Chapter 2
The Classical Bosonic String

Abstract Even though we will eventually be interested in a quantum theory of interacting strings, it will turn out to be useful to start two steps back and treat the free classical string. We will set up the Lagrangian formalism which is essential for the path integral quantization which we will treat in Chap. 3. We will then solve the classical equations of motion for single free closed and open strings. These solutions will be used for the canonical quantization which we will discuss in detail in the next chapter.

2.1 The Relativistic Particle

Before treating the relativistic string we will, as a warm up exercise, first study the free relativistic particle of mass m moving in a d-dimensional Minkowski space-time. Its action is simply the length of its world-line[1]

$$S = -m \int_{s_0}^{s_1} dx = -m \int_{\tau_0}^{\tau_1} d\tau \left[-\frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \eta_{\mu\nu} \right]^{1/2} , \tag{2.1}$$

where τ is an arbitrary parametrization along the world-line, whose embedding in d-dimensional Minkowski space is described by d real functions $x^\mu(\tau)$, $\mu = 0, \ldots, d-1$. We use the metric $\eta_{\mu\nu} = \text{diag}(-1, +1, \ldots, +1)$. The action (2.1) is invariant under τ-reparametrizations $\tau \to \tilde{\tau}(\tau)$. Under infinitesimal reparametrizations $\tau \to \tau + \xi(\tau)$, x^μ transforms like

$$\delta x^\mu(\tau) = -\xi(\tau)\, \partial_\tau x^\mu(\tau) \,. \tag{2.2}$$

[1]It is easy to generalize the action to the case of a particle moving in a curved background by simply replacing the Minkowski metric $\eta_{\mu\nu}$ by a general metric $G_{\mu\nu}(x)$.

R. Blumenhagen et al., *Basic Concepts of String Theory*, Theoretical and Mathematical Physics, DOI 10.1007/978-3-642-29497-6_2,

The action is invariant as long as $\xi(\tau_0) = \xi(\tau_1) = 0$. The momentum conjugate to $x^\mu(\tau)$ is

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = m \frac{\dot{x}^\mu}{\sqrt{-\dot{x}^2}}, \tag{2.3}$$

where $\dot{x} = \partial_\tau x$ and $\dot{x}^2 = \eta_{\mu\nu} x^\mu x^\nu$. Equation (2.3) immediately leads to the following constraint equation

$$\phi \equiv p^2 + m^2 = 0. \tag{2.4}$$

Constraints which, as the one above, follow from the definition of the conjugate momenta without the use of the equations of motion are called primary constraints. Their number equals the number of zero eigenvalues of the Hessian matrix $\frac{\partial p_\mu}{\partial \dot{x}^\nu} = \frac{\partial^2 L}{\partial \dot{x}^\mu \partial \dot{x}^\nu}$ which, in the case of the free relativistic particle, is one, the corresponding eigenvector being $\dot{x}^\mu$. The absence of zero eigenvalues is necessary (via the inverse function theorem) to express the 'velocities' $\dot{x}^\mu$ uniquely in terms of the 'momenta' and 'coordinates', p_μ and x^μ. Systems where the rank of $\frac{\partial^2 L}{\partial \dot{x}^\mu \partial \dot{x}^\nu}$ is not maximal, thus implying the existence of primary constraints, are called singular. For singular systems the τ-evolution is governed by the Hamiltonian $H = H_{\text{can}} + \sum c_k \phi_k$, where H_{can} is the canonical Hamiltonian, the ϕ_k an irreducible set of primary constraints and the c_k are constants in the coordinates and momenta. This is so since the Hamiltonian is well defined only on the submanifold of phase space defined by the primary constraints and can be arbitrarily extended off that submanifold. For the free relativistic particle we find that $H_{\text{can}} = \frac{\partial L}{\partial \dot{x}^\mu} \dot{x}^\mu - L$ vanishes identically and the dynamics is completely determined by the constraint Eq. (2.4). The condition $H_{\text{can}} \equiv 0$ implies the existence of a zero eigenvalue of the Hessian: $\frac{\partial^2 L}{\partial \dot{x}^\mu \partial \dot{x}^\nu} \dot{x}^\nu = \frac{\partial}{\partial \dot{x}^\mu} H_{\text{can}} = 0$. This is always the case for systems with 'time' reparametrization invariance and follows from the fact that the 'time' evolution of an arbitrary phase-space function $f(x, p)$, given by $\frac{df}{d\tau} = \frac{\partial f}{\partial \tau} + \{f, H\}_{\text{P.B.}}$, should also be valid for $\tilde{\tau} = \tilde{\tau}(\tau)$ on the constrained phase-space; here $\{\ ,\ \}_{\text{P.B.}}$ is the usual Poisson bracket, $\{f, g\}_{\text{P.B.}} = \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} \right)$. From this we also see that a particular choice of the constants c_n corresponds to a particular gauge choice which, for the relativistic particle, means a choice of the 'time' variable τ. We write

$$H = \frac{N}{2m}(p^2 + m^2) \tag{2.5}$$

and find that

$$\frac{dx^\mu}{d\tau} = \{x^\mu, H\}_{\text{P.B.}} = \frac{N}{m} p^\mu = \frac{N \dot{x}^\mu}{\sqrt{-\dot{x}^2}}, \tag{2.6}$$

from which $\dot{x}^2 = -N^2$ follows. For the choice $N = 1$ the parameter τ is the proper time of the particle.

At this point it is appropriate to introduce the concept of first and second class constraints. If $\{\phi_k\}$ is the collection of all constraints and if $\{\phi_a, \phi_k\}_{\text{P.B.}} = 0$, $\forall\, k$ upon application of the constraints, we say that ϕ_a is first class. Otherwise it is called second class. First class constraints are associated with gauge conditions.

For the relativistic particle the constraint given in Eq. (2.4) is trivially first class and reflects τ reparametrization invariance.

Classically, we can describe the free relativistic particle by an alternative action which has two advantages over Eq. (2.1): (1) it does not contain a square root, thus leading to simpler equations of motion and (2) it allows the generalization to the massless case. This is achieved by introducing an auxiliary variable $e(\tau)$, which should, however, not introduce new dynamical degrees of freedom. The action containing x^μ and e is

$$S = \frac{1}{2}\int_{\tau_0}^{\tau_1} e\left(e^{-2}\dot{x}^2 - m^2\right) d\tau\,. \tag{2.7}$$

e plays the role of an ein-bein on the world-line. To see that (2.7) is equivalent to (2.1), we derive the equations of motion

$$\begin{aligned} \frac{\delta S}{\delta e} &= 0 \quad \Rightarrow \quad \dot{x}^2 + e^2 m^2 = 0\,, \\ \frac{\delta S}{\delta x^\mu} &= 0 \quad \Rightarrow \quad \frac{d}{d\tau}\left(e^{-1}\dot{x}^\mu\right) = 0\,. \end{aligned} \tag{2.8}$$

Since the equation of motion for e is purely algebraic, e does not represent a new dynamical degree of freedom. We can solve for e and substitute it back into the action (2.7) to obtain (2.1), thus showing their classical equivalence.[2] We note that since $\frac{\partial^2 L}{\partial\dot{x}^\mu\partial\dot{x}^\nu} = e^{-1}\eta_{\mu\nu}$ has maximal rank, we now do not have primary constraints. The constraint equation $p^2 + m^2 = 0$ does not follow from the definition of the conjugate momenta alone; in addition one has to use the equations of motion. Constraints of this kind are called secondary constraints. But since it is first class, it implies a symmetry. Indeed, the action Eq. (2.7) is invariant under τ reparametrizations under which $x'^\mu(\tau') = x^\mu(\tau)$, $e'(\tau') = (\partial\tau'/\partial\tau)^{-1}e(\tau)$ or, in infinitesimal form with $\tau' = \tau + \xi$

$$\begin{aligned} \delta x^\mu &= -\xi\,\partial_\tau x^\mu\,, \\ \delta e &= -\partial_\tau(\xi e) \end{aligned} \tag{2.9}$$

and we can make a τ-reparametrization to go to the gauge $e = 1/m$. If we then naively used the gauge fixed action to find the equations of motion, we would find

[2] It is important to point out that classical equivalence does not necessarily imply quantum equivalence.

$\ddot{x}^\mu = 0$, whose solutions are all straight lines in Minkowski space, which we know to be incorrect. This simply means that we cannot use the reparametrization freedom to fix e and then forget about it. We rather have to use the gauge fixed equation of motion for e, $T \equiv \dot{x}^2 + 1 = 0$, as a constraint. This excludes all time-like and light-like lines and identifies the parameter τ in this particular gauge as the proper time of the particle. In the massless case we set $e = 1$ and have to supplement the equation $\ddot{x}^\mu = 0$ by the constraint $T \equiv \dot{x}^2 = 0$, which leaves only the light-like world-lines. Note that the equation of motion, $\ddot{x}^\mu = 0$, does not imply $T = 0$, but it implies that $\frac{dT}{d\tau} = 0$, i.e. $T = 0$ is a constraint on the initial data and is conserved.

