{i-1}Q_iD^-\Gamma_i^-, \quad \Gamma_iDD^-\Gamma_i^- = \Gamma_i\Gamma_i^- = I_{m-r_i}.$$

Finally we provide $\Gamma_0 \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^m, \mathbb{R}^{m-r_0}))$ and $\Gamma_0^- \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^{m-r_0}, \mathbb{R}^m))$ such that

$$\operatorname{im}\Gamma_0 = \mathbb{R}^{m-r_0}, \quad \ker\Gamma_0 = \ker Q_0, \quad \Gamma_0\Gamma_0^- = I_{m-r_0}, \quad \Gamma_0^-\Gamma_0 = Q_0.$$

Then we compose

$$\Gamma := \begin{bmatrix} \Gamma_d & \\ & \Gamma_{sub} \end{bmatrix}, \quad \Gamma^- := \begin{bmatrix} \Gamma_d^- & \\ & \Gamma_{sub}^- \end{bmatrix},$$

$$\Gamma_{sub} := \begin{bmatrix} \Gamma_0 & & & \\ & \Gamma_1D & & \\ & & \ddots & \\ & & & \Gamma_{\mu-1}D \end{bmatrix}, \quad \Gamma_{sub}^- := \begin{bmatrix} \Gamma_0^- & & & \\ & D^-\Gamma_1^- & & \\ & & \ddots & \\ & & & D^-\Gamma_{\mu-1}^- \end{bmatrix}$$

such that $\Gamma\Gamma^- = I_m, \quad \Gamma_{sub}\Gamma_{sub}^- = I_{m-d}$, and

$$\Gamma^-\Gamma = \begin{bmatrix} D\Pi_{\mu-1}D^- & & & & \\ & Q_0 & & & \\ & & \Pi_0Q_1 & & \\ & & & \ddots & \\ & & & & \Pi_{\mu-2}Q_{\mu-1} \end{bmatrix},$$

$$\Gamma_{sub}^-\Gamma_{sub} = \begin{bmatrix} Q_0 & & & \\ & \Pi_0Q_1 & & \\ & & \ddots & \\ & & & \Pi_{\mu-2}Q_{\mu-1} \end{bmatrix}.$$

Additionally we introduce

$$\Omega := \begin{bmatrix} 0 & & & \\ & \Gamma_1 & & \\ & & \ddots & \\ & & & \Gamma_{\mu-1} \end{bmatrix}, \quad \Omega^- := \begin{bmatrix} 0 & & & \\ & \Gamma_1^- & & \\ & & \ddots & \\ & & & \Gamma_{\mu-1}^- \end{bmatrix},$$

such that

$$\Omega^-\Omega = \begin{bmatrix} 0 & & & \\ & D\Pi_0 Q_1 D^- & & \\ & & \ddots & \\ & & & D\Pi_{\mu-2}Q_{\mu-1}D^- \end{bmatrix}, \quad \Omega\Omega^- = \begin{bmatrix} 0 & & & \\ & I_{m-r_1} & & \\ & & \ddots & \\ & & & I_{m-r_{\mu-1}} \end{bmatrix}.$$

For the coefficients of the inflated system (2.122) it follows that

$$\Gamma_{sub}^-\Gamma_{sub}\,\mathcal{N} = \mathcal{N}\Omega^-\Omega = \mathcal{N}, \quad \Gamma_{sub}^-\Gamma_{sub}\mathcal{M} = \mathcal{M}\Gamma_{sub}^-\Gamma_{sub}, \quad \mathcal{D} = \Omega^-\Gamma_{sub},$$

and further

$$\Gamma\mathfrak{A} = \begin{bmatrix} \Gamma_d D\Pi_{\mu-1}D^- & \\ & \Gamma_{sub}\mathcal{N} \end{bmatrix} = \begin{bmatrix} \Gamma_d & \\ & \Gamma_{sub}\mathcal{N}\Omega^-\Omega \end{bmatrix} = \begin{bmatrix} I_d & \\ & \Gamma_{sub}\mathcal{N}\Omega^- \end{bmatrix}\begin{bmatrix} \Gamma_d & \\ & \Omega \end{bmatrix},$$

$$\Gamma\mathfrak{B} = \begin{bmatrix} \Gamma_d\mathcal{W} & 0 \\ \Gamma_{sub}\mathcal{H}D^- & \Gamma_{sub}\mathcal{M} \end{bmatrix} = \begin{bmatrix} \Gamma_d\mathcal{W}\Gamma_d^-\Gamma_d & 0 \\ \Gamma_{sub}\mathcal{H}D^-\Gamma_d^-\Gamma_d & \Gamma_{sub}\mathcal{M}\Gamma_{sub}^-\Gamma_{sub} \end{bmatrix}$$

$$= \begin{bmatrix} \Gamma_d\mathcal{W}\Gamma_d^- & 0 \\ \Gamma_{sub}\mathcal{H}D^-\Gamma^- & \Gamma_{sub}\mathcal{M}\Gamma_{sub} \end{bmatrix}\begin{bmatrix} \Gamma_d & 0 \\ 0 & \Gamma_{sub} \end{bmatrix},$$

$$\mathfrak{D} = \begin{bmatrix} \Gamma_d^-\Gamma_d & 0 \\ 0 & \Omega^-\Gamma_{sub} \end{bmatrix} = \begin{bmatrix} \Gamma_d^- & 0 \\ 0 & \Omega^- \end{bmatrix}\begin{bmatrix} \Gamma_d & 0 \\ 0 & \Gamma_{sub} \end{bmatrix}.$$

Multiplying the inflated system (2.122) by the condensing matrix function Γ and introducing the new variables

$$\tilde{x} := \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} := \begin{bmatrix} \Gamma_d & 0 \\ 0 & \Gamma_{sub} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

gives

$$\underbrace{\begin{bmatrix} I & 0 \\ 0 & \Gamma_{sub}\mathcal{N}\Omega^- \end{bmatrix} \begin{bmatrix} \Gamma_d & 0 \\ 0 & \Omega \end{bmatrix}}_{\bar{A}} \Bigg(\underbrace{\begin{bmatrix} \Gamma_d^- & 0 \\ 0 & \Omega^- \end{bmatrix}}_{\bar{D}} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} \Bigg)'$$
$$+ \underbrace{\begin{bmatrix} \Gamma_d \mathcal{W} \Gamma_d^- & 0 \\ \Gamma_{sub}\mathcal{H}D^-\Gamma_d^- & \Gamma_{sub}\mathcal{M}\Gamma_{sub}^- \end{bmatrix}}_{\bar{B}} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} = \underbrace{\Gamma \begin{bmatrix} \mathcal{L}_d \\ \mathcal{L} \end{bmatrix}}_{\bar{L}} q.$$

This last DAE lives in $\mathbb{R}^m$, but the border space of its leading term is $\mathbb{R}^{n(\mu+1)}$. Because of

$$\ker \bar{A} = \ker \begin{bmatrix} \Gamma_d & 0 \\ 0 & \Omega \end{bmatrix} = \ker \bar{R}, \quad \operatorname{im} \bar{D} = \operatorname{im} \bar{R},$$

with the border projector $\bar{R} = \begin{bmatrix} D\Pi_{\mu-1}D^- & 0 \\ 0 & \Omega^-\Omega \end{bmatrix}$ the refactorization of the leading term (cf. Section 2.3) by means of

$$H := \begin{bmatrix} \Gamma_d^- & 0 \\ 0 & \Omega^- \end{bmatrix}, \quad H^- = \begin{bmatrix} \Gamma_d & 0 \\ 0 & \Omega \end{bmatrix}$$

suggests itself. H has constant rank d, and H^- is the reflexive generalized inverse with

$$H^-H = \begin{bmatrix} I_d & 0 \\ 0 & \Omega\Omega^- \end{bmatrix}, HH^- = \begin{bmatrix} D\Pi_{\mu-1}D^- & 0 \\ 0 & \Omega^-\Omega \end{bmatrix} = \bar{R},\ \bar{R}HH^-\bar{R} = \bar{R}.$$

This way we arrive at the DAE

$$\tilde{A}(\tilde{D}\tilde{x})' + \tilde{B}\tilde{x} = \bar{L}q,$$
$$\tilde{A} := \begin{bmatrix} I & 0 \\ 0 & \Gamma_{sub}\mathcal{N}\Omega^- \end{bmatrix}, \tilde{D} := \begin{bmatrix} I & 0 \\ 0 & \Omega\Omega^- \end{bmatrix}, \tilde{B} := \begin{bmatrix} \Gamma_d\mathcal{W}\Gamma_d^- - \Gamma_d'\Gamma_d^- & 0 \\ \Gamma_{sub}\mathcal{H}D^-\Gamma_d^- & \tilde{B}_{22} \end{bmatrix}.$$

The entry

$$\begin{aligned} \tilde{B}_{22} :&= \Gamma_{sub}\mathcal{M}\Gamma_{sub}^- - \Gamma_{sub}\mathcal{N}\Omega^-\Omega'\Omega^- \\ &= \Gamma_{sub}\Gamma_{sub}^- + \Gamma_{sub}(\mathcal{M} - I)\Gamma_{sub} - \Gamma_{sub}\mathcal{N}\Omega^-\Omega'\Omega^- =: I + \tilde{\mathcal{M}} \end{aligned}$$

has upper block triangular form, with identity diagonal blocks. $\tilde{\mathcal{M}}$ is strictly upper block triangular, and $I + \tilde{\mathcal{M}}$ remains nonsingular. Scaling the DAE by $\operatorname{diag}(I, (I+\tilde{\mathcal{M}})^{-1})$ yields

$$\begin{bmatrix} I & 0 \\ 0 & \tilde{\mathcal{N}} \end{bmatrix} \Big(\begin{bmatrix} I & 0 \\ 0 & \Omega\Omega^- \end{bmatrix} \tilde{x} \Big)' + \begin{bmatrix} \tilde{\mathcal{W}} & 0 \\ \tilde{\mathcal{H}} & I \end{bmatrix} \tilde{x} = \begin{bmatrix} I & 0 \\ 0 & (I+\tilde{\mathcal{M}})^{-1} \end{bmatrix} \bar{L}q, \tag{2.123}$$

with coefficients

$$\tilde{\mathcal{N}} := (I+\tilde{\mathcal{M}})^{-1}\Gamma_{sub}\mathcal{N}\Omega^{-}, \quad \tilde{\mathcal{H}} := (I+\tilde{\mathcal{M}})^{-1}\Gamma_{sub}\mathcal{H}D^{-}\Gamma_d^{-},$$
$$\tilde{\mathcal{W}} := \Gamma_d\mathcal{W}\Gamma_d^{-} - \Gamma_d'\Gamma_d^{-}.$$

The DAE (2.123) has T-canonical form, if the entries $\tilde{\mathcal{N}}_{i,i+1}$ have full column rank. Therefore, we take a closer look at these entries. Having in mind that $\tilde{\mathcal{M}}$ is strictly upper block triangular, we derive

$$\begin{aligned}\tilde{\mathcal{N}}_{i,i+1} &= (\Gamma_{sub}\mathcal{N}\Omega)_{i,i+1} = \Gamma_i D\mathcal{N}_{i,i+1}\Gamma_{i+1}^{-} = -\Gamma_i D\Pi_{i-1}Q_iQ_{i+1}D^{-}\Gamma_{i+1}^{-}\\ &= -\Gamma_i\Gamma_i^{-}\Gamma_i DQ_{i+1}D^{-}\Gamma_{i+1}^{-} = -\Gamma_i DQ_{i+1}D^{-}\Gamma_{i+1}^{-}.\end{aligned}$$

Then, $\tilde{\mathcal{N}}_{i,i+1}z = 0$ means $\Gamma_i D\mathcal{N}_{i,i+1}\Gamma_{i+1}^{-}z = 0$, thus $\mathcal{N}_{i,i+1}\Gamma_{i+1}^{-}z = 0$. Applying Proposition 2.29 (3) we find that $D\Pi_iQ_{i+1}D^{-}\Gamma_{i+1}^{-}z = \Gamma_{i+1}^{-}z \in \ker D\Pi_iQ_{i+1}D^{-}$, and hence $\Gamma_{i+1}^{-}z = 0$, therefore $z = 0$. This shows that $\tilde{\mathcal{N}}_{i,i+1}$ is injective for $i = 1,\ldots,\mu-2$. The injectivity of $\tilde{\mathcal{N}}_{0,1}$ follows analogously. We obtain in fact a T-canonical form. The resulting transformations are

$$L = \begin{bmatrix} I & 0\\ 0 & (I+\tilde{\mathcal{M}})^{-1}\end{bmatrix}\Gamma\begin{bmatrix}\mathcal{L}_d\\ \mathcal{L}\end{bmatrix} = \begin{bmatrix} I & 0\\ 0 & (I+\tilde{\mathcal{M}})^{-1}\end{bmatrix}\begin{bmatrix}\Gamma_d D\Pi_{\mu-1}\\ \Gamma_0 Q_0\\ \Gamma_1 D\Pi_0 Q_1\\ \vdots\\ \Gamma_{\mu-1}D\Pi_{\mu-2}Q_{\mu-1}\end{bmatrix}G_\mu^{-1}$$

and

$$K = \Gamma\begin{bmatrix} D\Pi_{\mu-1}\\ Q_0\\ \Pi_0Q_1\\ \vdots\\ \Pi_{\mu-2}Q_{\mu-1}\end{bmatrix} = \begin{bmatrix}\Gamma_d D\Pi_{\mu-1}\\ \Gamma_0Q_0\\ \Gamma_1 D\Pi_0Q_1\\ \vdots\\ \Gamma_{\mu-1}D\Pi_{\mu-2}Q_{\mu-1}\end{bmatrix}.$$

Both matrix functions K and L are continuous and pointwise nonsingular. This completes the proof of (1).
The assertion (2) now follows immediately, since $\mathcal{H} = 0$ implies $\tilde{\mathcal{H}} = 0$. □

2.9 Regularity intervals and critical points

Critical points *per se* attract much special interest and effort. In particular, to find out whether the ODE with a so-called singularity of the first kind (e.g. [123])

$$x'(t) = \frac{1}{t}M(t)x(t) + q(t),$$

has bounded solutions, standard ODE theory is of no avail, and one is in need of smarter tools using the eigenstructure of the matrix $M(0)$.

In the case of DAEs, the inherent ODE might be affected by singularities. For instance, the DAEs in [124] show inherent ODEs having a singularity of the first kind. The following example is taken from [124].

