Chapter 2
Social Welfare Functions and Income Distributions

2.1 Introduction

Old welfare economics, Pigou (1920), considered social welfare as a cardinal notion, while new welfare economics,[1] Little (1950) and Graaff (1957), consider social welfare as an ordinal notion. An in depth introduction to welfare economics and a discussion of the transition from old to new welfare economics, is expounded quite well by Samuelson (1947, Chap. 8, pp. 203–219, and pp. 249–252). For the purpose of our discourse, the ordinal notion will be adopted. To be clear, when speaking of individual welfare we refer to the utility associated to a single consumer or economic agent; when considering social welfare, or simply welfare, the notion is associated to society as a whole.

It is well known that, given a set of alternatives, a binary choice relation, or preference relation, endowed with some mathematical properties, can be selected as the basis to define a welfare function,[2] since, from a binary choice relation, generally, one can build[3] a welfare function which represents that binary choice relation. But, at least when the set of alternatives is very large, working with a welfare function is mathematically more operative than working with a preference relation. Hence, in this monograph we shall start directly by considering welfare functions.

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[1]At the end of nineteenth century, Pareto was the first scholar to consider utility (he called it "ophelimity") as an ordinal notion, then Robbins (1932, 1935) cogently argued the ordinal character of utility. Robbins expressly wrote: *There is no means of testing the magnitude of A's satisfaction as compared with B's.* (1935, pp. 139–140).

[2]See, for instance, Arrow et al. (1960).

[3]It is a well known fact that there are an infinite number of ways to generate a social welfare function from a given social preference relation.

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2.2 Social Welfare Functions

From an individual's point of view, it is common to assume that the utility enjoyed by a consumer is a function of the quantities of the various goods which he/she consumes in a specific time interval. Given all the various prices, these quantities are a function (usually increasing) of income, so that (indirect) utility becomes an increasing function of income.

It is important to underline that in this monograph no distinction is made between real and nominal income because money is never introduced hence prices play no role. But in making, for instance, international comparisons in living standards across countries, prices assume a very important role. Deaton (2010) provides a recent contribution on these international comparisons.

Since the plurality of goods is not considered, and individual income distribution[4] is dealt with directly, one can safely conclude that individual welfare is an increasing function of income, and thus, social welfare is an increasing function of each individual income.

Given a society composed of m *economic agents*, identified as families, or more generically as consumers, let y_j denote economic agent j's positive income, and $y = (y_1, y_2, \ldots, y_m) \in \Re^m_{++}$ *income distribution*. A *social welfare function*, or simply *welfare function*, denoted by W, is a function $W : \Re^m_{++} \to \Re$, namely $y \mapsto W(y)$, which is continuous and strictly increasing in every y_j.[5] According to the assumption that W is ordinal, any continuous and strictly increasing transformation of W is again a social welfare function, ordering income distributions in the same way.[6]

2.3 Symmetrical Social Welfare Functions

From a social point of view, one of the purposes of any economic policy in a democracy ought to be, as far as possible, neutrality among individuals. Thus, the social welfare function, W, must be symmetrical among individuals, namely: any permutation of the y_js must not change the value of $W(y_1, y_2, \ldots, y_m)$. This means that for every permutation matrix, P, and for every distribution of income, y, we

[4]Working directly with individual incomes bypasses a number of serious problems that occur when considering production. For instance the indivisibility of some goods, increasing returns to scale in production, externalities, non-convexities, … . It is obvious that general welfare economics must be concerned with such issues.

[5]On social welfare functions and the axioms which they are required to satisfy, see D'Aspremont and Gevers (2002). Also remember that there are some economists who try to show the impossibility of criteria for aggregating preferences; see Gul and Pesendorfer (2007).

[6]As usual in many branches of economic theory, the problem exists of the invulnerability to manipulation of an economic agent or coalition of agents when a social welfare function generates a social choice rule. On this see Campbell and Kelly (2006).

must have $W(yP) = W(y)$. Moreover, as already stated, W must be continuous and strictly increasing in every argument. These assumptions are satisfied, for instance, by this function[7]:

$$(y_1, y_2, \ldots, y_m) \mapsto W(y_1, y_2, \ldots, y_m) = \prod_{j=1}^{m} y_j. \tag{2.1}$$

Any increasing and continuous transformation of W must of course be permissible, due to the fact that the social welfare function has an ordinal character. In applications, for instance, instead of (2.1) it is easier to work with the equivalent form $W^*(y) = \log W(y) = \sum_j \log y_j$. We shall call function (2.1) *uniformly egalitarian welfare function* (UEW), to distinguish it from welfare functions such as (2.3). It is plain that the symmetry of function (2.1) automatically satisfies the criteria of fairness and reciprocity mentioned in Chap. 1, Sect. 1.2, despite the fact that nobody takes any explicit action in favour of fairness and reciprocity.