2.2 The Nambu-Goto Action

Let us now turn to the string. The generalization of Eq. (2.1) to a one-dimensional object is to take as its action the area of the world-sheet Σ swept out by the string, i.e.

$$\begin{aligned} S_{\mathrm{NG}} &= -T \int_\Sigma dA \\ &= -T \int_\Sigma d^2\sigma \left[-\det{}_{\alpha\beta} \left(\frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu} \right) \right]^{1/2} \\ &= -T \int_\Sigma d^2\sigma \left[\left(\dot{X} \cdot X' \right)^2 - \dot{X}^2 X'^2 \right]^{1/2} \\ &\equiv -T \int_\Sigma d^2\sigma \sqrt{-\Gamma}\,, \end{aligned} \tag{2.10}$$

where $\sigma^\alpha = (\sigma, \tau)$ are the two coordinates on the world-sheet; we choose them such that $\tau_i < \tau < \tau_f$ and $0 \leq \sigma < \ell$. The dot denotes derivative with respect to τ and the prime derivative with respect to σ. $X^\mu(\sigma, \tau)$, $\mu = 0, \ldots, d-1$ are maps of the world-sheet into d-dimensional Minkowski space and T is a constant of mass dimension two (mass/length), the string tension. Our conventions are such that X^μ has dimensions of length and so do σ and τ. $\Gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu}$ is the induced metric on the world-sheet, inherited from the ambient d-dimensional Minkowski space through which the string moves and $\Gamma < 0$ is its determinant. The requirement that Γ be negative means that at each point of the world-sheet there is one time-like or light-like and one space-like tangent vector. This is necessary for causal propagation of the string. Requiring $\dot{X}^\mu + \lambda X'^\mu$ to be time-like and space-like when λ is varied gives $\Gamma < 0$. The action Eq. (2.10) was first considered by Nambu and Goto, hence the subscript NG.

One distinguishes between open and closed strings. The world-sheet of a free open string has the topology of a strip while the world-sheet of a closed string has that of a cylinder. The string tension T is the only dimensionful quantity in string theory. Instead of the tension, one also uses the parameter

$$\alpha' = \frac{1}{2\pi T} \tag{2.11}$$

also called the Regge slope. α' has dimension (length)^2. The open and closed string tensions are the same because in the interacting theory an open string can close and become a closed string and vice versa.

It is also common to introduce the string length scale

$$\ell_s = 2\pi\sqrt{\alpha'} \tag{2.12}$$

and the string mass scale

$$M_s = (\alpha')^{-1/2}\,. \tag{2.13}$$

Being the area of the world-sheet, the Nambu-Goto action is invariant under reparametrizations under which X^μ transforms as a scalar[3]

$$\delta X^\mu(\sigma,\tau) = -\xi^\alpha\,\partial_\alpha X^\mu(\sigma,\tau)\,, \tag{2.14}$$

as long as $\xi^a = 0$ on the boundary of the world-sheet. In addition to local coordinate transformations, global Poincaré transformations of the space-time coordinates, $X^\mu \to X^\mu + a^\mu$, are also a symmetry of the action.

To derive the equations of motion for the string we vary its trajectory, keeping initial and final positions fixed, i.e. $\delta X^\mu(\sigma,\tau_i) = 0 = \delta X^\mu(\sigma,\tau_f)$. This gives

$$\frac{\partial}{\partial\tau}\frac{\partial\mathscr{L}}{\partial\dot{X}^\mu} + \frac{\partial}{\partial\sigma}\frac{\partial\mathscr{L}}{\partial X'^\mu} = 0 \tag{2.15}$$

together with the boundary conditions for the open string

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[3]A general tensor density of rank, say (1,1), and weight w transforms under reparametrizations $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma,\tau)$ of the world-sheet as

$$t_\alpha^\beta(\sigma,\tau) \to \tilde{t}_\alpha^\beta(\tilde{\sigma},\tilde{\tau}) = \left|\frac{\partial(\tilde{\sigma},\tilde{\tau})}{\partial(\sigma,\tau)}\right|^w \frac{\partial\sigma^\gamma}{\partial\tilde{\sigma}^\alpha}\frac{\partial\tilde{\sigma}^\beta}{\partial\sigma^\delta} t_\gamma^\delta(\sigma,\tau)\,,$$

where the first factor is the Jacobian of the transformation. For infinitesimal transformations $\tilde{\sigma}^\alpha(\sigma,\tau) \to \sigma^\alpha + \xi^\alpha(\sigma,\tau)$, this gives

$$\delta t_\alpha^\beta(\sigma,\tau) \equiv \tilde{t}_\alpha^\beta(\sigma,\tau) - t_\alpha^\beta(\sigma,\tau) = -(\xi^\gamma\partial_\gamma - w\partial_\gamma\xi^\gamma)t_\alpha^\beta - t_\gamma^\beta\partial_\alpha\xi^\gamma + t_\alpha^\delta\partial_\delta\xi^\beta\,.$$

The generalization to tensors of arbitrary rank is obvious.

$$\frac{\partial \mathscr{L}}{\partial X'^{\mu}} \delta X^{\mu} = 0 \qquad \text{at} \quad \sigma = 0, \ell \tag{2.16}$$

and the periodicity condition for the closed string

$$X^{\mu}(\sigma + \ell, \tau) = X^{\mu}(\sigma, \tau) \,. \tag{2.17}$$

For each coordinate direction μ and at each of the two ends of the open string there are two ways to satisfy the boundary condition of the open string:

1. we may impose Neumann boundary conditions which amounts to requiring δX^{μ} to be arbitrary at the boundary. This requires $\frac{\partial \mathscr{L}}{\partial X'^{\mu}} = 0$. Physically this conditions means that no momentum flows off the end of the string. This will be become clear below.
2. Alternatively, we may impose Dirichlet boundary conditions where we set $\delta X^{\mu} = 0$ at the boundary. In other words, we fix the position of the boundary of the string. Thus Dirichlet boundary condition breaks space-time translational invariance. We will discuss the consequences in Sect. 2.4

Due to the square root in the Lagrangian, the equations of motion are rather complicated. The canonical momentum is

$$\Pi_{\mu} = \frac{\partial \mathscr{L}}{\partial \dot{X}^{\mu}} = -T \frac{(\dot{X} \cdot X')\, X'_{\mu} - (X')^2\, \dot{X}_{\mu}}{\left[(X' \cdot \dot{X})^2 - \dot{X}^2\, X'^2\right]^{1/2}} \,. \tag{2.18}$$

The Hessian $\frac{\partial^2 \mathscr{L}}{\partial \dot{X}^{\nu} \partial \dot{X}^{\mu}} = \frac{\partial}{\partial \dot{X}^{\nu}} \Pi_{\mu}$ has, for each value of σ, two zero eigenvalues with eigenvectors $\dot{X}^{\mu}$ and X'^{μ}. The resulting primary constraints are

$$\Pi_{\mu}\, X'^{\mu} = 0 \tag{2.19}$$

and

$$\Pi^2 + T^2 X'^2 = 0 \,. \tag{2.20}$$

After gauge fixing they become non-trivial constraints on the dynamics and play an important role in string theory, as we will see later. The canonical Hamiltonian, $H_{\text{can}} = \int_0^L d\sigma (\dot{X} \cdot \Pi - \mathscr{L})$ is easily seen to vanish identically and hence the dynamics is completely governed by the constraints.

2.3 The Polyakov Action and Its Symmetries

Due to the occurrence of the square root, the Nambu-Goto action is difficult to deal with. As in the case of the relativistic particle, one can remove the square root at the expense of introducing an additional (auxiliary) field on the world-sheet. This field is a metric $h_{\alpha\beta}(\sigma,\tau)$ on the world-sheet with signature $(-,+)$. In the resulting action the d massless world-sheet scalars X^μ are coupled to two-dimensional gravity $h_{\alpha\beta}$:

$$\begin{aligned} S_{\mathrm{P}} &= -\frac{T}{2}\int_\Sigma d^2\sigma\,\sqrt{-h}\,h^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\eta_{\mu\nu} \\ &= -\frac{T}{2}\int_\Sigma d^2\sigma\,\sqrt{-h}\,h^{\alpha\beta}\,\Gamma_{\alpha\beta}\,, \end{aligned} \tag{2.21}$$

where $h=\det h_{\alpha\beta}$. This form of the string action is the starting point for the path integral quantization of Polyakov, hence the subscript P. Note that the world-sheet metric does not appear with derivatives, in accord with our requirement that it is not dynamical. The components of the metric play the role of Lagrange multipliers which impose the Virasoro constraints which are now no longer primary constraints.

The action is easy to generalize to a string moving in a curved background: one replaces the Minkowski metric $\eta_{\mu\nu}$ by a general metric $G_{\mu\nu}(X)$. In this general form, the action is that of a non-trivial, interacting field theory: a non-linear sigma-model. Choosing $G_{\mu\nu}=\eta_{\mu\nu}$ can be considered as the zeroth order term in a perturbative expansion around a flat background. This is of course a limitation and a complete theory should determine its own background in which the string propagates, much in the same way as in general relativity where the metric of space-time is determined by the matter content according to Einstein's equations. However, at this point this is simply a consequence of how the theory is formulated and it is not an inherent problem. We will discuss strings in non-trivial backgrounds in Chap. 14. For now we use (2.21).

We now define the energy-momentum tensor of the world-sheet theory in the usual way as the response of the system to changes in the metric under which $\delta S_{\mathrm{P}}=\frac{1}{4\pi}\int d^2\sigma\,\sqrt{-h}\,T_{\alpha\beta}\,\delta h^{\alpha\beta}$ ($\delta h^{\alpha\beta}=-h^{\alpha\gamma}h^{\beta\delta}\delta h_{\gamma\delta}$), i.e.

$$T_{\alpha\beta}=\frac{4\pi}{\sqrt{-h}}\frac{\delta S_{\mathrm{P}}}{\delta h^{\alpha\beta}} \tag{2.22}$$

is the world-sheet energy-momentum tensor. Using $\delta h=-h_{\alpha\beta}(\delta h^{\alpha\beta})\,h$ we find

$$T_{\alpha\beta}=-\frac{1}{\alpha'}\left(\partial_\alpha X^\mu\partial_\beta X_\mu-\frac{1}{2}h_{\alpha\beta}\,h^{\gamma\delta}\,\partial_\gamma X^\mu\,\partial_\delta X_\mu\right) \tag{2.23}$$

and the equations of motion are

$$T_{\alpha\beta}=0\,, \tag{2.24a}$$

$$\Box X^\mu=\frac{1}{\sqrt{-h}}\,\partial_\alpha(\sqrt{-h}\,h^{\alpha\beta}\,\partial_\beta X^\mu)=0 \tag{2.24b}$$

with the appropriate boundary and periodicity conditions:

$$X^\mu(\tau, \sigma + \ell) = X^\mu(\tau, \sigma) \tag{2.25}$$

for the closed string and

$$n^\alpha \partial_\alpha X^\mu \, \delta X_\mu|_{\sigma=0,\ell} = 0 \tag{2.26}$$

for the open string. Here n^α is a normal vector at the boundary. We require the boundary condition at each end of the string separately, since locality demands that we take δX^μ independently at the two ends.

