Chapter 2
Basic Concepts

The Hilbert space theory of weak boundary value problems forms the mathematical background of the FE-method. This theory requires some (rudimentary) knowledge of functional analysis and so the chapter starts with a short recap of the principal concepts as vector spaces, scalar product, linear functionals, projection, equivalent norms, Galerkin's method and the definition of Sobolev spaces. The main result is that in the weak theory Green's functions are the Riesz elements of linear functionals.

The exposition is based on the first and second Green's identities of the governing differential operator as these underlie the variational and energy principles of mathematical physics and their algebra is essential for the formulation of weak boundary value problems.

Green's identities are then used to derive the classical influence functions for displacements and force terms. The observation that influence functions for force terms can only be formulated with Betti's theorem (Green's second identity) but not with the principle of virtual forces (Green's first identity) leads to the distinction between strong and weak influence functions.

Physically Green's functions represent monopoles, dipoles and higher-order poles. Monopoles are like integral operators, they sum, while dipoles differentiate, they are sensitive to imbalances. Most Green's function, being the solution to point forces or higher-order terms, have infinite energy which makes that they lie outside the theory of weak boundary value problems. When and why that happens can be explained with Sobolev's Embedding Theorem.

In the algebra of linear problems Green's functions are identical with Lagrange multipliers. This observation allows to extend—formally—the concept of a Green's function also to nonlinear problems; not in the true sense of forming the kernel of an inverse operator but to speak of a Green's function at the linearization point of a stiffness matrix.

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2.1 Elements of Functional Analysis

The weak form of the classical boundary value problems of mechanics are of "quadratic type". What is meant by this is best explained by a simple spring.

In a linear spring with a stiffness k the displacement u is proportional to the applied force f

$$k\,u = f \tag{2.1}$$

and consequently the displacement u minimizes the function

$$\Pi(u) = \frac{1}{2}\,k\,u^2 - f\,u \tag{2.2}$$

because if u satisfies (2.1) then it is also a zero of $\Pi'(u) = k\,u - f$.

If u is a solution of (2.1) then multiplying both sides with any number v leaves the solution unchanged and therefore u is also the solution of the variational problem

Find a number u such that

$$\delta W_i := v\,k\,u = f\,v =: \delta W_e \quad \forall\, v \in \mathbb{R}\,. \tag{2.3}$$

The symbol $\forall$ means "for all". For all numbers v this equation must be true. This is the principle of virtual displacements: at the equilibrium position u the virtual strain energy δW_i of the spring equals the virtual exterior energy δW_e—and this holds true for all virtual displacements v.

Quadratic refers to the fact that the internal energy of the spring

$$W_i = \frac{1}{2}\,k\,u^2 =: a(u,u) \tag{2.4}$$

is a quadratic function, see Fig. 2.1, and this means it is positive definite

$$a(u,u) = k\,u^2 > 0 \quad u \neq 0. \tag{2.5}$$

With regard to the applied load f we can state that if it is not just infinitely large then the external work $f \cdot u$ is a *continuous function* of u. These two properties which we can attribute to $a(u,u)$ and the function $f \cdot u$ guarantee that the weak problem (2.3) has always a solution $u = f/k$. Ultimately the parabola $1/2\,k\,u^2$ will rise faster than the linear function. The point where the path of the parabola crosses the straight line $1/2\,fu$ is the equilibrium point u, the internal energy and external work balance at this point

$$W_i = \frac{1}{2}\,k\,u^2 = \frac{1}{2}\,fu = W_e \quad \text{at } u = f/k\,. \tag{2.6}$$

But at the start, when the spring begins to move, it is the straight line—representing the external work $1/2\,f\,u$—which grows faster (and it *must* grow faster!) than the

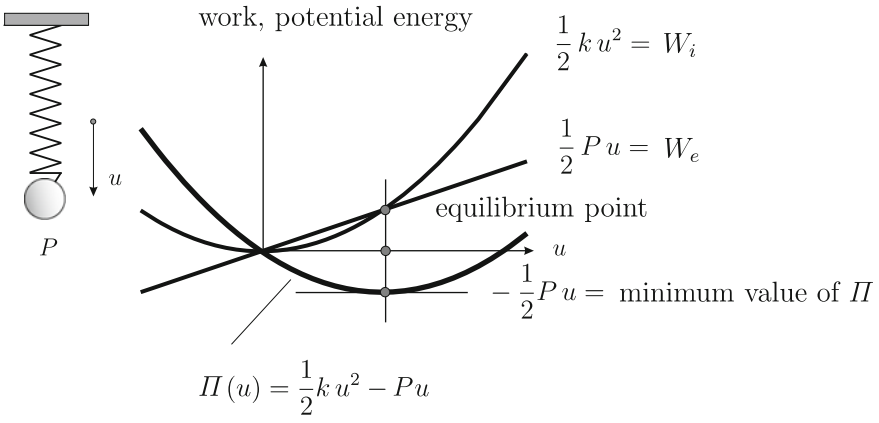

Fig. 2.1 Because the internal energy W_i grows quadratically with u, while the external work W_e grows only linearly, W_i always catches up with W_e, and there will always be an equilibrium point where $W_i = W_e$ [1]

parabola $1/2\,ku^2$ of the internal energy. If that were not the case then the spring would not move one *iota*! So in some sense—let $f = k = 1$—computational mechanics is rooted in the fact that $u > u^2$ in the interval $(0, 1)$, and that beyond this point the opposite is true. *The transition point $u = 1$ is the equilibrium point.*

In the continuous case, when the equilibrium position u of the mechanical system is a function, then technically things are more complicated but the arguments and the results are very nearly the same because the modern theory of partial differential equations is formulated in terms of weak boundary value problems on Hilbert spaces and this theory and its algebraic structure bears a close resemblance to what the engineer calls the *principle of virtual displacements*. In the following we will summarize the main results and concepts of this theory as far as we need them for the study of Green's functions and finite elements.

2.1.1 Notation

Small bold letters $\boldsymbol{u} = \{u_1, u_2, \ldots, u_n\}^T$ denote column vectors and capital bold letters $\boldsymbol{K} = [k_{ij}]$ matrices. The scalar product between two vectors $\boldsymbol{u}$ and $\boldsymbol{f}$ is written in two ways

$$\boldsymbol{u} \cdot \boldsymbol{f} = \boldsymbol{u}^T\,\boldsymbol{f} = u_1\,f_1 + u_2\,f_2 \tag{2.7}$$

and

$$\boldsymbol{S} \cdot \boldsymbol{E} = \sigma_{11}\,\varepsilon_{11} + \sigma_{12}\,\varepsilon_{12} + \sigma_{21}\,\varepsilon_{21} + \sigma_{22}\,\varepsilon_{22} \tag{2.8}$$

is the scalar product of matrices. A small dot, $3 \cdot 4$, denotes a multiplication. Green's functions are written $G(\boldsymbol{y}, \boldsymbol{x})$ or if the emphasis is only on the source point $\boldsymbol{x}$ then $G[\boldsymbol{x}]$. Eventually the same is done with the Dirac delta $\delta(\boldsymbol{y} - \boldsymbol{x})$ which has the short form $\delta[\boldsymbol{x}]$.

2.1.2 Vector Spaces and Scalar Product

A real vector space $\mathcal{V}$ is a set of objects, called vectors, on which are defined two operations, addition and scalar multiplication (scalar product). Associated with such a space is a field, the real numbers $\alpha \in \mathbb{R}$, such that if u is a member of $\mathcal{V}$ then also $\alpha\, u$ is a member of $\mathcal{V}$.[1]

The space $\mathbb{R}^n$, the set of all vectors with n components, is such a vector space. The sum of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathcal{V}$ is the vector $\boldsymbol{u} + \boldsymbol{v}$ and the scalar multiplication between two vectors

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1\, v_1 + u_2\, v_2 + \ldots + u_n\, v_n = |\boldsymbol{u}|\, |\boldsymbol{v}|\, \cos\varphi \tag{2.9}$$

is of course the scalar product of two vectors where φ is the angle between the two vectors.

Occasionally we will write the scalar product of two such vectors as the product of the row vector $\boldsymbol{v}^T$ and the column vector $\boldsymbol{u}$

$$\boldsymbol{v} \cdot \boldsymbol{u} = \boldsymbol{v}^T\, \boldsymbol{u} = (\boldsymbol{v}, \boldsymbol{u}) \tag{2.10}$$

or we will use the notation $(\boldsymbol{v}, \boldsymbol{u})$ or (u, v) respectively as a generic expression for the scalar product of two quantities.

For an expression (u, v) to be a scalar product it must have the following properties $(\alpha, \beta \in \mathbb{R})$

$$\begin{aligned}
&\text{Linearity:} && (\alpha\, u + \beta\, v, w) = \alpha\, (u, w) + \beta\, (v, w) && \text{(2.11a)}\\
&\text{Symmetry:} && (u, v) = (v, u) && \text{(2.11b)}\\
&\text{Definitness:} && (u, u) > 0 \quad \text{for } u \neq 0\,. && \text{(2.11c)}
\end{aligned}$$

With these properties it defines also the norm of an element

$$\|u\| := \sqrt{(u, u)}\,. \tag{2.12}$$

The square root makes that the element $2\, u$ has double the norm of u, $\|2\, u\| = 2\, \|u\|$.

[1] For not to confuse the vector space with the shear force V we write $\mathcal{V}$ and $\mathcal{V}_h$ as well.

A norm has the properties

$$\begin{aligned}
&\text{positive semi-definit:} && \|u\| = 0 \Leftrightarrow u = 0 \quad \text{or} \quad \|u\| \geq 0 \quad \forall\, u \in \mathcal{V} && \text{(2.13a)}\\
&\text{homogeneous:} && \|\lambda u\| = |\lambda| \|u\| && \text{(2.13b)}\\
&\text{triangular inequality:} && \|u + v\| \leq \|u\| + \|v\|\,. && \text{(2.13c)}
\end{aligned}$$

In particular does the first property guarantee that the norm can *separate* the elements of $\mathcal{V}$

$$\|u - v\| = 0 \quad \Rightarrow \quad u = v\,. \tag{2.14}$$

This is what makes a norm a norm.

A central property of a normed space $\mathcal{V}$ is the Cauchy-Schwarz inequality

$$|(u, v)| \leq \|u\|\, \|v\| \quad \forall\, u, v \in \mathcal{V}\,. \tag{2.15}$$

A vector space $\mathcal{V}$ with a norm is called a *normed vector space*. If the space is also complete, if the limit of any *Cauchy sequence* (the distance between consecutive elements gets smaller and smaller) is an element of $\mathcal{V}$ then it is said that the space is *complete*. Such normed vector spaces are called *Hilbert spaces*.

2.1.3 Linear Functionals

A linear functional $J(u)$ on $\mathcal{V}$ is a real-valued linear function

$$J(\alpha\, u + \beta\, v) = \alpha\, J(u) + \beta\, J(v) \quad \alpha, \beta \in \mathbb{R}\,. \tag{2.16}$$

It is a continuous functional on $\mathcal{V}$ if there exists a constant c such that

$$J(u) \leq c\, \|u\| \quad \forall\, u \in \mathcal{V}\,. \tag{2.17}$$

The set of all continuous linear functionals defined on $\mathcal{V}$ forms itself a normed vector space which is called the *dual space* of $\mathcal{V}$ and is denoted by $\mathcal{V}^*$. The norm on $\mathcal{V}^*$ is defined as

$$\|J\|_{\mathcal{V}^*} = \sup\left(\frac{J(u)}{\|u\|}\right) \tag{2.18}$$

which means that the norm of a functional is the maximum value the functional $J(u)$ attains in the unit ball (all u with norm $\|u\| = 1$) because dividing $J(u)$ by the norm $\|u\|$ of u is the same as first dividing u by its norm and then applying the functional

$$\frac{J(u)}{\|u\|} = J\left(\frac{u}{\|u\|}\right). \tag{2.19}$$

The following theorem asserts that we can associate with each linear and bounded functional $J(u)$ a so-called Riesz element v in the sense that $J(u) = (u, v)$.

Theorem 2.1 (Riesz representation theorem) *If $\mathcal{V}$ is a Hilbert space then its dual space $\mathcal{V}^*$ can be identified with $\mathcal{V}$:*

1. *For each $v \in \mathcal{V}$ the linear functional $J(u)$ defined by $J(u) = (u, v)$ belongs to $\mathcal{V}^*$ and*
$$\|J\|_{\mathcal{V}^*} = \|v\|_{\mathcal{V}} . \tag{2.20}$$
2. *For each $J \in \mathcal{V}^*$ there exists a unique $v \in \mathcal{V}$ such that*
$$\|J\|_{\mathcal{V}^*} = \|v\|_{\mathcal{V}} \tag{2.21}$$
and it is
$$J(u) = (u, v) \quad \forall\, u \in \mathcal{V} . \tag{2.22}$$

The Riesz element v in (2.22) is identical with what is later called the generalized Green's function of the functional $J(u)$.

Example 2.1 Let $\mathcal{V}$ be the Euclidean space $\mathbb{R}^3$. Let $\boldsymbol{v} = \{v_1, v_2, v_3\}^T$ be a vector anchored at the origin and pointing in a certain direction. Then the functional
$$J(\boldsymbol{u}) = \boldsymbol{v}^T \boldsymbol{u} \tag{2.23}$$
is simply the projection of a vector $\boldsymbol{u}$ onto the direction of the vector $\boldsymbol{v}$. Clearly the maximum value of $J(\boldsymbol{u})$, when it is restricted to the unit ball ($\|\boldsymbol{u}\| = 1$), is obtained for $\boldsymbol{u} = \boldsymbol{v}/\|\boldsymbol{v}\|$ and so indeed
$$\|J\|_{\mathcal{V}^*} = \|\boldsymbol{v}\|_{\mathcal{V}} . \tag{2.24}$$

Vice versa, let there be a linear functional $J(\boldsymbol{u})$ on $\mathbb{R}^3$. Let
$$j_i = J(\boldsymbol{e}_i) \quad i = 1, 2, 3 , \tag{2.25}$$
($\boldsymbol{e}_i$ = unit vector) then
$$J(\boldsymbol{u}) = u_1\, j_1 + u_2\, j_2 + u_3\, j_3 \tag{2.26}$$
and so the vector $\boldsymbol{j} = \{j_1, j_2, j_3\}^T$ represents the functional in the sense of (2.22)
$$J(\boldsymbol{u}) = \boldsymbol{j}^T \boldsymbol{u} . \tag{2.27}$$

Example 2.2 Let Ω be the unit disk and Γ the edge of the disk. On Ω we consider the following boundary value problem
$$-\Delta u = p \quad \text{on } \Omega \quad u = 0 \quad \text{on } \Gamma . \tag{2.28}$$

The space

$$\mathcal{V} = \{u \in H^1(\Omega) | u = 0 \text{ on } \Gamma\} \tag{2.29}$$

with the scalar product $a(u, v) := (\nabla u, \nabla v)$ and the norm $\|u\|_E := a(u, u)^{1/2}$ is a Hilbert space and so given any linear and bounded functional $J(u)$ there exists a Riesz element $G \in \mathcal{V}$ such that

Weak influence function

$$J(u) = a(G, u) \quad \forall\, u \in \mathcal{V} \tag{2.30}$$

where

$$a(G, u) = \int_\Omega \nabla G \cdot \nabla u \, d\Omega_{\mathbf{y}} = \int_\Omega (G_{,1}\ u_{,1} + G_{,2}\ u_{,2})\, d\Omega_{\mathbf{y}}\,. \tag{2.31}$$

Because of Green's first identity, see (2.86), this formula is equivalent to

$$J(u) = \int_\Omega G\,(-\Delta u)\, d\Omega_{\mathbf{y}} \quad \forall\, u \in \mathcal{V} \tag{2.32}$$

or if u is the solution of the boundary value problem (1.30) then holds

Strong influence function

$$J(u) = \int_\Omega G\ p\, d\Omega_{\mathbf{y}}\,. \tag{2.33}$$

That is the Riesz element is the Green's function of the functional. We call these two influence functions, (2.30) and (2.33), *weak* and *strong* influence functions.

2.1.4 Projection

When a vector $\boldsymbol{u} = \{u_x, u_y, u_z\}^T \in \mathbb{R}^3$ is projected onto the $x-y$-plane (or $\mathbb{R}^2$ for short)

$$\boldsymbol{u} \to \bar{\boldsymbol{u}} = \{u_x, u_y, 0\}^T \tag{2.34}$$

then the error vector $\boldsymbol{e} = \boldsymbol{u} - \bar{\boldsymbol{u}}$ is perpendicular to the plane

$$\boldsymbol{e} \cdot \boldsymbol{v} = 0 \quad \forall\, \boldsymbol{v} \in \mathbb{R}^2 \tag{2.35}$$

and so the (length)^2 of $\boldsymbol{u}$ is the (length)^2 of $\bar{\boldsymbol{u}}$ plus the (length)^2 of $\boldsymbol{e}$

$$\|\boldsymbol{u}\|^2 = \|\bar{\boldsymbol{u}}\|^2 + \|\boldsymbol{e}\|^2 . \tag{2.36}$$

This is the *Pythagorean theorem* which holds true in any inner product space for two orthogonal elements of the space, $(\boldsymbol{e}, \bar{\boldsymbol{u}}) = 0$, and so the idea of a projection can be extended to any inner product space.

Theorem 2.2 (Projection) *Suppose $\mathcal{V}$ is an inner product space and $\mathcal{V}_h$ is a finite-dimensional subspace of $\mathcal{V}$ and u is an element of $\mathcal{V}$. Then*

1. *there exists a unique element $u_h \in \mathcal{V}_h$ satisfying*

$$\|u - u_h\| < \|u - v_h\| \quad \forall\, v_h \in \mathcal{V}_h, v_h \neq u_h \tag{2.37}$$

   that is u_h wins the competition, it has the shortest distance to u and
2. *this statement is equivalent to*

$$(u - u_h, v_h) = 0 \quad \forall\, v_h \in \mathcal{V}_h . \tag{2.38}$$

   that is the error is orthogonal to $\mathcal{V}_h$.

2.1.5 Variational Problems

In the framework of the FE-method the classical boundary value problems of physics are formulated as variational problems on appropriate Hilbert spaces $\mathcal{V}$

$$u \in \mathcal{V}: \quad a(u, v) = J(v) \quad \forall\, v \in \mathcal{V} \tag{2.39}$$

where $a(u, v)$ is a (not necessarily symmetric) bilinear form and $J(v)$ is a linear functional [2]. The next theorem lists sufficient conditions for this problem to have a unique solution.

Theorem 2.3 (Lax-Milgram) *Suppose $\mathcal{V}$ is a Hilbert space and $a(u, v)$ is a bilinear form on $\mathcal{V}$ which is bounded and coercive (or $\mathcal{V}$-elliptic), that is there exist two constants c_E and c_B such that*

$$|a(u, v)| \leq c_B \|u\| \|v\| \quad \forall\, u, v \in \mathcal{V} \quad \textit{(bounded)} \tag{2.40}$$

$$a(u, u) \geq c_E \|u\|^2 \quad \forall\, u \in \mathcal{V} \quad \textit{($\mathcal{V}$-elliptic)} \tag{2.41}$$

Then given any $J \in \mathcal{V}^*$ there exists a unique $u \in \mathcal{V}$ such that

$$a(u, v) = J(v) \quad \forall\, v \in \mathcal{V} \tag{2.42}$$

and the solution u depends continuously on J that is

$$\|u\|_{\mathcal{V}} \leq \frac{1}{c_E} \|J\|_{V*} . \tag{2.43}$$

The difficult part in applying this theorem is to demonstrate that the bilinear form $a(u, v)$ in question is $\mathcal{V}$-elliptic.

Example 2.3 The energy of a beam, $EI\, u^{IV} = p$, is the integral

$$a(u, u) = \int_0^l EI\, (u'')^2\, dx \tag{2.44}$$

and the "energy space" is $H^2(0, l)$ with the norm

$$\|u\|_2^2 = \int_0^l (u^2 + (u')^2 + (u'')^2)\, dx\, . \tag{2.45}$$

Let

$$\mathcal{V} = \{u \,|u \in H^2(0, l), u(0) = u(l) = 0\} \tag{2.46}$$

be the solution space of a hinged beam. It can be shown that the energy is $\mathcal{V}$-elliptic

$$a(u, u) > c\, \|u\|_2^2 \quad \forall\, u \in \mathcal{V} \tag{2.47}$$

that is if $a(u, u)$ tends to zero then also the norm $\|u\|_2$ tends to zero and this means that

$$\int_0^l EI\, (u'')^2\, dx \to 0 \quad \Rightarrow \quad \int_0^l u^2\, dx \to 0\, , \int_0^l (u')^2\, dx \to 0\, . \tag{2.48}$$

This is a remarkable property because it shows that the bending moments of a properly supported beam control the beam. Without supports the beam could perform rigid body motions, $u(x) = a + b\, x$, and because the bending moment is then zero, $M = 0$, (2.47) would be violated.

2.1.6 Equivalent Norms

If the bilinear form $a(u, v)$ is $\mathcal{V}$-elliptic then it forms an alternative inner product on $\mathcal{V}$ and the expression

$$\|u\|_E := \sqrt{a(u, u)} \tag{2.49}$$

the so-called *energy norm*, is an equivalent norm[2] on $\mathcal{V}$, that is there exist two global constants c_E and c_B (not depending on the particular u) such that

$$\sqrt{c_E}\,\|u\|_{\mathcal{V}} \le \|u\|_E \le \sqrt{c_B}\,\|u\|_{\mathcal{V}} \tag{2.50}$$

which essentially means that the two norms generate the same topology on $\mathcal{V}$:

- if a sequence of elements converges in one norm to an element u then it converges also in the other norm to the same element,
- if $\mathcal{V}$ is complete then it is also complete with respect to the energy norm,
- if a linear functional $J(u)$ is bounded then it is also bounded in the energy norm,

In the FE-context $\mathcal{V}$ is the trial and solution space on which the variational problem

$$u \in \mathcal{V}: \quad a(u,v) = J(v) \quad \forall\, v \in \mathcal{V} \tag{2.51}$$

is posed. Because the Riesz representation theorem also holds true in the energy norm, the existence and uniqueness of the variational problem is settled.

In the same sense as $\|u\|$ and $\|u\|_E$ are equivalent norms on $\mathcal{V}$ the expression

$$||J||_{E^*} := \sup_{\substack{v\in\mathcal{V}\\ v\neq 0}} \frac{|J(v)|}{||v||_E} = \sup_{\substack{v\in\mathcal{V}\\ ||v||_E=1}} |J(v)| \tag{2.52}$$

is an equivalent norm on the dual $\mathcal{V}^*$, equivalent to $\|J\|_{\mathcal{V}^*}$. If u is the solution of (2.51) then

$$\frac{|J(v)|}{\|v\|_E} = \frac{|a(u,v)|}{\|v\|_E} \le \frac{\|u\|_E\,\|v\|_E}{\|v\|_E} = \|u\|_E \tag{2.53}$$

and so $\|u\|_E$ is an upper bound and because of

$$\frac{|J(u)|}{\|u\|_E} = \frac{|a(u,u)|}{\|u\|_E} = \|u\|_E \tag{2.54}$$

it is also the lowest upper bound. Hence the norm $\|J\|_{E^*}$ of the functional J is just the energy norm of the solution

$$\|J\|_{E^*} = \|u\|_E = \sqrt{a(u,u)}\,. \tag{2.55}$$

This implies that the exact solution u is that function in $\mathcal{V}$ which gets "the most mileage" out of J in the sense of (2.52), see Fig. 2.2.