A function like (2.1), treating everybody on a par, can be considered as ethically sound.[8] Of course, every function which does not change in value when its independent variables are subjected to permutations, respects an ethical principle, at least from the (narrow) economic stance. Functions (2.4) and (2.5) below are also ethically sound.

It is common to assume that individual welfare increases very considerably when individual income increases if the initial starting point was very low, while increments in high incomes only mildly increase individual welfare. To account for this, while preserving the egalitarian character of W, one can choose value, θ, satisfying $0 < \theta < 1$, and consider the following social welfare function:

$$(y_1, y_2, \ldots, y_m) \mapsto W_\theta(y_1, y_2, \ldots, y_m) = \prod_{j=1}^{m} y_j^{\theta}. \tag{2.2}$$

It is obvious that, as far as the ordering of income, functions W and W_θ are equivalent; but, when considering the policy implications of income redistribution, the choice of the value given to θ is relevant in evaluating different degrees of inequality. Indeed, for $0 < \theta < 1$, decrements in very low values of y_js sensibly reduce W_θ, while W_θ increases only mildly with greater values of y_js.[9]

The social welfare function W_θ in (2.2) can then be generalized as

$$(y_1, y_2, \ldots, y_m) \mapsto W_d(y_1, y_2, \ldots, y_m) = \prod_{j=1}^{m} y_j^{\theta_j}, \tag{2.3}$$

[7] See, for instance, Nash (1950).

[8] We can also qualify (2.1) as being neutral, but this adjective is too reductive.

[9] In some sense, this undermines the ordinal character of W.

for different positive values, θ_j s; but, in general, we shall limit ourselves to functions (2.1) and (2.2) as possible social welfare functions. Indeed, as pointed out by Sen (1970, p. 131), only (2.1) and (2.2) satisfy the Golden Rule stated in the Bible:

> Do unto others as ye would that others do unto you.
>
> (*Matthew 7,12*).

In my opinion, this sentence can be considered the greatest statement on fairness and reciprocity, at least in reference to people of good will.

Another egalitarian form of W is the function proposed by Atkinson (1970). With the notations previously introduced, this function can be written as

$$W_A(y) = \left(\sum_{j=1}^{m} \frac{1}{m} y_j^{\epsilon} \right)^{1/\epsilon} \qquad (\epsilon < 1,\ \epsilon \neq 0), \tag{2.4}$$

$$W_A^0(y) = \prod_{j=1}^{m} y_j^{1/m} \qquad (\epsilon = 0).$$

It is evident that Atkinson's welfare function is also symmetrical among economic agents; (2.4) amounts to reducing high income contributions to social welfare, as does (2.2). Note that $W_A^0(y)$ is the same as (2.2), with $\theta = 1/m$.

A third possible social welfare function is the one proposed by Kolm–Pollak[10]; it is expressed by the formula

$$W_{KP} = -\frac{1}{\delta} \log \left(\frac{1}{m} \sum_{j=1}^{m} \exp(-\delta y_j) \right), \tag{2.5}$$

for $\delta > 0$, which is again symmetrical among economic agents.

Both functions (2.4) and (2.5) satisfy the Golden Rule previously stated. Given total income, $\bar{Y}$, to be distributed, social welfare is maximized when $\bar{Y}$ is evenly distributed among all people.

2.4 Income Distribution and Social Welfare

With these premises, let's consider a community, or society, or economy, composed of population m, indexed by $j = 1, 2, \ldots, m$, with the initial incomes denoted by positive values $\bar{y}_j$ $(j = 1, 2, \ldots, m)$. We are not interested, here, in studying the functional distribution of incomes, i.e. how incomes are obtained by individuals

[10]See Pollak (1971).

as wages, profits and rents, in payment for the inputs supplied to obtain social production. But in strictly economic terms, it is an important task to study the link between the functional and the personal distribution of total income. Consider Dagum's (1999) very interesting analysis.