Energy-momentum conservation, $\nabla^\alpha T_{\alpha\beta} = 0$, which is a consequence of the diffeomorphism invariance of the Polyakov action, is easily verified with the help of the equations of motion for X^μ. ∇_α is a covariant derivative with the usual Christoffel connection $\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} h^{\gamma\delta}(\partial_\alpha h_{\delta\beta} + \partial_\beta h_{\alpha\delta} - \partial_\delta h_{\alpha\beta})$. From the vanishing of the energy-momentum tensor we derive $\det_{\alpha\beta}(\partial_\alpha X^\mu \partial_\beta X_\mu) = \frac{1}{4} h (h^{\gamma\delta} \partial_\gamma X_\mu \partial_\delta X^\mu)^2$ which, when inserted into S_{P}, shows the classical equivalence of the Polyakov and Nambu-Goto actions.

One checks that the constraints, Eqs. (2.19), (2.20), which were primary in the Nambu-Goto formulation, follow here only if we use the equation of motion $T_{\alpha\beta} = 0$, i.e. they are secondary. This is the same situation which we encountered in the case of the relativistic particle.

Note that we have introduced two metrics on the world-sheet, namely the metric inherited from the ambient space, i.e. the induced metric, $\Gamma_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$ which enters the Nambu-Goto action and the intrinsic metric $h_{\alpha\beta}$ which appears in the Polyakov action. They are, a priori, unrelated. The Polyakov action is not the area of the world-sheet measured with the intrinsic metric, which would simply be $\int d^2\sigma \sqrt{-h}$ and could be added to S_{P} as a cosmological term (see below). However, for any real symmetric 2×2 matrix A we have the inequality $(\operatorname{tr} A)^2 \geq 4 \det A$ with equality for $A \propto \mathbf{1}$. With the choice $A^\alpha{}_\beta = h^{\alpha\gamma}\Gamma_{\gamma\beta}$ it follows that $S_{\mathrm{P}} \geq S_{\mathrm{NG}}$. Equality holds if and only if $h_{\alpha\beta} \propto \Gamma_{\alpha\beta}$, i.e. if the two metrics are conformally related. This is the case if the equation of motion for $h_{\alpha\beta}$, Eq. (2.24a), is satisfied.

We can now ask whether there are other terms one could add to S_{P}. If we restrict ourselves to closed strings moving in Minkowski space-time without any other background fields, the only possibilities compatible with d-dimensional Poincaré invariance and power counting renormalizability (at most two derivatives) of the two-dimensional theory are[4]

$$S_1 = \lambda_1 \int_\Sigma d^2\sigma \, \sqrt{-h} \tag{2.27}$$

[4]For the open string with boundary $\partial\Sigma$ there are further possible terms besides S_1 and S_2, which are defined on the boundary of the world-sheet: $S_3 = \lambda_3 \int_{\partial\Sigma} ds$ and $S_4 = \lambda_4 \int_{\partial\Sigma} k \, ds$. Here k is the extrinsic curvature of the boundary. It turns out that these terms can also be discarded.

which is the cosmological term mentioned above, and

$$S_2 = \frac{\lambda_2}{4\pi}\int_\Sigma d^2\sigma\,\sqrt{-h}R = \lambda_2\,\chi(\Sigma) \tag{2.28}$$

where R is the curvature scalar for the metric $h_{\alpha\beta}$. S_2 is the two-dimensional Gauss-Bonnet term and χ the Euler number of the world-sheet, which is a topological invariant. The integrand is (locally) a total derivative and consequently does not contribute to the classical equations of motion. S_2 does, however, play a role in the organization of string perturbation theory. λ_2 turns out to be the constant background value of the dilaton field Φ, which is one of the massless excitations of the closed string and which couples to the world-sheet via $\frac{1}{4\pi}\int d^2\sigma\sqrt{-h}\Phi R$. Inclusion of the cosmological term S_1 would lead to the equation of motion $T_{\alpha\beta} = -\frac{\lambda_1}{2T}h_{\alpha\beta}$ from which we conclude that $\lambda_1 h^{\alpha\beta}h_{\alpha\beta} = 0$. This is unacceptable unless $\lambda_1 = 0$.[5] We will thus consider the action S_P, Eq. (2.21), which is the action of a collection of d massless real scalar fields (X^μ) coupled to gravity ($h_{\alpha\beta}$) in two dimensions.

Let us now discuss the symmetries of the Polyakov action.

1. Global symmetries:

   - Space-time Poincaré invariance:

$$\begin{aligned}\delta X^\mu &= a^\mu{}_\nu\, X^\nu + b^\mu \qquad\qquad (a_{\mu\nu} = -a_{\nu\mu})\,,\\ \delta h_{\alpha\beta} &= 0\end{aligned} \tag{2.29}$$

2. Local symmetries:

   - Reparametrization invariance

$$\begin{aligned}\delta X^\mu &= -\xi^\alpha\,\partial_\alpha X^\mu\,,\\ \delta h_{\alpha\beta} &= -(\xi^\gamma\,\partial_\gamma h_{\alpha\beta} + \partial_\alpha\xi^\gamma\,h_{\gamma\beta} + \partial_\beta\xi^\gamma\,h_{\alpha\gamma})\\ &= -(\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha)\,,\\ \delta\sqrt{-h} &= -\partial_\alpha(\xi^\alpha\,\sqrt{-h})\,.\end{aligned} \tag{2.30}$$

   - Weyl rescaling

$$\begin{aligned}\delta X^\mu &= 0\,,\\ \delta h_{\alpha\beta} &= 2\Lambda h_{\alpha\beta}\,.\end{aligned} \tag{2.31}$$

[5] Note that inclusion of $S_{1,2}$ breaks classical Weyl invariance. In the quantum theory the regularization procedure leads to an explicit breakdown of Weyl invariance and divergent counter-terms associated with S_1 and S_2 are generated.

Here ξ^α and Λ are arbitrary (infinitesimal) functions of (σ, τ) and $a_{\mu\nu}$ and b_μ are constants. From Eq. (2.29) we see that X^μ is a Minkowski space vector whereas $h_{\alpha\beta}$ is a scalar. Under reparametrizations of the world-sheet, Eq. (2.30), the X^μ are world-sheet scalars, $h_{\alpha\beta}$ a world-sheet tensor and $\sqrt{-h}$ a scalar density of weight -1. The scale transformations of the world-sheet metric, Eq. (2.31), is the infinitesimal version of $h_{\alpha\beta}(\sigma,\tau) \to \Omega^2(\sigma,\tau)h_{\alpha\beta}(\sigma,\tau)$ for $\Omega^2(\sigma,\tau) = e^{2\Lambda(\sigma,\tau)} \sim 1 + 2\Lambda(\sigma,\tau)$.

One immediate important consequence of Weyl invariance of the action is the tracelessness of the energy-momentum tensor:

$$T^\alpha{}_\alpha = h^{\alpha\beta}\, T_{\alpha\beta} = 0 \tag{2.32}$$

which is satisfied by the expression Eq. (2.23) without invoking the equations of motion. It is not difficult to see that this has to be so. Consider an action which depends on a metric and a collection of fields ϕ_i which transform under Weyl rescaling as $h_{\alpha\beta} \to e^{2\Lambda}h_{\alpha\beta}$ and $\phi_i \to e^{d_i\Lambda}\phi_i$. If the action is scale invariant, i.e. if $S[e^{2\Lambda}h_{\alpha\beta}, e^{d_i\Lambda}\phi_i] = S[h_{\alpha\beta},\phi_i]$, then

$$0 = \delta S = \int d^2\sigma \left\{ -2\frac{\delta S}{\delta h^{\alpha\beta}}\, h^{\alpha\beta} + \sum_i d_i \frac{\delta S}{\delta \phi_i}\phi_i \right\} \delta\Lambda\,. \tag{2.33}$$

If we now use the equations of motion for ϕ_i, $\frac{\delta S}{\delta\phi_i} = 0$ and the definition $T_{\alpha\beta} \propto \frac{\delta S}{\delta h^{\alpha\beta}}$, tracelessness of the energy-momentum tensor is immediate. We note that it follows without the use of the equations of motion if and only if $d_i = 0,\ \forall\, i$. This is, for instance, the case for the Polyakov action of the bosonic string (where $\{\phi_i\} = \{X^\mu\}$) but will not be satisfied for the fermionic string in Chap. 7.

The local invariances allow for a convenient gauge choice for the world-sheet metric $h_{\alpha\beta}$, called conformal or orthonormal gauge. Reparametrization invariance is used to choose coordinates such that locally $h_{\alpha\beta} = \Omega^2(\sigma,\tau)\eta_{\alpha\beta}$ with $\eta_{\alpha\beta}$ being the two-dimensional Minkowski metric defined by $ds^2 = -d\tau^2 + d\sigma^2$. It is not hard to show that this can always be done. Indeed, for any two-dimensional Lorentzian metric $h_{\alpha\beta}$, consider two null vectors at each point. In this way we get two vector fields and their integral curves which we label by σ^+ and σ^-. Then $ds^2 = -\Omega^2 d\sigma^+ d\sigma^-$; $h_{++} = h_{--} = 0$ since the curves are null. Now let

$$\sigma^\pm = \tau \pm \sigma\,, \tag{2.34}$$

from which it follows that $ds^2 = \Omega^2(-d\tau^2 + d\sigma^2)$. A choice of coordinate system in which the two-dimensional metric is conformally flat, i.e. in which

$$ds^2 = \Omega^2(-d\tau^2 + d\sigma^2) = -\Omega^2 d\sigma^+ d\sigma^- \tag{2.35}$$

is called a conformal gauge. The world-sheet coordinates $\sigma^\pm$ introduced above are called light-cone, isothermal or conformal coordinates. In these coordinates $\gamma_{\alpha\beta} \equiv \frac{h_{\alpha\beta}}{\sqrt{-h}} = \eta_{\alpha\beta}$. We can now use Weyl invariance to set $h_{\alpha\beta} = \eta_{\alpha\beta}$.