Most milage means: the load p to which a beam is subjected constitutes a functional

[2] The so called taxi norm—distance by grid lines (blocks)—and the Euclidean norm—as the raven flies—is also a pair of equivalent norms.

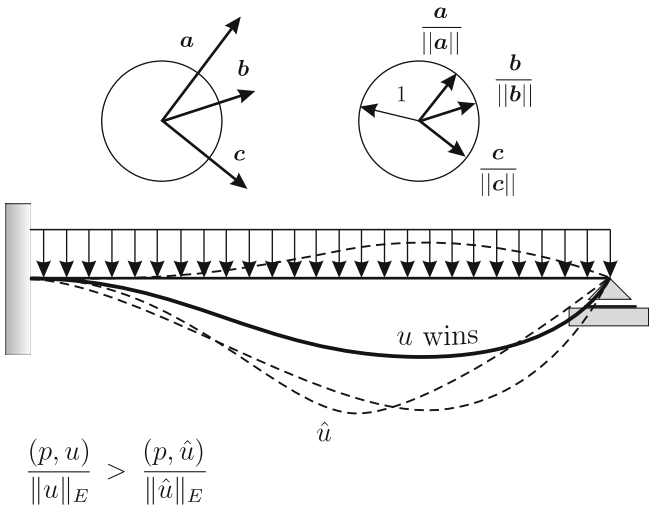

Fig. 2.2 In the unit ball $B_1 = \{u \mid ||u||_E = 1\}$ the normalized exact solution $u/||u||_E$ gets "the most mileage" out of p, i.e. the virtual work exceeds that of any other normalized deflection $\hat{u}/||\hat{u}||_E$ [1]

$$J(u) := \int_0^l p\, u\, dx \tag{2.56}$$

and the maximum value of $J(u/\|u\|)$ on the solution space

$$\mathcal{V} := \{u \in H^2(0,l) | u(0) = u'(0) = u(l) = 0\} \tag{2.57}$$

is attained if the function u is the deflection of the beam

$$EI\, u^{IV} = p \quad u(0) = u'(0) = u(l) = M(l) = 0\,. \tag{2.58}$$

This observation confirms that the principle of minimum potential energy is a maximum principle. The beam deflection u does not minimize the potential energy in the sense of "as small as possible" but it is the one function which *maximizes* the potential energy

$$\begin{aligned} \Pi(u) &= \frac{1}{2} \int_0^l EI\, (u'')^2\, dx - \int_0^l p\, u\, dx \\ &= -\frac{1}{2} \int_0^l p\, u\, dx\, |_{u=solution} = -\frac{1}{2}\, J(u) \end{aligned} \tag{2.59}$$

in the sense of having the largest possible distance from zero, $|\Pi(u)| = \max$, and it is the one function in $\mathcal{V}$ which gets the most mileage out of p.

2.1.7 Galerkin Method

The idea of the Galerkin method in solving the variational problem

$$u \in \mathcal{V}: \quad a(u, v) = J(v) \quad v \in \mathcal{V} \tag{2.60}$$

approximately is to construct a finite-dimensional subspace $\mathcal{V}_h$ which is spanned by a set of n shape functions $\varphi_i, i = 1, 2, \ldots n$, and to choose that function

$$u_h = \sum_{i=1}^{n} u_i \, \varphi_i(\boldsymbol{x}) \tag{2.61}$$

which satisfies (2.60) with regard to all shape functions φ_i. This leads to the $(n \times n)$ system

$$\boldsymbol{K}\,\boldsymbol{u} = \boldsymbol{f} \tag{2.62}$$

for the vector $\boldsymbol{u} = \{u_i\}$ where

$$k_{ij} = a(\varphi_i, \varphi_j) \quad f_i = J(\varphi_i)\,. \tag{2.63}$$

If $J(v)$ is a linear bounded functional and if the energy norm $\|u\|_E$ is an equivalent norm on $\mathcal{V}$ then the solution u is also the Riesz element of the functional $J(v)$.

2.1.8 Sobolev Spaces

The concept of a scalar product can be extended to functions defined over a domain Ω

$$(u, v)_{L_2} := \int_{\Omega} u\, v \; d\Omega \tag{2.64}$$

and this introduces the space $L_2(\Omega)$ as the set of all functions for which the integral

$$\int_{\Omega} u^2 \, d\Omega < \infty \tag{2.65}$$

is bounded. Given two such functions u and v in $L_2(\Omega)$ their scalar product is finite, $(u, v) < \infty$, and also (u, u) is finite and the expression

$$\|u\| := \sqrt{(u, u)} = \left(\int_{\Omega} u^2 \, d\Omega \right)^{1/2} \tag{2.66}$$

defines a norm on $L_2(\Omega)$.

The Sobolev-spaces are the extension of $L_2(\Omega)$ by incorporating also the weak partial derivatives of the functions up to a certain order m.

Before we can explain what a weak partial derivative is we must first explain what a multi-index is. A multi-index α is an array of non-negative integers which serves as short-hand for derivatives as in

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \partial_\alpha u \quad \alpha = (2, 1) \quad |\alpha| = 2 + 1 = 3 \tag{2.67}$$

$$\frac{\partial^4 u}{\partial x^2 \partial y^2} = \partial_\alpha u \quad \alpha = (2, 2) \quad |\alpha| = 2 + 2 = 4 \,. \tag{2.68}$$

Definition 2.1 (*weak derivative*) Let α a multi-index and $u \in L_2(\Omega)$. The function $u_\alpha = \partial_\alpha u \in L_2(\Omega)$ is called the weak derivative of u if

$$\int_{\Omega} u \, \partial_\alpha \varphi \, d\Omega = (-1)^{|\alpha|} \int_{\Omega} \partial_\alpha u \, \varphi \, d\Omega \quad \forall \, \varphi \in C_0^\infty(\Omega) \,. \tag{2.69}$$

Basically this is the integration by parts formula without boundary integrals because $\varphi \in C_0^\infty$ implies that φ and all its derivatives have zero boundary values and that φ is infinitely often differentiable on Ω. For "normal" functions such as $\sin x, e^x, x^2$ etc. weak derivatives and regular derivatives coincide.

This equation is of Betti-type: the work done by the "load" $\partial_\alpha \varphi$ on acting through u is the same as the work done by the "load" $\partial_\alpha u$ on acting through φ. For odd orders, $|\alpha| = 1, 3, 5, \ldots$, a (-1) factors in, symmetric and skew-symmetric operators.

Given a natural number m the inner product $(\cdot, , \cdot)_m$ of two functions u and v is the following sum

$$(u, v)_m := \sum_{|\alpha| \le m} (\partial_\alpha u, \partial_\alpha v)_{L_2(\Omega)} \tag{2.70}$$

and correspondingly is

$$\|u\|_m = \sqrt{\sum_{|\alpha| \le m} \|\partial_\alpha u\|^2_{L_2(\Omega)}} \tag{2.71}$$

the associated norm.

The *Sobolev-space* $H^m(\Omega)$ is the set of all m-times weakly differentiable functions $u \in L_2(\Omega)$, that is u and all its derivatives $\partial_\alpha u$ up to the order $|\alpha| = m$ are in $L_2(\Omega)$.

Example 2.4 Let $u(x) = \sin(x)$, $u'(x) = \cos(x)$, on the interval $(0, 2\pi)$. Then

$$\|u\|_0 = \sqrt{\int_0^{2\pi} \sin^2(x)\, dx} = \sqrt{\pi} \tag{2.72}$$

$$\|u\|_1 = \sqrt{\int_0^{2\pi} [\sin^2(x) + \cos^2(x)]\, dx} = \sqrt{\pi + \pi} \tag{2.73}$$

and evidently because of the cyclic nature of the derivatives

$$\|u\|_n = \sqrt{\int_0^{2\pi} [u^2(x) + (u'(x))^2 + \ldots (u^n(x))^2]\, dx} = \sqrt{n\,\pi}\,. \tag{2.74}$$

The space

$$\mathcal{V} = \{u \in H^1(\Omega) |\, u = 0 \text{ on } \Gamma\} \tag{2.75}$$

with the scalar product $a(u, v) := (\nabla u, \nabla v)$ and the norm $\|u\|_E := a(u, u)^{1/2}$ is a Hilbert space and therefore there exists for any linear and bounded functional $J(u)$ a Riesz element G such that

$$(\textit{weak infl. func.}) \quad J(u) = a(G, u) \quad \forall\, u \in \mathcal{V}\,. \tag{2.76}$$

Because of Green's first identity, see (2.86), this is equivalent to

$$J(u) = \int_\Omega G\,(-\Delta u)\, d\Omega_{\boldsymbol{y}} \quad \forall\, u \in \mathcal{V} \tag{2.77}$$

and if u is the solution of the boundary value problem (2.28) then follows

$$(\textit{strong infl. func.}) \quad J(u) = \int_\Omega G\, p\, d\Omega\,. \tag{2.78}$$

2.2 Green's Identities

Green's first and second identities—which are essentially applications of Gauss' Theorem—form two important corner stones of mathematical physics and of continuum mechanics. The classical energy and variational principles and conservation laws are based on these identities.

2.2.1 Gauss' Theorem

Let u and $\hat{u}$ be two arbitrary functions which have continuous first order derivatives on the interval $(0, l)$ then holds

$$\int_0^l u'\,\hat{u}\,dx = [u\,\hat{u}]_0^l - \int_0^l u\,\hat{u}'\,dx \tag{2.79}$$

and in higher dimensions, given a domain Ω and two functions u and $\hat{u}$ in $C^1(\Omega)$ (continuous first derivatives), as well

$$\int_\Omega u_{,x_i}\;\hat{u}\,d\Omega = \int_\Gamma u\,n_i\,\hat{u}\,ds - \int_\Omega u\,\hat{u}_{,x_i}\;d\Omega\,, \tag{2.80}$$

where n_i is the i-th component of the normal vector $\boldsymbol{n}$ on the edge Γ of the domain Ω.

The *divergence theorem* that connects the operators div and ∇ when applied to a vector field $\boldsymbol{u}$ and a function $\hat{u}$

$$\int_\Omega \operatorname{div}\boldsymbol{u}\,\hat{u}\,d\Omega = \int_\Gamma \boldsymbol{u}\cdot\boldsymbol{n}\,\hat{u}\,ds - \int_\Omega \boldsymbol{u}\cdot\nabla\,\hat{u}\,d\Omega \tag{2.81}$$

is an application of Gauss' theorem.

2.2.2 The Laplace Operator

Consider a thin membrane, stretched by a force H (uniformly in all directions), that seals an opening Ω and which is subjected to a pressure p, see Fig. 2.3. The deflection $u(\boldsymbol{x})$ of the membrane is the solution of the boundary value problem

$$-H\,\Delta u := -H\,(u_{,xx} + u_{,yy}) = p \quad \text{on } \Omega \quad u = 0 \quad \text{on } \Gamma\,. \tag{2.82}$$

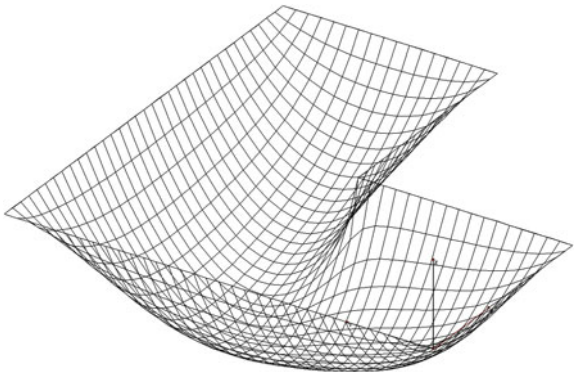

Fig. 2.3 Membrane under pressure load

Because H is a constant we can set $H = 1$ in the following.

Let $u \in C^2(\Omega)$ be a smooth (two times continuously differentiable) function on Ω. To this function we apply the Laplace operator

$$-\Delta u = -(u_{,xx} + u_{,yy}) \tag{2.83}$$

multiply the result with a second function $v \in C^1(\Omega)$, form the integral

$$\int_{\Omega} -\Delta u \, v \, d\Omega \tag{2.84}$$

and apply integration by parts to this integral

$$\int_{\Omega} -\Delta u \, v \, d\Omega = -\int_{\Gamma} \frac{\partial u}{\partial n} \, v \, ds + \int_{\Omega} \nabla u \cdot \nabla v \, d\Omega \, . \tag{2.85}$$

The result is Green's first identity

$$\mathcal{G}(u, v) = \underbrace{\int_{\Omega} -\Delta u \, v \, d\Omega + \int_{\Gamma} \frac{\partial u}{\partial n} \, v \, ds}_{\delta W_e} - \underbrace{\int_{\Omega} \nabla u \cdot \nabla v \, d\Omega}_{\delta W_i} = 0 \, . \tag{2.86}$$

Any pair of functions $\{u, v\}$ in $C^2(\Omega) \times C^1(\Omega)$ is a zero of this identity. If u is the solution to the boundary value problem (3.67) and v a virtual displacement of the membrane then the first two integrals represent the virtual exterior work δW_e and the integral

$$\int_{\Omega} \nabla u \cdot \nabla v \, d\Omega = \int_{\Omega} (u_{,x}\ v_{,x} + u_{,y}\ v_{,y}) \, d\Omega =: a(u, v) \tag{2.87}$$

is called the virtual internal energy δW_i or the *strain energy product* between u and v. It is often abbreviated as $a(u, v)$.

Choosing for v the translation $v = 1$ gives

$$\mathscr{G}(u, 1) = \int_{\Omega} p \cdot 1 \, d\Omega + \int_{\Gamma} \frac{\partial u}{\partial n} \cdot 1 \, ds = 0 \tag{2.88}$$

which is the equilibrium condition: the tractions $\partial u/\partial n$ on the edge balance the pressure distribution p; they are the support reactions of a membrane.

On the "diagonal", when $v = u$,

$$\mathscr{G}(u, u) = \underbrace{\int_{\Omega} -\Delta u \, u \, d\Omega}_{2\,W_e} - \underbrace{\int_{\Omega} \nabla u \cdot \nabla v \, d\Omega}_{2\,W_i} = 0 \tag{2.89}$$

Green's first identity formulates the *principle of conservation of energy* and on the "off-diagonals" when $v = \delta u$ is a virtual displacement which we assume to be compatible with the boundary conditions, $\delta u = 0$ on Γ,

$$\mathscr{G}(u, \delta u) = \underbrace{\int_{\Omega} -\Delta u \, \delta u \, d\Omega}_{\delta W_e} - \underbrace{\int_{\Omega} \nabla u \cdot \nabla \delta u \, d\Omega}_{\delta W_i} = 0 \tag{2.90}$$

it formulates the *principle of virtual displacements*.

The *principle of virtual forces* is the identity in "reverse order", that is with the places of u and δu interchanged

$$\begin{aligned} \mathscr{G}(\delta u, u) &= \delta W_e^* - \delta W_i^* \\ &= \underbrace{\int_{\Omega} -\Delta \delta u \, u \, d\Omega + \int_{\Gamma} \frac{\partial \delta u}{\partial n} \, u \, ds}_{\delta W_e^*} - \underbrace{a(u, \delta u)}_{\delta W_i^*} = 0 \,. \end{aligned} \tag{2.91}$$

Green's second identity is the expression

$$\mathscr{B}(u, v) = \mathscr{G}(u, v) - \mathscr{G}(v, u) = 0 - 0 = 0 \tag{2.92}$$

or

$$\mathscr{B}(u,v) = \int_{\Omega} -\Delta u\, v\, d\Omega + \int_{\Gamma} \frac{\partial u}{\partial n}\, v\, ds - \int_{\Gamma} \frac{\partial v}{\partial n}\, u\, ds - \int_{\Omega} u\,(-\Delta v)\, d\Omega = 0\,. \tag{2.93}$$

This is Betti's theorem which states that if a membrane is subjected to two different pressure distributions p_1 and p_2

$$-\Delta u_1 = p_1 \quad \text{on } \Omega \quad u_1 = 0 \quad \text{on } \Gamma \tag{2.94}$$

$$-\Delta u_2 = p_2 \quad \text{on } \Omega \quad u_2 = 0 \quad \text{on } \Gamma \tag{2.95}$$

then the reciprocal exterior work of the two solutions u_1 and u_2 is the same

$$\mathscr{B}(u_1,u_2) = \int_{\Omega} p_1\, u_2\, d\Omega - \int_{\Omega} u_1\, p_2\, d\Omega = 0\,. \tag{2.96}$$

FE-analysis operates with piecewise polynomials u_h and v_h which are smooth inside each element Ω_e, but the derivatives often jump at interelement boundaries and so the identities can only be formulated element for element

$$\mathscr{G}(u_h,v_h)_{\Omega_e} = \int_{\Omega_e} -\Delta\, u_h\ v_h\, d\Omega + \int_{\Gamma_e} \frac{\partial u_h}{\partial n}\, v_h\, ds - a(u_h,v_h)_{\Omega_e} = 0\,. \tag{2.97}$$

Adding this string of zeros

$$\mathscr{G}(u_h,v_h) := \sum_e \mathscr{G}(u_h,v_h)_{\Omega_e} = 0 + 0 + \ldots + 0 = 0 \tag{2.98}$$

results in

$$\mathscr{G}(u_h,v_h) := \sum_e \int_{\Omega_e} -\Delta\, u_h\, v_h\, d\Omega + \sum_k \int_{\Gamma_k} l_k\, v_h\, ds - a(u_h,v_h) = 0 \tag{2.99}$$

where

$$a(u_h,v_h) := \sum_e a(u_h,v_h)_{\Omega_e}\,. \tag{2.100}$$

The functions l_k on the mesh lines Γ_k represent the jumps of the normal derivative in between the elements

$$l_k := \frac{\partial u_h}{\partial n}_{+} - \frac{\partial u_h}{\partial n}_{-}\,. \tag{2.101}$$

If a side Γ_k of an element borders on the outer edge then l_k is simply the normal derivative on the edge.

2.2.3 Linear Self-Adjoint Operators

The Laplace operator is a linear self-adjoint differential operator of even order. Many operators in mathematical physics share this property and have the same mathematical structure so that in the following we can let L represent a generic operator of this type.

By applying integration by parts to the domain integral $(L\,u, \hat{u})$ the integral can be transformed into a series of boundary integrals $[\cdot\,,\cdot]$ and a symmetric bilinear form $a(u, \hat{u})$, the so called *strain energy product* or virtual strain energy, which is a domain integral. Collecting all terms on one side results in Green's first identity

$$\mathscr{G}(u, \hat{u}) = (L\,u, \hat{u}) + \sum_{i=0}^{m-1} (-1)^i [\partial_i u, \partial_{2m-1-i}\hat{u}] - a(u, \hat{u}) = 0 \tag{2.102}$$

and after a simple maneuver

$$\mathscr{B}(u, \hat{u}) := \mathscr{G}(u, \hat{u}) - \mathscr{G}(\hat{u}, u) = 0 - 0 = 0 \tag{2.103}$$

also in Green's second identity

$$\mathscr{B}(u, \hat{u}) = (L\,u, \hat{u}) + \sum_{i=0}^{2m-1} (-1)^i [\partial_i u, \partial_{2m-1-i}\hat{u}] - (u, L\hat{u}) = 0\,. \tag{2.104}$$

The $2\,m$ boundary functions

$$\partial_i u \quad i = 0, 1, \ldots\, 2\,m - 1 \tag{2.105}$$

in the boundary integrals are the *canonical boundary values*; they are also known as *Cauchy data*. They play a crucial role in the conservation laws, they register what happens on the boundary, what flows in and out of a region. Given these boundary functions and the right-hand side, $L\,u = p$, the function u inside Ω can be reconstructed from these data.

If you know the height of the end points, $u(0)$ and $u(l)$, of a rope and the curvature, $-u'' = p$, of the rope in between then you can draw the shape of the rope; this is the idea.

To the Laplace operator, $-\Delta u$, belong two boundary functions

$$\partial_0 u = u \quad \partial_1 u = \frac{\partial u}{\partial n} \tag{2.106}$$

and to the fourth-order beam equation $E\,I\,u^{IV} = p$ four boundary values

$$\partial_0 u = u \quad \partial_1 u = u' \quad \partial_2 u = -E\,I\,u'' = M \quad \partial_3 u = -E\,I\,u''' = V\,. \tag{2.107}$$

In the mechanical context the lower order terms, $\partial_i u$, $i < m$, are displacement terms while the higher order terms, $\partial_i u$, $m \leq i < 2m - 1$, are force terms and two such terms are called adjoint or conjugate, as for example the terms

$$\partial_0 u \, \partial_3 u = u \, V \quad 0 + 3 = 3 \tag{2.108}$$

$$\partial_1 u \, \partial_2 u = u' \, M \quad 1 + 2 = 3, \tag{2.109}$$

if the sum of the indices in the pair $\partial_i u \, \partial_j u$ equals $i + j = 2m - 1$ ($= 3$ in the case of a fourth order equation).

Obviously can these identities be extended to the Sobolev spaces as $H^2(\Omega)$ and $H^1(\Omega)$ by allowing the functions to have weak derivatives. But on two- and three-dimensional domains Ω this requires additional assumptions about the boundary values of such functions (*trace theorem*) which are very technical and which we do not want to discuss here in details.

For our purposes it suffices to assume that the identities are applicable to the standard FE-shape functions inside each element and that the extension to the whole domain, the whole mesh, can simply be done by adding zeros

$$0 + 0 + \ldots + 0 + 0 = 0 \, . \tag{2.110}$$

2.3 Duality

Imagine two opposite forces, P_l and P_r, that pull on both sides of a rod, so that equilibrium is maintained

$$- P_l + P_r = 0 \tag{2.111}$$

but then the equation can be multiplied with any number δu

$$\delta u \, (-P_l + P_r) = \delta u \cdot 0 = 0 \tag{2.112}$$

without changing the mathematics. That is one can slide the rod effortlessly across the table: the virtual exterior work is zero—for each displacement δu.

What looks like a child's play is in truth one of the most important steps in applied mathematics, it is the earliest manifestation of *duality*. A whole plethora of concepts and ideas such as *scalar product, adjoint quantities, virtual displacement, virtual work, Green's function, spectral theory, etc.* comes to light by this simple maneuver.

The common theme of all these concepts is perhaps best explained in terms of linear algebra.

2.3.1 Linear Algebra

Let

$$\boldsymbol{K}\,\boldsymbol{u} = \boldsymbol{f} \tag{2.113}$$

a linear system of equations where $\boldsymbol{K}$ is an $n \times n$ matrix and $\boldsymbol{u}$ and $\boldsymbol{f}$ are vectors in $\mathbb{R}^n$.