Example 2.66 (Rank drop in G_1 causes a singular inherent ODE). The DAE

$$\begin{bmatrix}1\\1\end{bmatrix}\left(\begin{bmatrix}1 & -1\end{bmatrix}x(t)\right)' + \begin{bmatrix}2 & 0\\0 & t+2\end{bmatrix}x(t) = q(t)$$

has a properly stated leading term on $[0,1]$. It is accompanied by the matrix functions

$$G_0(t) = \begin{bmatrix}1 & -1\\1 & -1\end{bmatrix}, \quad Q_0(t) = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}, \quad G_1(t) = \begin{bmatrix}2 & 0\\2+\frac{t}{2} & \frac{t}{2}\end{bmatrix},$$

such that the DAE is regular with tractability index 1 just on the interval $(0,1]$. The inherent ODE resulting there applies to $u(t) = x_1(t) - x_2(t)$, and it reads

$$u'(t) = -\frac{2}{t}(t+2)u(t) + \frac{1}{t}((t+2)q_1(t) - 2q_2(t)).$$

Observe that, in view of the closed interval $[0,1]$, this is no longer a regular ODE but an inherent explicit *singular* ODE (IESODE). Given a solution $u(\cdot)$ of the IESODE, a DAE solution is formed by

$$x(t) = \frac{1}{t}\begin{bmatrix}t+2\\2\end{bmatrix}u(t) + \frac{1}{t}\begin{bmatrix}-q_1(t)+q_2(t)\\-q_1(t)+q_2(t)\end{bmatrix}.$$

We refer to [124] for the specification of bounded solutions by means of boundary conditions as well as for collocation approximations. □

One could presume that rank changes in G_1 would always lead to singular inherent ODEs, but the situation is much more intricate. A rank drop of the matrix function G_1 is not necessarily accompanied by a singular inherent ODE, as the next example shows.

Example 2.67 (Rank drop in G_1 does not necessarily cause a singular inherent ODE). The DAE

$$\begin{bmatrix}1\\0\end{bmatrix}\left(\begin{bmatrix}t & 1\end{bmatrix}x(t)\right)' + \begin{bmatrix}\beta(t) & 0\\0 & 1\end{bmatrix}x(t) = q(t),$$

with an arbitrary continuous real function β, has a properly stated leading term on $(-\infty,\infty)$. Put

$$G_0(t) = \begin{bmatrix}t & 1\\0 & 0\end{bmatrix}, \quad D(t)^- = \frac{1}{1+t^2}\begin{bmatrix}t\\1\end{bmatrix}, \quad Q_0(t) = \frac{1}{1+t^2}\begin{bmatrix}1 & -t\\-t & t^2\end{bmatrix},$$

and compute

$$G_1(t) = \frac{1}{1+t^2}\begin{bmatrix}\beta(t)+t+t^3 & 1+t^2-t\beta(t)\\-t & t^2\end{bmatrix}, \quad \omega_1(t) := \det G_1(t) = t(1+t^2).$$

This DAE is regular with index 1 on the intervals $(-\infty,0)$ and $(0,\infty)$. The point $t_* = 0$ is a critical one. The inherent ODE reads, with $u(t) = tx_1(t) + x_2(t)$,

$$u'(t) = -\frac{\beta(t)}{t}u(t) + q_1(t) + \frac{\beta(t)}{t}q_2(t).$$

All DAE solutions have the form

$$x(t) = \frac{1}{t}\begin{bmatrix}1\\0\end{bmatrix}u(t) + \frac{1}{t}\begin{bmatrix}-q_2(t)\\tq_2(t)\end{bmatrix}.$$

Obviously, if the function β has a zero at $t_* = 0$, or if it actually vanishes identically, then there is no singularity within the inherent ODE, even though the matrix $G_1(t_*)$ becomes singular. Remember that the determinant ω_1 does not at all depend on the coefficient β.
We turn to a special case. Set q identically zero, $\beta(t) = t^\gamma$, with an integer $\gamma \geq 0$. The inherent ODE simplifies to

$$u'(t) = -t^{\gamma-1}u(t).$$

If $\gamma = 0$, this is a singular ODE, and its solutions have the form $u(t) = \frac{1}{t}c$. All nontrivial solutions grow unboundedly, if t approaches zero. In contrast, if $\gamma \geq 1$, the ODE is regular, and it has the solutions $u(t) = e^{-\frac{1}{\gamma}t^\gamma}u(0)$ which remain bounded. However, among the resulting nontrivial DAE solutions

$$x(t) = \frac{1}{t}\begin{bmatrix}1\\0\end{bmatrix}u(t)$$

there is no bounded one, even if $\gamma \geq 1$. □

As adumbrated by the above example, apart from the singularities concerning the inherent ODE, DAEs involve further sources of critical points which are unacquainted at all in explicit ODEs. In DAEs, not only the inherent ODE but also the associated subsystem (2.62) which constitutes the wrapping up, and which in higher index cases includes the differentiated parts, might be hit by singularities. In the previous two examples which show DAEs being almost overall index 1, a look at the solution representations supports this idea. The next example provides a first impression of a higher index case.

Example 2.68 (Rank drop in G_2). The DAE with properly stated leading term

$$\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}\left(\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}x(t)\right)' + \begin{bmatrix}0&0&\beta(t)\\1&1&0\\\gamma(t)&0&0\end{bmatrix}x(t) = q(t)$$

yields

$$G_0(t)=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},\quad Q_0(t)=\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix},\quad G_1(t)=\begin{bmatrix}1&0&\beta(t)\\0&1&0\\0&0&0\end{bmatrix},\quad \Pi_0(t)=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},$$

and further $\widehat{N_1}(t)=N_1(t)\cap N_0(t)=\{z\in\mathbb{R}^3: z_1=0, z_2=0, \beta(t)z_3=0\}$. Supposing $\beta(t)\neq 0$, for all t, we derive

$$Q_1(t)=\begin{bmatrix}1&0&0\\0&0&0\\-\frac{1}{\beta(t)}&0&0\end{bmatrix},\quad \Pi_0(t)Q_1(t)=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\quad G_2(t)=\begin{bmatrix}1&0&\beta(t)\\1&1&0\\\gamma(t)&0&0\end{bmatrix},$$

and $\omega_2(t):=\det G_2(t)=-\beta(t)\gamma(t)$. The projector functions Q_0, Q_1 are the widely orthogonal ones. Taking a look at the following equivalent formulation of the DAE,

$$\begin{aligned}
x_1(t)&=\frac{1}{\gamma(t)}q_3(t),\\
x_2'(t)+x_2(t)&=q_2(t)-\frac{1}{\gamma(t)}q_3(t),\\
x_3(t)&=\frac{1}{\beta(t)}(q_1(t)-(\frac{1}{\gamma(t)}q_3(t))'),
\end{aligned}$$

we see the correspondence of zeros of the function γ to rank drops in G_2, and to critical solution behavior.

Observe also that if we dispense with the demand that the function β has no zeros, and allow a zero at a certain point t_*, then the intersection $\widehat{N_1}(t_*)$ is nontrivial, $\widehat{N_1}(t_*)=N_0(t_*)$, and the above projector function $Q_1(t)$ grows unboundedly, if t approaches t_*. Nevertheless, since by construction G_2 depends just on the product $\Pi_1 Q_2$, we can continue forming the next matrix function G_2 considering the product $\Pi_0 Q_1$ that has a continuous extension. Then a zero of the function β also leads to a zero of $\det G_2$.

Apart from critical points, the resulting IERODE applies to

$$u=D\Pi_1 x=\begin{bmatrix}0\\x_2\end{bmatrix},$$

and it reads

$$u'+\underbrace{\begin{bmatrix}0&0\\0&1\end{bmatrix}}_{D\Pi_1 G_2^{-1}B_1D^-}u=\underbrace{\begin{bmatrix}0\\q_2-\frac{1}{\gamma}q_3\end{bmatrix}}_{D\Pi_1 G_2^{-1}q}.$$

Observe the coefficient $D\Pi_1 G_2^{-1}BD^-$ to be independent of the functions β and γ, while $D\Pi_1 G_2^{-1}$ does not depend on β. Therefore, the IERODE does not at all suffer from zeros of β.

Notice that, if one restricts interest to homogeneous DAEs only, then one cannot see the singular solution behavior in this example. □

Next we consider DAEs which fail to have a proper leading term on the entire given interval.

Example 2.69 (Rank drop in A causes a singular inherent ODE). Consider the DAE

$$\underbrace{\begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}}_{A} \Big(\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{D} x \Big)' + \underbrace{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}}_{B} x = q, \tag{2.124}$$

given on the interval $\mathcal{I} = [-1,1]$. The function α is continuous. Let b_{21} have no zeros. This system yields

$$\begin{aligned} x_1 &= \frac{1}{b_{21}}(q_2 - b_{22}x_2) \\ \alpha x_2' &= \underbrace{(\frac{b_{11}}{b_{21}} b_{11}b_{22} - b_{12})}_{=M} x_2 + q_1 - b_{11}q_2. \end{aligned}$$

For any $t_* \in \mathcal{I}$ with $\alpha(t_*) \neq 0$, there is an interval $\mathcal{I}_*$ around t_* such that α has no zeros on $\mathcal{I}_*$, and the DAE has a proper leading term there. On intervals where the function α has no zeros, one can write the ODE for x_2 in explicit form as

$$x_2' = \frac{1}{\alpha} M x_2 + \frac{1}{\alpha}(q_1 - b_{11}q_2). \tag{2.125}$$

Then, equation (2.125) is a well-defined continuous coefficient ODE on this interval so that standard solvability arguments apply.

In contrast, if $\alpha(t_*) = 0$, but $\alpha(t) \neq 0$, for $t \in \mathcal{I}$, $t \neq t_*$, then equation (2.125) becomes a singular ODE, more precisely, an explicit ODE with a singularity at t_*. For those kinds of equations special treatment is advisable. We have to expect a singular flow behavior of the component $x_2(t)$. The component $x_1(t)$ may inherit properties of $x_2(t)$ depending on the coefficient function b_{22}. Let us glance over typical situations.

Let $t_* = 0$, M be a constant function, $\alpha(t) = t$, $q(t) = 0$ on $\mathcal{I}$. Then the solutions of the singular ODE (2.125) are $x_2(t) = ct^M$, with a real constant c. The behavior of these solutions heavily depends on the sign of M. Figure 2.1 shows the different flow behavior of the component $x_2(t)$ in the cases $M = 2$, $M = -2$, $M = 0$, respectively:

If $M = 2$, then all solutions cross the origin, while no solution satisfies an initial condition $x_2(0) \neq 0$.

If $M = -2$, just the trivial solution passes the origin, and all other solutions grow unboundedly if t tends to zero. Again, there is no solution with $x_2(0) \neq 0$.

If $M = 0$, then every constant function solves the ODE, and every IVP is uniquely solvable. Derive, for $t \in [-1,0)$ and $t \in (0,1]$,

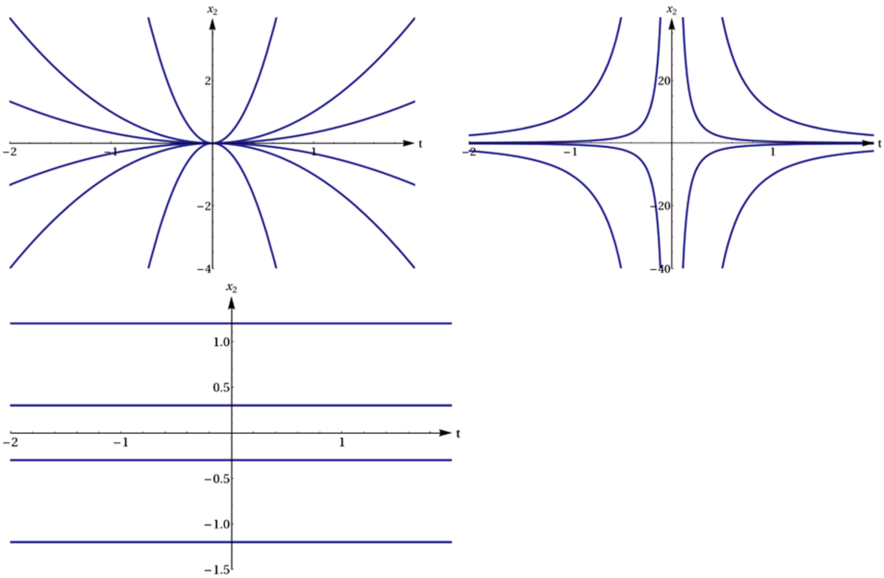


Fig. 2.1 $M = 2$, $M = -2$, $M = 0$

$$G_0(t) = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}, \; Q_0(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; G_1(t) = \begin{bmatrix} b_{11}(t) & t \\ b_{21}(t) & 0 \end{bmatrix}.$$

This shows the DAE to be regular with index 1 on both intervals $[-1,0)$ and $(0,1]$. □

Example 2.70 (Rank change in A causes an index change). Now we put the continuous entry (cf. Figure 2.2)

$$\alpha(t) = \begin{cases} 0 & \text{for } t \in [-1,0] \\ t^{\frac{1}{3}} & \text{for } t \in (0,1] \end{cases}$$

into the DAE

$$\begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x \right)' + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x = q, \tag{2.126}$$

which has a properly stated leading term merely on the subinterval $(0,1]$.
The admissible matrix function sequence

$$G_0 = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}, \; Q_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; G_1 = \begin{bmatrix} 1 & \alpha \\ 0 & 0 \end{bmatrix}, \; Q_1 = \begin{bmatrix} 0 & -\alpha \\ 0 & 1 \end{bmatrix}, \; G_1 = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix},$$

indicates the characteristic values $r_0 = 1, r_1 = 1$ and $r_2 = 2$ on $(0,1]$. The DAE is regular with index 2 there.
For every $q_1 \in \mathcal{C}((0,1],\mathbb{R})$, $q_2 \in \mathcal{C}^1((0,1],\mathbb{R})$, there is a unique solution $x \in \mathcal{C}^1_D((0,1],\mathbb{R}^2)$. The particular solution corresponding to $q_1(t) = 0$, $q_2(t) = t^{\frac{1}{3}}$,

$t \in (0,1]$, reads $x_1(t) = -\frac{1}{3}t^{-\frac{1}{3}}$, $x_2(t) = t^{\frac{1}{3}}$.
On the subinterval $[-1,0]$ the leading term is no longer properly stated, but we may turn to a proper reformulation if we replace D by $\bar{D} = 0$. Then, for $t \in [-1,0]$, it follows that

$$G_0(t) = 0,\ Q_0(t) = I,\ G_1(t) = I,\ r_0 = 0,\ r_1 = 2,$$

and the DAE is regular with index 1 on the interval $[-1,0]$. On this subinterval, for every continuous q, the solution is simply $x = q$. In particular, for $q_1(t) = 0$, $q_2(t) = -|t|^{\frac{1}{3}}$, the solution on this subinterval is $x_1(t) = 0$, $x_2(t) = -|t|^{\frac{1}{3}}$.
Altogether, combining now the segments, we have an excitation q that is continuous on the entire interval $\mathcal{I}$, and its second component is continuously differentiable on $(0,1]$. We have two solution segments. Can these segments be glued together to form a solution on the entire interval? While the second component has a continuous extension, the first has not, as shown in Figure 2.3.
Relaxing the minimal smoothness requirements for the excitation on the subintervals, and assuming more generously q to be continuous with a continuously differentiable second component on the whole interval $\mathcal{I}$, then, for every such q, there exists a unique solution $x \in C^1_D(\mathcal{I},\mathbb{R}^2)$. This means that, in the smoother setting, the critical point does not matter. Those kinds of critical points which can be healed by higher smoothness are said to be harmless. However, we stress once more that in a setting with minimal smoothness, these points are in fact critical. Written as