We collect some results about the world-sheet light-cone coordinates (2.34) which we will frequently use below. The components of the Minkowski metric in light-cone coordinates are

$$\begin{aligned} \eta_{+-} &= \eta_{-+} = -\frac{1}{2}, \quad \eta^{+-} = \eta^{-+} = -2\,, \\ \eta_{++} &= \eta_{--} = \eta^{++} = \eta^{--} = 0\,. \end{aligned} \tag{2.36}$$

We will also need

$$\partial_{\pm} = \frac{1}{2}(\partial_{\tau} \pm \partial_{\sigma}) \tag{2.37}$$

and indices are raised and lowered according to

$$\xi^{+} = -2\xi_{-} \quad \text{and} \quad \xi^{-} = -2\xi_{+}\,. \tag{2.38}$$

It is important to realize that reparametrizations which satisfy $\mathscr{L}_{\xi} h_{\alpha\beta} = -(\nabla_{\alpha}\xi_{\beta} + \nabla_{\beta}\xi_{\alpha}) \propto h_{\alpha\beta}$ can be compensated by a Weyl rescaling. Expressed in light-cone coordinates the conformal gauge preserving diffeomorphisms are those which satisfy $\partial_{+}\xi^{-} = \partial_{-}\xi^{+} = 0$, i.e. $\xi^{\pm} = \xi^{\pm}(\sigma^{\pm})$.[6] (Here we have used that $\nabla_{+}\xi_{+} = h_{+-}\nabla_{+}\xi^{-} = h_{+-}\partial_{+}\xi^{-}$ since the only non-vanishing Christoffel symbols in conformal gauge with $\Omega = e^{\Lambda}$ are $\Gamma^{+}_{++} = 2\partial_{+}\Lambda$ and $\Gamma^{-}_{--} = 2\partial_{-}\Lambda$.) Indeed, instead of $\sigma^{\pm}$ we could as well have chosen $\tilde{\sigma}^{\pm} = \tilde{\sigma}^{\pm}(\sigma^{\pm})$ or, in infinitesimal form, $\tilde{\sigma}^{\pm} = \sigma^{\pm} + \xi^{\pm}(\sigma^{\pm})$. Note that the transformation $\sigma^{\pm} \to \tilde{\sigma}^{\pm}(\sigma^{\pm})$ corresponds to $\begin{pmatrix}\tau\\ \sigma\end{pmatrix} \to \begin{pmatrix}\tilde{\tau}\\ \tilde{\sigma}\end{pmatrix} = \frac{1}{2}[\tilde{\sigma}^{+}(\tau+\sigma) \pm \tilde{\sigma}^{-}(\tau-\sigma)]$; i.e. any $\tilde{\tau}$ and $\tilde{\sigma}$ satisfying the two-dimensional wave equation will do the job.

Conformal gauge is unique to two dimensions. In $d > 0$ dimensions a metric $h_{\alpha\beta}$, being symmetric, has $\frac{1}{2}d(d+1)$ independent components. Reparametrization invariance allows to fix d of them, leaving $\frac{1}{2}d(d-1)$ components. In two dimensions this suffices to go to conformal gauge. The Polyakov action then still has one extra local symmetry, namely Weyl transformations, which allow us to eliminate the remaining metric component. This also shows that gravity in two dimensions is trivial in the sense that the graviton can be gauged away completely. For $d > 2$ Weyl invariance, even if present as for instance in conformal gravity, won't suffice to gauge away all metric degrees of freedom.[7]

The argument given above that conformal gauge is always possible was a local statement. We will now set up a global criterion and consider the general case with gauge condition

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[6] After Wick rotation to Euclidean signature on the world-sheet these are conformal transformations. More about this later.

[7] Note that the action for the relativistic particle was not Weyl invariant; there reparametrization invariance was sufficient to eliminate the one metric degree of freedom.

$$h_{\alpha\beta} = e^{2\phi}\,\hat{h}_{\alpha\beta}\,. \tag{2.39}$$

In conformal gauge $\hat{h}_{\alpha\beta} = \eta_{\alpha\beta}$. Under reparametrizations and Weyl rescaling the metric changes as

$$\begin{aligned}\delta h_{\alpha\beta} &= -(\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha) + 2\Lambda\, h_{\alpha\beta} \\ &\equiv -(P\xi)_{\alpha\beta} + 2\tilde{\Lambda}\, h_{\alpha\beta}\,,\end{aligned} \tag{2.40}$$

where the operator P maps vectors into symmetric traceless tensors according to

$$(P\xi)_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha - (\nabla_\gamma\, \xi^\gamma)\, h_{\alpha\beta}\,, \tag{2.41}$$

and we have defined $2\tilde{\Lambda} = 2\Lambda - \nabla_\gamma\, \xi^\gamma$. The decomposition into symmetric traceless and trace part is orthogonal with respect to the inner product $(\delta h^{(1)}|\delta h^{(2)}) = \int d^2\sigma \sqrt{-h} h^{\alpha\gamma} h^{\beta\delta} \delta h^{(1)}_{\alpha\beta} \delta h^{(2)}_{\gamma\delta}$. The trace part of $\delta h_{\alpha\beta}$ can always be cancelled by a suitable choice of Λ. It then follows that for the gauge Eq. (2.39) to be possible globally, there must exist a globally defined vector field ξ^α such that

$$(P\xi)_{\alpha\beta} = t_{\alpha\beta} \tag{2.42}$$

for arbitrary symmetric traceless $t_{\alpha\beta}$. If the operator P has zero modes, i.e. if there exist vector fields ξ_0 such that $P\xi_0 = 0$, then for any solution ξ we also have the solution $\xi + \xi_0$. In this case the gauge fixing is not complete and those reparametrizations which can be absorbed by a Weyl rescaling are still allowed, as we have already seen above.

The adjoint of P, $P^\dagger$, maps traceless symmetric tensors to vectors via

$$(P^\dagger t)_\alpha = -2\nabla^\beta\, t_{\alpha\beta}\,. \tag{2.43}$$

Zero modes of $P^\dagger$ are symmetric traceless tensors which cannot be written as $(P\xi)_{\alpha\beta}$ for any vector field ξ. Indeed, if $(P^\dagger t_0)_\alpha = 0$, then for all ξ^σ, $(\xi, P^\dagger t_0) = (P\xi, t_0) = 0$. This means that zero modes of $P^\dagger$ are metric deformations which cannot be absorbed by reparametrization and Weyl rescaling. If they do not exist, the gauge is possible globally. This applies in particular to the conformal gauge; there the condition is that the equations $\partial_- t_{++} = 0$ and $\partial_+ t_{--} = 0$ have no globally defined solutions. We will further discuss the solutions to these equations in Chap. 6. The equation

$$(P\xi)_{\alpha\beta} = 0 \tag{2.44}$$

is the conformal Killing equation and its solutions are called conformal Killing vectors. In contrast to Killing vectors which generate isometries, conformal Killing vectors generate Weyl rescalings of the metric; in particular, they preserve the conformal gauge.

In conformal gauge the Polyakov action simplifies to

$$\begin{aligned} S_P &= -\frac{T}{2}\int d^2\sigma\, \eta^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X_\mu \\ &= \frac{T}{2}\int d^2\sigma\,(\dot{X}^2 - X'^2) \\ &= 2T\int d^2\sigma\,\partial_+ X\cdot\partial_- X\ . \end{aligned} \tag{2.45}$$

Varying with respect to X^μ such that $\delta X^\mu(\tau_0) = 0 = \delta X^\mu(\tau_1)$ we obtain[8]

$$\delta S_p = T\int d^2\sigma\,\delta X^\mu\,(\partial_\sigma^2 - \partial_\tau^2)\,X_\mu - T\int_{\tau_0}^{\tau_1} d\tau\, X'_\mu\,\delta X^\mu\Big|_{\sigma=0}^{\sigma=\ell}\ . \tag{2.46}$$

The surface term is absent for the closed string for which we impose the periodicity condition[9]

$$\text{(closed string)}\qquad X^\mu(\sigma+\ell) = X^\mu(\sigma)\,. \tag{2.47}$$

To achieve the vanishing of the boundary term for the open string we have to impose either Dirichlet or Neumann boundary conditions for each X^μ and at each of the two ends of the string:

$$\partial_\sigma X^\mu|_{\sigma=0,\ell} = 0 \qquad \text{(Neumann)} \tag{2.48}$$

or (open string)

$$\delta X^\mu|_{\sigma=0,\ell} = 0 \qquad \text{(Dirichlet)}\ . \tag{2.49}$$

The Dirichlet boundary condition means that the end-point of the open string is fixed in space-time. This boundary condition thus breaks space-time Poincaré invariance. As we will discuss below, these boundary conditions have important implications.

The vanishing of (2.46) leads to the following equations of motion

$$(\partial_\sigma^2 - \partial_\tau^2)\,X^\mu = 4\partial_+\partial_- X^\mu = 0 \tag{2.50}$$

which have to be solved subject to (2.47) or (2.48), (2.49).

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[8] One can show that on the strip and the cylinder one can always go to conformal gauge and preserve $0 \le \sigma \le \ell$.

[9] More general periodicity conditions $X^\mu(\sigma+\ell) = M^\mu{}_\nu X^\nu(\sigma)$ for any constant $O(1, d-1)$ matrix M also leave the action invariant. If we want to interpret X^μ as coordinates in Minkowski space, only (2.47) is allowed, i.e. they are the only periodicity conditions which are invariant under d-dimensional Poincaré transformations. When we consider compactifications of the string we will consider so-called twisted boundary conditions for which M is non-trivial.