The transpose of a real number is the same number as the original number, $\pi^T = \pi$, and so given any two vectors $\boldsymbol{u}$ and $\hat{\boldsymbol{u}}$ the following expression is an identity

$$\hat{\boldsymbol{u}}^T\,\boldsymbol{K}\,\boldsymbol{u} = (\hat{\boldsymbol{u}}^T\,\boldsymbol{K}\,\boldsymbol{u})^T = \boldsymbol{u}^T\,\boldsymbol{K}^T\,\hat{\boldsymbol{u}} \tag{2.114}$$

or in a slightly different arrangement

$$\mathscr{B}(\boldsymbol{u}, \hat{\boldsymbol{u}}) = \hat{\boldsymbol{u}}^T\,\boldsymbol{K}\,\boldsymbol{u} - \boldsymbol{u}^T\,\boldsymbol{K}^T\,\hat{\boldsymbol{u}} = 0\,. \tag{2.115}$$

This identity implies that the system (2.113) can only have a solution if the right hand side $\boldsymbol{f}$ is orthogonal to all solutions $\hat{\boldsymbol{u}}$ of the homogeneous adjoint system $\boldsymbol{K}^T\,\hat{\boldsymbol{u}} = \boldsymbol{0}$ because

$$\mathscr{B}(\boldsymbol{u}, \hat{\boldsymbol{u}}) = \hat{\boldsymbol{u}}^T\,\boldsymbol{K}\,\boldsymbol{u} - \boldsymbol{u}^T\,\boldsymbol{K}^T\,\hat{\boldsymbol{u}} = \hat{\boldsymbol{u}}^T\,\boldsymbol{f} - 0 = \hat{\boldsymbol{u}}^T\,\boldsymbol{f} = 0\,. \tag{2.116}$$

And if (2.113) has a solution $\boldsymbol{u}$ then $\boldsymbol{u}$ is also a solution of the variational problem

$$\hat{\boldsymbol{u}}^T\,\boldsymbol{K}\,\boldsymbol{u} = \hat{\boldsymbol{u}}^T\,\boldsymbol{f} \quad \forall\,\hat{\boldsymbol{u}} \in \mathbb{R}^n \tag{2.117}$$

and vice versa which establishes that in the discrete case the *strong formulation*, (2.113), and the *weak formulation*, (2.117), are equivalent.

Next let the vector $\boldsymbol{g}_i$ be the solution of the adjoint equation if the right hand side is the i-th unit vector, $\boldsymbol{e}_i = \{0, 0, \ldots 1, \ldots 0, 0\}^T$

$$\boldsymbol{K}^T\,\boldsymbol{g}_i = \boldsymbol{e}_i \tag{2.118}$$

and let $\boldsymbol{u}$ the solution of (2.113) then follows

$$\mathscr{B}(\boldsymbol{u}, \boldsymbol{g}_i) = \boldsymbol{g}_i^T\,\boldsymbol{K}\,\boldsymbol{u} - \boldsymbol{u}^T\,\boldsymbol{K}^T\,\boldsymbol{g}_i = \boldsymbol{g}_i^T\,\boldsymbol{f} - \boldsymbol{u}^T\,\boldsymbol{e}_i = 0 \tag{2.119}$$

or

$$u_i = \boldsymbol{g}_i^T\,\boldsymbol{f}\,. \tag{2.120}$$

So the "point solutions" of the adjoint system and the solution $\boldsymbol{u}$ of the original problem are intrinsically intertwined. Projecting $\boldsymbol{f}$ on all the axes $\boldsymbol{g}_i$, $i = 1, 2 \ldots n$ allows to construct the solution $\boldsymbol{u} = u_1\,\boldsymbol{e}_1 + \ldots + u_n\,\boldsymbol{e}_n$ with respect to the basis $\boldsymbol{e}_i$ because projecting $\boldsymbol{f}$ onto the $\boldsymbol{g}_i$ is the same as projecting $\boldsymbol{u}$ onto the $\boldsymbol{e}_i$

$$u_i = \boldsymbol{g}_i^T \boldsymbol{f} = \boldsymbol{u}^T \boldsymbol{e}_i \,. \tag{2.121}$$

This is what is done when $\boldsymbol{f}$ is multiplied with the inverse matrix $\boldsymbol{K}^{-1}$

$$\boldsymbol{u} = \boldsymbol{K}^{-1} \boldsymbol{f} \tag{2.122}$$

because the rows of $\boldsymbol{K}^{-1}$ are just the vectors $\boldsymbol{g}_i$ and so

$$\boldsymbol{u} = (\boldsymbol{g}_1^T \boldsymbol{f})\, \boldsymbol{e}_1 + (\boldsymbol{g}_2^T \boldsymbol{f})\, \boldsymbol{e}_2 + \ldots + (\boldsymbol{g}_n^T \boldsymbol{f})\, \boldsymbol{e}_n \,. \tag{2.123}$$

In the continuous case, when the focus is on differential equations, as for example

$$-u''(x) = p(x) \quad u(0) = u(l) = 0 \,, \tag{2.124}$$

the matrix $\boldsymbol{K}$ has infinitely many columns and the unit vectors become Dirac deltas

$$-\frac{d^2}{dy^2} G(y, x) = \delta(y - x) \tag{2.125}$$

but the formalism is the same. By projecting the right-hand p side onto the solutions $G(y, x)$, that is by forming the L_2-scalar product (integral) of the two functions, the value of the solution can be recovered at any point x

$$u(x) = \underbrace{\int_0^l G(y, x)\, p(y)\, dy}_{\boldsymbol{g}_i^T \boldsymbol{f}} = \underbrace{\int_0^l \delta(y - x)\, u(y)\, dy}_{\boldsymbol{u}^T \boldsymbol{e}_i} \tag{2.126}$$

and the first integral is, as the equation shows, the same as the L_2-scalar product between the function u itself and the Dirac delta, so that these two equations are indeed equivalent to (2.121).

To the symmetry of the strain energy product

$$a(u, \hat{u}) = a(\hat{u}, u) \tag{2.127}$$

corresponds the symmetry of $\boldsymbol{K}$ and it so follows that if u and G are the variational solutions of the *primal* and the *dual* problem respectively

$$u \in \mathcal{V} \quad a(u, v) = (p, v) \quad \forall\, v \in \mathcal{V} \quad \textit{primal problem} \tag{2.128}$$

$$G \in \mathcal{V} \quad a(G, v) = (\delta, v) \quad \forall\, v \in \mathcal{V} \quad \textit{dual problem} \tag{2.129}$$

then this implies

$$u(\boldsymbol{x}) = (p, G) = (\delta, u) \,. \tag{2.130}$$

Equation (2.126) is the same result in a "longhand" notation.

Much more could be said at this point about linear self-adjoint operators with constant coefficients, Green's functions and matrices, about eigenvalues, eigenvectors, convolution and the construction of Green's function from eigenfunctions. Linear, time-invariant systems (signal processing) provide the most elementary application and they are also a good example for the transition from the discrete case to the continuous case, $\boldsymbol{K}_{(n\times n)} \rightarrow \boldsymbol{K}_{(\infty\times\infty)}$. Such systems are completely characterized by their response to an impulse which is either modeled as a Dirac delta function for continuous-time systems or as the Kronecker delta for discrete-time systems [3].

Remark 2.1 In signal processing the analysis is not done by integrating the impulse response function (the Green's function) and the input function but by multiplying the integral transforms of the two functions because a convolution in the time domain is equivalent to a multiplication in the frequency domain.

2.3.2 Vectors and Linear Functionals

In linear algebra, $\mathcal{V} = \mathbb{R}^n$, all linear functionals have the form of a scalar product

$$J(\boldsymbol{u}) = \boldsymbol{j}^T \boldsymbol{u} \tag{2.131}$$

between $\boldsymbol{u}$ and a certain vector $\boldsymbol{j}$ whose components are simply the values of the functional applied to the base vectors $\boldsymbol{e}_i, i = 1, 2, \ldots n$ of $\mathbb{R}^n$

$$j_i = J(\boldsymbol{e}_i)\,. \tag{2.132}$$

In the case of the functional $J(\boldsymbol{u}) = u_1$ which returns the first component of the input, the vector $\boldsymbol{j}$ is just the unit vector $\boldsymbol{e}_1$

$$J(\boldsymbol{u}) = \boldsymbol{e}_1^T \boldsymbol{u} = u_1\,. \tag{2.133}$$

To understand the notion of a Green's function or rather Green's vector $\boldsymbol{g}$ imagine a situation where the vector $\boldsymbol{u}$ is unknown and so $J(\boldsymbol{u})$ cannot be calculated directly. It is only known that the matrix $\boldsymbol{K}$ has mapped the vector $\boldsymbol{u}$ onto a vector $\boldsymbol{f} = \boldsymbol{K}\,\boldsymbol{u}$. Can the value $J(\boldsymbol{u})$, the first component u_1 of $\boldsymbol{u}$, be recovered from $\boldsymbol{f}$? Yes, this is possible. Solving the system

$$\boldsymbol{K}^T \boldsymbol{g} = \boldsymbol{j} = \boldsymbol{e}_1 \tag{2.134}$$

for the vector $\boldsymbol{g}$ and formulating the identity

$$\mathscr{B}(\boldsymbol{u}, \boldsymbol{g}) = \boldsymbol{g}^T\, \boldsymbol{K}\,\boldsymbol{u} - \boldsymbol{u}^T\, \boldsymbol{K}^T\, \boldsymbol{g} = \boldsymbol{g}^T\, \boldsymbol{f} - \boldsymbol{u}^T\, \boldsymbol{e}_1 = 0 \tag{2.135}$$

it is found that

$$J(\boldsymbol{u}) = u_1 = \boldsymbol{g}^T \boldsymbol{f} \,. \tag{2.136}$$

Note that the vector $\boldsymbol{g}$ depends on the matrix $\boldsymbol{K}$, that is to find the way back from $\boldsymbol{f}$ to $\boldsymbol{u}$, or here u_1, we must know which matrix $\boldsymbol{K}$ mapped $\boldsymbol{u}$ onto $\boldsymbol{f}$.

If a different matrix $\bar{\boldsymbol{K}}$ maps $\boldsymbol{u}$ onto a different vector $\bar{\boldsymbol{f}} = \bar{\boldsymbol{K}}\,\boldsymbol{u}$ then—even though the functional $J(\boldsymbol{u}) = u_1$ is the same—a different Green's vector is required

$$\bar{\boldsymbol{K}}^T \bar{\boldsymbol{g}} = \boldsymbol{j} = \boldsymbol{e}_1 \,. \tag{2.137}$$

The same logic applies in the continuous case. Take the function $u(x) = \sin(\pi\, x/l)$ and differentiate it two times or four times respectively

$$-u'' = \left(\frac{\pi}{l}\right)^2 \sin(\pi\, x/l) \quad = f(x) \tag{2.138}$$

$$E\,I\,u^{IV} = \left(\frac{\pi}{l}\right)^4 \sin(\pi\, x/l) \quad = \bar{f}(x) \,. \tag{2.139}$$

In the first case $u(x)$ is the deflection of a taut rope subjected to a lateral load $f(x)$ and in the second case it is the deflection of a beam subjected to a lateral load $\bar{f}(x)$.

To recover the deflection at mid-span, $u(l/2)$, from the right-hand sides $f(x)$ and $\bar{f}(x)$, different Green's functions are required though we are asking for the same value, $u(l/2)$. So the mapping is important. Which operator mapped u onto the right-hand side? Where do the data come from?

2.4 Influence Functions

Linear algebra is easy. Let us now turn to functions and let us replace the matrix $\boldsymbol{K}$ by the second-order differential operator in the equation for the rope

$$-\,u''(x) = p(x) \quad u(0) = u(l) = 0 \,. \tag{2.140}$$

The aim is to formulate two influence functions, one for the deflection $u(x)$ of the rope and and one for the transverse force $V(x) = H\,u'(x) = u'(x)$, by using Green's first identity (principle of virtual forces) and, alternatively, by using Green's second identity (Betti's theorem).

Green's first identity for the operator $-d^2/dx^2$ is the expression

$$\begin{aligned} &If\ \hat{u} \in C^2(0,l), u \in C^1(0,l)\ then \\ &\mathscr{G}(\hat{u},u) = \int_0^l -\hat{u}''\,u\,dx + [\hat{u}'\,u]_0^l - \int_0^l \hat{u}'\,u'\,dx = 0 \end{aligned} \tag{2.141}$$

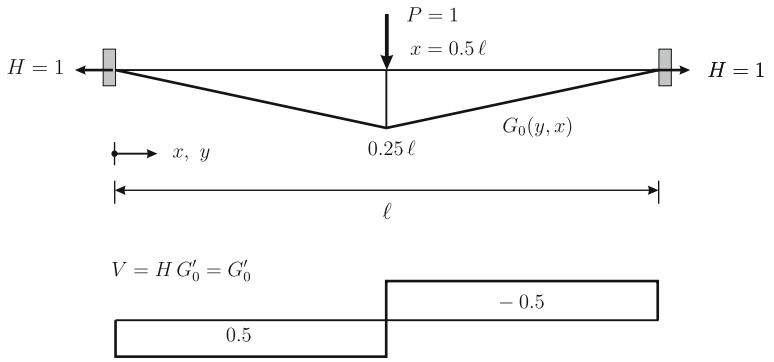

Fig. 2.4 Influence function G_0

and the second identity, $\mathscr{B}(\hat{u}, u) = \mathscr{G}(\hat{u}, u) - \mathscr{G}(u, \hat{u})$, is the expression

$$\begin{aligned} &If\ \hat{u} \in C^2(0,l), u \in C^2(0,l)\ then \\ &\mathscr{B}(\hat{u}, u) = \int_0^l -\hat{u}''\, u\, dx + [\hat{u}'\, u] - [\hat{u}\, u']_0^l - \int_0^l \hat{u}\, (-u'')\, dx = 0\,. \end{aligned} \tag{2.142}$$

Because these identities are based on integration by parts the functions u and $\hat{u}$ must meet certain regularity requirements. In the following, when we formulate these or other identities, we will tacitly assume that the functions satisfy these conditions or that we have partitioned the domain appropriately into subregions—typically the elements—to circumvent restrictions which apply because the shape functions are only piecewise C^2 and C^1 respectively.

2.4.1 Influence Function for u(x)

Let $G_0(y, x)$ be the deflection of the rope when a single force $P = 1$ acts at a point x of the rope. That is G_0 is the solution of the boundary value problem

$$-\frac{d^2}{dy^2} G_0(y, x) = \delta_0(y - x) \quad G_0(0, x) = G_0(l, x) = 0\,. \tag{2.143}$$

The index 0 on the Dirac delta is to indicate that it represents a point load. Higher-order Dirac deltas, $\delta_1, \delta_2, \ldots$, will represent moments, dislocations, etc. The Green's functions carry the same index as the Dirac delta to which they belong.

The Green's function has the form of a string plugged at x, see Fig. 2.4,

$$G_0(y,x) = \frac{1}{l}\begin{cases} y\,(l-x) & y \le x \\ x\,(l-y) & x \le y \end{cases} . \tag{2.144}$$

The first derivative jumps at x and the second derivative is the Delta function, so clearly $G_0(y,x)$ is not a C^2 function. To derive Betti's theorem Green's second identity is formulated on the punctured domain or—here—interval Ω_ε that is a small ε-neighborhood of the source point x is excluded from $[0,l]$

$$\Omega_\varepsilon := [0, x-\varepsilon] \cup [x+\varepsilon, l] \tag{2.145}$$

and we then let ε tend to zero.

The functions u and G_0 are C^2-functions on Ω_ε. The second derivative of $G_0(y,x)$ is zero on Ω_ε because G_0 is a linear function and so

$$\begin{aligned} \mathscr{B}(G_0,u)_{\Omega_\varepsilon} = & \underbrace{G_0'(x,y-\varepsilon)\,u(y-\varepsilon) - G_0'(x,y+\varepsilon)\,u(y+\varepsilon)}_{\lim_{\varepsilon\to 0}=1\cdot u(x)} \\ & - \underbrace{G_0(x,y-\varepsilon)\,u'(y-\varepsilon) + G_0(x,y+\varepsilon)\,u'(y+\varepsilon)}_{\lim_{\varepsilon\to 0}=0} \\ & - \int\limits_{\Omega_\varepsilon} G_0(y,x)\,p(y)\,dy = 0\,. \end{aligned} \tag{2.146}$$

Because the sum is zero for all $\varepsilon > 0$ the limit must be zero as well and so we conclude that

$$\lim_{\varepsilon\to 0} \mathscr{B}(G_0,u)_{\Omega_\varepsilon} = u(x) - \int\limits_0^l G_0(y,x)\,p(y)\,dy = 0\,. \tag{2.147}$$

The term $u(x)$—or to be more precise $1\cdot u(x)$—is the work done by the unit-jump in the derivative G_0' at the source point x on acting through $u(x)$; the derivative G_0' corresponds to the transverse force and to accommodate the point load the derivative must jump at x.

The expression on the second line in (2.146) is zero in the limit because both G_0 and u' are continuous at x.

Reordering the terms in (2.147) gives the influence function for $u(x)$

$$u(x) = \int\limits_0^l G_0(y,x)\,p(y)\,dy\,. \tag{2.148}$$

We can now also explain what it means that $G_0(y,x)$ solves the boundary value problem (2.143). This is an open question: a direct verification is not possible because

the second derivative of $G_0(y,x)$ is not defined at the source point x and also $\delta_0(y,x)$ is not a proper function.

The answer is that the differential equation (2.143) must be interpreted in the *distributional sense*. The function $G_0(y,x)$ is a solution of (2.143) if and only if

$$\int_0^l -\frac{d^2}{dy^2}\, G_0(y,x)\, u(y)\, dy = \int_0^l \delta_0(y,x)\, u(y)\, dy = u(x) \quad \forall\, u \in C_0^\infty(0,l)\,. \tag{2.149}$$

That is the two sides are weighted with a test function $u \in C_0^\infty(0,l)$ and the results on both sides should match. The test functions u are infinitely smooth functions and all their boundary values, $u, u', u'', \ldots$, are zero—at both ends.

The trick is to apply integration by parts to the left side, neglecting all warnings that this is not allowed because $-G_0''$ is not a proper function, and to shift the differential operator from G_0 onto the infinitely patient test function

$$\int_0^l -\frac{d^2}{dy^2}\, G_0(y,x)\, u(y)\, dy = \int_0^l G_0(y,x)\, (-u''(y))\, dy \tag{2.150}$$

and so it must be true that (2.149) is equivalent to

$$\int_0^l G_0(y,x)\, (-u''(y))\, dy = u(x) \quad \forall\, u \in C_0^\infty(0,l)\,. \tag{2.151}$$

And this was demonstrated above (substitute for $-u''$ the right hand side p).

2.4.2 Influence Function for $u'(x)$

Instead of a point load now a unit dislocation $\delta_1(y,x)$ is applied at the source point x so that the influence function $G_1(y,x)$ for the first derivative $u'(x)$ is the solution to the boundary value problem

$$-\frac{d^2}{dy^2}\, G_1(y,x) = \delta_1(y,x) \quad G_1(0,x) = G_1(l,x) = 0 \tag{2.152}$$

and consequently

$$\mathscr{B}(G_1,u)_{\Omega_\varepsilon} = \underbrace{G_1'(x,y-\varepsilon)\,u(y-\varepsilon) - G_1'(x,y+\varepsilon)\,u(y+\varepsilon)}_{\lim_{\varepsilon\to 0}=0}$$
$$\underbrace{- G_1(x,y-\varepsilon)\,u'(y-\varepsilon) + G_1(x,y+\varepsilon)\,u'(y+\varepsilon)}_{\lim_{\varepsilon\to 0}=1\cdot u'(x)}$$
$$- \int_{\Omega_\varepsilon} G_1(y,x)\,p(y)\,dy = 0 \tag{2.153}$$

and therefore as well

$$\lim_{\varepsilon\to 0}\mathscr{B}(G_1,u)_{\Omega_\varepsilon} = u'(x) - \int_0^l G_1(y,x)\,p(y)\,dy = 0 \tag{2.154}$$

which is the influence function for $u'(x)$

$$u'(x) = \int_0^l G_1(y,x)\,p(y)\,dy\,. \tag{2.155}$$

Remark 2.2 The Dirac delta δ_1

$$\delta_1(y,x) = 0 \quad x \neq y \tag{2.156}$$

$$\int_0^l \delta_1(y,x)\,u(y)\,dy = u'(x) \quad x \in (0,l) \tag{2.157}$$

can be considered the first derivative of δ_0 because integration by parts allows to write (note that $u(0) = u(l) = 0$)

$$\int_0^l \delta_0'(y,x)\,u(y)\,dy = \int_0^l \delta_0(y,x)\,u'(y)\,dy = u'(x) \tag{2.158}$$

so that δ_0' performs the same action as δ_1. For a very detailed discussion of these shifts and other transformation rules applicable to Dirac deltas see [4].

2.4.3 Weak Influence Function for $u(x)$

Next Green's first identity (or the principle of virtual forces) will be used to formulate an alternative influence function for $u(x)$. On the punctured domain the starting point is the expression

$$\mathscr{G}(G_0, u)_{\Omega_\varepsilon} = \underbrace{G_0'(x, y-\varepsilon)\, u(y-\varepsilon) - G_0'(x, y+\varepsilon)\, u(y+\varepsilon)}_{\lim_{\varepsilon\to 0} = u(x)} - a(G_0(y, x), u)_{\Omega_\varepsilon} = 0 \tag{2.159}$$

and taking the limit

$$\lim_{\varepsilon\to 0} \mathscr{G}(G_0, u)_{\Omega_\varepsilon} = u(x) - a(G_0, u) = 0 \tag{2.160}$$

gives

$$u(x) = a(G_0, u) := \int_0^l G_0'\, u'\, dy\,. \tag{2.161}$$

We call this influence function for $u(x)$ a *weak influence function* in contrast to (2.148) which we call a *strong influence function*.

A weak influence function is as good as a strong influence function. Weak means that it is based on an evaluation of the stresses and strains, which are of the order m if $2\,m$ is the order of the differential equation, while a strong influence function extracts $u(x)$ (or possibly other values) from the right-hand side p which—in terms of the solution u—is of order $2\,m$.

Weak influence functions can safely be formulated for displacement terms, the lower order terms (derivatives less than m if the differential equation is of order $2\,m$), but not for force terms (the higher derivatives), the result is nil, is zero.

To verify this we try to derive a weak influence function for the first derivative u' of a rope. The starting point is as before a formulation on the punctured interval

$$\mathscr{G}(G_1, u)_{\Omega_\varepsilon} = -G_1'(x, y-\varepsilon)\, u(y-\varepsilon) + G_1'(x, y+\varepsilon)\, u(y+\varepsilon) - a(G_1(y, x), u)_{\Omega_\varepsilon} = 0\,. \tag{2.162}$$

The limit of the first two terms is zero

$$\lim_{\varepsilon\to 0} \left[-G_1'(x, y-\varepsilon)\, u(y-\varepsilon) + G_1'(x, y+\varepsilon)\, u(y+\varepsilon)\right] = 0 \tag{2.163}$$

because both functions, the slope G_1' of the Green's function G_1 and the solution u, are continuous at x. But if the limit of the first two terms is zero (the exterior work) then the limit of the strain energy product between G_1 and u (the interior work) must

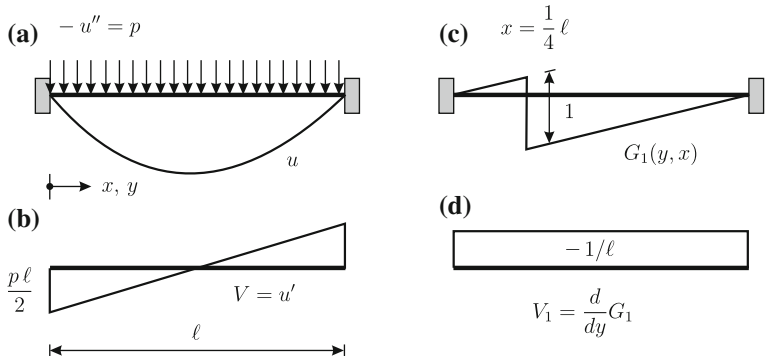

Fig. 2.5 Rope **a** deflection u, **b** shear force $V = u'$, **c** influence function G_1 for $u'(x)$, **d** derivative of G_1

be zero as well

$$\lim_{\varepsilon\to 0} \mathscr{G}(G_1, u)_{\Omega_\varepsilon} = 0 - \lim_{\varepsilon\to 0} a(G_1, u)_{\Omega_\varepsilon} = 0\,, \tag{2.164}$$

which is a correct but useless result. Stated separately the limit is

$$a(G_1, u) := \lim_{\varepsilon\to 0} a(G_1, u)_{\Omega_\varepsilon} = 0 \tag{2.165}$$

and not, as one might have guessed, $u'(x)$.