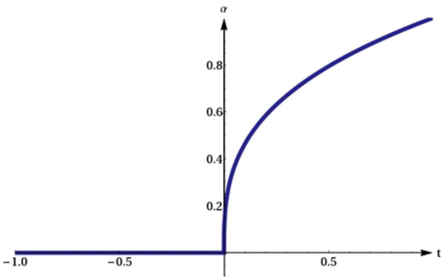


Fig. 2.2 Continuous function α of Example 2.70

$$\begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} x' + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x = q, \tag{2.127}$$

the DAE (2.126) yields a special DAE in standard canonical form (SCF). To ensure continuously differentiable solutions on the entire interval, one now has to suppose not only that q is continuously differentiable, but also that αq_2 is so. □

Example 2.71 (Index drop in A yielding a harmless critical point). We replace the function α in (2.126) by a different one and turn to

$$\begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x \right)' + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x = q, \tag{2.128}$$

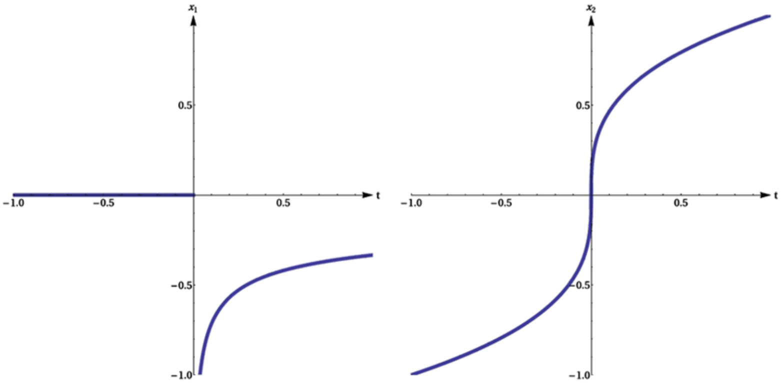


Fig. 2.3 Solution segments of x_1, x_2 in Example 2.70

with

$$\alpha(t) = \begin{cases} -|t|^{\frac{1}{3}} & \text{for } t \in [-1,0) \\ t^{\frac{1}{3}} & \text{for } t \in [0,1]. \end{cases}$$

The DAE (2.128) has a properly stated leading term on the two subintervals $[-1,0)$ and $(0,1]$, but on the entire interval $[-1,1]$ the leading term fails to be properly stated. The point $t_* = 0$ is a critical one.
The matrix function sequence

$$G_0 = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}, \; Q_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; G_1 = \begin{bmatrix} 1 & \alpha \\ 0 & 0 \end{bmatrix}, \; Q_1 = \begin{bmatrix} 0 & -\alpha \\ 0 & 1 \end{bmatrix}, \; G_1 = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix},$$

is admissible with characteristic values $r_0 = 1, r_1 = 1$ and $r_2 = 2$ on the intervals $[-1,0)$ and $(0,1]$ which indicates the DAE to be regular with index 2 there.
We apply the excitation $q_1 \equiv 0$, $q_2 = \alpha$ on both subintervals. As in the previous example, the first components of the solution segments cannot be glued together, as is sketched in Figure 2.4. For smoother excitations, we again obtain solutions belonging to $\mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^2)$. Furthermore, if q and αq_2 are smooth, then the SCF version (cf. 2.127) has $\mathcal{C}^1$-solutions. Those critical points which disappear in smoother settings are said to be harmless. □

Equations (2.124), (2.126) and (2.128) possess the following property independently of the behavior of the function α: There is a subspace $N_A \subseteq \mathbb{R}^2$, such that

$$N_A \oplus \operatorname{im} D = \mathbb{R}^2, \quad N_A = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \subseteq \ker A.$$

This property motivates the following generalization of proper leading terms for DAEs (2.1) having continuous coefficients as before.

Definition 2.72. For the DAE (2.1), let the time-varying subspace $\operatorname{im} D$ be a $\mathcal{C}^1$-subspace on $\mathcal{I}$, and let a further $\mathcal{C}^1$-subspace N_A exist such that

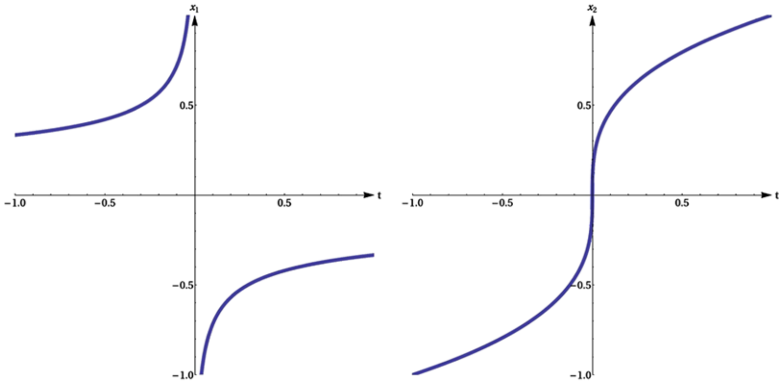


Fig. 2.4 Solution segments of x_1, x_2 in Example 2.71

$$N_A(t) \oplus \operatorname{im} D(t) = \mathbb{R}^n, \quad N_A(t) \subseteq \ker A(t), \quad t \in \mathcal{I}. \tag{2.129}$$

(1) If $N_A(t) = \ker A(t)$ for all t from a dense subset of the interval $\mathcal{I}$, then we speak of a DAE with an *almost proper leading term.*
(2) If $\dim \ker D(t) \geqslant 1$, then equation (2.1) is called a DAE with a *quasi-proper leading term* on $\mathcal{I}$.

DAEs with proper leading terms constitute a particular case of DAEs with almost-proper leading terms, if $N_A(t) = \ker A(t)$ holds true for all $t \in \mathcal{I}$.
The DAEs in Examples 2.69–2.71 have quasi-proper leading terms on the entire given interval $\mathcal{I}$. Examples 2.69 and 2.71 show even almost proper leading terms.
Example 2.70 represents a simple case of a DAE in SCF. Large classes of DAEs with those quasi-proper leading term, including the DAEs in SCF, are treated in detail in Chapter 9.
To some extent, the quasi-proper DAE form is quite comfortable. However, we should be aware that, in the general setting of quasi-proper DAEs, there is no way of indicating basic level critical points as in Examples 2.69, 2.70, and 2.71. This is why we prefer properly stated leading terms.

Our examples clearly account for the correspondence between singular solution behavior and points at which the matrix function sequence loses one of the required constant-rank properties. Roughly speaking, at all points where the matrix function sequence determining regularity cannot be built, we expect a critical (in some sense) solution behavior. We refer to [194] for a closer view of the relevant literature. As in [194], we consider critical (in [194] named *singular*) points to be the counterparts of regular points. Therefore, in this section, we deal with square DAEs (2.1) the coefficients A of which do not necessarily show constant rank. We recall Examples 2.69, 2.70, and 2.71 once more, which demonstrate critical solution behavior corresponding to the rank changes of A.
The DAE in Example 2.70 fails to have a proper leading term on the subinterval $[-1,0)$. On this subinterval, the special matrix function $A = 0$ has constant rank 0 and $\ker A = \mathbb{R}^2$ is a C^1-subspace on $[-1,0)$. For this reason, in this special case,

we could do with a proper refactorization to a proper leading term. Such proper refactorizations apply also in general cases as the next proposition says.

Proposition 2.73. *Let the DAE (2.1) have a quasi-proper leading term on the given interval $\mathcal{I}$. Let $\tilde{\mathcal{I}} \subseteq \mathcal{I}$ be a subinterval such that* $\ker A$ *is a $\mathcal{C}^1$-subspace on $\tilde{\mathcal{I}}$. Then $\tilde{R} := A^+A$ is continuously differentiable on $\tilde{\mathcal{I}}$ and the DAE has there the reformulation with a proper leading term*

$$A(\tilde{R}Dx)' + (B - A\tilde{R}'D)x = q, \quad t \in \tilde{\mathcal{I}}. \tag{2.130}$$

Proof. The matrix function $\tilde{R}$ is continuously differentiable as an orthoprojector function along a $\mathcal{C}^1$-subspace. We rewrite the leading term in the DAE (2.1) on the subinterval as

$$A(Dx)' = A\tilde{R}(Dx)' = A(\tilde{R}Dx)' - A\tilde{R}'Dx,$$

which leads to (2.130).
Introduce R as the projector function onto $\operatorname{im} D$ along N_A so that $\operatorname{im} D = \operatorname{im} R$ and $N_A = \operatorname{im}(I - R)$. Owing to condition (2.129), R is well defined. Additionally, it holds that $\operatorname{im}(I-R) = N_A \subseteq \ker A = \ker \tilde{R}$, and hence $\tilde{R}(I-R) = 0$, thus $\tilde{R} = \tilde{R}R$. This implies $\operatorname{im} \tilde{R}D = \operatorname{im} \tilde{R}R = \operatorname{im} \tilde{R}$, which proves the spaces $\ker A = \ker \tilde{R}$ and $\operatorname{im} \tilde{R}D = \operatorname{im} \tilde{R}$ to be transversal $\mathcal{C}^1$-subspaces on $\tilde{\mathcal{I}}$. Therefore, the DAE (2.130) has a proper leading term. □

Definition 2.74. Let the DAE (2.1), with $m = k$, have a quasi-proper leading term. Then, $t_* \in \mathcal{I}$ is said to be a *regular point* of the DAE, if there is an open interval $\mathcal{I}_*$ containing t_* such that either the original DAE is regular on $\tilde{\mathcal{I}} := \mathcal{I} \cap \mathcal{I}_*$ or $\ker A$ is a $\mathcal{C}^1$-subspace on $\tilde{\mathcal{I}}$ and the proper reformulation (2.130) is a regular DAE on $\tilde{\mathcal{I}}$.
Otherwise, t_* is said to be a *critical point*.
Each open interval on which the DAE is regular is called a *regularity interval*. Denote by $\mathcal{I}_{reg}$ the set of all $t \in \mathcal{I}$ being regular points of the DAE.

In this sense, $t_* = 0$ is the only critical point of the DAEs in Examples 2.66, 2.67, 2.69, 2.70, and 2.71, while in Example 2.68 the set of critical points is formed by the zeros of the functions β and γ. The left boundary point in Example 2.66 is a critical point while the right boundary point is regular.
By definition, Example 2.66 shows the regularity interval $(0,1)$ but $\mathcal{I}_{reg} = (0,1]$. We find the regularity intervals $(-\infty,0)$ and $(0,\infty)$ in Example 2.67, whereby the characteristic values are on both sides $r_0 = 1, r_1 = 2$ and $\mu = 2$.
In Example 2.68, regularity intervals exist around all inner points t of the given interval where $\beta(t)\gamma(t) \neq 0$, with uniform characteristics $r_0 = 2, r_1 = 2, r_2 = 3$ and $\mu = 2$.
The peculiarity of Example 2.70 consists of the different characteristic values on the regularity intervals $(-1,0)$ and $(0,1)$.

Each regularity interval consists of regular points, exclusively. All subintervals of a regularity interval inherit the characteristic values. If there are intersecting regularity intervals, then the DAE has common characteristic values on these intervals, and the union of regularity intervals is a regularity interval, again ([173], applying

widely orthogonal projector functions one can simplify the proof given there).
The set $\mathcal{I}_{reg} \subseteq \mathcal{I}$ may be described as the union of disjoint regularity intervals, eventually completed by the regular boundary points. By definition, $\mathcal{I} \setminus \mathcal{I}_{reg}$ is the set of critical points of the DAE (2.1).

The regularity notion (cf. Definitions 2.6 and 2.25) involves several constant-rank conditions. In particular, the proper leading term brings the matrix function $G_0 = AD$ with constant rank $r_0 = r$. Further, the existence of regular admissible matrix functions includes that, at each level $k = 1, \ldots, \mu - 1$,

(A) the matrix function G_k has constant rank r_k, and

(B) the intersection $\widehat{N}_k$ is trivial, i.e., $\widehat{N}_k = \{0\}$.

Owing to Proposition 2.7 we have $\ker \Pi_{k-1} = N_0 + \cdots + N_{k-1}$, and hence

$$\widehat{N}_k = N_k \cap (N_0 + \cdots + N_{k-1}) = \ker G_k \cap \ker \Pi_{k-1}.$$

Then, the intersection $\widehat{N}_k$ is trivial, exactly if the matrix function

$$\begin{bmatrix} G_k \\ \Pi_{k-1} \end{bmatrix} \tag{2.131}$$

has full column rank m. This means that condition (B) also represents a rank condition.

Suppose the coefficients A, D and B of the DAE are sufficiently smooth (at most class $\mathcal{C}^{m-1}$ will do). Then, if the algebraic *rank conditions* are fulfilled, the requirements for the projector functions Π_k and $D\Pi_k D^-$ to be continuous respectively continuously differentiable, can be satisfied at one level after the other. In consequence (cf. [173, 174, 194]), a critical point can be formally characterized as the location where the coefficient A has a rank drop, or where one of the constant-rank conditions type (A) or type (B), at a level $k \geq 1$, is violated first.

Definition 2.75. Let the DAE (2.44) have a quasi-proper leading term, and t_* be a critical point. Then, t_* is called

(1) a *critical point of type* 0, if $\operatorname{rank} G_0(t_*) < r := \operatorname{rank} D(t_*)$,
(2) a *critical point of type A at level* $k \geq 1$ (briefly, type k-A), if there are admissible projectors functions $Q_0, \ldots, Q_{k-1}$, and G_k changes its rank at t_*,
(3) a *critical point of type B at level* $k \geq 1$ (briefly, type k-B), if there are admissible projector functions $Q_0, \ldots, Q_{k-1}$, the matrix function G_k has constant rank, but the full-rank condition for the matrix function (2.131) is violated at t_*.

It is worth emphasizing that the proposed typification of critical points remains invariant with respect to transformations and refactorizations (Section 2.3), and also with respect to the choice of admissible projector functions (Subsection 2.2.2).