Eq. (2.50) is the two-dimensional massless wave equation with the general solution

$$X^\mu(\sigma,\tau) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-)\,. \tag{2.51}$$

Here $X^\mu_{L,R}$ are arbitrary functions of their respective arguments, subject only to periodicity or boundary conditions. They describe the "left"- and "right"-moving modes of the string, respectively. In the case of the closed string the left- and right-moving components are completely independent for the unconstrained system, an observation which is crucial for the formulation of the heterotic string. This is however not the case for the open string where the boundary condition mixes left- with right-movers through reflection at the ends of the string. We will present explicit Fourier series solutions for all possible boundary conditions in the next two subsection.

On a solution of the equations of motion we still have to impose the constraints resulting from the gauge fixed equations of motion for the metric: we have to require that the energy-momentum tensor vanishes; i.e.

$$T_{01} = T_{10} = -2\pi T\,(\dot X\cdot X') = 0\,, \tag{2.52a}$$

$$T_{00} = T_{11} = -\pi T\,(\dot X^2 + X'^2) = 0 \tag{2.52b}$$

which can be alternatively expressed as

$$(\dot X \pm X')^2 = 0\,. \tag{2.53}$$

In light-cone coordinates they become

$$T_{++} = -2\pi T(\partial_+ X\cdot\partial_+ X) = 0\,, \tag{2.54a}$$

$$T_{--} = -2\pi T(\partial_- X\cdot\partial_- X) = 0\,, \tag{2.54b}$$

$$T_{+-} = T_{-+} = 0\,, \tag{2.54c}$$

where $T_{++} = \frac{1}{2}(T_{00}+T_{01})$, $T_{--} = \frac{1}{2}(T_{00}-T_{01})$; Eq. (2.54c) expresses the tracelessness of the energy-momentum tensor. In terms of the left- and right-movers the constraints Eqs. (2.54a), (2.54b) become $\dot X_R^2 = \dot X_L^2 = 0$. Energy-momentum conservation, i.e. $\nabla^\alpha T_{\alpha\beta} = 0$ becomes

$$\partial_- T_{++} + \partial_+ T_{-+} = 0\,, \tag{2.55a}$$

$$\partial_+ T_{--} + \partial_- T_{+-} = 0 \tag{2.55b}$$

which, using Eq. (2.54c), simply states that

$$\partial_- T_{++} = 0\,, \tag{2.56a}$$

$$\partial_+ T_{--} = 0 \tag{2.56b}$$

i.e.

$$T_{++} = T_{++}(\sigma^+) \quad \text{and} \quad T_{--} = T_{--}(\sigma^-)\,. \tag{2.57}$$

The conservation equations (2.56) imply the existence of an infinite number of conserved charges. In fact, for any function $f(\sigma^+)$ we have $\partial_-(f(\sigma^+)\,T_{++}) = 0$ and the corresponding conserved charges are

$$L_f = 2T\int_0^\ell d\sigma\; f(\sigma^+)\,T_{++}(\sigma^+) \tag{2.58}$$

and likewise for the right-movers.

The Hamiltonian for the string in conformal gauge is

$$\begin{aligned} H &= \int_0^\ell d\sigma\,(\dot{X}\cdot\Pi - \mathscr{L}) \\ &= \frac{T}{2}\int_0^\ell d\sigma\,(\dot{X}^2 + X'^2) \\ &= T\int_0^\ell d\sigma\,\left((\partial_+X)^2 + (\partial_-X)^2\right)\,, \end{aligned} \tag{2.59}$$

where, as before, the canonical momentum is $\Pi^\mu = \partial\mathscr{L}/\partial\dot{X}_\mu = T\dot{X}^\mu$. We note that the Hamiltonian is just one of the constraints. This was to be expected from our discussion of constrained systems in the context of the relativistic particle. Indeed, we saw that the canonical Hamiltonian derived from the Nambu-Goto action vanishes identically and the τ-evolution is completely governed by the constraints, i.e.

$$H = \int_0^\ell d\sigma\,\left\{N_1(\sigma,\tau)\,\Pi\cdot X' + N_2(\sigma,\tau)\,(\Pi^2 + T^2X'^2)\right\}\,, \tag{2.60}$$

where N_1 and N_2 are arbitrary functions of σ and τ. Using the basic equal τ Poisson brackets

$$\begin{aligned} &\{X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\}_{\text{P.B.}} = \{\Pi^\mu(\sigma,\tau), \Pi^\nu(\sigma',\tau)\}_{\text{P.B.}} = 0\,, \\ &\{X^\mu(\sigma,\tau), \Pi^\nu(\sigma',\tau)\}_{\text{P.B.}} = \eta^{\mu\nu}\,\delta(\sigma-\sigma') \end{aligned} \tag{2.61}$$

we find

$$\dot{X}^\mu = N_1\,X'^\mu + 2\,N_2\,\Pi^\mu \tag{2.62}$$

and

$$\dot{\Pi}^\mu = \partial_\sigma(N_1\,\Pi^\mu + 2T^2\,N_2\,X'^\mu)\,. \tag{2.63}$$

If we choose $N_1 = 0$ and $N_2 = \frac{1}{2T}$, Eqs. (2.62) and (2.63) lead to the equation of motion $(\partial_\sigma^2 - \partial_\tau^2)X^\mu = 0$ which we have obtained previously from the action in conformal gauge. This means that choosing $N_1 = 0$ and $N_2 = \frac{1}{2T}$ is equivalent to fixing the conformal gauge. With this choice for the functions N_1 and N_2 we also get the Hamiltonian (2.59).

In conformal gauge the Poisson brackets are

$$\begin{aligned}\{X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\}_{\text{P.B.}} &= \{\dot{X}^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)\}_{\text{P.B.}} = 0\,, \\ \{X^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)\}_{\text{P.B.}} &= \frac{1}{T}\,\eta^{\mu\nu}\,\delta(\sigma-\sigma')\,. \end{aligned} \tag{2.64}$$

With their help one readily shows that $-T\int \dot{X}\cdot X' d\sigma$ and $\frac{1}{2}T\int(\dot{X}^2 + X'^2)d\sigma$ generate constant σ- and τ-translations, respectively. More generally, using the explicit expression for T_{++} one finds that the charges L_f of Eq. (2.58) generate transformations $\sigma^+ \to \sigma^+ + f(\sigma^+)$, i.e. those reparametrizations which do not lead out of conformal gauge:

$$\{L_f, X(\sigma)\}_{\text{P.B.}} = -f(\sigma^+)\,\partial_+ X(\sigma)\,. \tag{2.65}$$

So far we have only discussed issues connected with world-sheet symmetries. However, invariance under d-dimensional global Poincaré transformations, Eq. (2.29), leads, via the Noether theorem, to two conserved currents; invariance under translations gives the energy-momentum current

$$P^\alpha_\mu = -T\sqrt{h}\,h^{\alpha\beta}\,\partial_\beta X_\mu\ , \tag{2.66}$$

whereas invariance under Lorentz transformations gives the angular momentum current

$$J^\alpha_{\mu\nu} = -T\sqrt{h}\,h^{\alpha\beta}\,(X_\mu\,\partial_\beta X_\nu - X_\nu\,\partial_\beta X_\mu) = X_\mu P^\alpha_\nu - X_\nu P^\alpha_\mu\ . \tag{2.67}$$

Using the equations of motion, it is easy to check conservation of P^α_μ and $J^\alpha_{\mu\nu}$. The total conserved charges (momentum and angular momentum) are obtained by integrating the currents over a space-like section of the world-sheet, say $\tau = 0$. Then the total momentum in conformal gauge is

$$P_\mu = \int_0^\ell d\sigma\ P^\tau_\mu = T\int_0^\ell d\sigma\ \partial_\tau X_\mu(\sigma) \tag{2.68}$$

and the total angular momentum is

$$J_{\mu\nu} = \int_0^\ell d\sigma\ J^\tau_{\mu\nu} = T\int_0^\ell d\sigma\,(X_\mu\partial_\tau X_\nu - X_\nu\partial_\tau X_\mu)\,. \tag{2.69}$$

It is straightforward to see that P_μ and $J_{\mu\nu}$ are conserved for the closed string. Indeed, $\frac{\partial P_\mu}{\partial\tau} = \int_0^\ell d\sigma\, \partial_\tau^2 X_\mu = \int_0^\ell d\sigma\, \partial_\sigma^2 X_\mu = \partial_\sigma X_\mu(\sigma = \ell) - \partial_\sigma X_\mu(\sigma = 0)$ which vanishes for the closed string by periodicity. For the open string it only vanishes if we impose Neumann boundary conditions at both ends. Hence our earlier statement that Neumann boundary conditions have the physical interpretation that no momentum flows off the ends of the string. This is not the case, however, for Dirichlet boundary conditions. They break Poincaré invariance and consequently space-time momentum is not conserved. Conservation of the total angular momentum is also easy to check for closed strings and open strings with Neumann boundary conditions at both ends.

With the help of the Poisson brackets Eq. (2.64) it is straightforward to verify that P^μ and $J^{\mu\nu}$ generate the Poincaré algebra:

$$\begin{aligned}
\{P^\mu, P^\nu\}_{\text{P.B.}} &= 0\,,\\
\{P^\mu, J^{\rho\sigma}\}_{\text{P.B.}} &= \eta^{\mu\sigma}\, P^\rho - \eta^{\mu\rho}\, P^\sigma\,,\\
\{J^{\mu\nu}, J^{\rho\sigma}\}_{\text{P.B.}} &= \eta^{\mu\rho}\, J^{\nu\sigma} + \eta^{\nu\sigma}\, J^{\mu\rho} - \eta^{\nu\rho}\, J^{\mu\sigma} - \eta^{\mu\sigma}\, J^{\nu\rho}\,.
\end{aligned} \tag{2.70}$$

2.4 Oscillator Expansions

Let us now solve the classical equations of motion of the string in conformal gauge, taking into account the boundary conditions. We will do this for the unconstrained system. The constraints then have to be imposed on the solutions. We have to distinguish between the closed and the open string and will treat them in turn.