An example may prove the point. We try to formulate a weak influence function for the first derivative $V = u'$ of the rope, see Fig. 2.5, at the point $x = 0.25\,\ell$. But the strain energy product between the slope $V = u'$ of the rope and the slope of the Green's function G_1 evidently is zero

$$a(G_1, u) = \int_0^l \frac{d}{dy} G_1(y, x)\, u'(y)\, dy = 0 \tag{2.166}$$

while the true value of V at the quarter point is $pl/4$.

2.4.3.1 Background

Does this result not contradict the claim that any linear bounded functional $J(u)$ allows such a "weak representation" by a Riesz element (G_1),

$$J(u) = u'(x) = a(G_1, u) \quad ? \tag{2.167}$$

No it does not. Taking the derivative of a function is a bounded functional only if the three indices

- $i =$ the order of the derivative
- $m =$ the order of the energy space $H^m(\Omega)$
- $n =$ dimension of the continuum $\Omega \subset \mathbb{R}^n$

satisfy the inequality

$$m - i > \frac{n}{2}\,. \tag{2.168}$$

The differential operators, typically, have the order $2m = 2$ or $2m = 4$ and the associated energy space is $H^1(\Omega)$ and $H^2(\Omega)$ respectively, that is the order m of the energy space is half the value of the order $2m$ of the differential equation.

The solution space of the rope, $-u'' = p$, is $H^1(0, l)$ and on this space the zero-order derivative is a bounded functional but the first-order derivative is not

$$\underbrace{J(u) = u(x)}_{bounded} \quad \underbrace{J(u) = u'(x)}_{unbounded} \tag{2.169}$$

while on $H^2(0, l)$ (beam problems, $EI\,u^{IV} = p$, $2m = 4$) the derivatives are classified as follows

$$\underbrace{J(u) = u(x) \quad J(u) = u'(x)}_{bounded} \quad \underbrace{J(u) = u''(x) \quad J(u) = u'''(x)}_{unbounded}\,. \tag{2.170}$$

There is nothing wrong with higher order derivatives. It all depends on the definition of the underlying solution space. If the index m of the Sobolev space H^m is high enough, if it satisfies the inequality

$$m > i + \frac{n}{2} \tag{2.171}$$

then also higher order derivatives are bounded functionals.

To be precise, in 1-D problems the ith order derivative is a bounded functional on $H^m(0, l)$

$$|J(u)| = \left|\frac{d^i u}{dx^i}(x)\right| < c\,\|u\|_m \tag{2.172}$$

if $C^i(0, l) \subset H^m(0, l)$ ("continuous embedding") and according to Sobolev's Embedding Theorem this is the case if $m > i + 1/2$. So for the third derivative, $J(u) = u'''(x)$, to be bounded the solution space H^m must have an index $m > 3+1/2$, that is on $H^4(0, l)$ the third derivative is a continuous functional. In 2-D the necessary condition would be $m > i + 2/2$ and in 3-D $m > i + 3/2$, see p. xx.

Note that it would not do to choose very smooth shape functions, $\varphi_i \in H^4(0, l)$, and then to claim that the problem

$$a(G_h, \varphi_i) = J(\varphi_i) = \varphi_i'''(x) \tag{2.173}$$

is well-posed, that it is in agreement with the Riesz-representation theorem. For that the strain energy

$$a(u, u) = \int_0^l (u')^2 \, dx \tag{2.174}$$

must be equivalent on $\mathcal{V}$ to the norm on $H^4(0, l)$

$$\|u\|_4 := \left\{ \int_0^l (u^{IV})^2 + (u''')^2 + (u'')^2 + (u')^2 + u^2] \, dx \right\}^{1/2} \tag{2.175}$$

but because the strain energy disregards the higher-order derivatives it cannot really separate the elements of $H^4(0, l)$, that is it cannot be a norm on $H^4(0, l)$. The correct weak form for the Green's function would be

$$(G_h, \varphi_i)_4 = \varphi_i'''(x) \quad \forall \varphi_i \in \mathcal{V}_h \tag{2.176}$$

where $(\cdot, \cdot)_4$ is the scalar-product on $H^4(0, l)$ and in $u_h'''(x) = (G_h, p)$ the load p would be the eighth-order derivative of u or something similar to that.

Remark 2.3 An engineer will not wonder about this negative outcome, that the strain energy $a(G_1, u)$ is zero, because for him a weak influence functions such as

$$u(x) = \int_0^l G'(y, x) \, u'(y) \, dy \equiv a(G, u) \quad (-u'' = p) \tag{2.177}$$

is based on the *principle of virtual forces* and which virtual force do you apply, so he would ask, to calculate a force term?

To clarify this issue: What the engineer calls the principle of virtual forces is Green's first identity with δu and u in reverse order

$$\mathscr{G}(\delta u, u) = 0 \, . \tag{2.178}$$

The virtual displacement comes first and the solution is second while in the *principle of virtual displacements* the sequence is: first u and then δu.

The reverse order makes that ("virtual") forces, $\delta u''$ and δV $(= \delta u')$, act on u

$$\mathscr{G}(\delta u, u) = \int_0^l -\delta u''\, u\, dx + [\delta V\, u]_0^l - a(\delta u, v) = 0 \quad \text{(rope)}\,. \tag{2.179}$$

The single terms $u(x) = \ldots$ or $V(x) = \ldots$ in the influence functions are the limits of certain "boundary integrals"[3]

$$\lim_{\varepsilon\to 0}\left\{[V(G_0)\, u]_0^{x-\varepsilon} + [V(G_0)\, u]_{x-\varepsilon}^l\right\} = 1 \cdot u(x) \tag{2.180}$$

$$\lim_{\varepsilon\to 0}\left\{[G_1\, V]_0^{x-\varepsilon} + [G_1\, V]_{x-\varepsilon}^l\right\} = 1 \cdot V(x)\,. \tag{2.181}$$

But the force term $V(x)$ could only jump out of the boundary integral $[\delta u\, V]$ and this is not a part of (2.179). It is contained in Green's second identity and this is why Betti can do it but the principle of virtual forces can not.

2.4.4 A Sequence that Converges to G_1

The jump in the Green's function G_1 at x forced us to exclude an ε-neighborhood of the source point from the interval $[0, l]$.

Alternatively we could study what happens if a sequence of continuous and piecewise polynomial functions $G_1^\varepsilon(y, x)$ tends to $G_1(y, x)$. Such a sequence is easily constructed: we place two point forces $P = \pm 1/\Delta x$ at each side of the source point x, an unit Δx apart, and we then let the distance Δx shrink to zero so that the sequence of shapes $G_1^{\Delta x}$ converges to the exact Green's function $G_1(y, x)$, see Fig. 2.6.

Let us see how we go about the formulation of the identities in this case. For a particular value of Δx Green's first identity reads

$$\mathscr{G}(G_1^{\Delta x}, u) = \frac{1}{\Delta x}\, u(x + 0.5\,\Delta x) - \frac{1}{\Delta x}\, u(x - 0.5\,\Delta x) - a(G_1^{\Delta x}, u) = 0\,. \tag{2.182}$$

The limit of the first term is $u'(x)$

$$\lim_{\Delta x\to 0}\left[\frac{1}{\Delta x}\, u(x + 0.5\,\Delta x) - \frac{1}{\Delta x}\, u(x - 0.5\,\Delta x)\right] = u'(x) \tag{2.183}$$

and because of

$$\lim_{\Delta x\to 0} \mathscr{G}(G_1^{\Delta x}, u) = u'(x) - \lim_{\Delta x\to 0} a(G_1^{\Delta x}, u) = 0 \tag{2.184}$$

the second term must have the same limit

[3] In higher dimensions these terms would be genuine boundary integrals.

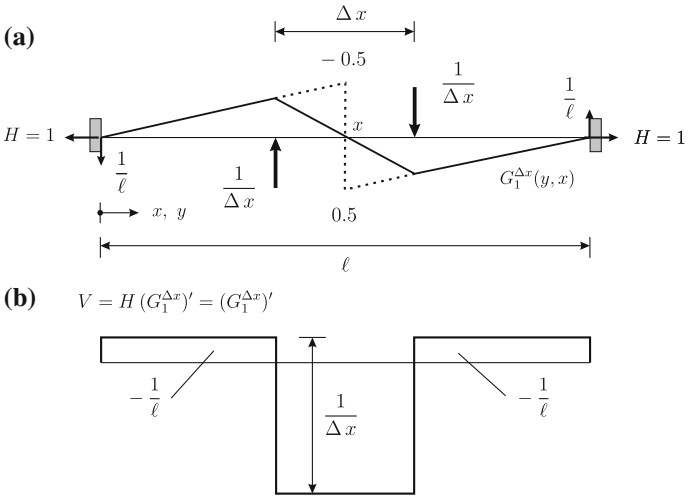

Fig. 2.6 Influence function G_1 (*dashed line*) for the first derivative $V = H\,u' = u'$ **a** FE-approximation $G_1^{\Delta x}$ and **b** its derivative

$$\lim_{\Delta x \to 0} a(G_1^{\Delta x}, u) = u'(x)\,. \tag{2.185}$$

This is a remarkable result because we know that

$$a(G_1, u) := \lim_{\varepsilon \to 0} a(G_1, u)_{\Omega_\varepsilon} = 0\,. \tag{2.186}$$

So the sequence, $\Delta x \to 0$, of the integrals $a(G_1^{\Delta x}, u)$ converges to $u'(x)$ but when we substitute for $G_1^{\Delta x}$ the limit function G_1 then the integral is zero.

Numerically this is easily verified: the slope of G_1 is $1/\ell$ and so

$$a(G_1, u) = \int_0^\ell \frac{1}{\ell}\, u'(x)\, dx = u(\ell) - u(0) = 0 - 0 = 0\,. \tag{2.187}$$

The Green's function G_1 of the rope seemingly has a bounded energy, but all the (infinite) energy is concentrated in one point, the point where the deflection jumps, $[[G]] = 1$. But because the energy integral $a(G_1, u)$, when we follow the definition (2.186), is insensitive to issues at points (1-D) the infinity is non-existing in the energy metric.

The same happens in 2-D and 3-D problems: a stair welded together from flat steel panels has zero energy but when a flat sheet of metal is folded into the same shape then the energy of the stair is infinite though the energy hiding in the folds (= lines) of the plate is "lost" because the strain energy is an 2-D integral which

is insensitive to sudden jumps of the integrand along curves (= the folds) because curves have measure zero.

Remark 2.4 The situation in Fig. 2.6 is the following: the two forces $\pm 1/\Delta x$ are statically equivalent to a unit couple $1/\Delta x \cdot \Delta x = 1$ and so the support reactions, $\pm 1/\ell$, are independent of Δx. The transverse force V is equal to the support reaction, $V = -1/\ell$, and on passing the foot of the force $-1/\Delta x$ the transverse force V jumps by $1/\Delta x$.

Basically what happens when $\Delta x \to 0$ is that the two infinite forces tear the rope apart but surprisingly the discontinuity, the dislocation, is finite; the gap has unit length. If G_1 were a finite element shape function we would say that it is a non-conforming shape function. Such functions have an infinite strain energy.

To see this let us assume, for simplicity, that the length ℓ of the rope is very large then $1/\ell \simeq 0$ and so the strain energy in the rope is approximately

$$a(G_1^{\Delta x}, G_1^{\Delta x}) = \int_0^l H\,[(G_1^{\Delta x})']^2\,dy \simeq \int_{x-0.5\,\Delta x}^{x+0.5\,\Delta x} H\left(\frac{1}{\Delta x}\right)^2 dy$$
$$= H\left(\frac{1}{\Delta x}\right)^2 \Delta x = \frac{H}{\Delta x} \tag{2.188}$$

and this term becomes infinite when $\Delta x \to 0$. If $1/\ell$ is not negligible then the transverse force in the middle is $1/\Delta x - 1/\ell$ but the leading term in the square is the same as before

$$\int_{x-0.5\,\Delta x}^{x+0.5\,\Delta x} H\left(\frac{1}{\Delta x} - \frac{1}{\ell}\right)^2 dy = H\left(\frac{1}{\Delta x} - \frac{1}{\ell}\right)^2 \Delta x = \frac{H}{\Delta x} + \ldots \tag{2.189}$$

and so the energy becomes infinite as well.

2.4.5 Elevators and Escalators

Weak influence functions are like escalators while strong influence functions resemble elevators. Depict the Green's function as a mountain range where the height above sea level indicates the influence a point load $P = 1$ on top of a mountain will have on $u(x)$. The lift $G(y, x)$ takes us directly to the mountain top while the escalator $a(G, u)$ does one step dx at a time, calculates the increment $du = u'\,G'\,dx$—the derivative G' plays the role of a weight—and accumulates all the single increments du to reach the value $u(x)$.

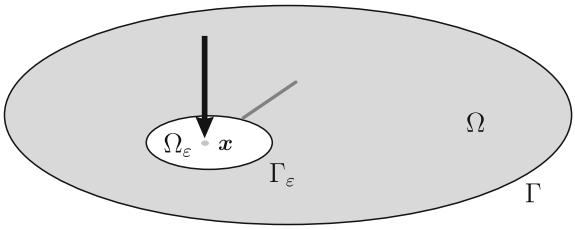

Fig. 2.7 Punctured domain with source point

The classical example of an equation of this incremental type is the formula

$$\int_{\Omega} \operatorname{div} \boldsymbol{u}\, d\Omega = \int_{\Gamma} \boldsymbol{u} \cdot \boldsymbol{n}\, ds = J(\boldsymbol{u}) \tag{2.190}$$

which is the influence function for the net flow, $J(\boldsymbol{u})$, in and out of a planar region Ω bounded by a curve Γ. This influence function counts all the positive and negative increments of the velocity field $\boldsymbol{u}$ inside Ω and if the average value of the divergence of the field $\boldsymbol{u}$

$$\operatorname{div} \boldsymbol{u} = u_{x,x} + u_{y,y} \tag{2.191}$$

is zero then the net flow $J(\boldsymbol{u})$ is also zero (Fig. 2.7).

Remark 2.5 Later in Chap. 3 it will be seen that the FE-method does not distinguish between weak and strong influence functions. In FE-analysis the approximate Green's function is the solution of the variational problem

$$G_h \in \mathcal{V}_h : \quad a(G_h, \varphi_i) = J(\varphi_i) \quad \varphi_i \in \mathcal{V}_h \tag{2.192}$$

and G_h gives the same result with both formulas

$$J(u_h) = \underbrace{\int_0^l G_h(y,x)\, p(y)\, dy}_{strong} = \underbrace{a(G_h, u_h)}_{weak}, \tag{2.193}$$

because on $\mathcal{V}_h$ the two formulas—strong and weak—coincide

$$J(u_h) = \int_0^l G_h(y,x)\, p(y)\, dy = \boldsymbol{g}^T\, \boldsymbol{f} = \boldsymbol{g}^T\, \boldsymbol{K}\, \boldsymbol{u} = a(G_h, u_h)\,. \tag{2.194}$$

In a matrix formulation the difference between strong and weak influence functions only depends on how one reads the equations

$$J(u_h) = \underbrace{\boldsymbol{g}^T \boldsymbol{f}}_{strong} = \underbrace{\boldsymbol{g}^T \boldsymbol{K}\boldsymbol{u}}_{weak} . \tag{2.195}$$

In the weak formulation $\boldsymbol{g}^T \boldsymbol{K}\boldsymbol{u}$ we sum over all entries k_{ij} which procedure is like the domain integral of the divergence. While the strong formulation $\boldsymbol{g}^T \boldsymbol{f}$ in contrast weights the nodal displacements with the vector $\boldsymbol{f}$. If $\boldsymbol{f}$ has only one non-zero component f_i then this formulation can be very economically.

Note that in (2.190) the divergence is weighted with $g = 1$ while here the gradient ∇u_h is weighted with the gradient ∇G_h.

Remark 2.6 In beam analysis engineers use the zero result of the weak influence function for the force term $M(x) = -EI\, u''(x)$

$$a(G_2, u) = \int_0^l \frac{M_2\, M}{EI}\, dy = 0 \tag{2.196}$$

as a check that the derivative u' of the beam deflection u is continuous at a point x (orthogonality of M_i belonging to the redundant X_i, and M in the force method). If the result is not zero then the deflection curve has a kink at x and so M cannot be the correct bending moment distribution in the beam.

2.4.6 Influence Functions in Higher Dimensions

In higher dimensions the technique to formulate influence functions is basically the same. To exemplify this we consider the boundary value problem

$$-\Delta u = p \quad \text{on } \Omega \quad u = 0 \quad \text{on } \Gamma . \tag{2.197}$$

The Green's function for $u(\boldsymbol{x})$ is the solution of the boundary value problem

$$-\Delta G_0 = \Delta_0(\boldsymbol{y} - \boldsymbol{x}) \quad G_0(\boldsymbol{y}, \boldsymbol{x}) = 0 \quad \boldsymbol{y} \in \Gamma \tag{2.198}$$

and Green's second identity for the Laplace operator reads

$$\mathscr{B}(u, v) = \int_\Omega -\Delta u\, v\, d\Omega + \int_\Gamma \frac{\partial u}{\partial n}\, v\, ds - \int_\Gamma \frac{\partial v}{\partial n}\, u\, ds - \int_\Omega u\, (-\Delta v)\, d\Omega = 0 . \tag{2.199}$$

It is well known that the Green's function has a logarithmic singularity at the source point $\boldsymbol{x}$

$$G(\boldsymbol{y}, \boldsymbol{x}) = -\frac{1}{2\,\pi} \ln r + \text{regular terms} \tag{2.200}$$

and so the integral identity must be formulated on the punctured domain, see Fig. 2.7,

$$\Omega_\varepsilon = \Omega - N_\varepsilon(\boldsymbol{x}) \tag{2.201}$$

that is a small circular neighborhood N_ε of the source point $\boldsymbol{x}$ is spared from the domain and then the argument is the following: because Green's second identity is zero for all $\varepsilon > 0$

$$\mathscr{B}(G_0, u)_{\Omega_\varepsilon} = \underbrace{\int\limits_{\Gamma_{N_\varepsilon}} \frac{\partial G_0}{\partial n}\, u\, ds_{\boldsymbol{y}}}_{\lim_{\varepsilon\to 0}=u(\boldsymbol{x})} - \underbrace{\int\limits_{\Gamma_{N_\varepsilon}} \frac{\partial u}{\partial n}\, G_0\, ds_{\boldsymbol{y}}}_{\lim_{\varepsilon\to 0}=0} - \int\limits_{\Omega_\varepsilon} G_0\, p\, d\Omega_{\boldsymbol{y}} = 0 \tag{2.202}$$

it must be also zero in the limit, $\varepsilon \to 0$,

$$\lim_{\varepsilon\to 0} \mathscr{B}(u, G_0[\boldsymbol{x}])_{\Omega_\varepsilon} = u(\boldsymbol{x}) - \int\limits_{\Omega} G_0\, p\, d\Omega_{\boldsymbol{y}} = 0 \tag{2.203}$$

or

$$u(\boldsymbol{x}) = \int\limits_{\Omega} G_0\, p\, d\Omega_{\boldsymbol{y}} \tag{2.204}$$

which is the influence function for the solution $u(\boldsymbol{x})$.

The derivation of the influence function for the slope in the direction of an arbitrary unit vector $\boldsymbol{m}$

$$\frac{\partial u}{\partial m}(\boldsymbol{x}) = \nabla u(\boldsymbol{x}) \cdot \boldsymbol{m} \tag{2.205}$$

follows the same lines. In this case the Green's function G_1 is the solution of the boundary value problem

$$-\Delta G_1 = \delta_1(\boldsymbol{y} - \boldsymbol{x}) \quad G_1(\boldsymbol{y}, \boldsymbol{x}) = 0 \quad \boldsymbol{y} \in \Gamma \tag{2.206}$$

where the Dirac function has the properties

$$\delta_1(\boldsymbol{y} - \boldsymbol{x}) = 0 \qquad \boldsymbol{y} \neq \boldsymbol{x} \tag{2.207}$$

$$\int\limits_{\Omega} \delta_1(\boldsymbol{y} - \boldsymbol{x})\, u(\boldsymbol{y})\, d\Omega_{\boldsymbol{y}} = \frac{\partial u}{\partial m}(\boldsymbol{x}) \qquad \boldsymbol{x} \in \Omega\,. \tag{2.208}$$

Evidently the Green's function is

$$G_1(\boldsymbol{y}, \boldsymbol{x}) = \nabla_{\boldsymbol{x}}\, G_0(\boldsymbol{y}, \boldsymbol{x}) \cdot \boldsymbol{m} = G_{0,x_1}\, m_1 + G_{0,x_2}\, m_2\,. \tag{2.209}$$

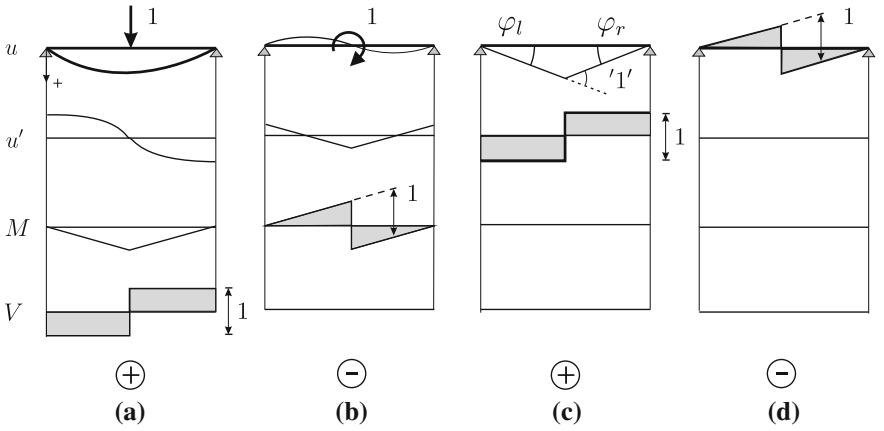

Fig. 2.8 The four singularities of a beam, '1' $\equiv \tan\varphi_l + \tan\varphi_r = 1$, generate the four influence functions (uppermost row) for **a** u, **b** u', **c** $M = -EI\,u''$ and **d** $V = -EIu'''$ at the center of the beam; $+$ = inf. func integrates, $-$ = inf. func. differentiates

As before the starting point is

$$\mathscr{B}(G_1, u)_{\Omega_\varepsilon} = \underbrace{\int\limits_{\Gamma_{N_\varepsilon}} \frac{\partial G_1}{\partial n}\, u\, ds_{\boldsymbol{y}}}_{\lim_{\varepsilon\to 0}=\frac{1}{2}\frac{\partial u}{\partial m}(\boldsymbol{x})} - \underbrace{\int\limits_{\Gamma_{N_\varepsilon}} \frac{\partial u}{\partial n}\, G_1\, ds_{\boldsymbol{y}}}_{\lim_{\varepsilon\to 0}=-\frac{1}{2}\frac{\partial u}{\partial m}(\boldsymbol{x})} - \int\limits_{\Omega_\varepsilon} G_1\, p\, d\Omega_{\boldsymbol{y}} = 0 \qquad (2.210)$$

and after taking the limit

$$\lim_{\varepsilon\to 0} \mathscr{B}(u, G_1[\boldsymbol{x}])_{\Omega_\varepsilon} = \frac{\partial u}{\partial m}(\boldsymbol{x}) - \int\limits_{\Omega} G_1\, p\, d\Omega_{\boldsymbol{y}} = 0\,, \qquad (2.211)$$

we recover

$$\frac{\partial u}{\partial m}(\boldsymbol{x}) = \int\limits_{\Omega} G_1(\boldsymbol{y}, \boldsymbol{x})\, p(\boldsymbol{y})\, d\Omega_{\boldsymbol{y}} \qquad (2.212)$$

which is the influence function for the slope. Basically it is the same logic as in the 1-D case, see Fig. 2.8.