Let's assume the existence of a *Public Authority*[11] whose task is to redistribute income, $\bar{y}_j$s, by means of taxes and subsidies, among m citizens.[12] It is reasonable to suppose that the action of the P.A., in choosing taxes and subsidies, aims to redistribute income to maximize a social welfare function $W : \Re^m_{++} \rightarrow \Re$, under the following constraint

$$\sum_{j=1}^{m} y_j = \sum_{j=1}^{m} \bar{y}_j = \bar{Y}. \tag{2.6}$$

If we consider the social welfare function given by formula (2.1), it is easy to show that its maximum value, under constraint (2.6), is given by the even distribution of $y_j = \bar{Y}/m$ $(j = 1, 2, \ldots, m)$.[13] Of course, this result does not take into consideration the peculiarities of the initial point of income distribution.

2.5 Optimal, Equivalent and Even Income Distributions

Let's consider income distribution, $\bar{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m) \in \Re^m_{++}$, with the mean value $\mu(\bar{y})$, and a continuous and strictly increasing social welfare function, $W : \Re^m_{++} \rightarrow \Re$, that is not necessarily symmetrical. *Optimal income distribution* is, therefore, income distribution, $\hat{y} \in \Re^m_{++}$, that resolves:

$$\max W(y) \texttt{ under } \sum_{j=1}^{m} y_j = m\mu(\bar{y}),$$

where $m\mu(\bar{y}) = \sum_j \bar{y}_j$. Namely, optimal income distribution gives to each citizen an income which maximizes social welfare under the constraint that total income is preserved. Due to the continuity of W, and to the compactness of the constraint, there is at least one solution to this problem.

Denoting the sum vector by s, i.e. $s = (1, 1, \ldots, 1)$, an *equivalent income distribution* is income distribution, $y^* = \alpha s = (\alpha, \alpha, \ldots, \alpha) \in \Re^m_{++}$, which gives the same income, α, to everybody, while preserving the same initial value of social

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[11] For instance, a government democratically elected by all citizens.

[12] On problems of progressive income taxation and their redistributional effects see Lambert (1993, Chap. 6) and Lambert (1999). An early paper on the mathematical theory of optimum income taxation is Mirrlees (1971), who proposes decreasing marginal tax rates. A different viewpoint in favour of increasing marginal tax rates is presented by Diamond (1998).

[13] The same result is obtained also for social welfare functions expressed by the Atkinson formulae (2.4) and by (2.5).

welfare, namely:

$$W(\alpha s) = W(\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m).$$

Again due to the continuity of W, this problem has of course a unique solution, because W is assumed to be strictly increasing in its arguments. Generally, for an uneven initial income distribution, $\bar{y}$, and a symmetrical social welfare function, we have $W(\hat{y}) \geq W(\alpha s)$; indeed we have $m\alpha \leq m\mu(\bar{y})$.

Given W, the value α is a continuous function of $\bar{y}$; hence it is possible to write

$$\alpha = f_W(\bar{y}). \tag{2.7}$$

It is important to note that f_W is insensitive to increasing transformations of W; hence, the values taken by this function are meaningful in comparisons between income distributions. Function f_W is continuous and increasing with respect to $\mu(\bar{y})$, according to the properties stated for W. This implies that f_W and W are monotonic transformations of each other, namely: for every pair y and $y' \in \Re^m_{++}$ it is true that

$$f_W(y) \geq f_W(y') \texttt{ is equivalent to } W(y) \geq W(y'), \tag{2.8}$$

a property which can be useful in some applications.

Optimal income distribution generally is not unique, while equivalent income distribution is unique. Optimal income distribution allocates the whole of disposable income, $\bar{Y}$, to economic agents, while equivalent income distribution generally does not. Thus, we can eventually increase the value of α to α^*, to exhaust total disposable income. Let's call α^* *even income distribution*. Of course, we always have:

$$W(\alpha s) \leq W(\alpha^* s) \leq W(\hat{y}).$$

The notion of equivalent income distribution however, can be very misleading as a redistributive criterion. Assume $m = 3$, $\bar{y} = (\varepsilon, \varepsilon, 1000 - 2\varepsilon)$, and W given by (2.1); then we have $W(\bar{y}) = (1000 - 2\varepsilon)\varepsilon^2$, which is a very "low" value when ε takes a low value. Correspondingly, equivalent income distribution is determined by $\alpha = \sqrt[3]{(1000 - 2\varepsilon)\varepsilon^2}$, while even income distribution is expressed by $\alpha^* = 1000/3$, from which we obtain $W(\alpha^* s) = 1000^3/27$. Of course, the values taken by W are not significant per se, but the corresponding differences among income distributions look meaningful.