The DAEs in Examples 2.66 and 2.67 have the type 1-A critical point $t_* = 0$. In Example 2.68, the zeros of the function γ are type 2-A critical points, while the zeros of the function β yield type 1-B critical points. Examples 2.69, 2.70 and 2.71 show different cases of type 0 critical points.

While the zero of the function α in Example 2.71 yields a harmless critical point, in contrast, in Example 2.69, the zero of α causes a singular inherent ODE.
How do harmless critical points differ from the other critical points? As suggested by Example 2.71, we prove the nonsingularity of the matrix function G_μ to indicate harmless critical points in general.

Let the DAE (2.44) have an almost proper leading term. For simplicity, let DD^* be continuously differentiable such that the widely orthogonal projector functions can be used. Assume the set of regular points $\mathcal{I}_{reg}$ to be dense in $\mathcal{I}$.
Let Q_0 be the orthogonal projector function onto $\ker D =: N_0$, which is continuous on the entire interval $\mathcal{I}$, since D has constant rank r there. Set $G_0 = AD$, $B_0 = B$, $G_1 = G_0 + BQ_0$. These functions are also continuous on $\mathcal{I}$. For all $t \in \mathcal{I}_{reg}$ it holds further that $\operatorname{rank} G_0(t) = r$. On each regularity interval, which is a regularity region, we construct the matrix function sequence by means of widely orthogonal projector functions up to G_μ, whereby μ denotes the lowest index such that $G_\mu(t)$ is nonsingular for all $t \in \mathcal{I}_{reg}$. In particular, $\Pi_1, \dots, \Pi_{\mu-1}$ are defined and continuous on each part of $\mathcal{I}_{reg}$. Assume now that

$$\Pi_1, \dots, \Pi_{\mu-1} \quad \text{have continuous extensions on } \mathcal{I}, \tag{2.132}$$

and we keep the same denotation for the extensions. Additionally, suppose

$$D\Pi_1 D^-, \dots, D\Pi_{\mu-1}D^- \quad \text{are continuously differentiable on } \mathcal{I}.$$

Then, the projector functions $\Pi_{i-1}Q_i = \Pi_{i-1} - \Pi_i$, $i = 1, \dots, \mu-1$, have continuous extensions, too, and the matrix function sequence (cf. (2.5)–(2.8), and Proposition 2.7)

$$\begin{aligned} B_i &= B_{i-1}\Pi_{i-1} - G_i D^-(D\Pi_i D^-)' D\Pi_{i-1}, \\ G_{i+1} &= G_i + B_i \Pi_{i-1} Q_i, \quad i = 1, \dots, \mu-1, \end{aligned}$$

is defined and continuous on the entire interval $\mathcal{I}$. In contrast to the regular case, where the matrix functions G_j have constant rank on the entire interval $\mathcal{I}$, now, for the time being, the projector functions Q_j are given on $\mathcal{I}_{reg}$ only, and

$$N_i(t) = \operatorname{im} Q_i(t) = \ker G_i(t), \quad \text{for all} \quad t \in \mathcal{I}_{reg}.$$

The projector function $\Pi_0 = P_0$ inherits the constant rank $r = \operatorname{rank} D$ from D. On each of the regularity intervals, the rank r_0 of G_0 coincides with the rank of D, and hence we are aware of the uniform characteristic value $r_0 = r$ on all regularity intervals, that is on $\mathcal{I}_{reg}$.
Owing to its continuity, the projector function Π_1 has constant rank on $\mathcal{I}$. Taking into account the relations

$$\ker \Pi_1(t) = N_0(t) \oplus N_1(t), \quad \dim N_0(t) = m - r_0, \quad \dim N_1(t) = m - r_1, \ t \in \mathcal{I}_{reg}$$

we recognize the characteristic value $r_1 = \operatorname{rank} G_1$ to be also uniform on $\mathcal{I}_{reg}$, and so on. In this way we find out that all characteristics

$$r_0 \le \cdots \le r_{\mu-1} < r_\mu = m \quad \text{are uniform on } \mathcal{I}_{reg}.$$

In particular, the DAE has index μ on $\mathcal{I}_{reg}$.
Denote by $G_\mu(t)^{adj}$ the matrix of cofactors to $G_\mu(t)$, and introduce the determinant $\omega_\mu(t) := \det G_\mu(t)$, such that

$$\omega_\mu(t) G_\mu(t)^{-1} = G_\mu(t)^{adj}, \quad t \in \mathcal{I}_{reg}.$$

By construction, it results that $G_\mu Q_i = B_i Q_i = B_i \Pi_{i-1} Q_i$, for $i = 1, \ldots, \mu - 1$, thus

$$\omega_\mu(t) Q_i(t) = G_\mu(t)^{adj} B_i(t) \Pi_{i-1}(t) Q_i(t), \quad i = 1, \ldots, \mu - 1, \quad t \in \mathcal{I}_{reg}. \tag{2.133}$$

The last expression possesses a continuous extension, and hence $\omega_\mu Q_i = G_\mu^{adj} B_i \Pi_{i-1} Q_i$ is valid on $\mathcal{I}$.
Observe that a nonsingular $G_\mu(t_*)$ also indicates that the projector functions $Q_1, \ldots, Q_{\mu-1}$ have continuous extensions over the critical point t_*. In this case, the decoupling formulas (2.51), (2.62) keep their value for the continuous extensions, and it is evident that the critical point is a harmless one.
In contrast, if G_μ has a rank drop at the critical point t_*, then the decoupling formulas actually indicate different but singular solution phenomena. Additionally, several projector functions Q_j may suffer discontinuities, as is the case in Example 2.68.

Next, by means of the widely orthogonal projector functions, on each regularity interval, we apply the basic decoupling (see Subsection 2.4.2, Theorem 2.30) of a regular DAE into the IERODE (2.51) and the subsystem (2.62). In order to safely obtain coefficients that are continuous on the entire interval $\mathcal{I}$, we multiply the IERODE (2.51) by ω_μ, the first row of (2.62) by ω_μ^μ, the second by $\omega_\mu^{\mu-1}$, and so on up to the last line which we multiply by ω_μ. With regard to assumption (2.132) and relation (2.133), the expressions $\omega_\mu G_\mu^{-1}$ and $\omega_\mu \mathcal{K}$, $\omega_\mu \mathcal{M}_{l+1}$ (cf. (2.54), (2.55)) are continuous on $\mathcal{I}$, and so are all the coefficients of the subsystem resulting from (2.62). Instead of the IERODE (2.51) we are now confronted with the equation

$$\omega_\mu u' - \omega_\mu (D\Pi_{\mu-1}D^-)' u + D\Pi_{\mu-1} G_\mu^{adj} B_\mu D^- u = D\Pi_{\mu-1} G_\mu^{adj} q, \tag{2.134}$$

which is rather a scalarly implicit inherent ODE or an inherent explicit singular ODE (IESODE). As is proved for regular DAEs by Theorem 2.30, the equivalence of the DAE and the system decoupled in this way is given. We refer to [194, Subsection 4.2.2] for a detailed description in a slightly different way. Here we take a look at the simplest lower index cases only.
The case $\mu = 1$ corresponds to the solution decomposition $x = D^- u + Q_0 x$, the inherent ODE

$$\omega_1 u' - \omega_1 R' u + DG_1^{adj} B_1 D^- u = DG_1^{adj} q, \tag{2.135}$$

and the subsystem

$$\omega_1 Q_0 x = -Q_0 G_1^{adj} B_1 D^- u + Q_0 G_1^{adj} q. \tag{2.136}$$

For $\mu = 2$, we apply the solution decomposition $x = D^- u + \Pi_0 Q_1 x + Q_0 x$. The inherent ODE reads

$$\omega_2 u' - \omega_2 (D\Pi_1 D^-)' u + D\Pi_1 G_2^{adj} B_1 D^- u = D\Pi_1 G_2^{adj} q, \tag{2.137}$$

and we have to add the subsystem

$$\begin{aligned} &\begin{bmatrix} -\omega_2 Q_0 \omega_2 Q_1 D^- (D\Pi_0 Q_1 x)' \\ 0 \end{bmatrix} + \begin{bmatrix} \omega_2^2 Q_0 x \\ \omega_2 \Pi_0 Q_1 x \end{bmatrix} \\ &\qquad + \begin{bmatrix} Q_0 \omega_2 P_1 \omega_2 \mathcal{K} \Pi_1 \\ \Pi_0 Q_1 \omega_2 \mathcal{K} \Pi_1 \end{bmatrix} D^- u = \begin{bmatrix} Q_0 \omega_2 P_1 G_2^{adj} \\ \Pi_0 Q_1 G_2^{adj} \end{bmatrix} q. \end{aligned} \tag{2.138}$$

A careful inspection of our examples proves that these formulas comprise a worst case scenario. For instance, in Example 2.68, not only is $D\Pi_1 G_2^{adj} B_1 D^-$ continuous but already $D\Pi_1 G_2^{-1} B_1 D^-$ can be extended continuously. However, as in Example 2.66, the worst case can well happen.

Proposition 2.76. *Let the DAE (2.1) have an almost proper leading term, and DD^* be continuously differentiable. Let the set of regular points $\mathcal{I}_{reg}$ be dense in $\mathcal{I}$. If the projector functions $\Pi_1, \ldots, \Pi_{\mu-1}$ associated with the widely orthogonal projector functions have continuous extensions on the entire interval $\mathcal{I}$, and $D\Pi_1 D^-, \ldots, D\Pi_{\mu-1} D^-$ are continuously differentiable, then the following holds true:*

(1) *The DAE has on $\mathcal{I}_{reg}$ uniform characteristics $r_0 \le \cdots \le r_{\mu-1} < r_\mu = m$.*
(2) *If $G_\mu(t_*)$ is nonsingular at the critical point t_*, then the widely orthogonal projector functions $Q_0, \ldots, Q_{\mu-1}$ themselves have continuous extensions over t_*. If the coefficients A, D, and B are sufficiently smooth, then t_* is a harmless critical point.*
(3) *If $G_\mu(t_*)$ is nonsingular at the critical point t_*, then $G_{\mu-1}(t)$ has necessarily constant rank $r_{\mu-1}$ on a neighborhood including t_*.*
(4) *If the DAE has index 1 on $\mathcal{I}_{reg}$, then its critical points fail to be harmless.*
(5) *A critical point of type B leads necessarily to a singular G_μ, and hence it can never been harmless.*

Proof. Assertion (1) is already verified. Assertion (2) follows immediately by making use of the decoupling. If A, D, B are smooth, then the coefficients of the subsystem (2.62) are also sufficiently smooth, and allow for the respective solutions.
Turn to (3). Owing to (2), $Q_{\mu-1}$ is continuous, and $\operatorname{rank} Q_{\mu-1}(t_*) = m - r_{\mu-1}$, $G_{\mu-1}(t_*) Q_{\mu-1}(t_*) = 0$ are valid, thus $\operatorname{rank} G_{\mu-1}(t_*) \le r_{\mu-1}$. The existence of a $z \in \ker G_{\mu-1}(t_*)$, $P_{\mu-1}(t_*) z = z \neq 0$, would imply $G_{\mu-1}(t_*) z = 0$, and hence would

contradict the nonsingularity of $G_{\mu-1}(t_*)$.
(4) is a direct consequence of (3).
For proving Assertion (5) we remember the relation

$$\Pi_{j-1}(t)Q_j(t) = \Pi_{j-1}(t)Q_j(t)\Pi_{j-1}(t), \quad t \in \mathcal{I}_{reg}.$$

These relations remain valid for the continuous extensions, that is, for $t \in \mathcal{I}$. Consider a type $k-B$ critical point t_*, and a nontrivial $z \in N_k(t_*) \cap (N_0(t_*) + \cdots + N_{\mu-1}(t_*))$, which means $G_k(t_*)z = 0$, $\Pi_{k-1}(t_*)z = 0$. This yields

$$\begin{aligned} G_\mu(t_*)z = G_k(t_*)z + B_k(t_*)Q_k(t_*)\Pi_{k-1}(t_*)z \\ + \cdots + B_{\mu-1}(t_*)\Pi_{\mu-2}(t_*)Q_{\mu-1}(t_*)\Pi_{k-1}(t_*)z = 0, \end{aligned}$$

and hence, $G_\mu(t_*)$ is singular. □

2.10 Strangeness versus tractability

2.10.1 Canonical forms

Among the traditional goals of the theory of linear time-varying DAEs are appropriate generalizations of the Weierstraß–Kronecker canonical form and equivalence transformations into these canonical forms. So far, except for the T-canonical form which applies to both standard form DAEs and DAEs with properly stated leading term (cf. Subsection 2.8), reduction to canonical forms is developed for standard form DAEs (e.g. [39], [25], [127]).
While equivalence transformations for DAEs with properly stated leading term include transformations K of the unknown, scalings L and refactorizations H of the leading term (cf. Section 2.3), equivalence transformations for standard form DAEs combine only the transformations K of the unknowns and the scalings L.
Transforming the unknown function by $x = K\tilde{x}$ and scaling the standard form DAE (2.112) by L yields the equivalent DAE

$$\underbrace{LEK}_{\tilde{E}}\tilde{x}' + \underbrace{(LFK + LEK')}_{\tilde{F}}\tilde{x} = Lq.$$

Therefore the transformation matrix functions K must be continuously differentiable.

In the remaining part of this subsection we use the letters K and H also for special entries in the matrix functions describing the coefficients of the canonical forms below. No confusion will arise from this.

Definition 2.77. The structured DAE with continuous coefficients

$$\begin{bmatrix} I_{m-l} & K \\ 0 & N \end{bmatrix} x' + \begin{bmatrix} W & 0 \\ H & I_l \end{bmatrix} x = q, \tag{2.139}$$

$0 \leq l \leq m$, is said to be in

(1) *standard canonical form* (SCF), if $H = 0,\ K = 0$, and N is strictly upper triangular,
(2) *strong standard canonical form* (SSCF), if $H = 0,\ K = 0$, and N is a constant, strictly upper triangular matrix,
(3) *S-canonical form*, if $H = 0$, $K = [0\ K_1 \dots K_\kappa]$, and

$$N = \begin{bmatrix} 0 & N_{1,2} & \cdots & N_{1,\kappa} \\ & \ddots & & \vdots \\ & & \ddots & N_{\kappa-1,\kappa} \\ & & & 0 \end{bmatrix} \begin{matrix} \}l_1 \\ \\ \}l_{\kappa-1} \\ \}l_\kappa \end{matrix} ,$$

is strictly upper block triangular with full row rank entries $N_{i,i+1},\ i = 1,\dots,\kappa-1$,
(4) *T-canonical form*, if $K = 0$ and N is strictly upper block triangular with full column rank entries $N_{i,i+1},\ i = 1,\dots,\kappa-1$.