2.4.7 Weak Influence Functions

As in the 1-D case we can derive a weak influence function for $u(\boldsymbol{x})$

$$\lim_{\varepsilon\to 0} \mathscr{G}(G_0, u)_{\Omega_\varepsilon} = u(\boldsymbol{x}) - a(G_0, u) = 0 \qquad (2.213)$$

but not for the slope (Fig. 2.8).

2.4.8 Non-Zero Boundary Values

To be complete a word must be said about influence functions for problems where the displacements on the boundary are not zero as in

$$-\Delta u = 0 \quad u = \bar{u} \quad \text{on } \Gamma\,. \tag{2.214}$$

As before Green's second identity is formulated with the solution u and a test function $v \in \mathcal{V}$ ($v = 0$ on Γ)

$$\mathscr{B}(u, v) = \int_{\Omega} 0 \cdot v \, d\Omega + \int_{\Gamma} \frac{\partial u}{\partial n} \cdot 0 \, ds - \int_{\Gamma} \bar{u} \frac{\partial v}{\partial n} \, ds - \int_{\Omega} u(-\Delta v) \, d\Omega = 0 \tag{2.215}$$

and so it follows that if the Green's function solves the boundary value problem

$$-\Delta G(\boldsymbol{y}, \boldsymbol{x}) = \delta(\boldsymbol{y} - \boldsymbol{x}) \quad \text{on } \Omega \quad G(\boldsymbol{y}, \boldsymbol{x}) = 0 \quad \boldsymbol{y} \in \Gamma \tag{2.216}$$

then $\mathscr{B}(u, G) = 0$ is equivalent to

$$u(\boldsymbol{x}) = \int_{\Gamma} \frac{\partial G(\boldsymbol{y}, \boldsymbol{x})}{\partial n} \bar{u}(\boldsymbol{y}) \, ds_{\boldsymbol{y}}\,. \tag{2.217}$$

By weighting the boundary displacements $\bar{u}$ with the slope of the Green's function—which is the term conjugated to the displacement $u = \bar{u}$—the value of $u(\boldsymbol{x})$ can be calculated.

In a boundary value problem such as

$$-\Delta u = p \quad u = \bar{u} \quad \text{on } \Gamma_D \quad \frac{\partial u}{\partial n} = \bar{t} \quad \text{on } \Gamma_N \tag{2.218}$$

the test functions $v \in \mathcal{V}$ must vanish on the part Γ_D (D as in Dirichlet) of $\Gamma = \Gamma_D \cup \Gamma_N$ (N as in Neumann) and so if the Green's function solves the boundary value problem

$$-\Delta G(\boldsymbol{y}, \boldsymbol{x}) = \delta(\boldsymbol{y} - \boldsymbol{x}) \quad G(\boldsymbol{y}, \boldsymbol{x}) = 0 \quad \boldsymbol{y} \in \Gamma_D \tag{2.219}$$

then

$$\begin{aligned} u(\boldsymbol{x}) = &\int_{\Omega} G(\boldsymbol{y}, \boldsymbol{x}) \, p(\boldsymbol{y}) \, d\Omega_{\boldsymbol{y}} + \int_{\Gamma_D} \frac{\partial G(\boldsymbol{y}, \boldsymbol{x})}{\partial n} \bar{u}(\boldsymbol{y}) \, ds_{\boldsymbol{y}} \\ &+ \int_{\Gamma_N} G(\boldsymbol{y}, \boldsymbol{x}) \, \bar{t}(\boldsymbol{y}) \, ds_{\boldsymbol{y}} \end{aligned} \tag{2.220}$$

is the influence function. All the input $\{p, \bar{t}, \bar{u}\}$ contributes to $u(\boldsymbol{x})$.

Example 2.5 If a rigid punch indents the surface of the half-space then this is such a boundary value problem with a prescribed inhomogeneous displacement $\bar{u}$ along the Dirichlet part Γ_D of the soil surface, see Fig. 2.9, and so all influence functions must have zero displacements on Γ_D.

By analogy with (2.220) it follows that the influence function for the stress σ_{yy} at a point $\boldsymbol{x}$ inside the half-space must be

$$\sigma_{yy}(\boldsymbol{x}) = \bar{u} \int_{\Gamma_D} t(\boldsymbol{y}, \boldsymbol{x})\, ds_{\boldsymbol{y}}\,. \tag{2.221}$$

What in the Laplace equation is the slope $\partial G/\partial n$ are here the tractions $t(\boldsymbol{y}, \boldsymbol{x})$ (vertical stress) directly under the rigid punch due to a unit dislocation in vertical direction at the source point $\boldsymbol{x}$ (= influence function for σ_{yy}). We know from experience that the stress σ_{yy} at the ends of the punch is singular and so we conclude that the integral of the tractions $t(\boldsymbol{y}, \boldsymbol{x})$ must be unbounded and not measurable when the dislocation is applied directly under the edge of the punch.

Example 2.6 In 1-D problems the boundary consists of points. Supports are point supports. If a support settles by $\bar{u}$ units the influence $J(u)$ on any observable quantity is

$$J(u)(x) = R(x)\,\bar{u} \tag{2.222}$$

where $R(x)$ is the support reaction in the direction of the displacement $\bar{u}$ due to the action of the Dirac delta associated with the functional $J(u)(x)$.

2.4.9 Average Values of Stresses

The integration by parts formula

$$\int_0^l u'\, v\, dx = [u\, v]_0^l - \int_0^l u\, v'\, dx \tag{2.223}$$

is also a statement about the average value of a derivative u'

$$\frac{1}{l}\int_0^l u'\, dx = \frac{1}{l}[u \cdot 1]_0^l = \frac{1}{l}\,(u(l) - u(0)) \tag{2.224}$$

which is just the slope between the end points.

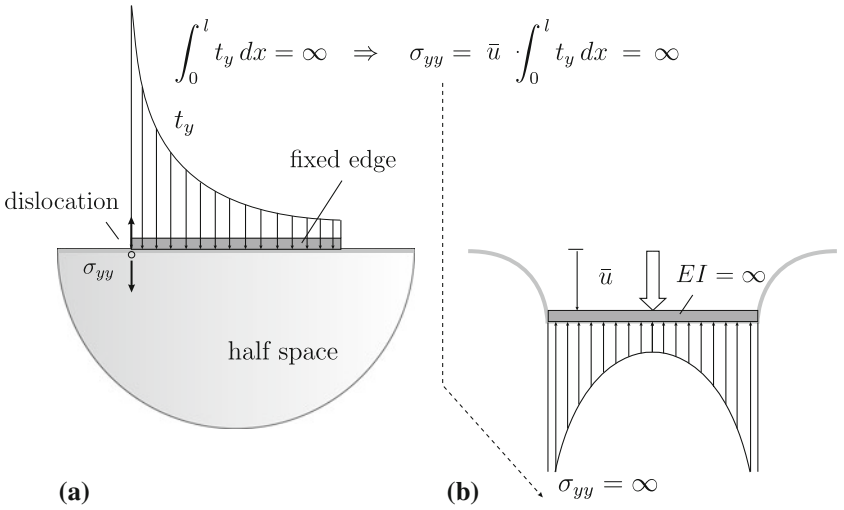

Fig. 2.9 Why the stress under a corner point of a rigid punch becomes infinite, **a** the influence function for σ_{yy} is generated by a unit dislocation while the punch is kept fixed and **b** the sum of the forces needed to hold the punch fixed times the indentation $\bar{u}$ of the punch is the stress σ_{yy} at the edge of the punch

In the same sense the average value of the stress

$$\sigma_{xx} = E\,(\varepsilon_{xx} + \nu\,\varepsilon_{yy}) = E\,(u_{x,x} + \nu\,u_{y,y}) \tag{2.225}$$

in a plate element Ω_e of size $|\Omega_e|$ can be expressed by a boundary integral

$$\frac{1}{|\Omega_e|}\int\limits_{\Omega_e} E\,(u_{x,x} + \nu\,u_{y,y})\,d\Omega = \frac{E}{|\Omega_e|}\int\limits_{\Gamma_e} (u_x\,n_x + \nu\,u_y\,n_y)\,ds\,. \tag{2.226}$$

The influence function for the boundary integral is the displacement field when line forces $E/|\Omega_e|\,n_x$ and $E\,\nu/|\Omega_e|\,n_y$ respectively pull on the edge Γ_e of the element, see Fig. 2.10 [1].

If the element Ω_e is embedded into a rather soft matrix, $E_M \ll 1$, then the forces can stretch the element unhindered, the influence function will extend quiet far, and so much of the load will flow through the element. But when the matrix is very stiff, $E_M \gg 1$, and the element in contrast is very soft, $E \ll 1$, then the influence function will almost be zero and so hardly any stresses will flow through Ω_e.

There exists a close connection between these influence functions and the hydrostatic pressure distribution in a plate. The edge forces which generate the influence function for the average value of $\sigma_{xx} + \sigma_{yy}$

$$t_x = \frac{E}{|\Omega_e|\,(1+\nu)}\,n_x \quad t_y = \frac{E}{|\Omega_e|\,(1+\nu)}\,n_y \tag{2.227}$$

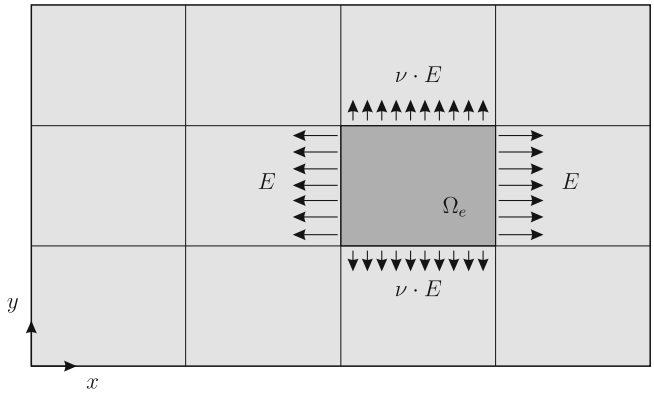

Fig. 2.10 These forces generate the influence function for the average value of σ_{xx} in the element Ω_e

are essentially the same forces a plate which is submerged in a fluid experiences along its edge

$$t_x = p\, n_x \quad t_y = p\, n_y \quad p = \text{water pressure}\,. \tag{2.228}$$

The stress tensor in the plate, $\boldsymbol{S} = p\, \boldsymbol{I}$, is a diagonal matrix and so the core displacement field is simply $u_x = \alpha\, x, u_y = \alpha\, y, \alpha = p/(E + \nu)$, see Fig. 2.11. This is also the influence function for the average value of $\sigma_{xx} + \sigma_{yy}$ when the factor $E/|\Omega_e|(1 + \nu)$ replaces p.

Core displacement field means that an arbitrary vector $\boldsymbol{u}_0$ can be added to the solution. This has no effect on the average value of the stresses because if the plate is freely floating, if it is untethered, then the applied load must satisfy the equilibrium condition

$$\int_{\Omega} p \cdot \boldsymbol{r}\, d\Omega + \int_{\Gamma} \boldsymbol{t} \cdot \boldsymbol{r}\, ds = 0 \tag{2.229}$$

where $\boldsymbol{r} = \boldsymbol{a} + \boldsymbol{x} \times \boldsymbol{\omega}$ is a rigid body motion of the plate and this property guarantees that the effect of $\boldsymbol{u}_0$ is nil when the influence function is evaluated.

Remark 2.7 The fluid pressure is a pure Neumann problem for the submerged plate but because the integral of the normal vector is zero along any closed contour

$$\int_{\Gamma} n_x\, ds = 0 \quad \int_{\Gamma} n_y\, ds = 0 \tag{2.230}$$

the problem is well posed, are the equilibrium conditions satisfied.

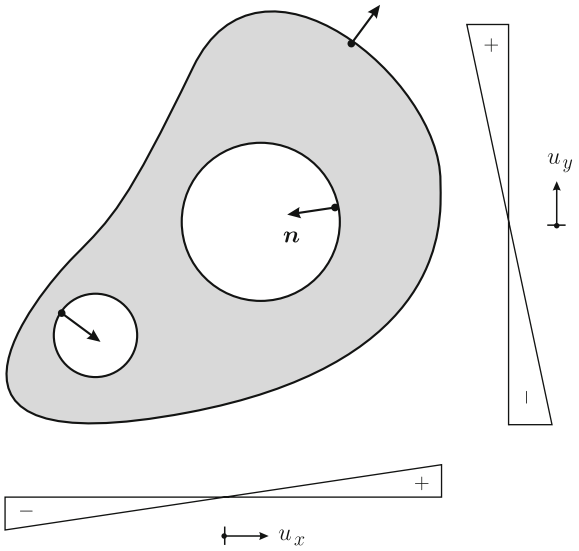

Fig. 2.11 Edge forces in the direction of the normal vector, as water pressure, generate a very simple displacement field in any plate; the field is determined up to arbitrary constants

Remark 2.8 The average values of the stresses (derivatives) must be zero if the edge of a plate is fixed all around because the forces $E/|\Omega_e|\, n_x$ and $E/|\Omega_e|\, n_y$ respectively applied to a fixed edge will effect nothing.

Remark 2.9 In beams the bending stresses σ_x typically exhibit an antisymmetric linear distribution in vertical direction (y) so that the average value of these stresses over the length of the beam (= plate)

$$\int_0^l \int_0^h \sigma_x(y,x)\, dy\, dx = 0 \tag{2.231}$$

is zero. This is easily understood by looking at the forces which generate the influence function for this integral, see Fig. 2.12. If $\nu = 0$ then only lateral loads E will act on the beam but because of $\nu = 0$ they will have no effects on the horizontal edges. If $\nu \neq 0$ then the stretching of the beam by the lateral forces $\pm E$ translates into a constriction of the beam but this is compensated by an opposite displacement produced by the vertical forces $\pm\nu\, E$ so that again the displacement of the horizontal edges is zero.

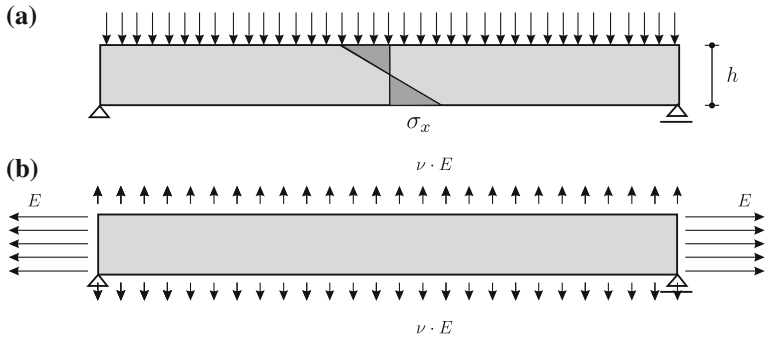

Fig. 2.12 Beam **a** bending stresses σ_x, **b** these forces generate the influence function for the average value of σ_x in the beam

2.5 Properties of Green's Functions

What is a point load mathematically, how do we define a Dirac delta? The engineer may help who defines point loads via the equilibrium conditions.

A point load $P = 1$ placed on a rope at a point x makes that the shear force V jumps by one unit

$$\lim_{\varepsilon \to 0} [V(x + 0.5\,\varepsilon) - V(x - 0.5\,\varepsilon)] = 1\,. \tag{2.232}$$

When the same force is placed on a membrane, see Fig. 2.13, then the *integral* of the slope $\partial u/\partial n$ over concentric circles Γ_{N_ε} must be 1 in the limit

$$\lim_{\varepsilon \to 0} \int\limits_{\Gamma_{N_\varepsilon}} \frac{\partial u}{\partial n}(\boldsymbol{y})\, ds_{\boldsymbol{y}} = \lim_{\varepsilon \to 0} \int\limits_{\Gamma_{N_\varepsilon}} \frac{\partial u}{\partial n}(\boldsymbol{x} + \varepsilon\, \nabla r)\, ds_{\boldsymbol{y}} = 1 \tag{2.233}$$

where

$$\nabla r = \nabla(\boldsymbol{y} - \boldsymbol{x}) = \begin{bmatrix} \cos\,\varphi \\ \sin\,\varphi \end{bmatrix} \tag{2.234}$$

is a "pointer" of length one which—like a compass needle—shows the direction to the points $\boldsymbol{y} = (\varepsilon, \varphi)$ (polar coordinates) on the edge Γ_ε.

Because the size of Γ_ε shrinks like $2\,\pi\,\varepsilon$ the slope of u must counterbalance this tendency by increasing as ε^{-1}. This is indeed the case: the displacement of a membrane near a unit point load has the form

$$u = -\frac{1}{2\,\pi}\, \ln\, r + \text{regular terms} \tag{2.235}$$

and the slope of the singular term on a concentric circle Γ_r with radius r is

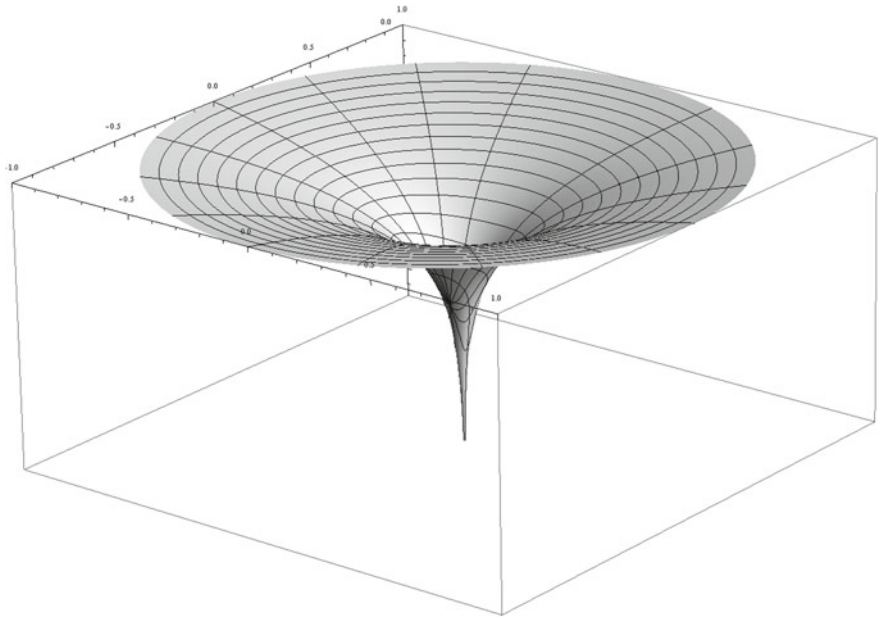

Fig. 2.13 A point load acting at the center of a circular membrane, Poisson equation

$$-\frac{1}{2\,\pi}\,\frac{\partial \ln r}{\partial n} = -\frac{1}{r}\nabla r \cdot \boldsymbol{n} = \frac{1}{r}\nabla r \cdot \nabla r = \frac{1}{2\,\pi\,r} \cdot 1 \tag{2.236}$$

Note that the outward normal on the edge Γ_ε points towards the center $\boldsymbol{x}$, that is it has the opposite direction of ∇r, and this makes that $\nabla r \cdot \boldsymbol{n} = -\nabla r \cdot \nabla r$.

So by taking the limit of the singular term the point load at the center of the shrinking circle is recovered, see Fig. 2.14a,

$$\lim_{\varepsilon \to 0} \int\limits_{\Gamma_{N_\varepsilon}} \frac{\partial u}{\partial n}(\boldsymbol{y})\, ds_{\boldsymbol{y}} = \lim_{\varepsilon \to 0} \int\limits_{0}^{2\,\pi} \frac{1}{2\,\pi\,\varepsilon}\,\varepsilon\, d\varphi = 1\,. \tag{2.237}$$

The contribution of the regular terms to this integral is zero because a bounded function integrated over a shrinking circle gives zero in the limit.

This is the engineering approach to Dirac deltas. If there is a "black hole" then the neighborhood of the black hole must show traces of its presence. When you circle the singularity once then you get a lift of one unit in the relevant quantity, the integral of the vertical forces or the integral of the deflection.

Note that the dimension n of the space, 2-D or 3-D, determines the strength $O(r^{-n})$ of the singularity. In 2-D the circle that surrounds the point load is of size $2\,\pi\,r$ and in 3-D it is a sphere of size $4\,\pi\,r^2$. This dimension argument is the reason

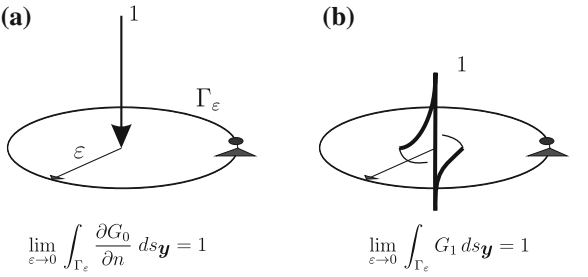

Fig. 2.14 A point load, solution G_0, and a dislocation, solution G_1

why the gravitational forces or electrostatic forces near a point mass or a point charge have the same order r^{-2}.