Closed Strings

The general solution of the two-dimensional wave equation compatible with the periodicity condition $X^\mu(\sigma,\tau) = X^\mu(\sigma+\ell,\tau)$ is[10]

$$X^\mu(\sigma,\tau) = X_R^\mu(\tau-\sigma) + X_L^\mu(\tau+\sigma) \tag{2.71}$$

where

$$X_R^\mu(\tau-\sigma) = \frac{1}{2}(x^\mu - c^\mu) + \frac{\pi\alpha'}{\ell}p^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-\frac{2\pi}{\ell}in(\tau-\sigma)}, \tag{2.72a}$$

[10] A more general condition which guarantees that the boundary term in (2.46) vanishes is $X^\mu(\tau,\sigma) = X^\mu(\tau+\Delta,\sigma+\ell)$. In fact, the periodicity of the solution is not preserved under a world-sheet Lorentz transformation. We can always find a Lorentz frame in which the more general periodicity condition reduces to the usual one.

$$X_L^\mu(\tau+\sigma) = \frac{1}{2}(x^\mu + c^\mu) + \frac{\pi\alpha'}{\ell} p^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \overline{\alpha}_n^\mu \, e^{-\frac{2\pi}{\ell} in(\tau+\sigma)} \tag{2.72b}$$

with $n \in \mathbb{Z}$ and arbitrary Fourier modes α_n^μ and $\overline{\alpha}_n^\mu$. The normalizations have been chosen for later convenience, and we have also introduced the parameters c^μ, which will become relevant in Chap. 10 when we discuss toroidal compactifications. Here we can choose the zero mode part of the expansion (2.72a) and (2.72b) left-right symmetric and set $c^\mu = 0$. Our notation is such that the α_n^μ are positive frequency modes for $n < 0$ and negative frequency modes for $n > 0$. Note that the left- and right-moving parts are only coupled through the zero modes x^μ and p^μ. The requirement that $X^\mu(\sigma,\tau)$ be a real function implies that x^μ and p^μ are real and that

$$\alpha_{-n}^\mu = (\alpha_n^\mu)^* \quad \text{and} \quad \overline{\alpha}_{-n}^\mu = (\overline{\alpha}_n^\mu)^* \,. \tag{2.73}$$

If we define

$$\alpha_0^\mu = \overline{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu \,, \tag{2.74}$$

we can write

$$\partial_- X^\mu = \dot{X}_R^\mu = \frac{2\pi}{\ell}\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{+\infty} \alpha_n^\mu \, e^{-\frac{2\pi}{\ell} in(\tau-\sigma)} \,, \tag{2.75a}$$

$$\partial_+ X^\mu = \dot{X}_L^\mu = \frac{2\pi}{\ell}\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{+\infty} \overline{\alpha}_n^\mu \, e^{-\frac{2\pi}{\ell} in(\tau+\sigma)} \,. \tag{2.75b}$$

From

$$P^\mu = \int_0^\ell d\sigma\, \Pi^\mu = \frac{1}{2\pi\alpha'} \int_0^\ell d\sigma\, \dot{X}^\mu = p^\mu \,, \tag{2.76}$$

we conclude that p^μ is the total space-time momentum of the string. From

$$q^\mu(\tau) \equiv \frac{1}{\ell} \int_0^\ell d\sigma\, X^\mu = x^\mu + \frac{2\pi\alpha'}{\ell} p^\mu \tau \,, \tag{2.77}$$

we learn that x^μ is the 'center of mass' position of the string at $\tau = 0$. Using the expression for the total angular momentum, we find

$$\begin{aligned} J^{\mu\nu} &= \int_0^\ell d\sigma\, (X^\mu \Pi^\nu - X^\nu \Pi^\mu) = \frac{1}{2\pi\alpha'} \int_0^\ell d\sigma\, (X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu) \\ &= l^{\mu\nu} + E^{\mu\nu} + \overline{E}^{\mu\nu} \end{aligned} \tag{2.78}$$

with

$$l^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu \tag{2.79}$$

and

$$E^{\mu\nu} = -i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu) \tag{2.80}$$

with a similar expression for $\overline{E}^{\mu\nu}$.

From the Poisson brackets Eq. (2.64) we derive the brackets for the α_n^μ, $\overline{\alpha}_n^\mu$, x^μ and p^μ:

$$\{\alpha_m^\mu, \alpha_n^\nu\}_{\text{P.B.}} = \{\overline{\alpha}_m^\mu, \overline{\alpha}_n^\nu\}_{\text{P.B.}} = -im\delta_{m+n}\, \eta^{\mu\nu} , \tag{2.81a}$$

$$\{\overline{\alpha}_m^\mu, \alpha_n^\nu\}_{\text{P.B.}} = 0 , \tag{2.81b}$$

$$\{x^\mu, p^\nu\}_{\text{P.B.}} = \eta^{\mu\nu} . \tag{2.81c}$$

We have introduced the notation $\delta_m = \delta_{m,0}$. x^μ and p^μ, the center of mass position and momentum, are canonically conjugate. The Hamiltonian (2.59), expressed in terms of oscillators, is

$$H = \frac{\pi}{\ell} \sum_{n=-\infty}^{+\infty} (\alpha_{-n} \cdot \alpha_n + \overline{\alpha}_{-n} \cdot \overline{\alpha}_n) . \tag{2.82}$$

We have seen above that the constraints (2.54a) and (2.54b), together with energy-momentum conservation, give rise to an infinite number of conserved charges Eq. (2.58), with a similar expression for the right-movers. We now choose for the functions $f(\sigma^\pm)$ a complete set satisfying the periodicity condition appropriate for the closed string: $f_m(\sigma^\pm) = \exp(\frac{2\pi i}{\ell} m\sigma^\pm)$ for all integers m. We then define the Virasoro generators as the corresponding charges at $\tau = 0$ [11]

$$L_n = -\frac{\ell}{4\pi^2} \int_0^\ell d\sigma\; e^{-\frac{2\pi i}{\ell} n\sigma}\; T_{--} = \frac{1}{2} \sum_m \alpha_{n-m} \cdot \alpha_m ,$$

$$\bar{L}_n = -\frac{\ell}{4\pi^2} \int_0^\ell d\sigma\; e^{+\frac{2\pi i}{\ell} n\sigma}\; T_{++} = \frac{1}{2} \sum_m \bar{\alpha}_{n-m} \cdot \bar{\alpha}_m . \tag{2.83}$$

With the help of the representation of the periodic δ-function

[11] Since the Hamiltonian is one of the constraints and the constraints form a closed algebra under Poisson brackets (i.e. they are first class; cf. below), it is clear that the L_n are constant in τ (modulo the constraints); this is indeed easily verified.

$$\frac{1}{\ell}\sum_{n\in\mathbb{Z}} e^{\frac{2\pi i}{\ell}n(\sigma-\sigma')} = \delta(\sigma-\sigma')\,, \tag{2.84}$$

one can invert the above definitions:

$$T_{--}(\sigma) = -\left(\frac{2\pi}{\ell}\right)^2 \sum_n L_n\, e^{\frac{2\pi i}{\ell}n\sigma} \tag{2.85}$$

and likewise for T_{++}.

The L_m's satisfy the reality condition

$$L_n = L_{-n}^*, \quad \overline{L}_n = \overline{L}_{-n}^* \,. \tag{2.86}$$

Comparing with Eq. (2.83), we find that the Hamiltonian is simply

$$H = \frac{2\pi}{\ell}(L_0 + \overline{L}_0)\,. \tag{2.87}$$

The general τ evolution operator would have been $H = \sum_n (c_n L_n + \overline{c}_n \overline{L}_n)$; the choice implied by Eq. (2.87), $c_n = \overline{c}_n = \delta_n$ is the conformal gauge. Since the constraint $T\int_0^\ell d\sigma\, \dot{X}\cdot X' = \frac{2\pi}{\ell}(L_0 - \bar{L}_0)$ generates rigid σ-translations and since on a closed string no point is special, we need to require that $L_0 - \overline{L}_0 = 0$. It is through this condition that the left-movers know about the right-movers. The Virasoro generators satisfy an algebra, called the (centerless) Virasoro algebra:

$$\begin{aligned}
\{L_m, L_n\}_{\text{P.B.}} &= -i(m-n)\, L_{m+n}\,, \\
\{\overline{L}_m, \overline{L}_n\}_{\text{P.B.}} &= -i(m-n)\, \overline{L}_{m+n}\,, \\
\{\overline{L}_m, L_n\}_{\text{P.B.}} &= 0\,.
\end{aligned} \tag{2.88}$$

In the mathematical literature this algebra is called Witt algebra. Equation (2.88) is straightforward to verify. It is nothing but the Fourier decomposition of the (equal τ) algebra of the Virasoro constraints:

$$\begin{aligned}
\{T_{--}(\sigma), T_{--}(\sigma')\}_{\text{P.B.}} &= +2\pi[T_{--}(\sigma) + T_{--}(\sigma')]\,\partial_\sigma\delta(\sigma-\sigma')\,, \\
\{T_{++}(\sigma), T_{++}(\sigma')\}_{\text{P.B.}} &= -2\pi[T_{++}(\sigma) + T_{++}(\sigma')]\,\partial_\sigma\delta(\sigma-\sigma')\,, \\
\{T_{++}(\sigma), T_{--}(\sigma')\}_{\text{P.B.}} &= 0\,.
\end{aligned} \tag{2.89}$$

It is useful to recognize that if we replace the Poisson brackets by Lie brackets, a realization of the Virasoro algebra is furnished by the vector fields $\overline{L}_n = e^{\frac{2\pi i}{\ell}n\sigma^+}\partial_+$ and $L_n = e^{\frac{2\pi i}{\ell}n\sigma^-}\partial_-$. They are the generators of the reparametrizations $\sigma^\pm \to \sigma^\pm +$

$f_n(\sigma^\pm)$. If we define the variable $z = e^{\frac{2\pi i}{\ell}\sigma^-} \in S^1$, we get $L_n = i z^{n+1}\partial_z$, which are reparametrizations of the circle S^1. The algebra (2.89) expresses the conformal invariance of the classical string theory. Its quantum version will be one of the central themes in the following chapters.