In the case of time-invariant coefficients, these four canonical forms are obviously equivalent. However, this is no longer true for time-varying coefficients.
The matrix function N is nilpotent in all four canonical forms, and N has uniform nilpotency index κ in (3) and (4). N and all its powers N^k have constant rank in (2), (3) and (4). In contrast, in (1), the nilpotency index and the rank of N may vary with time. The S-canonical form is associated with DAEs with regular strangeness index $\zeta = \kappa - 1$ (cf. [127]), while the T-canonical form is associated with regular DAEs with tractability index $\mu = \kappa$ (cf. Subsection 2.8). The classification into SCF and SSCF goes back to [39] (cf. also [25]). We treat DAEs being transformable into SCF as quasi-regular DAEs in Chapter 9. Here we concentrate on the S-canonical form. We prove that each DAE being transformable into S-canonical form is regular with tractability index $\mu = \kappa$, and hence, each DAE with well-defined regular strangeness index ζ is a regular DAE with tractability index $\mu = \zeta + 1$. All the above canonical forms are given in standard form. For the T-canonical form, a version with properly stated leading term is straightforward (cf. Definition 2.64).

The strangeness index concept applies to standard form DAEs (2.112) with sufficiently smooth coefficients. A reader who is not familiar with this concept will find a short introduction in the next subsection. For the moment, *we interpret DAEs with regular strangeness index as those being transformable into S-canonical form.* This is justified by an equivalence result of [127], which is reflected by Theorem 2.78 below.

The regular strangeness index ζ is supported by a sequence of *characteristic values* $\bar{r}_i, \bar{a}_i, \bar{s}_i,\ i = 0,\dots,\zeta$, which are associated with constant-rank conditions for matrix functions, and which describe the detailed size of the S-canonical form. By definition, $s_\zeta = 0$ (cf. Subsection 2.10.2). These characteristic values are invariant with respect to the equivalence transformations, however, they are not independent of each other.

Theorem 2.78. *Each DAE (2.112) with smooth coefficients, well-defined strangeness index ζ and characteristic values $\bar{r}_i, \bar{a}_i, \bar{s}_i$, $i = 0, \ldots, \zeta$, is equivalent to a DAE in S-canonical form with $\kappa = \zeta + 1$, $l = l_1 + \cdots + l_\kappa$, $m - l = \bar{r}_\zeta$, and*

$$l_1 \leq \cdots \leq l_\kappa, \quad l_1 = \bar{s}_{\kappa-2} = \bar{s}_{\zeta-1},\, l_2 = \bar{s}_{\kappa-3}, \ldots, l_{\kappa-1} = \bar{s}_0,\, l_\kappa = \bar{s}_0 + \bar{a}_0.$$

Proof. This assertion comprises the regular case of [127, Theorem 12] which considers more general equations having also underdetermined parts (indicated by nontrivial further characteristic values $\bar{u}_i$). □

By the next assertion, which represents the main result of this subsection, we prove each DAE with regular strangeness index ζ to be at the same time a regular DAE with tractability index $\mu = \zeta + 1$. Therefore, the tractability index concept applies at least to the entire class of DAEs which are accessible by the strangeness index concept. Both concepts are associated with characteristic values being invariant under equivalence transformations, and, of course, we would like to know how these characteristic values are related to each other. In particular, the question arises whether the constant-rank conditions supporting the strangeness index coincide with the constant-rank conditions supporting the tractability index.

Theorem 2.79. (1) *Let the standard form DAE (2.112) have smooth coefficients, regular strangeness index ζ and characteristic values $\bar{r}_i$, $\bar{a}_i$, $\bar{s}_i$, $i = 0, \ldots, \zeta$. Then this DAE is regular with tractability index $\mu = \zeta + 1$ and associated characteristic values*

$$r_0 = \bar{r}_0, \quad r_j = m - \bar{s}_{j-1}, \quad j = 1, \ldots, \mu.$$

(2) *Each DAE in S-canonical form with smooth coefficients can be transformed into T-canonical form with $H = 0$.*

Proof. (1) We prove the assertion by constructing a matrix function sequence and admissible projector functions associated with the tractability index framework for the resulting S-canonical form described by Theorem 2.78.
The matrix function N within the S-canonical form has constant rank $l - l_\kappa$. Exploiting the structure of N we compose a projector function $Q_0^{[N]}$ onto $\ker N$, which is upper block triangular, too. Then we set

$$P_0 := \begin{bmatrix} I_{m-l} & KQ_0^{[N]} \\ 0 & P_0^{[N]} \end{bmatrix}, \quad \text{such that} \quad \ker P_0 = \ker \begin{bmatrix} I_{m-l} & K \\ 0 & N \end{bmatrix}.$$

P_0 is a projector function. The DAE coefficients are supposed to be smooth enough so that P_0 is continuously differentiable. Then we can turn to the following properly stated version of the S-canonical form:

$$\begin{bmatrix} I_{m-l} & K \\ 0 & N \end{bmatrix} (P_0 x)' + \underbrace{\left(\begin{bmatrix} W & 0 \\ 0 & I_l \end{bmatrix} - \begin{bmatrix} I_{m-l} & K \\ 0 & N \end{bmatrix} P_0' \right)}_{\begin{bmatrix} W & -K'Q_0^{[N]} \\ 0 & I_l - N(P_0^{[N]})' \end{bmatrix}} x = q. \tag{2.140}$$

The product $NP_0^{[N]'}$ is again strictly upper block triangular, and $I_l - N(P_0^{[N]})'$ is nonsingular. Scaling the DAE by

$$\begin{bmatrix} I_{m-l} & 0 \\ 0 & (I_l - N(P_0^{[N]})')^{-1} \end{bmatrix}$$

yields

$$\begin{bmatrix} I_{m-l} & K \\ 0 & M_0 \end{bmatrix} (P_0 x)' + \begin{bmatrix} W & -K'Q_0^{[N]} \\ & I_l \end{bmatrix} x = q. \tag{2.141}$$

The matrix function M_0 has the same structure as N, and $\ker M_0 = \ker N$. For the subsystem corresponding to the second line of (2.141)

$$M_0(P_0^{[N]} v)' + v = q_2.$$

Proposition B.2 in Appendix B provides a matrix function sequence $G_j^{[N]}$, $j = 0, \dots, \kappa$, and admissible projector functions $Q_0^{[N]}, \dots, Q_{\kappa-1}^{[N]}$ such that this subsystem is a regular DAE with tractability index $\mu^{[N]} = \kappa$ and characteristic values

$$r_i^{[N]} = l - l_{\kappa-i}, \quad i = 0, \dots, \kappa - 1, \quad r_\kappa^{[N]} = l.$$

Now we compose a matrix function sequence and admissible projector functions for the DAE (2.141). We begin with $D = D^- = R = P_0$, and build successively for $i = 0, \dots, \kappa$

$$G_i = \begin{bmatrix} I_{m-l} & * \\ 0 & G_i^{[N]} \end{bmatrix}, \quad Q_i = \begin{bmatrix} 0 & * \\ 0 & Q_i^{[N]} \end{bmatrix}, \quad \Pi_i = \begin{bmatrix} I_{m-l} & * \\ 0 & \Pi_i^{[N]} \end{bmatrix}, \quad B_i = \begin{bmatrix} W & * \\ 0 & B_i^{[N]} \end{bmatrix}.$$

The coefficients are supposed to be smooth enough so that the Π_i are continuously differentiable. It follows that the matrix functions G_i have constant ranks

$$r_i = m - l + r_i^{[N]} = m - l + l - l_{\kappa-i} = m - l_{\kappa-i}, \ i = 0, \dots, \kappa - 1, \ r_\kappa = m - l + r_\kappa^{[N]} = m.$$

This confirms that the DAE is regular with tractability index $\mu = \kappa$. Applying again Theorem 2.78, we express $r_i = m - l_{\kappa-i} = \bar{s}_{i-1}$ for $i = 1, \dots, \kappa - 1$, further $r_0 = m - (\bar{s}_0 + \bar{a}_0) = \bar{r}_0$, and this completes the proof of (1).

(2) This is a consequence of assertion (1), and the fact that each regular DAE with tractability index μ can be transformed into T-canonical form (with $\kappa = \mu$, cf. Theorem 2.65). □

2.10.2 Strangeness reduction

The original strangeness index concept is a special reduction technique for standard form DAEs (2.112)

$$E(t)x'(t) + F(t)x(t) = q(t)$$

with sufficiently smooth coefficients on a compact interval $\mathcal{I}$. We repeat the basic reduction step from [127]. For more details and a comprehensive discussion of reduction techniques we refer to [130] and [189].

As mentioned before, the strangeness index is supported by several constant-rank conditions. In particular, the matrix E in (2.112) is assumed to have constant rank $\bar{r}$. This allows us to construct continuous injective matrix functions T, Z, and $\bar{T}$ such that

$$\operatorname{im} T = \ker E, \quad \operatorname{im} \bar{T} = (\ker E)^{\perp}, \quad \operatorname{im} Z = (\operatorname{im} E)^{\perp}.$$

The columns of $T, \bar{T}$, and Z are basis functions of the corresponding subspaces. Supposing Z^*FT to have constant rank $\bar{a}$, we find a continuous injective matrix function V such that

$$\operatorname{im} V = (\operatorname{im} Z^*FT)^{\perp}.$$

If, additionally, $V^*Z^*F\bar{T}$ has constant rank $\bar{s}$, then one can construct pointwise nonsingular matrix functions K and L, such that the transformation $x = K\tilde{x}$, and scaling the DAE (2.112) by L leads to

$$\begin{bmatrix} I_{\bar{s}} & & & & \\ & I_{\bar{d}} & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix} \tilde{x}' + \begin{bmatrix} 0 & \tilde{F}_{1,2} & 0 & \tilde{F}_{1,4} & \tilde{F}_{1,5} \\ 0 & 0 & 0 & \tilde{F}_{2,4} & \tilde{F}_{2,5} \\ 0 & 0 & I_{\bar{a}} & 0 & 0 \\ I_{\bar{s}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tilde{x} = Lq, \tag{2.142}$$

with $\bar{d} := \bar{r} - \bar{s}$.
The system (2.142) consists of $m = \bar{s} + \bar{d} + \bar{a} + \bar{s} + \bar{u}$ equations, $\bar{u} := m - \bar{r} - \bar{a} - \bar{s}$. The construction of K and L involves three smooth factorizations of matrix functions and the solution of a classical linear IVP (see [130]).
The fourth equation in (2.142) is simply $\tilde{x}_1 = (Lq)_4$, which gives rise to replacement of the derivative $\tilde{x}_1'$ in the first line by $(Lq)_4'$. Doing so we attain the new DAE

$$\underbrace{\begin{bmatrix} 0 & & & & \\ & I_{\bar{d}} & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}}_{E_{new}} \tilde{x}' + \underbrace{\begin{bmatrix} 0 & \tilde{F}_{1,2} & 0 & \tilde{F}_{1,4} & \tilde{F}_{1,5} \\ 0 & 0 & 0 & \tilde{F}_{2,4} & \tilde{F}_{2,5} \\ 0 & 0 & I_{\bar{a}} & 0 & 0 \\ I_{\bar{s}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{F_{new}} \tilde{x} = Lq - \begin{bmatrix} (Lq)_4' \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{2.143}$$

which is expected to have a lower index since the mentioned differentiation of $\tilde{x}_1$ is carried out analytically.

This *reduction step* is supported by the three rank conditions

$$\operatorname{rank} E = \bar{r}, \quad \operatorname{rank} Z^* F T = \bar{a}, \quad \operatorname{rank} V^* Z^* F \bar{T} = \bar{s}. \tag{2.144}$$

The following proposition guarantees these constant-rank conditions to be valid, if the DAE under consideration is regular in the tractability sense.

Proposition 2.80. *Let the DAE (2.112) be regular with tractability index μ and characteristic values $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu$. Then the constant-rank conditions (2.144) are valid,*

$$\bar{r} = r_0, \quad \bar{a} = r_1 - r_0, \quad \bar{s} = m - r_1,$$

so that the reduction step is feasible.

Proof. We choose symmetric projector functions $\mathcal{W}_0$, Q_0 and $\mathcal{W}_1$, and verify the relations

$$\operatorname{rank} Z^* B T = \operatorname{rank} \mathcal{W}_0 B Q_0 = r_1 - r_0, \quad \operatorname{rank} V^* Z^* F \bar{T} = \operatorname{rank} \mathcal{W}_1 B = m - r_1.$$

□

The reduction from $\{E, F\}$ to $\{E_{new}, F_{new}\}$ can be repeated as long as the constant-rank conditions are given. This leads to an iterative reduction procedure. One starts with $\{E_0, F_0\}$:=$\{E, F\}$ and forms, for each $i \geq 0$, a new pair $\{E_{i+1}, F_{i+1}\}$ to $\{E_i, F_i\}$. This works as long as the three constant-rank conditions

$$\bar{r}_i = \operatorname{rank} E_i, \quad \bar{a}_i = \operatorname{rank} Z_i^* F_i T_i, \quad \bar{s}_i = \operatorname{rank} V_i^* Z_i^* F_i \bar{T}_i, \tag{2.145}$$

hold true.
The *strangeness index* $\zeta \in \mathbb{N} \cup \{0\}$ is defined to be

$$\zeta := \min\{i \in \mathbb{N} \cup \{0\} : \bar{s}_i = 0\}.$$

The strangeness index is the minimal index such that the so-called strangeness disappears. ζ is named the *regular strangeness index*, if there are no so-called underdetermined parts during the iteration such that $\bar{u}_i = 0$ and $\bar{r}_i + \bar{a}_i + \bar{s}_i = m$ for all $i = 0, \ldots, \zeta$.
The values $\bar{r}_i$, $\bar{a}_i$, $\bar{s}_i$, $i \geq 0$, and several additional ones, are called *characteristic values* associated with the strangeness index concept.
If the original DAE (2.112) has regular strangeness index ζ, then the reduction procedure ends up with the DAE

$$\begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} \tilde{\tilde{x}}' + \begin{bmatrix} 0 & 0 \\ 0 & I_a \end{bmatrix} \tilde{\tilde{x}} = \tilde{\tilde{q}},$$

with $d = \bar{d}_\zeta$, $a = \bar{a}_\zeta$.

Remark 2.81. Turn for a moment back to time-invariant DAEs and constant matrix pairs. If the matrix pair $\{E, F\}$ is regular with Kronecker index μ (which is the same as the tractability index μ), and characteristic values $r_0 \le \cdots \le r_{\mu-1} < r_\mu = m$, then this pair has the regular strangeness index $\zeta = \mu - 1$. The characteristic values associated with the strangeness index can then be obtained from the $r_0, \ldots, r_\mu$ by means of the formulas

$$\bar{r}_i = m - \sum_{j=0}^{i} (m - r_j),$$
$$\bar{a}_i = \sum_{j=0}^{i} (m - r_j) - (m - r_{i+1}),$$
$$\bar{s}_i = m - r_{i+1}, \quad i = 0, \ldots, \zeta.$$

The same relations apply to DAEs with time-varying coefficients, too (cf. [139]).