To summarize the main properties of Green's function let $\partial_i u,\ i = 0, 1, 2, 3$ be the boundary values of an abstract fourth-order equation $L\,u = p$ as for example of a Kirchhoff plate

$$\partial_0 u = u \quad \partial_1 u = \nabla u \cdot \boldsymbol{n} \quad \partial_2 u = m_n(u) \quad \partial_3 u = v_n(u)\,. \tag{2.238}$$

What is needed for an influence function of $\partial_1 u(\boldsymbol{x})$ (e.g.) is

- an integral identity such as $\mathscr{B}(u, v)$ which contains a boundary integral

$$\int_\Gamma \partial_1 u\,\partial_2 v\,ds \quad 1 + 2 = 2\,m - 1 = 3 \tag{2.239}$$

with $\partial_1 u$ and the term conjugated to it, $\partial_2 v$,

- and a homogeneous solution $G(\boldsymbol{y}, \boldsymbol{x})$ of the governing equation with the property

$$\lim_{\varepsilon\to 0}\int_{\Gamma_{N_\varepsilon}} \partial_2 G(\boldsymbol{y}, \boldsymbol{x})\,ds_{\boldsymbol{y}} = 1 \tag{2.240}$$

because this provides in the limit the Dirac delta effect

$$\partial_1 u(\boldsymbol{x}) = \lim_{\varepsilon\to 0}\int_{\Gamma_{N_\varepsilon}} \partial_2 G(\boldsymbol{y}, \boldsymbol{x})\,\partial_1 u(\boldsymbol{y})\,ds_{\boldsymbol{y}}\,. \tag{2.241}$$

Second order differential equations possess two Green's functions

$$G_0(\boldsymbol{y}, \boldsymbol{x}) = \text{single force} \quad G_1(\boldsymbol{y}, \boldsymbol{x}) = \text{dislocation} \tag{2.242}$$

and fourth order equations four such functions,

$$G_0(\boldsymbol{y},\boldsymbol{x}) = \text{single force} \qquad G_1(\boldsymbol{y},\boldsymbol{x}) = \text{moment} \tag{2.243}$$

$$G_2(\boldsymbol{y},\boldsymbol{x}) = \text{twist} \qquad G_3(\boldsymbol{y},\boldsymbol{x}) = \text{dislocation}\,. \tag{2.244}$$

2.5.1 Modern Approach

What we presented here was the classical theory. In FE-analysis things are much simpler, we must not be concerned with limits over shrinking circles and alike, we have more freedom and more room to invent a nearly infinite variety of Green's functions. It is only required that the functional $J(u)$ is linear. The nodal values g_i of the Green's function

$$G_h(\boldsymbol{y},\boldsymbol{x}) = \sum_i g_i(\boldsymbol{x})\,\varphi_i(\boldsymbol{y}) \tag{2.245}$$

are the solution of the system $\boldsymbol{K}\,\boldsymbol{g} = \boldsymbol{j}$ where $j_i = J(\varphi_i)$ and once we have solved this system we can map p onto the value $J(u_h)$ of the FE-solution with ease

$$J(u_h) = \int_{\Omega} G_h(\boldsymbol{y},\boldsymbol{x})\,p(\boldsymbol{y})\,d\Omega_{\boldsymbol{y}} = \boldsymbol{g}^T\,\boldsymbol{f}\,. \tag{2.246}$$

2.5.2 Maxwell

If a unit point load applied at one point $\boldsymbol{x}_1$ of an elastic structure results in a given deflection at another point $\boldsymbol{x}_2$, then the same load applied at $\boldsymbol{x}_2$ will result in the same deflection at $\boldsymbol{x}_1$.

This is Maxwell's theorem and the underlying principle[4]

$$\int_0^l G_0(y,x_2)\,\delta_0(y-x_1)\,dy = \int_0^l G_0(y,x_1)\,\delta_0(y-x_2)\,dy \tag{2.247}$$

can be extended to any pair $\{i, j\}$ of Green's functions $G_i(y,x_1)$ and $G_j(y,x_2)$

$$\int_0^l G_i(y,x_2)\,\delta_j(y-x_1)\,dy = \int_0^l G_j(y,x_1)\,\delta_i(y-x_2)\,dy\,. \tag{2.248}$$

[4] The indices on the Green's functions and the Dirac deltas are to distinguish the different functions, see Sect. 2.4.1.

Say $G_1(y, x_2)$ is the influence function for the shear force $V = H\,u'(x)$ in a rope at the point x_2

$$-H\frac{d^2}{dy^2}G_1(y, x_2) = \delta_1(y - x_2) \quad G_1(0, x_2) = G_1(l, x_2) = 0 \tag{2.249}$$

and $G_0(y, x_1)$ is the influence function for the deflection $u(x)$ of the rope at another point x_1

$$-H\frac{d^2}{dy^2}G_0(y, x_1) = \delta_0(y - x_1) \quad G_0(0, x_1) = G_0(l, x_1) = 0 \tag{2.250}$$

then

$$\int_0^l G_1(y, x_2)\,\delta_0(y - x_1)\,dy = \int_0^l G_0(y, x_1)\,\delta_1(y - x_2)\,dy \tag{2.251}$$

or

$$G_1(x_1, x_2) = V(G_0(x_2, x_1)) \tag{2.252}$$

that is the magnitude of G_1 at x_1 is the same as the magnitude of the shear force belonging to G_0 at the point x_2.

Formally this allows to derive any influence function $G_i(y, \xi)$ from $G_0(y, x)$ by the maneuver

$$\int_0^l G_0(y, x)\,\delta_i(y - \xi)\,dy = \int_0^l G_i(y, \xi)\,\delta_0(y - x)\,dy = G_i(x, \xi)\,. \tag{2.253}$$

But this is a rather cumbersome approach. To plot the influence function for the shear force $G_1(y, x)$ in a rope you would subdivide the rope into ten equally spaced points x_i and place first a unit force $P = 1$ at the point x_1, calculate the shear force V at x, then place the force at the point x_2, again calculate the shear force at x, etc. A linear plot of all these values would be a—for engineering purposes often good enough—first approximation of the influence function.

2.5.3 Modes of Decay

Generating an influence function can be compared to throwing a stone into a water basin and watching the waves slowly ebb away. If the basin is an intricate system of ponds and channels then the energy transport will be influenced by the width of the channels and the curvature of the channels.

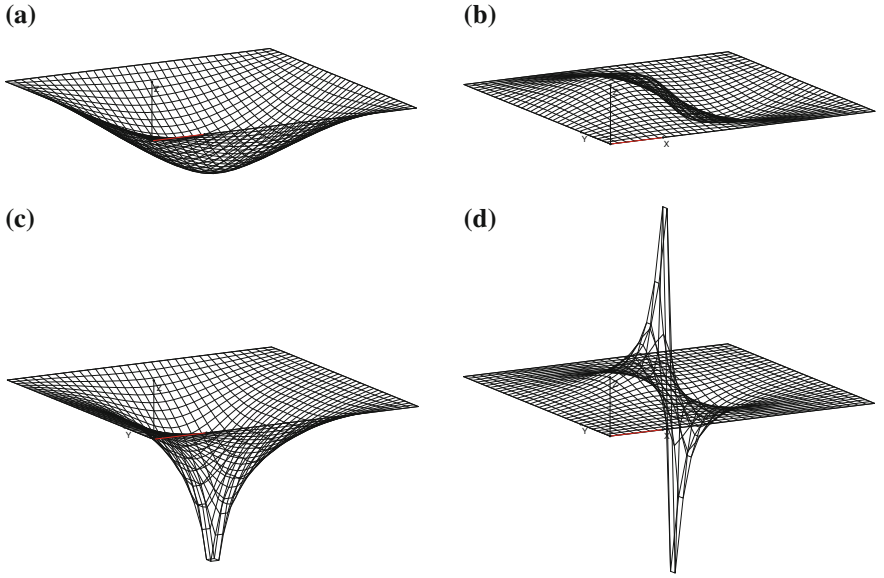

Fig. 2.15 Influence functions on a slab ($K\,\Delta\Delta w = p$), **a** deflection, **b** rotation, **c** moment m_{xx}, **d** shear force q_x

But the mode of decay, s. Fig. 2.15, primarily depends on the character of the influence function, and this in turn depends on the function value (here of a beam)

$$u(x) \quad u'(x) \quad u''(x) \quad u'''(x) \tag{2.254}$$

which the influence function extracts from the right-hand side, the fourth order derivative, $EIu^{IV} = p$,

$$\frac{d^i u(x)}{dx^i} = \int_0^l G_i(y,x)\, p(y)\, dy \quad i = 0, 1, 2, 3\,. \tag{2.255}$$

That is an influence function can be considered an *integral operator*. The more a kernel $G_i(y,x)$ integrates the more "volume" it has, the more it extends in all directions and the slower the rate of its decay, s. Fig. 2.15.

The kernel $G_0(y,x)$ maps p onto $u(x)$, it integrates four times! The opposite is the Dirac delta δ_0 which is a so-called *reproducing kernel*. It neither integrates nor differentiates; it has zero spread. It measures at one point only and the output is equal to the input

$$\int_0^l \delta_0(y,x)\, p(y)\, dy = p(x)\,. \tag{2.256}$$

In the list of the Green's functions of the beam, see (2.255), the Dirac delta would be the fifth kernel, $G_4 \equiv \delta_0$ which maps p onto itself—no gain.

2.5.4 Dipoles and Monopoles

Consider two opposite electric point charges a distance Δx apart and with a strength $\pm 1/\Delta x$ which is inversely proportional to their distance Δx. If the distance between the two charges shrinks to zero, $\Delta x \to 0$, they become a *dipole*.

In solid mechanics point charges are point loads $\pm 1/\Delta x$ and while an electric field needs no carrier, is "invisible" (to our limited senses), a dipole in a solid makes its presence felt by a gap, a unit dislocation.[5] Such a dislocation generates the influence function for a stress in a solid or—in the one-dimensional case—the normal force N in a bar.

Of a somewhat different type is the influence function for a displacement because it is generated by a point load $P = 1$, a *monopole*. Influence functions which are generated by monopoles sum, they add, while influence functions generated by dipoles measure differences, they are sensitive to imbalances.[6] Each of the influence functions in Fig. 2.15 belongs to either one of these two types:

- G.F. for displacements and bending moments *sum*.
- G.F. for rotations, stresses and shear forces *differentiate*

A monopole generates a dent, a pit. Anything that falls into the pit makes that the displacement increases. The pit sums, it integrates. A dipole instead represents a shear deformation and these two opposite movements differentiate.

The influence function for the slope u' in a beam comes from a dipole, a couple $M = 1$, which is the limit of two opposite point forces

$$M = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \Delta x = 1 \,. \tag{2.257}$$

If the resulting shear deformation is perfectly antisymmetric as it is in the middle of a hinged beam and if the load happens to be symmetric then the slope is zero; no need for the cross-section of the beam to rotate to counterbalance the effect of uneven forces. Also the influence function for a shear force V is of dipole-type, is a *"high-pass filter"* while the influence function for the bending moment M is of monopole-type is a *"low-pass filter"*. The latter influence function is generated by two moments $M = \pm 1/\Delta x$ which rotate inward—towards the bent—and so generate a symmetric deflection but with a sharp bent, a jump in the first derivative, at the source point.

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[5] You see a gap only in 1-D. In 2-D you can only sense it if you circle the gap once. Then you will experience a displacement shift in the direction of the dislocation.

[6] We restrain at this point from introducing quadrupoles or even octupoles [5].

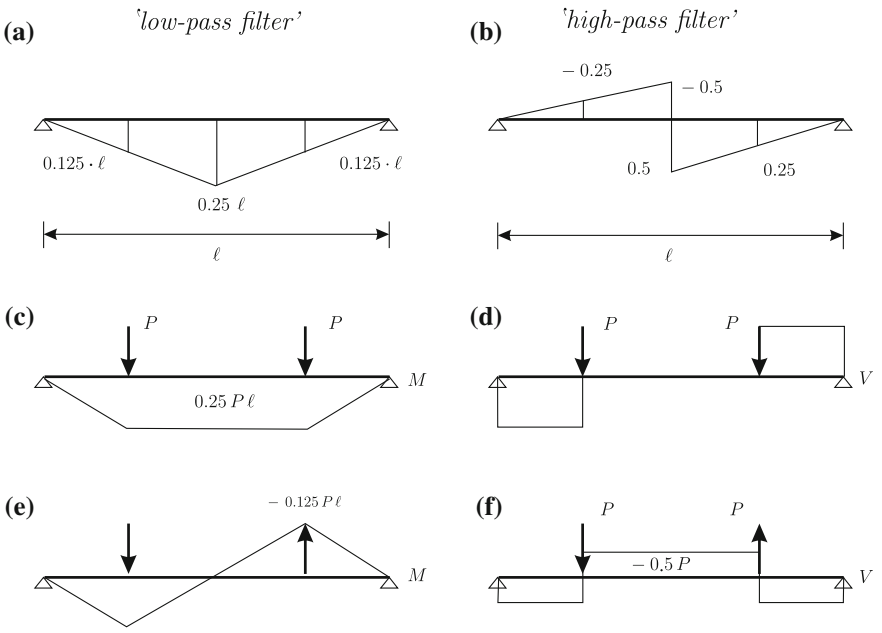

Fig. 2.16 Influence functions (uppermost row) for **a** the bending moment and **b** the shear force at the center of the beam, **c** and **d** bending moments and shear forces of a symmetric load and of an antisymmetric load, (**e**) and (**f**)

Now it is clear what happens: the maximum effect is observed if the loading and the influence are of the same type (symmetric—symmetric or antisymmetric—antisymmetric) and the minimum effect if the two have opposite "signs" are of opposite type, s. Fig. 2.16.

The difference between monopoles and dipoles also explains why displacements and bending moments are easier to approximate than stresses and shear forces. It is the difference between numerical integration and numerical differentiation, see Fig. 2.17 and also Figs. 2.18 and 2.19.

Remark 2.10 All influence functions for support reactions integrate though the support reactions are normal forces (stresses) or shear forces respectively and so we would expect that the influence functions differentiate. But at a fixed support one half of the shear deformation (= influence function) is hindered by the foundation. So with one half-wave being zero and the other half-wave making a full swing to produce the required dislocation $[[u]] = 1$ the influence function turns into an one-sided integration formula.

Remark 2.11 Not all influence functions decay! If parts of the released structure (released = after an N-, V- or M-hinge is built in) can perform rigid body movements then the opposite may be true, s. Fig. 2.20b.

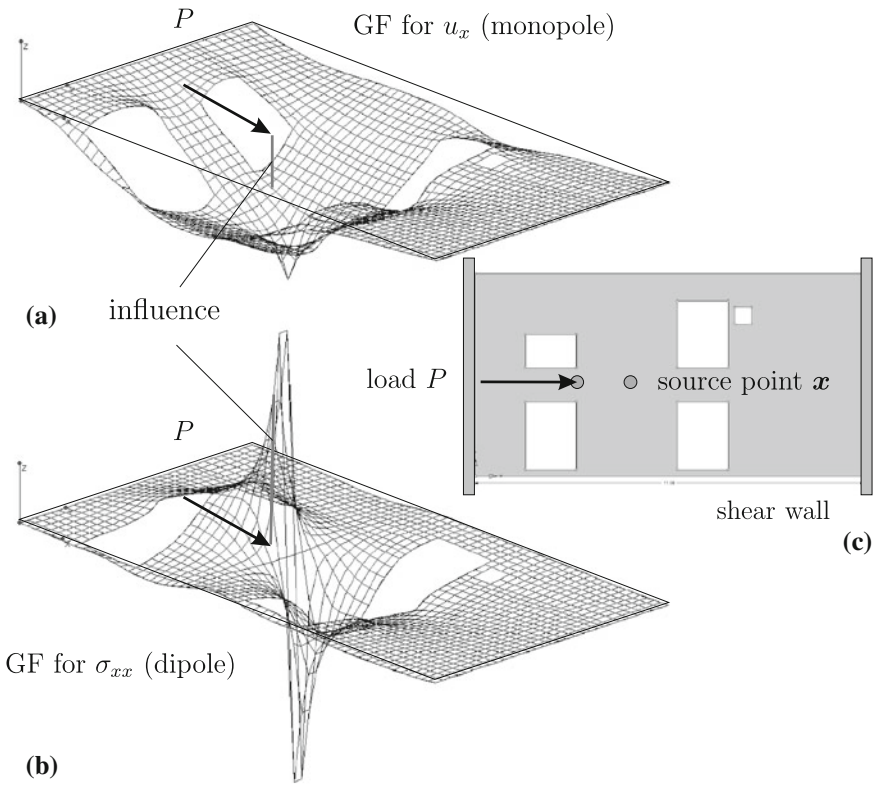

Fig. 2.17 Monopoles and dipoles, **a** influence function for the horizonal displacement and **b** the stress σ_{xx} in a shear wall **c** displayed are only the horizontal components of the two influence functions

2.5.5 Multipole Expansion

To be complete we mention the *multipole expansion* of influence functions whereby large complex problems in acoustics or in electromagnetic and gravitational field theory can be solved in reasonable time.

The gravitational potential generated by a body Ω with mass density $\rho(\boldsymbol{x})$ is the integral

$$u(\boldsymbol{x}) = -G \int_{\Omega} \frac{1}{r}\, \rho(\boldsymbol{y})\, d\Omega_{\boldsymbol{y}} \quad r = |\boldsymbol{y} - \boldsymbol{x}| \tag{2.258}$$

where the kernel $1/r$ is the free-space Green's function (fundamental solution) of the Laplacian

$$-\Delta \frac{1}{r} = 4\,\pi\,\delta(\boldsymbol{y} - \boldsymbol{x}) \tag{2.259}$$

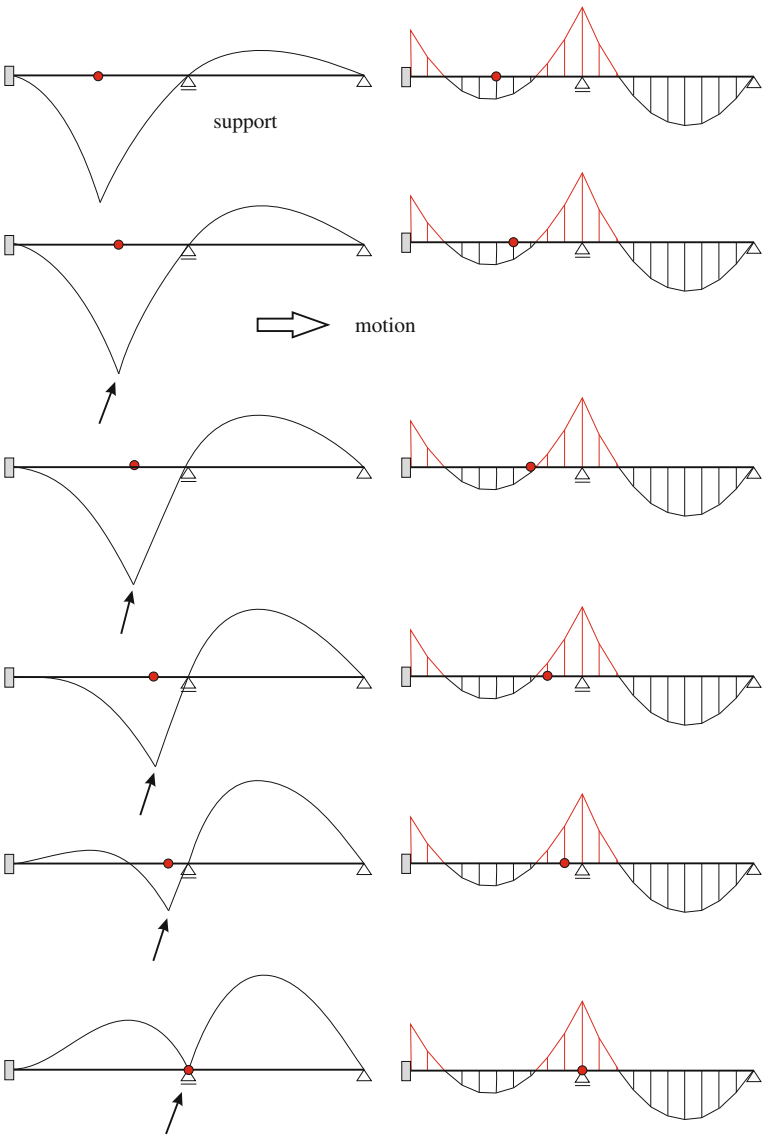

Fig. 2.18 How the influence functions for the bending moment $M = -E\,I\,u''$ wanders along the beam and keeps its shape

and G is the gravitational constant.[7]

By doing a Taylor expansion of the kernel

[7] Actually the complete Green's function is $1/(4\,\pi\,r)$. But in the literature the potential is always given in the form (2.258), that is the $1/4\,\pi$ must be contained in the constant G.

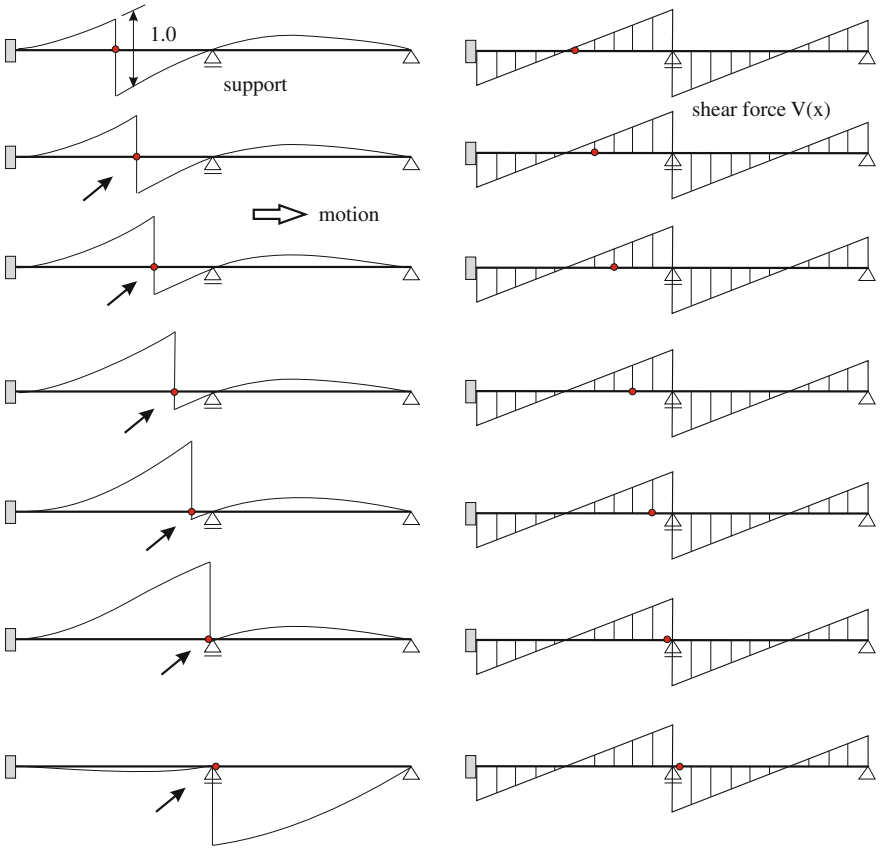

Fig. 2.19 How the influence function for the shear force $V = -EI\,u'''$ wanders along the beam but essentially stays the same. Note that near the fixed support the two-sided dislocation becomes a one-sided swing and so produces the maximal effect

$$\frac{1}{|\boldsymbol{y}-\boldsymbol{x}|} = \frac{1}{r}\left[1 + \frac{1}{r}\,\boldsymbol{e_x}\cdot\boldsymbol{y} + \frac{1}{2\,r^2}(3\,(\boldsymbol{e_x}\cdot\boldsymbol{y})^2 - |\boldsymbol{y}|^2) + O\left(\frac{|\boldsymbol{y}|}{|\boldsymbol{x}|}\right)^3\right], \quad (2.260)$$

where the unit vector $\boldsymbol{e_x}$ signals the direction to $\boldsymbol{x} = r\,\boldsymbol{e_x}$, the potential can be written as

$$u(\boldsymbol{x}) = u_{mon}(\boldsymbol{x}) + u_{dip}(\boldsymbol{x}) + u_{quad}(\boldsymbol{x}) + \ldots \quad (2.261)$$

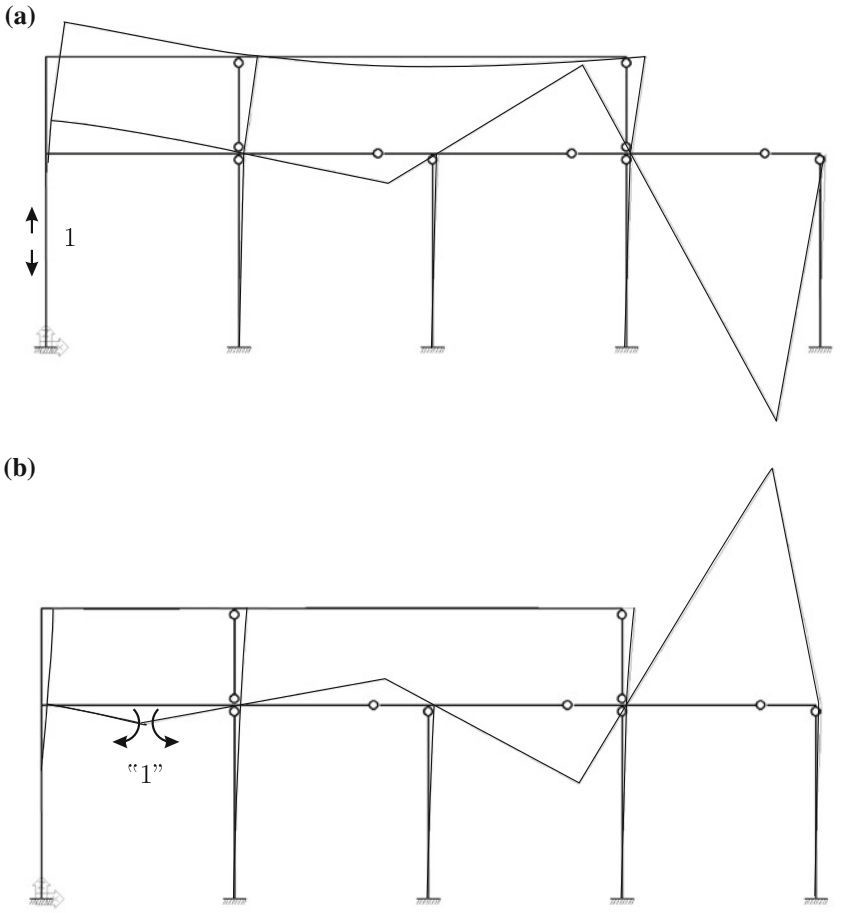

Fig. 2.20 **a** Influence function for the normal force N and **b** the bending moment M in a frame. Not all influence functions decay!