Open Strings

Next we discuss open strings, where we have to distinguish between Neumann and Dirichlet boundary conditions. Because the boundary reflects left- into right-movers, and vice versa, the open string solutions have only one set of oscillator modes.

For Neumann boundary conditions at both ends, we have to require $X'^\mu = 0$ at $\sigma = 0$ and $\sigma = \ell$. The general solution of the wave equation subject to these boundary conditions is

$$\text{(NN)} \qquad X^\mu(\sigma,\tau) = x^\mu + \frac{2\pi\alpha'}{\ell} p^\mu \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu \, e^{-i\frac{\pi}{\ell}n\tau} \cos\left(\frac{n\pi\sigma}{\ell}\right) \tag{2.90}$$

from which we get

$$\partial_\pm X^\mu = \frac{1}{2}(\dot{X}^\mu \pm X'^\mu) = \frac{\pi}{\ell}\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{+\infty} \alpha_n^\mu \, e^{-\frac{i\pi n}{\ell}(\tau\pm\sigma)} . \tag{2.91}$$

We have defined

$$\alpha_0^\mu = \sqrt{2\alpha'}\, p^\mu . \tag{2.92}$$

As in the case of the closed string we easily show that x^μ and p^μ are the center of mass position and total space-time momentum of the open string. The total angular momentum is

$$J^{\mu\nu} = \frac{1}{2\pi\alpha'} \int_0^\pi d\sigma \, (X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu) = l^{\mu\nu} + E^{\mu\nu} \tag{2.93}$$

with $l^{\mu\nu}$ and $E^{\mu\nu}$ as in Eqs. (2.79) and (2.80). We again find

$$\{\alpha_m^\mu, \alpha_n^\nu\}_{\text{P.B.}} = -im\delta_{m+n}\,\eta^{\mu\nu} , \tag{2.94a}$$

$$\{x^\mu, p^\nu\}_{\text{P.B.}} = \eta^{\mu\nu} . \tag{2.94b}$$

In terms of the oscillators the Hamiltonian for the open string is

$$H = \frac{\pi}{2\ell} \sum_{n=-\infty}^{+\infty} \alpha_{-n} \cdot \alpha_n . \tag{2.95}$$

The easiest way to derive this is to use the doubling trick and to write

$$H = \frac{1}{2\pi\alpha'}\int_0^\ell d\sigma\big((\partial_+X)^2 + (\partial_-X)^2\big) = \frac{1}{2\pi\alpha'}\int_{-\ell}^\ell d\sigma(\partial_+X)^2 \tag{2.96}$$

which is possible because of $X'(\sigma) = -X'(-\sigma)$. On the interval $-\ell \le \sigma \le \ell$ the functions $e^{i\pi m\sigma/\ell}$ are periodic.

The open string boundary conditions mix left- with right-movers and consequently T_{++} with T_{--}. We define the Virasoro generators for the open string as (again at $\tau = 0$)

$$\begin{aligned} L_m &= -\frac{\ell}{2\pi^2}\int_0^\ell d\sigma\left(e^{\frac{i\pi}{\ell}m\sigma}\,T_{++} + e^{-\frac{i\pi}{\ell}m\sigma}\,T_{--}\right) \\ &= \frac{\ell}{2\pi^2\alpha'}\int_0^\ell d\sigma\,\left(e^{\frac{i\pi}{\ell}m\sigma}\,(\partial_+X)^2 + e^{-\frac{i\pi}{\ell}m\sigma}\,(\partial_-X)^2\right) \\ &= \frac{\ell}{2\pi^2\alpha'}\int_{-\ell}^{+\ell} d\sigma\, e^{\frac{i\pi}{\ell}m\sigma}\,(\partial_+X)^2 \\ &= \frac{1}{2}\sum_{n=-\infty}^{+\infty}\alpha_{m-n}\cdot\alpha_n\ . \end{aligned} \tag{2.97}$$

The L_m are a complete set of conserved charges respecting the open string boundary conditions. Comparison with Eq. (2.96) gives

$$H = \frac{\pi}{\ell}L_0\ , \tag{2.98}$$

which, as in the closed string case, reflects the fact that we are in conformal gauge. The L_m satisfy the Virasoro algebra

$$\{L_m, L_n\}_{\text{P.B.}} = -i\,(m-n)\,L_{m+n}\ . \tag{2.99}$$

The second choice of boundary conditions are Dirichlet conditions at both ends of the string. We impose them by requiring $\dot{X}^\mu = 0$ at $\sigma = 0$ and at $\sigma = \ell$. The positions of the ends are fixed at $X^\mu(\sigma = 0, \tau) = x_0^\mu$, $X^\mu(\sigma = \ell, \tau) = x_1^\mu$. The general solution of the wave equation, subject to these boundary conditions, is

$$\begin{aligned} \text{(DD)}\quad X^\mu(\sigma,\tau) &= x_0^\mu + \frac{1}{\ell}(x_1^\mu - x_0^\mu)\,\sigma \\ &\quad + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\,\alpha_n^\mu\; e^{-\frac{i\pi}{\ell}n\tau}\,\sin\left(\frac{\pi n\sigma}{\ell}\right) \end{aligned} \tag{2.100}$$

with $n \in \mathbb{Z}$. There is no center of mass momentum. From (2.100) we derive

$$\partial_\pm X^\mu = \pm\frac{\pi}{\ell}\sqrt{\frac{\alpha'}{2}}\sum_{n=-\infty}^{\infty}\alpha_n^\mu e^{-\frac{i\pi n}{\ell}(\tau\pm\sigma)}, \tag{2.101}$$

where

$$\alpha_0^\mu = \frac{1}{\sqrt{2\alpha'}}\frac{1}{\pi}\left(x_1^\mu - x_0^\mu\right). \tag{2.102}$$

The oscillator modes satisfy Eq. (2.94a). There is no center of mass momentum. The center of mass position is

$$q^\mu = \frac{1}{\ell}\int_0^\ell d\sigma\, X^\mu(\sigma,\tau) = \left(\frac{x_0^\mu + x_1^\mu}{2}\right). \tag{2.103}$$

For the Virasoro generators L_m for $m \neq 0$ one gets the same expressions as for Neumann boundary conditions, Eq. (2.97). The L_m also satisfy the algebra (2.99). With $H = \frac{\pi}{\ell}L_0$ we find for the Hamiltonian

$$H = \frac{T}{2\ell}\left(x_1^\mu - x_0^\mu\right)^2 + \frac{\pi}{2\ell}\sum_{n\neq 0}\alpha_{-n}\cdot\alpha_n\,. \tag{2.104}$$

The first term is the potential energy of the stretched string.

We can also impose mixed boundary conditions, i.e. different boundary conditions at the two ends of the open string. For Neumann boundary conditions at $\sigma = 0$ and Dirichlet boundary condition at $\sigma = \ell$ the general solution reads

$$(\text{ND}) \quad X^\mu(\sigma,\tau) = x^\mu + i\sqrt{2\alpha'}\sum_{r\in\mathbb{Z}+\frac{1}{2}}\frac{1}{r}\alpha_r^\mu\, e^{-\frac{i\pi}{\ell}r\tau}\cos\left(\frac{\pi r\sigma}{\ell}\right), \tag{2.105}$$

where x^μ is the position of the $\sigma = \ell$ end of the open string. Note that the center of mass momentum vanishes and that the oscillators carry half-integer modes. They also satisfy Eq. (2.94a). For completeness we also give the last possible combination of boundary conditions

$$(\text{DN}) \quad X^\mu(\sigma,\tau) = x^\mu + \sqrt{2\alpha'}\sum_{r\in\mathbb{Z}+\frac{1}{2}}\frac{1}{r}\alpha_r^\mu\, e^{-\frac{i\pi}{\ell}r\tau}\sin\left(\frac{\pi r\sigma}{\ell}\right), \tag{2.106}$$

where x^μ is the position of the $\sigma = 0$ end of the open string. We have $(\alpha_n^\mu)^* = \alpha_{-n}^\mu$ for all possible boundary conditions. From (2.105) and (2.106) one derives

$$\partial_\pm X^\mu = \begin{cases} \dfrac{\pi}{\ell}\sqrt{\dfrac{\alpha'}{2}}\sum_r \alpha_r^\mu e^{-\frac{i\pi r}{\ell}(\tau\pm\sigma)} & \text{(ND)} \\ \pm\dfrac{\pi}{\ell}\sqrt{\dfrac{\alpha'}{2}}\sum_r \alpha_r^\mu e^{-\frac{i\pi r}{\ell}(\tau\pm\sigma)} & \text{(DN)}\,. \end{cases} \tag{2.107}$$

For all four boundary conditions one can use the doubling trick and combine $\partial_\pm X$ into one field, say left moving which is defined on the doubled interval $0 \le \sigma \le 2\ell$:

$$\begin{aligned} \partial_+ X^\mu &= \begin{cases} \partial_+ X^\mu(\sigma)\,, & 0 \le \sigma \le \ell \\ \pm\partial_- X^\mu(2\ell-\sigma)\,, & \ell \le \sigma \le 2\ell \end{cases} \quad \begin{cases} +\text{sign for (NN) and (ND)}\,, \\ -\text{sign for (DD) and (DN)}\,, \end{cases} \\ &= \frac{\pi}{\ell}\sqrt{\frac{\alpha'}{2}}\sum \alpha_n^\mu e^{-\frac{i\pi n}{\ell}(\tau+\sigma)}\,, \quad 0 \le \sigma \le 2\ell \quad \begin{cases} n \in \mathbb{Z} & \text{for NN and DD}\,, \\ n \in \mathbb{Z}+\frac{1}{2} & \text{for DN and ND}\,. \end{cases} \end{aligned} \tag{2.108}$$