2.10.3 Projector based reduction

Although linear regular higher index DAEs are well understood, they are not accessible to direct numerical integration as pointed out in Chapter 8. Especially for this reason, different kinds of index reduction have their meaning.
We formulate a reduction step for the DAE (2.44) with properly stated leading term, i.e.,

$$A(Dx)' + Bx = q,$$

by applying the projector function $\mathcal{W}_1$ associated with the first terms of the matrix function sequence. $\mathcal{W}_1$ projects along $\operatorname{im} G_1 = \operatorname{im} G_0 \oplus \operatorname{im} \mathcal{W}_0 B Q_0$, and, because of $\operatorname{im} A \subseteq \operatorname{im} G_0 \subseteq \operatorname{im} G_1$, multiplication of the DAE by $\mathcal{W}_1$ leads to the derivative-free equations

$$\mathcal{W}_1 Bx = \mathcal{W}_1 q. \tag{2.146}$$

We emphasize that these equations are just a part of the derivative-free equations, except for the case $\mathcal{W}_0 = \mathcal{W}_1$, which is given in Hessenberg systems, and in Example 2.82 below. The complete set is described by

$$\mathcal{W}_0 Bx = \mathcal{W}_0 q. \tag{2.147}$$

We suppose the matrix function $\mathcal{W}_1$ to have constant rank $m - r_1$, which is at least ensured in regular DAEs. For regular DAEs the subspace

$$S_1 = \ker \mathcal{W}_1 B$$

is known to have dimension r_1.
Introduce a continuous reflexive generalized inverse $(\mathcal{W}_1 B)^-$, and put

$$Z_1 := I - (\mathcal{W}_1 B)^- \mathcal{W}_1 B.$$

Z_1 is a continuous projector function onto S_1. Because of $\mathcal{W}_1 BQ_0 = 0$ the following properties hold true:

$$\begin{aligned}
Z_1 Q_0 &= Q_0 \\
DZ_1 &= DZ_1 P_0 = DZ_1 D^- D \\
DZ_1 D^- &= DZ_1 D^- DZ_1 D^- \\
\operatorname{im} DZ_1 D^- &= \operatorname{im} DZ_1 = DS_1 = DS_0.
\end{aligned}$$

$DZ_1 D^-$ is a priori a continuous projector function. Assuming the DAE coefficients to be sufficiently smooth, it becomes continuously differentiable, and we do so. In consequence, for each function $x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ it follows that

$$DZ_1 x = DZ_1 D^- Dx \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n), \quad D(I - Z_1)x = Dx - DZ_1 x \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n),$$

which allows us to write the DAE as

$$A(DZ_1 x)' + A(D(I - Z_1)x)' + Bx = q. \tag{2.148}$$

Equation (2.146) is consistent, since, for reasons of dimensions, $\operatorname{im} \mathcal{W}_1 B = \operatorname{im} \mathcal{W}_1$. It follows that

$$(I - Z_1)x = (\mathcal{W}_1 B)^- \mathcal{W}_1 q. \tag{2.149}$$

This allows us to remove the derivative $(D(I - Z_1)x)'$ from the DAE, and to replace it by the exact solution part derived from (2.146). The resulting new DAE

$$A(DZ_1 x)' + Bx = q - A(D(\mathcal{W}_1 B)^- \mathcal{W}_1 q)'$$

has no properly stated lading term. This is why we express $A(DZ_1 x)' = A\{DZ_1 D^-(DZ_1 x)' + (DZ_1 D^-)' DZ_1 x\}$, and turn to the new DAE with a properly stated leading term

$$\underbrace{ADZ_1 D^-}_{A_{new}}(\underbrace{DZ_1}_{D_{new}} x)' + \underbrace{(A(DZ_1 D^-)' DZ_1 + B)}_{B_{new}} x = q - A(D(\mathcal{W}_1 B)^- \mathcal{W}_1 q)' \tag{2.150}$$

which has the same solutions as the original DAE (2.44) has, and which is expected to have a lower index (cf. [138]).

Example 2.82 (Index reduction step). We reconsider the DAE (2.10) from Example 2.4,

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & -t & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{A(t)} \Bigg(\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} x(t) \Bigg)' + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -t & 1 \end{bmatrix}}_{B(t)} x(t) = q(t), \quad t \in \mathbb{R},$$

where an admissible matrix function sequence for this DAE is generated. This DAE is regular with tractability index 3. Now compute

$$\mathcal{W}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{W}_1 B(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -t & 1 \end{bmatrix}.$$

Since $\mathcal{W}_1 B$ is already a projector function, we can set $(\mathcal{W}_1 B)^- = \mathcal{W}_1 B$. This implies

$$Z_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 0 \end{bmatrix}, \quad D(t) Z_1(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 0 \end{bmatrix},$$

and finally the special DAE (2.150)

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_{new}(t)} \Bigg(\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 0 \end{bmatrix}}_{D_{new}(t)} x(t) \Bigg)' + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -t & 1 \end{bmatrix}}_{B_{new}(t)} x(t) = \begin{bmatrix} q_1(t) \\ q_2(t) - q_3'(t) \\ q_3(t) \end{bmatrix}, \quad t \in \mathbb{R},$$

which is indeed regular with tractability index 2. □

For the special choice $(\mathcal{W}_1 B)^- = (\mathcal{W}_1 B)^+$, the resulting Z_1 is the orthoprojector function onto S_1. This version is the counterpart to the strangeness reduction step from Subsection 2.10.2.

At first glance it seems to be somewhat arbitrary to figure out just the equations (2.146) for reduction. However, after the explanations below it will be seen as a nice option.

An analogous reduction step can be arranged by choosing the complete set of derivative-free equations (2.147) as a candidate. For regular DAEs, the subspace $\ker \mathcal{W}_0 B = S_0$ has dimension r_0, and we again obtain consistency, as well as the projector $Z_0 := I - (\mathcal{W}_0 B)^- \mathcal{W}_0 B$ onto S_0. From (2.147) it follows that

$$(I - Z_0) x = (\mathcal{W}_0 B)^- \mathcal{W}_0 q.$$

Now we need a smoother solution x to be able to differentiate this expression. To be more transparent we assume at least D and Z_0, as well as the solution x, to be continuously differentiable, and turn to the standard form

$$\underbrace{AD}_{E} x' + \underbrace{(B - AD')}_{F} x = q.$$

Here we express

$$x' = (Z_0 x)' + ((\mathcal{W}_0 B)^- \mathcal{W}_0 q)' = Z_0 x' + Z_0' x + ((\mathcal{W}_0 B)^- \mathcal{W}_0 q)',$$

such that we arrive at the new DAE

$$\underbrace{E Z_0}_{E_{new}} x' + \underbrace{(F + E Z_0')}_{F_{new}} x = q - E((\mathcal{W}_0 B)^- \mathcal{W}_0 q)'. \tag{2.151}$$

This kind of reduction is in essence the procedure described in [189]. The description in [189] concentrates on the coefficient pairs, and one turns to a condensed version of the pair $\{EZ_0, (I-\mathcal{W}_0)(F+EZ_0')\}$.

In the following we do not provide a precise proof of the index reduction, but explain the idea behind it. Assume the DAE (2.44) to be regular with tractability index μ and characteristic values $r_0 \leq \cdots \leq r_{\mu-1} = r_\mu = m$, and take a further look to the completely decoupled version consisting of the IERODE (2.51) and the subsystem (cf. (2.63))

$$\mathcal{N}(\mathcal{D}v)' + \mathcal{M}v = \mathcal{L}q. \tag{2.152}$$

This subsystem comprises the inherent differentiations. It reads in detail

$$\begin{bmatrix} 0 & \mathcal{N}_{0,1} & \cdots & \mathcal{N}_{0,\mu-1} \\ & 0 & \ddots & \vdots \\ & & \ddots & \mathcal{N}_{\mu-2,\mu-1} \\ & & & 0 \end{bmatrix} \begin{bmatrix} 0 \\ (D\Pi_0 Q_1 x)' \\ \vdots \\ (D\Pi_{\mu-2}Q_{\mu-1}x)' \end{bmatrix} \tag{2.153}$$

$$+ \begin{bmatrix} I & \mathcal{M}_{0,1} & \cdots & \mathcal{M}_{0,\mu-1} \\ & I & \ddots & \vdots \\ & & \ddots & \mathcal{M}_{\mu-2,\mu-1} \\ & & & I \end{bmatrix} \begin{bmatrix} Q_0 x \\ \Pi_0 Q_1 x \\ \vdots \\ \Pi_{\mu-2}Q_{\mu-1}x \end{bmatrix} = \begin{bmatrix} \mathcal{L}_0 q \\ \mathcal{L}_1 q \\ \vdots \\ \mathcal{L}_{\mu-1} q \end{bmatrix}.$$

We see that if we replace the derivative term $(D\Pi_{\mu-2}Q_{\mu-1}x)'$ by its exact solution part $(D\mathcal{L}_{\mu-1}q)'$ we arrive at the system

$$\mathcal{N}_{new} \begin{bmatrix} 0 \\ (D\Pi_0 Q_1 x)' \\ \vdots \\ (D\Pi_{\mu-3}Q_{\mu-2}x)' \\ 0 \end{bmatrix} + \begin{bmatrix} I & \mathcal{M}_{0,1} & \cdots & \mathcal{M}_{0,\mu-1} \\ & I & \ddots & \vdots \\ & & \ddots & \mathcal{M}_{\mu-2,\mu-1} \\ & & & I \end{bmatrix} \begin{bmatrix} Q_0 x \\ \Pi_0 Q_1 x \\ \vdots \\ \Pi_{\mu-2}Q_{\mu-1}x \end{bmatrix} \tag{2.154}$$

$$= \begin{bmatrix} \mathcal{L}_0 q - \mathcal{N}_{0,\mu-1}(\mathcal{L}_{\mu-1}q)' \\ \mathcal{L}_1 q - \mathcal{N}_{1,\mu-1}(\mathcal{L}_{\mu-1}q)' \\ \vdots \\ \mathcal{L}_{\mu-2} q - \mathcal{N}_{\mu-2,\mu-1}(\mathcal{L}_{\mu-1}q)' \\ \mathcal{L}_{\mu-1} q \end{bmatrix}.$$

While the matrix function $\mathcal{N}$ has nilpotency index μ, the new matrix function

$$\mathcal{N}_{new} = \begin{bmatrix} 0 & \mathcal{N}_{0,1} & \cdots & \mathcal{N}_{0,\mu-2} & 0 \\ & 0 & \ddots & \vdots & 0 \\ & & \ddots & \mathcal{N}_{\mu-3,\mu-2} & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{bmatrix}$$

has nilpotency index $\mu - 1$ (cf. Proposition 2.29). That means, replacing the derivative $(D\Pi_{\mu-2}Q_{\mu-1}x)'$ by the true solution term reduces the index by one. Clearly, replacing further derivatives and successively solving the subsystem for $(I-\Pi_{\mu-1})x = Q_0x+\Pi_0Q_1x+\cdots+\Pi_{\mu-2}Q_{\mu-1}x$ reduces the index up to one. We keep in mind that, replacing at least the derivative $(D\Pi_{\mu-2}Q_{\mu-1}x)'$ reduces the index by at least one. However, in practice, we are not given the decoupled system. How can we otherwise make sure that this derivative is replaced?
Consider for a moment the equation

$$\mathcal{W}_{\mu-1}Bx = \mathcal{W}_{\mu-1}q \tag{2.155}$$

that is also a part of the derivative-free equations of our DAE. Since the subspace $S_{\mu-1} = \ker \mathcal{W}_{\mu-1}$ has dimension $r_{\mu-1}$, the matrix function $\mathcal{W}_{\mu-1}B$ has constant rank $m - r_{\mu-1}$, and equation (2.155) is consistent, we obtain with $Z_{\mu-1} := I - (\mathcal{W}_{\mu-1}B)^{-}\mathcal{W}_{\mu-1}B$ a continuous projector function onto $S_{\mu-1}$, and it follows that

$$(I-Z_{\mu-1})x = (\mathcal{W}_{\mu-1}B)^{-}\mathcal{W}_{\mu-1}q.$$

Since we use completely decoupling projector functions $Q_0,\ldots,Q_{\mu-1}$, we know that $\Pi_{\mu-2}Q_{\mu-1}$ is the projector function onto $\operatorname{im}\Pi_{\mu-2}Q_{\mu-1}$ along $S_{\mu-1}$. Therefore, with $I-Z_{\mu-1}$ and $\Pi_{\mu-2}Q_{\mu-1}$ we have two projector functions along $S_{\mu-1}$. This yields

$$I-Z_{\mu-1} = (I-Z_{\mu-1})\Pi_{\mu-2}Q_{\mu-1}, \quad \Pi_{\mu-2}Q_{\mu-1} = \Pi_{\mu-2}Q_{\mu-1}(I-Z_{\mu-1}),$$

and therefore, by replacing $(D(I-Z_{\mu-1})x)'$ we replace at the same time $(D\Pi_{\mu-2}Q_{\mu-1}x)'$. This means that turning from the original DAE (2.44) to

$$ADZ_{\mu-1}D^{-}(DZ_{\mu-1}x)' + (A(DZ_{\mu-1}D^{-})'DZ_{\mu-1}+B)x = q - A(D(\mathcal{W}_{\mu-1}B)^{-}\mathcal{W}_{\mu-1}q)'$$

indeed reduces the index by one. However, the use of $Z_{\mu-1}$ is rather a theoretical option, since $\mathcal{W}_{\mu-1}$ is not easy to obtain. The point is that working instead with (2.146) and Z_1 as described above, and differentiating the extra components $D(I-Z_1)x$, includes the differentiation of the component $D(I-Z_{\mu-1})x$ as part of it. In this way, the reduction step from (2.44) to (2.150) seems to be a reasonable compromise from both theoretical and practical viewpoints.
At this point we emphasize that there are various possibilities to compose special reduction techniques.

2.11 Generalized solutions

We continue to consider linear DAEs

$$A(t)(D(t)x(t))' + B(t)x(t) = q(t), \quad t \in \mathcal{I}, \tag{2.156}$$

with coefficients $A \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^n, \mathbb{R}^m))$, $D \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^m, \mathbb{R}^n))$, $B \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^m))$. $\mathcal{I} \subseteq \mathbb{R}$ denotes an interval. Here we focus on IVPs. Let $t_0 \in \mathcal{I}$ be fixed. We state the initial condition in the form

$$Cx(t_0) = z, \tag{2.157}$$

by means of a matrix $C \in L(\mathbb{R}^m, \mathbb{R}^d)$ which satisfies the condition

$$C = CD(t_0)^- D(t_0), \tag{2.158}$$

and which is further specified below (cf. Theorem 2.52) where appropriate.
A classical solution of the IVP (2.156), (2.157) is a continuous function x which possesses a continuously differentiable component Dx and satisfies the initial condition as well as pointwise the DAE. Excitations corresponding to classical solutions are at least continuous.