The potentials

$$u_{mon}(\boldsymbol{x}) = -\frac{G}{r}\int_{\Omega}\rho(\boldsymbol{y})\,d\Omega \tag{2.262}$$

$$u_{dip}(\boldsymbol{x}) = -\frac{G}{r^2}\int_{\Omega}\rho(\boldsymbol{y})\,\boldsymbol{e_x}\cdot\boldsymbol{y}\,d\Omega \tag{2.263}$$

$$u_{quad}(\boldsymbol{x}) = -\frac{G}{2\,r^3}\int_{\Omega}\rho(\boldsymbol{y})(3\,(\boldsymbol{e_x}\cdot\boldsymbol{y})^2 - |\boldsymbol{y}|^2)\,d\Omega \tag{2.264}$$

represent monopoles (+), dipoles (+−) and quadrupoles (+ − +−) respectively. If the Earth were a perfect sphere with a uniform density then its higher potentials would be zero, $u(\boldsymbol{x}) = u_{mon}(\boldsymbol{x})$. This expansion is often coupled with a multi-level clustering of the boundary elements, that is the cells or panels into which the surface of a vibrating machine is subdivided, and so the two techniques combined allow to solve exterior problems at "quasi-linear" costs.

2.5.6 Infinite Energy

The strain energy product $a(u, v)$ of two functions is an integral. On the diagonal, $v = u$, it is, up to the factor $1/2$, the internal energy

$$W_i = \frac{1}{2}\, a(u, u) \tag{2.265}$$

of the function u. If the integral does not exist, if the integral is infinite, it is said that the function has infinite energy. Most Green's functions have infinite energy.

Apply for example a point load $P = 1$ at the center, $\boldsymbol{x} = 0$, of a circular membrane with radius $R = 1$, see Fig. 2.13,

$$-\Delta\, G = P \cdot \delta(\boldsymbol{y} - \boldsymbol{0}) \quad G(\boldsymbol{y}, \boldsymbol{0}) = 0 \quad \boldsymbol{y} \in \Gamma\,. \tag{2.266}$$

The solution to this problem, the Green's function for $u(\boldsymbol{0})$

$$G(\boldsymbol{y}, \boldsymbol{0}) = -P\, \frac{1}{2\,\pi} \ln r \tag{2.267}$$

has a singularity at $\boldsymbol{y} = \boldsymbol{0}$. This means that the point force once placed on the center of the membrane will sink and sink and not stop before it has reached the point ∞, so that the exterior work done by the load is infinite

$$W_e = \int\limits_{\Omega} -\Delta\, G \cdot G\; d\Omega_y = P \int\limits_{\Omega} \delta(\boldsymbol{y} - \boldsymbol{0})\, G(\boldsymbol{y}, \boldsymbol{0})\, d\Omega_y = P \cdot \infty\,. \tag{2.268}$$

According to the principle of conservation of energy $W_e = W_i$ which is identical with Green's first identity

$$\mathscr{G}(u, u) = \underbrace{\int\limits_{\Omega} -\Delta u\, u\, d\Omega + \int\limits_{\Gamma} \frac{\partial u}{\partial n}\, u\, ds}_{2\,W_e} - \underbrace{a(u, u)}_{2\,W_i} = 0\,, \tag{2.269}$$

the internal energy ($d\Omega = r\, dr\, d\varphi$) must be infinite as well, which indeed it is

$$a(G,G) = \int_{\Omega} \nabla G \cdot \nabla G \, d\Omega = \frac{1}{2\pi} \int_{\Omega} \frac{1}{r^2} (r_{,x}^2 + r_{,y}^2) \, d\Omega$$
$$= \frac{1}{2\pi} \int_0^1 \frac{1}{r} \, dr \int_0^{2\pi} (\cos^2 \varphi + \sin^2 \varphi) \, d\varphi = \int_0^1 \frac{1}{r} \, dr = \infty \,, \tag{2.270}$$

and so we have in the end the—fitting—result

$$\mathscr{G}(G,G) = \infty - \infty = 0 \,. \tag{2.271}$$

But when a guitar string is plugged with a finger

$$-H\,G'' = \delta(y-x) \quad G(0,x) = G(l,x) = 0 \tag{2.272}$$

which produces a triangular shape G, then the internal energy of the string is bounded

$$a(G,G) = \int_0^l H\,(G')^2 \, dy < \infty \tag{2.273}$$

because G' is piecewise constant and finite.

The important point is that if the energy of the Green's functions G is bounded then the functional $J(u)$ which is associated with a Green's function

$$J(u) = \int_{\Omega} G(\boldsymbol{y}, \boldsymbol{x}) \, p(\boldsymbol{y}) \, d\Omega_{\boldsymbol{y}} \tag{2.274}$$

is also bounded and vice versa.

In the case of linear functionals bounded and continuous is the same. A functional on a space $\mathcal{V}$ is continuous if there exists a constant c, which is independent of the single u, such that for all u

$$|J(u)| < c \, \|u\| \,. \tag{2.275}$$

2.5.7 Genealogy of Influence Functions

Theoretically it suffices to know the Green's function of the functional

$$J(u) = u(x) = \int_0^l G(y,x) \, p(y) \, dy \tag{2.276}$$

because the Green's function $\tilde{G}(y,x)$ of any other linear functional $\tilde{J}(u)$ is simply

$$\tilde{G} = \tilde{J}(G) \tag{2.277}$$

as follows by direct substitution

$$\tilde{J}(u) = \int_0^l \tilde{J}(G)(y,x)\, p(y)\, dy = \int_0^l \tilde{G}(y,x)\, p(y)\, dy\,. \tag{2.278}$$

Though one must be careful because applying a functional $\tilde{J}$ under the integral sign is not a trivial task. There are some rules which apply to such maneuvers.

2.6 Sobolev's Embedding Theorem

The question whether a Green's functions has finite energy is answered by the following theorem.

Theorem 2.4 (Sobolev's Embedding Theorem) *Assume Ω is a bounded domain of $\mathbb{R}^n$ with a piecewise smooth edge (as in engineering applications). For $2(m-i) > n$ we have the inclusion*

$$C^i(\bar{\Omega}) \subset H^{i+m}(\Omega) \tag{2.279}$$

and there exist constants $c_i < \infty$ such that for all $u \in H^{i+m}(\Omega)$ the following estimate holds

$$\|u\|_{C^i(\bar{\Omega})} \leq c_i \, \|u\|_{H^{i+m}(\Omega)}\,. \tag{2.280}$$

The norm of a function u

$$\|u\|_{C^i(\bar{\Omega})} := \max_{0\leq|j|\leq i} \left| \frac{\partial^{|j|} u(\boldsymbol{x})}{\partial x^j} \right| \tag{2.281}$$

is the maximum value of $|u|$ and its derivatives up to the order i on $\bar{\Omega}$.

This theorem implies that the strain energy induced by the point load is bounded and the conjugated quantity (the effect the point load produces) is finite and continuous if the three indices satisfy the inequality [6],

$$m - i > \frac{n}{2}\,. \tag{2.282}$$

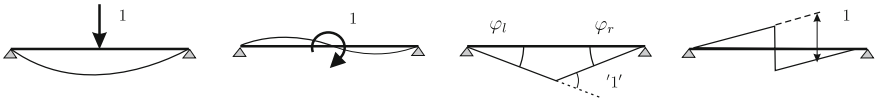

Fig. 2.21 The four singularities of a beam, '1' $\equiv \tan\varphi_l + \tan\varphi_r = 1$

Table 2.1 Energy is bounded (Yes) or (No)

singularity $m=1$	$n=1$ rope, bar, Timoshenko beam	$n=2$ plate, Reissner–Mindlin	$n=3$ 3-D
$i=0$: ↓	Yes	No	No
$i=1$: ⌐	No	No	No
singularity $m=2$	Euler–Bernoulli beam	Kirchhoff plate	
$i=0$: ↓	Yes	Yes	
$i=1$: ↷	Yes	No	
$i=2$: ∨	No	No	
$i=3$: ⌐	No	No	

The order of the energy is

$m=1$ Timoshenko beams, Reissner–Mindlin plates, elastic solids
$m=2$ Euler–Bernoulli beams, Kirchhoff plates

and the singularities in a second order equation ($2m=2$) have the indices

$$i=0 \quad \text{force} \quad i=1 \quad \text{dislocation}$$

and in a fourth order equation ($2m=4$), see Fig. 2.21,

$$i=0 \quad \text{force} \quad i=1 \quad \text{moment}$$
$$i=2 \quad \text{bend} \quad i=3 \quad \text{dislocation.}$$

The index $n=1,2,3$ corresponds to the space dimension. Table 2.1 summarizes the inequality (2.282).

What makes this theorem so important is the link between integrals and point values that is which index m must a Sobolev space $H^m(\Omega)$ have for it to contain $C(\Omega)$

$$C(\Omega) \subset H^m(\Omega) \tag{2.283}$$

or else when does $\|u\|_m < \infty$ imply that u is continuous (or to be precise: equivalent to a continuous function) on Ω. This is the case if $m > n/2$.

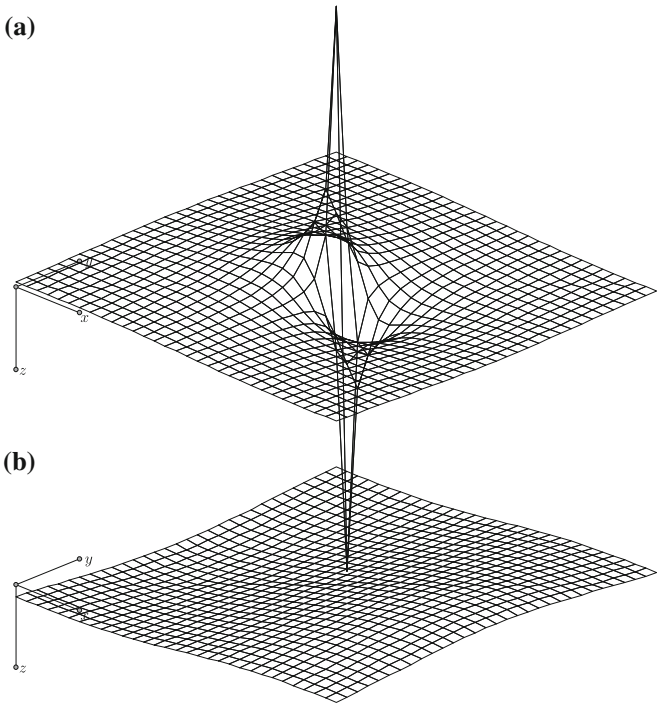

Fig. 2.22 Split of the Green's function for the first derivative $u_{,x}$ on a rectangular domain into **a** the fundamental solution $g_0(\boldsymbol{y}, \boldsymbol{x})$ and **b** the regular part $u_R(\boldsymbol{y}, \boldsymbol{x})$

If for example Ω is a (Kirchhoff) plate with energy space $H^2(\Omega)$, corresponding to $\Delta\Delta u = p$, and a function u lies in $H^2(\Omega)$—if all its derivatives up to the order 2 are square integrable on Ω—then u must be continuous, no sudden jumps are allowed, and it must also be bounded, $|u(\boldsymbol{x})| < \infty$, on Ω. There is guaranteed to be an upper limit to the deflection.

The space $H^1(\Omega)$ does have this property only in 1-D, $1 > 1/2$, but not in 2-D, $1 \not> 2/2$, and not in 3-D, $1 \not> 3/2$.

Remark 2.12 In 2-D the H^1-functions can have at most singularities at isolated points but they cannot be discontinuous along whole lines while in 3-D the H^1-functions can have singularities both at isolated points and along curves [7] A4.

2.7 Fundamental Solutions

Each Green's function

$$G(\boldsymbol{y}, \boldsymbol{x}) = g(\boldsymbol{y}, \boldsymbol{x}) + u_R(\boldsymbol{y}, \boldsymbol{x}) \tag{2.284}$$

can be split into a fundamental solution—in the case of the 2-D Poisson equation this would be the function

$$g(\boldsymbol{y}, \boldsymbol{x}) = -\frac{1}{2\pi} \ln r \tag{2.285}$$

and a regular solution $u_R(\boldsymbol{y}, \boldsymbol{x})$. The fundamental solution exhibits the particular feature which is characteristic for the Green's function, in this case

$$-\Delta g(\boldsymbol{y}, \boldsymbol{x}) = \delta(\boldsymbol{y} - \boldsymbol{x}) \tag{2.286}$$

while the regular part, which is a homogeneous solution of the governing equation,

$$-\Delta u_R(\boldsymbol{y}, \boldsymbol{x}) = 0\,, \tag{2.287}$$

makes that the sum $G = g + u_R$ satisfies the boundary conditions, see Fig. 2.22.

Because of their disregard for boundary conditions fundamental solutions are also called *free-space Green's functions*.

2.7.1 Influence Function

When Green's second identity is formulated with the fundamental solution (2.285) and a sufficiently smooth function u

$$\lim_{\varepsilon \to 0} B(g[\boldsymbol{x}], u) = 0 \tag{2.288}$$

and this expression is solved for $u(\boldsymbol{x})$

$$u(\boldsymbol{x}) = \int_\Gamma [g(\boldsymbol{y}, \boldsymbol{x}) \frac{\partial u}{\partial n}(\boldsymbol{y}) - \frac{\partial}{\partial n} g(\boldsymbol{y}, \boldsymbol{x})\, u(\boldsymbol{y})]\, ds_y + \int_\Omega g(\boldsymbol{y}, \boldsymbol{x})\, p(\boldsymbol{y})\, d\Omega_y \tag{2.289}$$

then this is an alternative influence function for $u(\boldsymbol{x})$. The boundary element method is based on this type of influence functions [8, 9]. It operates with fundamental solutions instead of Green's functions because fundamental solutions are problem independent.

But given a boundary value problem such as

$$-\Delta u = p \quad u = 0 \quad \text{on } \Gamma \tag{2.290}$$

the influence function (2.289) cannot be applied directly

$$u(\boldsymbol{x}) = \int_\Gamma g(\boldsymbol{y}, \boldsymbol{x}) \underset{?}{\frac{\partial u}{\partial n}}(\boldsymbol{y})\, ds_y + \int_\Omega g(\boldsymbol{y}, \boldsymbol{x})\, p(\boldsymbol{y})\, d\Omega_y \tag{2.291}$$

because the slope is unknown on Γ and must first be determined by an integral equation (apply the previous equation to points on the boundary)

$$\int_\Gamma g(\boldsymbol{y},\boldsymbol{x})\, s_h(\boldsymbol{y})\, ds_{\boldsymbol{y}} = -\int_\Omega g(\boldsymbol{y},\boldsymbol{x})\, p(\boldsymbol{y})\, d\Omega_{\boldsymbol{y}} \quad \boldsymbol{x} \in \Gamma \tag{2.292}$$

where $s_h(\boldsymbol{y})$ is a boundary element approximation of the slope.

2.8 Ill-Posed Problems

The Green's function of the Laplace operator

$$-\Delta G(\boldsymbol{y},\boldsymbol{x}) = \delta(\boldsymbol{y}-\boldsymbol{x}) \quad G(\boldsymbol{y},\boldsymbol{x}) = 0 \quad \boldsymbol{y} \in \Gamma\,, \tag{2.293}$$

has—as most Green's functions do—infinite energy. This poses a problem because it is no longer possible to claim that the FE-solution minimizes the distance in the energy to the exact solution

$$\|G - G_h\|_E \le \|G - v_h\|_E \quad \forall\ v_h \in \mathcal{V}_h \tag{2.294}$$

because the distance to the point $\|G\| = \infty$ is always infinite—regardless of which point $G_h \in \mathcal{V}_h$ is picked as best approximation to G.

In the literature on goal-oriented adaptive refinement the Dirac deltas, the point loads $P = 1$, are therefore always replaced by equivalent surface loads $p = 1/|\Omega_\varepsilon|$ spread over a tiny disc Ω_ε centered at the source point so that the response of the medium, let us call this the function $\tilde{G}$, is almost the Green's function G. Numerically there is of course no difference between nodal forces f_i coming from P and p. The authors do this only to stay inside the bounds of the theory and to justify the estimate, see (4.20),

$$|u(\boldsymbol{x}) - u_h(\boldsymbol{x})| \le \|\tilde{G} - \tilde{G}_h\|_E\, \|u - u_h\|_E \tag{2.295}$$

which (theoretically) breaks down when instead of $\tilde{G}$ the exact Green's function is substituted because G has infinite energy.

To a large extent mathematical analysis is about inequalities about bounds. When you can provide bounds for an error term then you are in a much better position because you can then *control* the error and this is why the inequality above is invaluable. If you can control something then this means that you understand on what this something depends.

From the perspective of the theory of weak boundary value problems the search for an approximation G_h is an ill-posed problem because (1) the exact solution G does not lie in the energy space $H^1(\Omega)$ of the Laplace operator and (2) the functional $J(v)$ is not bounded on $H^1(\Omega)$.

But the finite element method simply disregards such warnings and operates with Green's functions just as if they happened to have finite energy and the success proves the FE-method right. Which certainly is a curious situation: *In an FE-program the input is mostly processed by kernel functions which are solutions of ill-posed problems*.

Remark 2.13 The functional $J(u) = u(\boldsymbol{x})$ would be a bounded functional on $H^1(\Omega)$ if there would exist a constant $c < \infty$, independent of the single u, such that

$$|J(u)| \leq c \, \|u\|_1 \quad \forall \, u \in H^1(\Omega) \, . \tag{2.296}$$

This would be true if $C(\Omega)$ can be "continuously embedded" into $H^1(\Omega)$

$$C(\Omega) \subset H^1(\Omega) \tag{2.297}$$

which is another way of saying that there exists a constant $c < \infty$ such that

$$\max_{\boldsymbol{x} \in \Omega} |u(\boldsymbol{x})| \leq c \, \|u\|_1 \tag{2.298}$$

for all functions u in $H^1(\Omega)$. Think of it this way: if the integral of u squared and its first-order derivatives squared, ($\sim$ *energy*), tends to zero then u uniformly tends to zero—pointwise (!). When the stored heat (= energy) in a small box Ω is nearly zero then the temperature $u(\boldsymbol{x})$ at each point of Ω must be near absolute zero. Or what is the same: if two functions u and v are close in the H^1-norm then they are also close in the C-norm which means

$$\max_{\boldsymbol{x} \in \Omega} |u(\boldsymbol{x}) - v(\boldsymbol{x})| \leq c \, \|u - v\|_1 \, . \tag{2.299}$$

But the function $u = -\ln(-\ln^{-1} r)$ has a finite H^1-norm on a disc Ω_ρ with radius $\rho = 0.5$ and centered at $\boldsymbol{x}$

$$\|u\|_1^2 = \int\limits_{\Omega} (u^2 + u_{,x}^2 + u_{,y}^2) \, d\Omega = \int\limits_0^{2\pi} \int\limits_0^{\rho} \left(u^2 + \frac{4}{r^2 \, \ln^2 r^2} \right) r \, dr \, d\varphi < \infty \tag{2.300}$$

but becomes infinite at the center, $r = 0$ and so the functional $J(u) = u(0)$ cannot be a bounded functional on $H^1(\Omega_\rho)$.

2.9 Nonlinear Problems

Green's functions are only applicable to linear problems because scalar products are linear and so the superposition principle is limited to linear problems

$$u_1(\boldsymbol{x}) + u_2(\boldsymbol{x}) = \int_{\Omega} G(\boldsymbol{y}, \boldsymbol{x})\,(p_1(\boldsymbol{y}) + p_2(\boldsymbol{y}))\,d\Omega_{\boldsymbol{y}}\,. \tag{2.301}$$

But two observations suggest to extend at least the concept of a Green's function to nonlinear problems:

- Functionals $J(u)$ are not limited to linear problems.
- In linear problems Green's functions are identical with the Lagrange multiplier λ

$$\mathcal{L}(u, \lambda) := J(u) - (a(u, \lambda) - (p, \lambda)) \tag{2.302}$$

because

$$\mathcal{L}_{,u} = J(v) - a(v, \lambda) = 0 \qquad \forall\, v \in \mathcal{V} \quad \Rightarrow \lambda = G \tag{2.303}$$

$$\mathcal{L}_{,\lambda} = a(u, v) - (p, v) = 0 \qquad \forall\, v \in \mathcal{V} \tag{2.304}$$

and in the nonlinear case λ corresponds to the Green's function at the linearization point. So Lagrange multipliers inherently carry over the concept of a Green's function also to nonlinear problems although one must be fully aware that a nonlinear solution does *not* allow such a representation as in (2.301). But the very same idea which has been applied so successfully to linear problems, *goal-oriented adaptive refinement*, see Sect. 4.7, can also be applied to nonlinear problems and in this regard the extension to nonlinear problems is important.

2.9.1 Lagrange Multiplier

Assume a function $f(x, y)$ is to be minimized under the side condition that the point (x, y) satisfies an equation $g(x, y) = 0$. With the two functions f and g and the Lagrange multiplier λ we form the expression

$$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda\, g(x, y)\,, \tag{2.305}$$

and the point $\{x, y, \lambda\}$ at which L becomes stationary

$$d\mathcal{L} = \mathcal{L}_{,x}\; dx + \mathcal{L}_{,y}\; dy + \mathcal{L}_{,\lambda}\; d\lambda = 0 \tag{2.306}$$

is also the point x, y at which $f(x, y)$ attains its minimum value.