The signs are chosen to have continuity at $\sigma = \ell$. Clearly

$$\begin{aligned} \partial_+ X^\mu(\sigma+2\ell) &= \partial_+ X^\mu(\sigma) && \text{for (NN) and (DD)}\,. \\ \partial_+ X^\mu(\sigma+2\ell) &= -\partial_+ X^\mu(\sigma) && \text{for (DN) and (ND)}\,. \end{aligned} \tag{2.109}$$

If the string moves in d space-time dimensions, one can combine various boundary conditions. For instance, one can have open strings with $(p+1)$ Neumann directions and $(d-p-1)$ Dirichlet directions. The end points of the open string are then confined to $(p+1)$ dimensional subspaces of the d dimensional target space. Space-time translation symmetry along the $(d-p-1)$ transverse directions is broken by these solutions. As we have already remarked, this means that the space-time momentum in the Dirichlet directions, which is carried by the open string, is not conserved; it can flow off the ends of the string. Since the translation invariance is spontaneously broken, momentum must be conserved. One is thus forced to consider the subspaces, to which the endpoints are attached, as dynamical objects which exchange momentum with the open strings ending on them. These objects are called Dp-branes. The world-volume of a Dp-brane is $(p+1)$-dimensional. String endpoints can move along them (these are the Neumann directions), but cannot leave them. In other words, open string cannot simply end in free space. They are always attached to D-branes.

We will further analyze D-branes in Chaps. 6 and 9. In particular, we will show that they also have tension and therefore a mass density. However, as we will see, the tension scales like $1/g_s$ with the string coupling constant, which indicates that these objects are not visible in string perturbation theory, but should be considered as non-perturbative objects. That means they are string theory analogues of monopoles or instantons.

By choosing N boundary conditions in some and D boundary conditions in the remaining coordinate directions, we obtained static D-branes at fixed transverse position and of infinite extent. Since branes carry a finite tension, which we will compute in Chap. 6, branes of infinite extent are therefore infinitely heavy and can absorb any amount of space-time momentum. But in general they are dynamical objects and their dynamics is governed by world-volume actions which are $(p+1)$-dimensional generalizations of the Nambu-Goto action, to be discussed in Chap. 16. The transverse fluctuations of a D-brane correspond to massless scalar fields in this $(p+1)$-dimensional field theory. These are the Goldstone bosons of the spontaneously broken translation invariance.

If we impose N boundary conditions in all space-time directions, we obtain space-time filling D-branes. The other extreme is a D-instanton, which only exists at one space-time point. A D1-brane is also called a D-string, to be distinguished from the fundamental string that we have studied so far and which might end on a D-string. D-branes play a central role in all recent developments of string theory and we will learn more about them as we go along.

2.5 Examples of Classical String Solutions

The solutions to the wave equations satisfying various periodicity and boundary conditions which we have found in the previous section are still subject to the Virasoro constraints: $T_{00} = T_{01} = 0$. We will now construct simple explicit solutions of the classical equations of motion which satisfy the constraints.

Since in conformal gauge the coordinate functions X^μ are solutions of the wave equation, we can use the remaining gauge freedom to set $X^0 = t = \kappa\tau$ for some constant κ. The X^i, $i = 1, \ldots, d-1$ then satisfy

$$(\partial_\sigma^2 - \partial_\tau^2)X^i = 0 \tag{2.110}$$

with solution

$$X^i(\sigma,\tau) = \frac{1}{2}a^i(\sigma+\tau) + \frac{1}{2}b^i(\sigma-\tau) \,. \tag{2.111}$$

The constraint $\dot{X}\cdot X' = -\dot{X}^0 X'^0 + \dot{X}^i X'^i = 0$ leads to $a'^2 = b'^2$ and $\dot{X}^2 + X'^2 = 0$ to $\frac{1}{2}(a'^2 + b'^2) = \kappa^2$. Combined this gives

$$a'^2 = b'^2 = \kappa^2 \,. \tag{2.112}$$

The simplest example of an open string with N boundary conditions is

$$X^1 = L\cos\left(\frac{\pi\sigma}{\ell}\right)\cos\left(\frac{\pi\tau}{\ell}\right) \qquad X^0 = t = \frac{\pi L}{\ell}\tau \,,$$

$$
\begin{aligned}
X^2 &= L\cos\left(\frac{\pi\sigma}{\ell}\right)\,\sin\left(\frac{\pi\tau}{\ell}\right)\,, \\
X^i &= 0, \qquad i = 3,\dots,d-1\,.
\end{aligned}
\tag{2.113}
$$

It clearly satisfies the constraints. It is a straight string of length $2L$ rotating around its midpoint in the (X^1, X^2)-plane. Its total (spatial) momentum vanishes and its energy is $E = L\pi T$ from which we derive the mass $M^2 = -P^\mu P_\mu = (L\pi T)^2$. The angular momentum is $J = J_{12} = \frac{1}{2}L^2\pi T$ and we find that $J = \frac{1}{2\pi T}M^2 = \alpha' M^2$. This is a straight line in the (M^2, J) plane with slope $\alpha' = (2\pi T)^{-1}$, called a Regge trajectory. It can actually be shown that for any classical open string solution $J < \alpha' M^2$. (In the gauge chosen here and in the center of mass frame $J^2 = \frac{1}{2}J_{ij}J^{ij}$, $i, j = 1,\dots,d-1$.) The velocity of the string is $\boldsymbol{v}^2 = \cos^2(\frac{\pi\sigma}{\ell})$. It is one at both endpoints. This is an immediate consequence of the constraint $\dot{X}^2 + X'^2 = 0$ and holds for any open string with Neumann boundary conditions ($X' = 0$ at the ends).

A second simple open string solution is

$$
X^0 = t = \tau \quad X^1 = vt\,, \quad X^2 = \frac{1}{\ell}L\sigma\,, \quad X^3 = \cdots = 0\,. \tag{2.114}
$$

This describes a string which is spanned between D-branes which are a distance L apart. The string moves rigidly along the X^1 direction with a velocity v. It satisfies N b.c. along X^0 and X^1 and Dirichlet boundary conditions along X^2. The constraints are satisfied if $L = \ell\sqrt{1-v^2}$. This is simply the relativistic length contraction.

For the closed string the periodicity requirement leads to $a(\sigma+\ell) = a(\sigma)$ and $b(\sigma+\ell) = b(\sigma)$. From

$$
\begin{aligned}
X\left(\sigma + \frac{\ell}{2}, \tau + \frac{\ell}{2}\right) &= \frac{1}{2}a(\sigma+\tau+\ell) + \frac{1}{2}b(\sigma-\tau) \\
&= \frac{1}{2}a(\sigma+\tau) + \frac{1}{2}b(\sigma-\tau)\,,
\end{aligned}
\tag{2.115}
$$

we find that the period of a classical closed string is $\ell/2$. For an initially static closed string configuration, i.e. one that satisfies $\dot{X}(\sigma, \tau = 0) = 0$, we find $X(\sigma,\tau) = \frac{1}{2}(a(\sigma+\tau) + a(\sigma-\tau))$. After half a period, i.e. at $\tau = \frac{\ell}{4}$, $X(\sigma, \frac{\ell}{4}) = X(\sigma + \frac{\ell}{2}, \frac{\ell}{4})$, the loop doubles up and goes around itself twice: $X(\sigma) = X(\sigma + \frac{\ell}{2})$. A simple closed string configuration is

$$
\begin{aligned}
X^0 &= t = \frac{2\pi}{\ell}R\tau\,, \\
X^1 &= \frac{1}{2}R\Big[\cos\Big(\frac{2\pi}{\ell}(\sigma+\tau)\Big) + \cos\Big(\frac{2\pi}{\ell}(\sigma-\tau)\Big)\Big] = R\cos\left(\frac{2\pi\sigma}{\ell}\right)\cos\left(\frac{2\pi\tau}{\ell}\right),
\end{aligned}
$$

$$X^2 = \frac{1}{2}R\Big[\sin\Big(\frac{2\pi}{\ell}(\sigma+\tau)\Big) + \sin\Big(\frac{2\pi}{\ell}(\sigma-\tau)\Big)\Big] = R\sin\Big(\frac{2\pi\sigma}{\ell}\Big)\cos\Big(\frac{2\pi\tau}{\ell}\Big)\,. \tag{2.116}$$

At $t = 0$ it represents a circular string of radius R in the (X^1, X^2)-plane, centered around the origin. Its energy is $E = 2\pi RT$. Linear and angular momentum vanish. At $\tau = \frac{\ell}{4}$ $(t = \frac{\pi}{2}R)$ it has collapsed to a point and at $\tau = \frac{\ell}{2}$ $(t = \pi R)$ it has expanded again to its original size. Similar to the open string case, one can show that a general classical closed string configuration satisfies $J \le \frac{1}{2}\alpha' M^2$.

Further Reading

Constrained systems with applications to string theory are discussed in

- K. Sundermeyer, *Constrained Dynamics*. Lecture Notes in Physics, vol. 169 (Springer, Heidelberg, 1982)
- A. Hanson, T. Regge, C. Teitelboim, *Constrained Hamiltonian Systems* (Accademia Nazionale de Lincei, Roma, 1976), also available at https://scholarworks.iu.edu/dspace/handle/2022/3108
- J. Govaerts, *Halitonian Quantization and Constrained Dynamics* (Leuven University Press, Leuven, 1991)

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- O. Alvarez, Theory of strings with boundaries. Nucl. Phys. B **216**, 125 (1983)

Classical string solutions:

- P. Shellard, A. Vilenkin, *Cosmic Strings and Other Topological Defects* (Cambridge University Press, Cambridge, 1994)

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2013, XII, 784 p., Hardcover
ISBN: 978-3-642-29496-9