2.11.1 *Measurable solutions*

A straightforward generalization is now to turn to measurable solution functions x such that the part Dx is absolutely continuous, the initial condition makes sense owing to condition (2.158), and the DAE is satisfied for almost every $t \in \mathcal{I}$. The corresponding right-hand sides q are also measurable functions.
DAEs with excitations $q \in L_2(\mathcal{I}, \mathbb{R}^m)$ result, e.g., from Galerkin approximations of PDAEs (cf. [207], [203]). Furthermore, in optimization problems one usually applies measurable control functions.

We point out that regularity of the DAE, its characteristic values and the tractability index are determined from the coefficient functions A, D and B alone. Also the decoupling procedure is given in terms of these coefficient functions. Therefore, the regularity notion, the tractability index, characteristic values and the decoupling procedure retain their meaning also if we change the nature of the solutions and excitations.

We use the function space

$$H_D^1(\mathcal{I}, \mathbb{R}^m) := \{x \in L_2(\mathcal{I}, \mathbb{R}^m) : Dx \in H^1(\mathcal{I}, \mathbb{R}^n)\}$$

to accommodate the generalized solutions. For $x \in H_D^1(\mathcal{I}, \mathbb{R}^m)$ the resulting defect $q := A(Dx)' + Bx$ belongs to $L_2(\mathcal{I}, \mathbb{R}^m)$. Conversely, given an excitation $q \in L_2(\mathcal{I}, \mathbb{R}^m)$, it seems to make sense if we ask for IVP solutions from $H_D^1(\mathcal{I}, \mathbb{R}^m)$. The following proposition is a counterpart of Proposition 2.50. Roughly speaking, in higher index cases, excitations are directed to the inherent regular ODE and the nullspace component only, which is ensured by means of the filtering projector $G_\mu P_1 \cdots P_{\mu-1} G_\mu^{-1}$.

Proposition 2.83. *Let the DAE (2.156) be fine with tractability index μ and characteristic values $0 \le r_0 \le \cdots \le r_{\mu-1} < r_\mu = m$. Put $d = m - \sum_{j=1}^{\mu}(m - r_{j-1})$, Let $Q_0, \ldots, Q_{\mu-1}$ be completely decoupling projectors. Set $V_1 := I$ and $V_\mu := G_\mu P_1 \cdots P_{\mu-1} G_\mu^{-1}$ if $\mu > 1$.*
Let the matrix C in the initial condition 2.157 have the property $\ker C = N_{can}(t_0)$.
Then, for every $z \in \operatorname{im} C$ and $q = V_\mu p$, $p \in L_2(\mathcal{I}, \mathbb{R}^m)$, the IVP (2.156), (2.157) has exactly one solution $x \in H_D^1(\mathcal{I}, \mathbb{R}^m)$.
If, additionally, the component $Q_0 P_1 \cdots P_{\mu-1} G_\mu^{-1} q = Q_0 P_1 \cdots P_{\mu-1} G_\mu^{-1} p$ is continuous, then the solution x is continuous.

Proof. We refer to Section 2.6 for details. Applying the decoupling and regarding condition $q = V_\mu p$, we arrive at the inherent regular ODE

$$u' - (D\Pi_{can} D^-)' u + D\Pi_{can} G_\mu^{-1} B D^- u = D\Pi_{can} G_\mu^{-1} p \tag{2.159}$$

and the solution expression

$$x = D^- u + Q_0 P_1 \cdots P_{\mu-1} G_\mu^{-1} p. \tag{2.160}$$

The initial value problem for (2.159) with the initial condition

$$u(t_0) = D(t_0) C^- z$$

has (cf. [79, pp. 166–167]) a continuous solution u with u' in $L_2(\mathcal{I}, \mathbb{R}^n)$, and which satisfies the ODE for almost all t. Then, by (2.160), x is in $H_D^1(\mathcal{I}, \mathbb{R}^m)$, and the initial condition (2.157) is fulfilled:

$$Cx(t_0) = CD(t_0)^- u(t_0) = CD(t_0)^- D(t_0) C^- z = CC^- z = z.$$

The second part of the assertion follows from the solution representation (2.160). □

If the tractability index equals 1, then $V_1 = I$ is valid, and hence the operator

$$L : H_D^1(\mathcal{I}, \mathbb{R}^m) \to L_2(\mathcal{I}, \mathbb{R}^m), \quad Lx := A(Dx)' + Bx,$$

is surjective. The corresponding IVP operator $\mathcal{L}$ is then a bijection.

If the DAE (2.156) is regular with tractability index 2, then by means of completely decoupling projectors we obtain the solution expression

$$x = D^- u + \Pi_0 Q_1 G_2^{-1} q + Q_0 P_1 G_2^{-1} q + Q_0 Q_1 D^- (D\Pi_0 Q_1 G_2^{-1} q)',$$

where u satisfies

$$u' - (D\Pi_1 D^-)' u + D\Pi_1 G_2^{-1} B D^- u = D\Pi_1 G_2^{-1} q, \quad u(t_0) = D(t_0) C^- z,$$

for the IVP (2.156), (2.157). In this way, the excitation q is supposed to be from the function space

$$\{q \in L_2(\mathcal{I}, \mathbb{R}^m) : D\Pi_0 Q_1 G_2^{-1} q \in H^1(\mathcal{I}, \mathbb{R}^n)\}.$$

Because of $V_2 = G_2 P_1 G_2^{-1}$ and $D\Pi_0 Q_1 G_2^{-1} V_2 = 0$, the excitation $q = V_2 p$ in the above proposition belongs to this space for trivial reasons.
Dealing with piecewise smooth excitations q, the solution expression shows how jumps are passed onto the solution.
We refer to Chapter 12 for a discussion of abstract differential equations, which also includes related results.
In all the above cases, suitably posed initial conditions play their role. If one replaces the initial condition (2.157) by the condition $x(t_0) = x_0$ we used to apply for regular ODEs, and which makes sense only for solutions being continuous, then the value x_0 must be consistent. Otherwise solvability is lost. As discussed in Subsection 2.6.2, a consistent value depends on the excitation.

2.11.2 Distributional solutions

The theory of *distributional solutions* allows us to elude the problem with inconsistent initial values and to consider discontinuous excitations q. We briefly address DAEs having $\mathcal{C}^\infty$-coefficients and a distributional excitation. For facts on generalized functions we refer to [201], [215].

Let $\mathfrak{D}$ denote the space of functions from $\mathcal{C}^\infty(\mathcal{I}, \mathbb{R})$ with compact support in $\mathcal{I}$, and $\mathfrak{D}'$ its dual space. The elements of $\mathfrak{D}'$ are said to be generalized functions or distributions. Denote by $\langle \cdot, \cdot \rangle$ the dual pairing between $\mathfrak{D}'$ and $\mathfrak{D}$.
For $y \in [\mathfrak{D}']^k$ and $\varphi \in [\mathfrak{D}]^k$, $k \in \mathbb{N}$, we define

$$\langle y, \varphi \rangle := \sum_{j=1}^{k} \langle y_j, \varphi_j \rangle.$$

For a matrix function $M \in \mathcal{C}^\infty(\mathcal{I}, L(\mathbb{R}^k, \mathbb{R}^l))$, $l, k \in \mathbb{N}$, and $y \in [\mathfrak{D}']^k$ we define the product $My \in [\mathfrak{D}']^l$ by

$$\langle My, \varphi \rangle = \langle y, M^* \varphi \rangle, \quad \forall \varphi \in [\mathfrak{D}]^l.$$

This product is well defined since $M^* \varphi$ belongs to $[\mathfrak{D}]^k$. For this, the $\mathcal{C}^\infty$ property of M is crucial.
Any distribution $y \in [\mathfrak{D}']^k$ possesses the distributional derivative $y' \in [\mathfrak{D}']^k$ defined by means of

$$\langle y', \varphi \rangle = -\langle y, \varphi' \rangle, \quad \forall \varphi \in [\mathfrak{D}]^k.$$

The product rule $(My)' = M'y + My'$ is valid.

Now we are prepared to consider distributional solutions of the given DAE (2.156) supposing its coefficient functions A, D, B have all entries belonging to $\mathcal{C}^\infty(\mathcal{I}, \mathbb{R})$.
Given a distributional excitation $q \in [\mathfrak{D}']^m$, a distribution $x \in [\mathfrak{D}']^m$ is said to be a *distributional solution* of the (generalized) DAE (2.156) if

$$\langle A(Dx)' + Bx, \varphi \rangle = \langle q, \varphi \rangle, \quad \forall \varphi \in [\mathfrak{D}]^m, \tag{2.161}$$

or, equivalently,

$$\langle x, -D^*(A^*\varphi)' + B^*\varphi \rangle = \langle q, \varphi \rangle, \quad \forall \varphi \in [\mathfrak{D}]^m. \tag{2.162}$$

Since the entries of A, D, B belong to $\mathcal{C}^\infty(\mathcal{I}, \mathbb{R})$, for regular DAEs, all admissible matrix functions and admissible projector functions have those entries, too. And hence the decoupling procedure described in Section 2.4 keeps its value also for the distributional solution. Every regular DAE possesses distributional solutions.

2.12 Notes and references

(1) For constant coefficient DAEs

$$\bar{E}\bar{x}'(t) + \bar{F}\bar{x}(t) = \bar{q}(t), \tag{2.163}$$

the Kronecker index and regularity are well defined via the properties of the matrix pencil $\{\bar{E}, \bar{F}\}$, and these characteristics are of particular importance in view of an appropriate numerical treatment. From about 1970, challenged by circuit simulation problems, numerical analysts and experts in circuit simulation begun to devote much work to the numerical integration of larger systems of implicit ODEs and DAEs (e.g., [86], [64], [202], [89]). In particular, linear variable coefficient DAEs

$$\bar{E}(t)\bar{x}'(t) + \bar{F}(t)\bar{x}(t) = \bar{q}(t) \tag{2.164}$$

were tackled by the implicit Euler method

$$\bar{E}(t_l)\frac{1}{h}(\bar{x}_l - \bar{x}_{l-1}) + \bar{F}(t_l)\bar{x}_l = \bar{q}(t_l).$$

Obviously, for the method to be just feasible, the matrix $\frac{1}{h}\bar{E}(t_l) + \bar{F}(t_l)$ must be nonsingular, but this can be guaranteed for all steps t_l and all sufficiently small stepsizes h, if one requires the so-called *local matrix pencils* $\{\bar{E}(t), \bar{F}(t)\}$ to be regular on the given interval (we mention at this point, that feasibility is by far not sufficient for a numerical integration method to work well). However, it was already discovered in [84] that the local pencils are not at all relevant characteristics of more general DAEs. Except for regular index-1 DAEs, local matrix pencils may change their index and lose their regularity under smooth regular transformations of the variables. That means that the local matrix pencils $\{E(t), F(t)\}$ of the DAE

$$E(t)x'(t) + F(t)x(t) = q(t), \quad t \in \mathcal{I}, \tag{2.165}$$

which result from transforming $\bar{x}(t) = K(t)x(t)$ in the DAE (2.164), with a pointwise nonsingular continuously differentiable matrix function K, may have completely different characteristics from the local pencils $\{\bar{E}(t), \bar{F}(t)\}$. Nevertheless, the DAEs are equivalent, and hence, the local matrix pencils are irrelevant for determining the characteristics of a DAE. The coefficients of equivalent DAEs (2.164) and (2.165) are related by the formulas $E(t) = \bar{E}(t)K(t)$, $F(t) = \bar{F}(t)K(t) + \bar{E}(t)K'(t)$, which gives the impression that one can manipulate the resulting local pencil almost arbitrarily by choosing different transforms K.
In DAEs of the form

$$\bar{A}(t)(\bar{D}(t)\bar{x}(t))' + \bar{B}(t)\bar{x}(t) = \bar{q}(t), \tag{2.166}$$

the transformation $\bar{x}(t) = K(t)x(t)$ leads to the equivalent DAE

$$A(t)(D(t)x(t))' + B(t)x(t) = q(t). \tag{2.167}$$

The coefficients are related by $A(t) = \bar{A}(t)$, $D(t) = \bar{D}(t)K(t)$ and $B(t) = \bar{B}(t)K(t)$, and the local pencils $\{\bar{A}(t)\bar{D}(t), \bar{B}(t)\}$ and $\{A(t)D(t), B(t)\} = \{\bar{A}(t)\bar{D}(t)K(t), \bar{B}(t)K(t)\}$ are now equivalent. However, we do not consider this to justify the local pencils of the DAE (2.167) as relevant carriers of DAE essentials. For the DAE (2.167), also so-called *refactorizations of the leading term* yield equivalent DAEs, and any serious concept incorporates this fact. For instance, inserting $(Dx)' = (DD^+Dx)' = D(D^+Dx)' + D'D^+Dx$ does not really change the DAE (2.167), however, the local matrix pencils may change their nature as the following example demonstrates. This rules out the local pencils again. The DAE

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\tilde{A}} \Bigg(\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -t & 1 \end{bmatrix}}_{\tilde{D}(t)} x(t) \Bigg)' + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -t & 1 \end{bmatrix}}_{\tilde{B}(t)} x(t) = q(t), \quad t \in \mathbb{R}, \tag{2.168}$$

has the local pencil $\{\tilde{A}\tilde{D}(t), \tilde{B}(t)\}$ which is regular with index 3. However, deriving

$$(\tilde{D}(t)x(t))' = (\tilde{D}(t) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t))' = \tilde{D}(t) \Bigg(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t) \Bigg)' + \tilde{D}'(t) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t)$$

yields the equivalent DAE

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & -t & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{A(t)} \Bigg(\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} x(t) \Bigg)' + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -t & 1 \end{bmatrix}}_{B(t)} x(t) = q(t), \quad t \in \mathbb{R}, \tag{2.169}$$

the local matrix pencils $\{A(t)D, B(t)\} = \{E(t), F(t)\}$ of which are singular for all $t \in \mathbb{R}$.
We see, aiming for the characterization of a variable coefficient DAE, that it does not make sense to check regularity and index of the local pencils, neither for standard form DAEs nor for DAEs with properly stated leading term.

(2) Although in applications one commonly already has DAEs with proper leading term or standard form DAEs (2.165) the leading coefficient E of which has constant rank, there might be a different view of the constant-rank requirement for E seeing it as a drawback. In the early work on DAEs (cf. [39, 25]), the *standard canonical form* (SCF) of a DAE plays its role. By definition, the DAE (2.165) is in SCF, if it is in the form

$$\begin{bmatrix} I & 0 \\ 0 & N(t) \end{bmatrix} x'(t) + \begin{bmatrix} W(t) & 0 \\ 0 & I \end{bmatrix} x(t) = q(t), \tag{2.170}$$

where $N(t)$ is strictly lower (or upper) triangular. We emphasize that $N(t)$, and consequently $E(t)$, need not have constant rank on the given interval. Supposing the excitation q and the matrix function N are sufficiently smooth, this DAE has continuously differentiable solutions, and the flow does not show critical behavior.