Lagrange multipliers are not restricted to optimization problems. Imagine a function $f(x)$ is to be evaluated at a point x and x is subject to the constraint $g(x) = 0$ as in

$$f(x) = \sin^2(x) \quad g(x) = x - 1 = 0\,. \tag{2.307}$$

Then the expression

$$\mathcal{L} = f(x) + \lambda\, g(x) \tag{2.308}$$

is stationary for a certain value of λ at the point $\{x, \lambda\}$, that is

$$d\mathcal{L} = \mathcal{L}_{,x}\, dx + \mathcal{L}_{,\lambda}\, d\lambda = 0\,. \tag{2.309}$$

The point $\{x, \lambda\}$ is found by solving the two equations

$$\mathcal{L}_{,x} = f'(x) + \lambda\, g'(x) = 0 \qquad \text{(this determines } \lambda) \tag{2.310}$$

$$\mathcal{L}_{,\lambda} = g(x) = 0 \qquad \text{(this determines } x)\,. \tag{2.311}$$

In the following the function f will be a functional $J(\boldsymbol{u})$ and the vector $\boldsymbol{u}$ is subject to the constraint $\boldsymbol{K}\,\boldsymbol{u} = \boldsymbol{f}$.

2.9.2 Lagrange Multiplier and Linear Algebra

Lagrange multipliers are denoted by λ or $\boldsymbol{\lambda}$ and we follow this convention here but in essence they are identical with the nodal vector $\boldsymbol{g}$ of the Green's function, so $\boldsymbol{\lambda} \equiv \boldsymbol{g}$.

Let

$$J(\boldsymbol{u}) = \boldsymbol{j}^T\, \boldsymbol{u} \tag{2.312}$$

a linear functional on $\mathbb{R}^n$ which is to be evaluated under the side condition that the vector $\boldsymbol{u}$ satisfies the symmetric $(n \times n)$ system

$$\boldsymbol{K}\,\boldsymbol{u} = \boldsymbol{f}\,. \tag{2.313}$$

With a third vector $\boldsymbol{\lambda} \in \mathbb{R}^n$ we form the Lagrange functional

$$\mathcal{L}(\boldsymbol{u}, \boldsymbol{\lambda}) = J(\boldsymbol{u}) - \boldsymbol{\lambda}^T\, (\boldsymbol{K}\,\boldsymbol{u} - \boldsymbol{f}) \tag{2.314}$$

and the point $\{\boldsymbol{u}, \boldsymbol{\lambda}\}$ at which the functional $\mathcal{L}$ is stationary

$$d\mathcal{L} = \mathcal{L}_{,u_i}\; du_i - \mathcal{L}_{,\lambda_i}\; d\lambda_i = 0 \tag{2.315}$$

is determined by the two equations

$$\boldsymbol{K}\,\boldsymbol{u} = \boldsymbol{f} \quad \boldsymbol{K}\,\boldsymbol{\lambda} = \boldsymbol{j} \tag{2.316}$$

and so it follows that

$$J(\boldsymbol{u}) = \boldsymbol{\lambda}^T\, \boldsymbol{f} \tag{2.317}$$

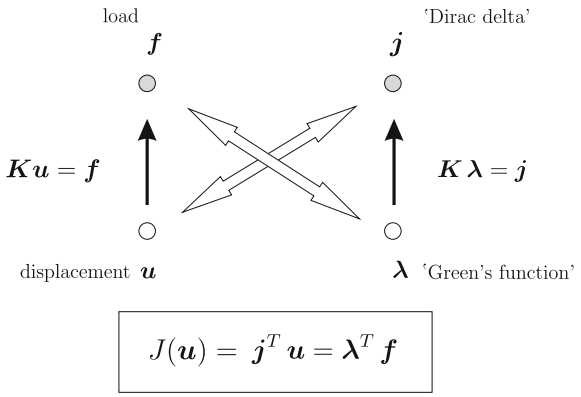

Fig. 2.23 The Lagrange multiplier method is an application of Betti's theorem that is of the duality inherent in the scalar product, $\pi^T = \pi$

which means that the Lagrange multiplier plays the same role als the nodal vector $\boldsymbol{g}$ of the Green's function in FE-analysis, see Fig. 2.23.

The Lagrange multiplier method as presented here is identical with Betti's theorem that is it is a simple application of the identity ($\pi^T = \pi$)

$$\mathscr{B}(\boldsymbol{u}, \boldsymbol{\lambda}) = \boldsymbol{\lambda}^T \, \boldsymbol{K} \, \boldsymbol{u} - \boldsymbol{u}^T \, \boldsymbol{K} \, \boldsymbol{\lambda} = 0 \tag{2.318}$$

which holds true for any two vectors $\boldsymbol{u}, \boldsymbol{\lambda}$ and a symmetric matrix $\boldsymbol{K}$. This immediately implies that if the two vectors are solutions of (2.316) then

$$\mathscr{B}(\boldsymbol{u}, \boldsymbol{\lambda}) = \boldsymbol{\lambda}^T \, \boldsymbol{f} - \boldsymbol{u}^T \, \boldsymbol{j} = 0 \tag{2.319}$$

or $J(\boldsymbol{u}) = \boldsymbol{\lambda}^T \, \boldsymbol{f}$.

The stationarity condition $d\mathcal{L} = 0$ only serves to derive the two equations (2.316).

2.9.3 Nonlinear Functionals

Next let $J(u)$ a nonlinear functional. We only presuppose that $J(0) = 0$. The expression

$$J'(u; v) := \frac{d}{d\varepsilon} J(u + \varepsilon \, v)|_{\varepsilon=0} \tag{2.320}$$

is the Gateaux derivative of the functional $J(u)$. Because of the chain rule of the Calculus this expression is linear in the second argument, in v. Let $f(s)$ be the function

$$f(s) := J(u + s \, e) \quad e = u - u_h \,, \;\; 0 \leq s \leq 1 \,. \tag{2.321}$$

According to the Fundamental Theorem of the Calculus we have

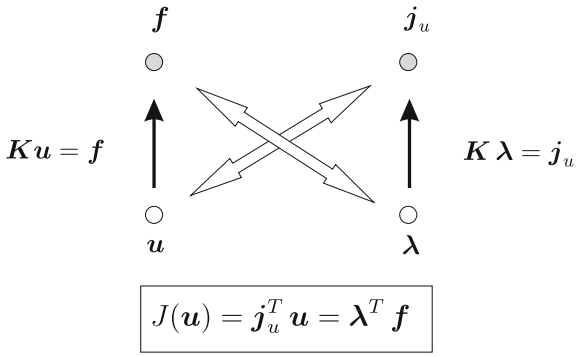

Fig. 2.24 The Lagrange multiplier method applied to a non-linear functional

$$f(1) - f(0) = J(u) - J(u_h) = \int_0^1 f'(s)\,ds = \int_0^1 J'(u_h + s\,e; e)\,ds \qquad (2.322)$$

and in particular, if $u = 0$,

$$-J(u_h) = \int_0^1 J'(u_h - s\,u_h; -u_h)\,ds = -\int_0^1 J'(u_h - s\,u_h; u_h)\,ds\,. \qquad (2.323)$$

In FE-analysis the function u_h is a weighted sum of the shape functions

$$u_h(\boldsymbol{x}) = \sum_{i=1}^{n} u_i\,\varphi_i(\boldsymbol{x}) \qquad (2.324)$$

and so, see Fig. 2.24,

$$J(u_h) = J(\boldsymbol{u}) := \boldsymbol{j}_u^T\,\boldsymbol{u} = \sum_{i=1}^{n} j_{u\,i}\,u_i \quad j_{ui} := \int_0^1 J'(\boldsymbol{u}_h - s\,\boldsymbol{u}_h; \varphi_i)\,ds\,. \qquad (2.325)$$

The functional $J(\boldsymbol{u})$ ($\boldsymbol{u}$ is a vector) has the same value as $J(u_h)$ (u_h is a function) if $\boldsymbol{u}$ is the nodal vector of u_h as in (2.324). The point $\{\boldsymbol{u}, \boldsymbol{\lambda}\}$ at which the Lagrange functional

$$\mathcal{L}(\boldsymbol{u}, \boldsymbol{\lambda}) = J(\boldsymbol{u}) - \boldsymbol{\lambda}^T\,(\boldsymbol{K}\,\boldsymbol{u} - \boldsymbol{f}) \qquad (2.326)$$

becomes stationary is determined by the two equations

$$\boldsymbol{K}\,\boldsymbol{u} = \boldsymbol{f} \quad \boldsymbol{K}\,\boldsymbol{\lambda} = \boldsymbol{j}_u\,, \qquad (2.327)$$

and we have

$$J(\boldsymbol{u}) = \boldsymbol{j}_u^T \boldsymbol{u} = \boldsymbol{\lambda}^T \boldsymbol{K}\, \boldsymbol{u} = \boldsymbol{\lambda}^T \boldsymbol{f}\,. \tag{2.328}$$

This looks like magic: seemingly a nonlinear functional can be given the form of a scalar product. But the vector $\boldsymbol{\lambda}$ which multiplies $\boldsymbol{f}$ depends on the argument u_h. Only in linear problems is $\boldsymbol{\lambda}$ truly a constant vector. The (pseudo) linearity in (2.328) is due to the chain rule of the Calculus.

Example 2.7 Let

$$J(u) = \int_0^l u^2\, dx \tag{2.329}$$

then

$$J'(u; v) = \int_0^l 2\, u\, v\, dx \quad j_{ui} = \int_0^l u_h\, \varphi_i\, dx \tag{2.330}$$

and

$$J(u_h) = \boldsymbol{\lambda}^T \boldsymbol{f} = \boldsymbol{j}_u^T \boldsymbol{K}^{-1} \boldsymbol{f} = \boldsymbol{j}_u^T \boldsymbol{u}\,. \tag{2.331}$$

A direct evaluation of $J(u_h)$ gives

$$J(u_h) = J(u_1\, \varphi_1 + \ldots + u_n\, \varphi_n) = \boldsymbol{u}^T \boldsymbol{M}\, \boldsymbol{u} \quad m_{ij} = \int_0^l \varphi_i\, \varphi_j\, dx \tag{2.332}$$

where $\boldsymbol{M} = [m_{ij}]$ is the "mass matrix" and we see that $\boldsymbol{j}_u = \boldsymbol{M}\, \boldsymbol{u}$.

Because $\boldsymbol{\lambda} = \boldsymbol{K}^{-1}\, \boldsymbol{j}_u$ depends on $\boldsymbol{u}$ the direct evaluation (2.332) is probably always faster than (2.331).

The real value of the vector $\boldsymbol{\lambda} = \boldsymbol{K}^{-1}\, \boldsymbol{j}_u$ is that it can be plotted. It represents the sensitivity of the functional $J(\boldsymbol{u})$ with respect to the vector $\boldsymbol{f}$. Imagine a planar mesh with two degrees of freedom at each node. At each node $\boldsymbol{x}_i$ the two corresponding values of $\boldsymbol{\lambda}$ form a small vector $\boldsymbol{\lambda}_i$ as for example in Figs. 3.20, 3.21 and 3.22 in Sect. 3.10.2. Nodes where $\boldsymbol{\lambda}_i \equiv \boldsymbol{g}_i$ is orthogonal to the vector $\boldsymbol{f}_i$ (which is the local representative of $\boldsymbol{f}$ at node $\boldsymbol{x}_i$) do not contribute to $J(\boldsymbol{u})$.

2.9.4 Nonlinear Problems

Next let us assume that the boundary value problem itself is nonlinear

$$a(u_h; \varphi_i) = (p, \varphi_i) \quad i = 1, 2, \ldots n\,, \tag{2.333}$$

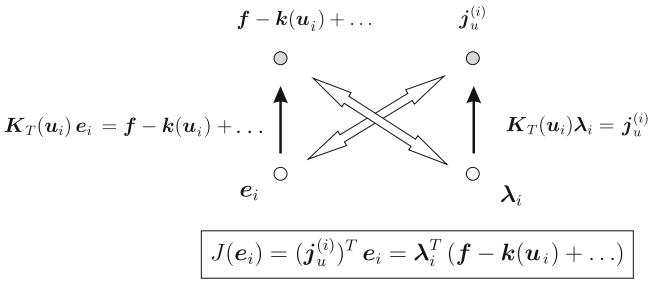

Fig. 2.25 The Lagrange multiplier method applied to a nonlinear problem

and so the vector $\boldsymbol{u} = \{u_i\}$ of nodal values of the FE-solution

$$u_h(\boldsymbol{x}) = \sum_{i=1}^{n} u_i\,\varphi_i(\boldsymbol{x}) \tag{2.334}$$

satisfies the nonlinear vector-valued equation

$$\boldsymbol{k}(\boldsymbol{u}) = \boldsymbol{f} \tag{2.335}$$

where

$$k_i(\boldsymbol{u}) := a(u_h;\varphi_i) \quad f_i = (p,\varphi_i) \quad i = 1,2,\ldots n\,. \tag{2.336}$$

We allow, as before, that $J(u)$ is a nonlinear functional

$$J(\boldsymbol{u}) = \boldsymbol{j}_u^T\,\boldsymbol{u}\,. \tag{2.337}$$

Hence the Lagrange functional is

$$\mathcal{L}(\boldsymbol{u},\boldsymbol{\lambda}) = \boldsymbol{u}^T\,\boldsymbol{j}_u - \boldsymbol{\lambda}^T(\boldsymbol{k}(\boldsymbol{u}) - \boldsymbol{f}) \tag{2.338}$$

and we conclude that the stationary point $\{\boldsymbol{u},\boldsymbol{\lambda}\}$ is determined by the two equations

$$\boldsymbol{k}(\boldsymbol{u}) = \boldsymbol{f} \quad \boldsymbol{K}_T(\boldsymbol{u})\,\boldsymbol{\lambda} = \boldsymbol{j}_u \tag{2.339}$$

where $\boldsymbol{K}_T$ is the tangent stiffness matrix at the point $\boldsymbol{u}$.

This result is no longer symmetric and not what we had hoped for. If the first equation were of the same form as the second

$$\boldsymbol{K}_T(\boldsymbol{u})\,\boldsymbol{u} = \boldsymbol{f} \tag{2.340}$$

then we could apply immediately the previous logic and write $J(\boldsymbol{u}) = \boldsymbol{\lambda}^T\,\boldsymbol{f}$.

So let us assume for a moment it were symmetric. With an arbitrary vector $\boldsymbol{u}_i$ we can construct the matrix $\boldsymbol{K}_T(\boldsymbol{u}_i)$ and therewith the identity ($\pi^T = \pi$)

$$\mathscr{B}(\boldsymbol{e}, \boldsymbol{\lambda}) = \boldsymbol{\lambda}^T \, \boldsymbol{K}_T(\boldsymbol{u}_i) \, \boldsymbol{e} - \boldsymbol{e}^T \, \boldsymbol{K}_T(\boldsymbol{u}_i) \, \boldsymbol{\lambda} = 0 \tag{2.341}$$

which holds true for all vectors $\boldsymbol{e}$ and $\boldsymbol{\lambda}$.

The solution of $\boldsymbol{k}(\boldsymbol{u}) = \boldsymbol{f}$ is a zero of the vector-valued function

$$\boldsymbol{s}(\boldsymbol{u}) := \boldsymbol{k}(\boldsymbol{u}) - \boldsymbol{f} \tag{2.342}$$

and so if we follow the logic of Newton's algorithm

$$\boldsymbol{s}(\boldsymbol{u}_{i+1}) = \boldsymbol{0} = \boldsymbol{s}(\boldsymbol{u}_i) + \boldsymbol{s}'(\boldsymbol{u}_i) \, (\boldsymbol{u}_{i+1} - \boldsymbol{u}_i) + \cdots \tag{2.343}$$

and if we let

$$\boldsymbol{e}_{i+1} = \boldsymbol{u}_{i+1} - \boldsymbol{u}_i \tag{2.344}$$

then we are led to

$$\boldsymbol{K}_T(\boldsymbol{u}_i) \, \boldsymbol{e}_{i+1} \simeq \boldsymbol{f} - \boldsymbol{k}(\boldsymbol{u}_i) \, . \tag{2.345}$$

Denoting

$$\boldsymbol{K}_T(\boldsymbol{u}_i) \, \boldsymbol{\lambda}_i = \boldsymbol{j}_u^{(i)} \tag{2.346}$$

the identity (4.47) implies

$$\mathscr{B}(\boldsymbol{e}_{i+1}, \boldsymbol{\lambda}_i) = \underbrace{\boldsymbol{\lambda}_i^T \, \boldsymbol{K}_T(\boldsymbol{u}_i)}_{(\boldsymbol{j}_u^{(i)})^T} \, \boldsymbol{e}_{i+1} - \boldsymbol{e}_{i+1}^T \, \boldsymbol{K}_T(\boldsymbol{u}_i) \, \boldsymbol{\lambda}_i = 0 \tag{2.347}$$

or

$$J(\boldsymbol{e}_{i+1}) = (\boldsymbol{j}_u^{(i)})^T \, \boldsymbol{e}_{i+1} = \boldsymbol{e}_{i+1}^T \, \boldsymbol{K}_T(\boldsymbol{u}_i) \, \boldsymbol{\lambda}_i \simeq (\boldsymbol{f} - \boldsymbol{k}(\boldsymbol{u}_i))^T \, \boldsymbol{\lambda}_i \tag{2.348}$$

which means that the vector $\boldsymbol{\lambda}_i$ and the residual $\boldsymbol{k}(\boldsymbol{u}_i) - \boldsymbol{f}$ at the linearization point allow to approximate the error $J(\boldsymbol{e}_{i+1})$, see Fig. 2.25.

This basically is the algebra by which goal-oriented adaptive refinement can be extended to nonlinear problems. For more details see Sect. 4.7.

Remark 2.14 We assumed the tangent stiffness matrix $\boldsymbol{K}_T$ to be symmetric but the extension to non-symmetric matrices is obvious: replace the second matrix $\boldsymbol{K}_T$ in (4.47) by $\boldsymbol{K}_T^T$.

2.10 Mixed Problems

The boundary value problem

$$-\Delta u = p \quad \text{on } \Omega \quad u = 0 \quad \text{on } \Gamma \tag{2.349}$$

can be split into a coupled system

$$\nabla u - \boldsymbol{\sigma} = \boldsymbol{\sigma}_0 \tag{2.350}$$

$$-\text{div}\,\boldsymbol{\sigma} = p \tag{2.351}$$

for the two functions u and $\boldsymbol{\sigma}$ or $\boldsymbol{v} = \{u, \boldsymbol{\sigma}\}^T$. We have added an additional term $\boldsymbol{\sigma}_0$ on the right-hand side to account for possible initial stresses.

To this system belongs the identity

$$\mathscr{G}(\boldsymbol{v}, \hat{\boldsymbol{v}}) = \int_\Omega \underbrace{\left[(\nabla u - \boldsymbol{\sigma}) \cdot \hat{\boldsymbol{\sigma}} - \text{div}\,\boldsymbol{\sigma}\, \hat{u}\right]}_{\boldsymbol{L}\,\boldsymbol{v}\cdot\hat{\boldsymbol{v}}} d\Omega + \int_\Gamma \boldsymbol{\sigma}\cdot\boldsymbol{n}\,\hat{u}\, ds$$
$$- \int_\Omega (\nabla u \cdot \hat{\boldsymbol{\sigma}} + \nabla \hat{u} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \hat{\boldsymbol{\sigma}})\, d\Omega = 0 \tag{2.352}$$

and Green's second identity is the expression

$$\mathscr{B}(\boldsymbol{v}, \hat{\boldsymbol{v}}) = \int_\Omega \boldsymbol{L}\,\boldsymbol{v}\cdot\hat{\boldsymbol{v}}\, d\Omega + \int_\Gamma \boldsymbol{\sigma}\cdot\boldsymbol{n}\,\hat{u}\, ds - \int_\Gamma \hat{\boldsymbol{\sigma}}\cdot\boldsymbol{n}\, u\, ds - \int_\Omega \boldsymbol{v}\cdot\boldsymbol{L}(\hat{\boldsymbol{v}})\, d\Omega = 0\,. \tag{2.353}$$

Let G be the Green's function for the value of $u(\boldsymbol{x})$ at some interior point of Ω

$$-\Delta G(\boldsymbol{y}, \boldsymbol{x}) = \delta(\boldsymbol{y} - \boldsymbol{x}) \quad G(\boldsymbol{y}, \boldsymbol{x}) = 0 \quad \boldsymbol{y} \in \Gamma \tag{2.354}$$

and let the "mixed representation" of this function be

$$\boldsymbol{g} = \{G, \boldsymbol{\sigma}_G\}^T\,, \quad \boldsymbol{\sigma}_G = \nabla G \tag{2.355}$$

which is a homogeneous solution of $\boldsymbol{L}\,\boldsymbol{v} = \boldsymbol{0}$ at all points except at $\boldsymbol{x}$, then in the limit

$$\lim_{\varepsilon \to 0} \mathscr{B}(\boldsymbol{g}, \boldsymbol{v})_{\Omega_\varepsilon} = 0 \tag{2.356}$$

out of

$$\lim_{\varepsilon \to 0} \int_{\Gamma_{N_\varepsilon(\boldsymbol{x})}} \boldsymbol{\sigma}_G(\boldsymbol{y}, \boldsymbol{x}) \cdot \boldsymbol{n}\, u\, ds = u(\boldsymbol{x}) \tag{2.357}$$

would pop $u(\boldsymbol{x})$ and so we have found an influence function

$$u(\boldsymbol{x}) = \int_{\Omega} (G(\boldsymbol{y}, \boldsymbol{x})\, p(\boldsymbol{y}) + \boldsymbol{\sigma}_G(\boldsymbol{y}, \boldsymbol{x}) \cdot \boldsymbol{\sigma}_0(\boldsymbol{y}))\, d\Omega_{\boldsymbol{y}}\,. \tag{2.358}$$

The kernels in the influence function for σ_1 and σ_2

$$G^{(1)}(\boldsymbol{y}, \boldsymbol{x}) = \frac{\partial G}{\partial x_1}(\boldsymbol{y}, \boldsymbol{x}) \quad G^{(2)} = \frac{\partial G}{\partial x_2}(\boldsymbol{y}, \boldsymbol{x}) \tag{2.359}$$

represent unit dislocations in x_1 and x_2 direction respectively and the σ_i are limit values of the integrals

$$\lim_{\varepsilon \to 0} \int_{\Gamma_{N_\varepsilon(\boldsymbol{x})}} \boldsymbol{\sigma} \cdot \boldsymbol{n}\, G^{(i)}\, ds = \sigma_i(\boldsymbol{x}) \tag{2.360}$$

and so

$$\sigma_i(\boldsymbol{x}) = \int_{\Omega} (G^{(i)}(\boldsymbol{y}, \boldsymbol{x})\, p(\boldsymbol{y}) + \boldsymbol{\sigma}_{G^{(i)}}(\boldsymbol{y}, \boldsymbol{x}) \cdot \boldsymbol{\sigma}_0(\boldsymbol{y}))\, d\Omega_{\boldsymbol{y}} \tag{2.361}$$

where

$$\boldsymbol{\sigma}_{G^{(i)}} = \nabla G^{(i)}\,. \tag{2.362}$$

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