In contrast, in our analysis, several constant-rank conditions play their role, in particular, each rank-changing point of $A(t)D(t)$ or $E(t)$ is considered as a critical point, that is, as a candidate for a point where something extraordinary with the solutions may happen. We motivate this opinion by Examples 2.69–2.71, among them also DAEs in SCF.

(3) The ambition to allow for matrix coefficients E with variable rank in a more general DAE theory is closely related to the SCF as well as to the derivative array approach (cf. [41]).
Given a DAE (2.165) with $m = k$ and coefficients $E, F \in \mathcal{C}^{2m}(\mathcal{I}, L(\mathbb{R}^m))$, one considers the derivative array system (also, the prolongated or expanded system)

$$\underbrace{\begin{bmatrix} E(t) & 0 & . & . & . & 0 \\ E'(t)+F(t) & E(t) & 0 & . & . & . \\ * & * & E(t) & . & . & . \\ . & . & . & . & . & . \\ * & * & * & * & * & E(t) \end{bmatrix}}_{\mathcal{J}_\kappa(t)} \begin{bmatrix} x^1 \\ x^2 \\ . \\ . \\ x^{\kappa+1} \end{bmatrix} = - \begin{bmatrix} F(t) \\ F'(t) \\ . \\ . \\ F^{(\kappa)}(t) \end{bmatrix} x + \begin{bmatrix} q(t) \\ q'(t) \\ . \\ . \\ q^{(\kappa)}(t) \end{bmatrix}, \tag{2.171}$$

which results from (2.165) by formally differentiating this equation κ times, collecting all these equations, and replacing the derivative values $x^{(j)}(t)$ by jet variables x^j. The $(\kappa+1)m \times (\kappa+1)m$ matrix function $\mathcal{J}_\kappa$ is said to be *smoothly 1-full* on $\mathcal{I}$ ([25, Definition 2.4.7]), if there is a smooth nonsingular matrix function $\mathcal{R}_\kappa$ such that

$$\mathcal{R}_\kappa(t)\mathcal{J}_\kappa(t) = \begin{bmatrix} I_m & 0 \\ 0 & \mathcal{K}(t) \end{bmatrix}.$$

If $\mathcal{J}_\kappa$ is smoothly 1-full, then an explicit vector field can be extracted from the derivative array system (2.171), say

$$x^1 = \mathcal{C}(t)x + \sum_{j=0}^{\kappa} \mathcal{D}_j(t) q^{(j)}(t).$$

The solution set of the DAE (2.165) is embedded into the solution set of the explicit ODE

$$x'(t) = \mathcal{C}(t)x(t) + \sum_{j=0}^{\kappa} \mathcal{D}_j(t) q^{(j)}(t), \tag{2.172}$$

which is called a *completion ODE* associated with the DAE, often also the *underlying ODE*.
In this context, one speaks (cf. [25, 41]) about *solvable systems* (2.165), if for every $q \in \mathcal{C}^m(\mathcal{I}, \mathbb{R}^m)$ there exists at least one solution $x \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^m)$, which is uniquely determined by its value at any $t \in \mathcal{I}$. Any DAE that is transformable into SCF is solvable in this sense. For every such solvable system, there is an index $\kappa \leq m$ such that the derivative array matrix function $\mathcal{J}_\kappa$ has constant rank and is smoothly 1-full. The reverse statement is true under additional assumptions (cf. [25, 41]).

If N in the SCF (2.170) is the zero matrix, then the leading coefficient of this DAE has constant rank. Correspondingly, if the matrix function $\mathcal{J}_1$ has constant rank and is smoothly 1-full on $\mathcal{I}$, then E has constant rank. Namely, we have here

$$\mathcal{R}\mathcal{J}_1 = \mathcal{R}\begin{bmatrix} E & 0 \\ E' + F & E \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \mathcal{K} \end{bmatrix}.$$

The block $\mathcal{K}$ has constant rank since $\mathcal{J}_1$ has. Now, E has constant rank because of

$$\mathcal{R}\begin{bmatrix} 0 \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ \mathcal{K} \end{bmatrix}.$$

It becomes clear that the leading coefficient E of a *solvable system* (2.165) may undergo rank changes only in so-called higher index cases, that is, if $\kappa \geq 2$ is the lowest index such that $\mathcal{J}_\kappa$ has constant rank and is smoothly 1-full.
To illustrate what is going on we revisit the simple SCF-DAE

$$\begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} x' + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x = q$$

given on the interval $\mathcal{I} = [-1, 1]$. The function α is a strictly positive on $(0, 1]$ and vanishes identically on $[-1, 0]$. Suppose α to be four times continuously differentiable. Notice that, in contrast, in Example 2.70 we only need continuous α. We form the derivative array functions

$$\mathcal{J}_1 = \begin{bmatrix} 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & \alpha' & 0 & \alpha \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathcal{J}_2 = \begin{bmatrix} 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \alpha' & 0 & \alpha & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha'' & 1 & \alpha' + \alpha'' & 0 & \alpha \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

These matrix functions have constant-rank two, respectively four. Multiplication by the smooth nonsingular matrix functions

$$\mathcal{R}_1 = \begin{bmatrix} 0 & 0 & 1 & -\alpha' \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -\alpha \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathcal{R}_2 = \begin{bmatrix} 0 & 0 & 1 & -\alpha' & 0 & -\alpha \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -\alpha & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha'' & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

yields

$$\mathcal{R}_1\mathcal{J}_1 = \begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{R}_2\mathcal{J}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \alpha' + \alpha'' & 0 & \alpha \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The derivative array function $\mathcal{J}_2$ is smoothly 1-full on the entire interval $\mathcal{I} = [-1,1]$ independently of the behavior of the function α. On the other hand, 1-fullness on $\mathcal{I}$ does not apply to $\mathcal{J}_1$. However, $\mathcal{J}_1$ is smoothly 1-full on the subinterval $[-1,0)$, where α vanishes identically. It becomes clear that the restriction of the DAE onto a subinterval does not necessarily show the same characteristic. A more rigorous characterization of the DAE would depend on the interval.

We stress once again that we aim for a DAE analysis including a regularity notion, which meets the lowest possible smoothness demands, and that we have good reasons for treating the rank-changing points of the leading coefficient as critical points and for regarding regularity intervals.
From our point of view, *regularity* of linear DAEs comprises the following three main aspects (cf. Definition 2.25):

(a) The homogeneous equation has a solution space of finite dimension d.
(b) Equations restricted to subintervals inherit property (a) with the same d.
(c) Equations restricted to subintervals inherit the further characteristics $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m$.

(4) Regularity is an often applied notion in mathematics to characterize quite diverse features. Also, different regularity notions are already known for linear DAEs.

They refer to different intentions and are not consistent with each other. We pick up some of them.

Repeatedly (e.g., [129, 130]) regularity of linear DAEs is bound to the unique solvability of initial value problems for *every sufficiently smooth excitation* and consistent initial conditions. Note that this property is named *solvability*, e.g., in [41, 25].
In [25] the linear DAE is said to be regular, if the local matrix pencils remain regular, a property that is helpful for numerical integration.
In [189] the ordered matrix function pair $\{E, F\}$ is said to be regular, if $E(t)$ has constant rank $r < m$ and $E(t)E(t)^* + F(t)F(t)^*$ is nonsingular, a property that is useful for the reduction procedure. A DAE showing a regular coefficient pair is then named *reducible*. So, for instance, the constant coefficient pair

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\}$$

is regular in [189], but fails to be regular in [25].
Moreover, apart from higher smoothness demands, *complete reducibility* of the DAE and *complete regularity* of the pair $\{E, F\}$ in [189] are consistent with our regularity notion. Namely, it is proved in [189] by a careful inspection of all involved constant-rank requirements that complete reducibility is in full agreement with the conditions yielding a *well-defined regular strangeness index* (cf. Section 2.10). In turn, as shown in Section 2.10, the rank conditions supporting regularity in the tractability index context are also in agreement with those from the regular strangeness index.

(5) Distributional solutions of linear DAEs with constant coefficients were studied very early, e.g., [53]. Generalized (distributional) solutions of linear DAEs with smooth variable coefficients have been worked out in [187] (also [189, Chapter III]), whereby so-called impulsive-smooth distributions play a central role. Recently, also time-varying DAEs with discontinuous coefficients are on the agenda, see [208].

(6) Further solution generalizations such as *weak solutions* result from the special settings of partial differential-algebraic equations (PDAEs) and abstract differential-algebraic equations (ADAEs), see, e.g., [207, 191] and references therein.

(7) There are simple interrelations between standard form DAEs (2.165) and DAEs (2.167) with proper leading term.
If D is continuously differentiable, we rewrite the DAE (2.167) as

$$A(t)D(t)x'(t) + (B(t) + A(t)D'(t))x(t) = q(t), \tag{2.173}$$

which has standard form. If equation (2.167) has a properly stated leading term, the resulting matrix function $E = AD$ has constant-rank and the variable subspace $\ker E = \ker D$ is a $\mathcal{C}^1$-subspace.
Conversely, if a standard form DAE (2.165) with a constant rank matrix function E is given and, additionally, $\ker E$ is a $\mathcal{C}^1$-subspace, then, taking a continuously differentiable projector valued function P, $\ker P = \ker E$, we may write

$Ex' = EPx' = E(Px)' - EP'x$. In this way we obtain

$$E(t)(P(t)x(t))' + (F(t) - E(t)P'(t))x(t) = q(t), \tag{2.174}$$

which is a DAE with properly stated leading term. Now it is evident that any DAE (2.167) with a properly stated leading term and a continuously differentiable matrix function D yields a standard form DAE (2.165) such that the leading coefficient E has constant rank and $\ker E$ is a $\mathcal{C}^1$-subspace, and vice versa.

Moreover, there are various possibilities to factorize a given matrix function E, and to rewrite a standard form DAE as a DAE with proper leading term.
If the matrix function E itself is continuously differentiable and has constant rank, then its nullspace is necessarily a $\mathcal{C}^1$-subspace, so that we may use equation (2.174). Additionally in this case, by taking any continuously differentiable generalized inverse E^- and by writing $Ex' = EE^-Ex' = EE^-(Ex)' - EE^-E'x$ we form

$$E(t)E(t)^-(E(t)x(t))' + (F(t) - E(t)E(t)^-E'(t))x(t) = q(t)$$

which is also a DAE with properly stated leading term.
Furthermore, very often the original DAE consists of two kinds of equations, those containing derivatives and those which are derivative-free. Then, the matrix function E has the special form

$$E(t) = \begin{bmatrix} E_1(t) \\ 0 \end{bmatrix}, \quad \operatorname{rank} E_1(t) = \operatorname{rank} E(t),$$

or can be easily brought into this form. In this case, we can simply turn to

$$\begin{bmatrix} I \\ 0 \end{bmatrix} \big(E_1(t)x(t)\big)' + \Big(F(t) - \begin{bmatrix} E_1'(t) \\ 0 \end{bmatrix}\Big)x(t) = q(t).$$

We also point out the following full-rank factorization on a compact interval $\mathcal{I}$, which is provided by a continuous singular value decomposition (e.g. [49]),

$$E(t) = \begin{bmatrix} U_{11}(t) & U_{12}(t) \\ U_{21}(t) & U_{22}(t) \end{bmatrix} \begin{bmatrix} \Sigma(t) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{11}(t) & V_{12}(t) \\ V_{21}(t) & V_{22}(t) \end{bmatrix}^* = \underbrace{\begin{bmatrix} U_{11}(t) \\ U_{21}(t) \end{bmatrix} \Sigma(t)}_{A(t)} \underbrace{\begin{bmatrix} V_{11}^*(t) & V_{21}^*(t) \end{bmatrix}}_{D(t)},$$

$\operatorname{rank} \Sigma(t) = \operatorname{rank} E(t) =: r$, $n = r$. The factors U, Σ and V are continuously differentiable, supposing E is so. Then, $A(t)$ has full column rank n and $D(t)$ has full row rank n.

As in the constant coefficient case, the special form of the factorization does not matter for the nature of the solution. Only the nullspace $\ker D = \ker E$ specifies what a solution is. Namely, for every two matrix functions $D \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n)$ and $\bar{D} \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^{\bar{n}})$ with constant rank and the common nullspace $N := \ker D = \ker \bar{D}$, it holds that

$$\mathcal{C}^1_{\bar{D}}(\mathcal{I}, \mathbb{R}^m) = \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m).$$

Since the Moore–Penrose inverses D^+ and $\bar{D}^+$ are continuously differentiable, too, for any $x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ we find $\bar{D}x = \bar{D}\bar{D}^+\bar{D}x = \bar{D}D^+Dx \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^{\bar{n}})$, and hence $x \in \mathcal{C}^1_{\bar{D}}(\mathcal{I}, \mathbb{R}^m)$.

(8) Our stability analysis of linear DAEs as well as the stability issues in Part II concerning nonlinear index-1 DAEs carry forward the early ideas of [146, 96] as well as fruitful discussions on a series of special conferences on stability issues. The basic tool of our analysis is the explicit description of the unique IERODE defined by fine decouplings. We emphasize that we do not transform the given DAE, but work with the originally given data. In [17] they try another way of considering DAEs via transformation into SCF and proposing Lyapunov functions.

(9) In Section 2.2 we provide the admissible matrix function sequences and admissible projector functions together with their main properties. This part generalizes the ideas of [167], [170]. While [167], [170] are restricted to regular DAEs, we now give an adequate generalization for systems of k equations for m unknowns.
The new preliminary rearrangement of the DAE terms for better structural insight in Subsection 2.4.1 is also valid for nonregular DAEs. We discuss this topic in Chapter 10 for over- and underdetermined DAEs. We emphasize once again that we only rearrange terms in the given setting, but we do not transform the DAE at all.

The discussion of regular and fine DAEs renews the ideas of [170] and [169], while the discussion of critical points reflects parts of [173, 174, 194], but we apply a relaxed notion of regular points by the introduction of quasi-proper leading terms. [194] is the first monograph offering a comprehensive introduction to the projector based decoupling of regular linear DAEs, both in standard form and with proper leading term. Moreover, this book considers critical points in the context of DAEs having almost overall uniform characteristics.

(10) In the present chapter we describe harmless critical points somewhat loosely as those which disappear in a smoother setting. A precise investigation on the background of the concept of *quasi-regular* DAEs (cf. Chapter 9) can be found in [59].

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