2.6 Social Welfare and Status Quo

While aiming to improve the distribution of income by means of transfers (taxes and subsidies), the P.A. cannot dismiss the status quo, namely the initial point of income distribution, $\bar{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m)$. Thus, it seems sensible to introduce an *equity parameter*, q, chosen by the P.A. where $0 < q < 1$, which is not to be exceeded for reasons of distributive equity and social peace. Namely, the after transfer income, y_js, must satisfy inequalities

$$y_j \geq (1-q)\bar{y}_j \qquad (j = 1, 2, \ldots, m). \tag{2.9}$$

In a sense, these inequalities help preserve the initial order of income after redistribution.

Given W, $\bar{y}$, and q, the problem is maximizing W under constraints (2.9) and

$$\sum_{j=1}^{m} y_j = \bar{Y}. \tag{2.10}$$

The Lagrangian L for this problem, denoted by ν_js for the non negative multipliers associated to (2.9), and λ associated to constraint (2.10), is

$$L(y, \lambda, \nu_1, \nu_2, \ldots, \nu_m) = W(y) + \sum_{j=1}^{m} \nu_j[y_j - (1-q)\bar{y}_j] + \lambda\left(\bar{Y} - \sum_{j=1}^{m} y_j\right).$$

From L we obtain the following necessary condition for an interior constrained maximum of social welfare

$$\frac{\partial L}{\partial y_j} = \frac{\partial W}{\partial y_j} + \nu_j - \lambda = 0 \qquad (j = 1, 2, \ldots, m), \tag{2.11}$$

$$\frac{\partial L}{\partial \nu_j} = y_j - (1-q)\bar{y}_j \geq 0, \quad \nu_j[y_j - (1-q)\bar{y}_j] = 0 \quad (j = 1, 2, \ldots, m), \tag{2.12}$$

$$\frac{\partial L}{\partial \lambda} = \bar{Y} - \sum_{j=1}^{m} y_j = 0. \tag{2.13}$$

While (2.11) and (2.13) must hold as equalities, some, but not all, of relations (2.9) can be strict inequalities.

If W has egalitarian form (2.1), and, to simplify calculations, we take $W^* = \log W$ instead of W, then (2.11) becomes

$$\frac{1}{y_j} + \nu_j - \lambda = 0 \qquad (j = 1, 2, \ldots, m). \tag{2.14}$$

From this we obtain

$$y_j = \frac{1}{\lambda - \nu_j} \qquad (j = 1, 2, \ldots .m). \tag{2.15}$$

Clearly, the values of the multipliers, ν_js, must verify $\lambda - \nu_j > 0$ for all js, because after all the transfers, individual incomes must be positive.

To obtain the values of the $m+1$ multipliers, we must insert (2.15) into constraints (2.9) and (2.10), keeping in mind that some relations in (2.9) may be strict inequalities. Thus, let $M = \{1, 2, \ldots, m\}$ and

$$M_1 = \left\{ j : \frac{1}{\lambda - \nu_j} > (1-q)\bar{y}_j \right\}, \qquad M_2 = \left\{ j : \frac{1}{\lambda - \nu_j} = (1-q)\bar{y}_j \right\},$$

with $M_1 \cup M_2 = M$. It is plain that we have $\nu_j = 0$ for every $j \in M_1$, and $\nu_j = \lambda - \frac{1}{(1-q)\bar{y}_j}$ for every $j \in M_2$. Letting $|M_1| = m_1$ and $|M_2| = m_2$ denoting the number of elements in sets M_1 and M_2 from relations (2.10) and (2.15) we obtain:

$$\lambda = \frac{m_1}{\bar{Y} - (1-q)\sum_{j \in M_2} \bar{y}_j} > 0; \tag{2.16}$$

thus, we have:

$$y_j = \frac{\bar{Y} - (1-q)\sum_{j \in M_2} \bar{y}_j}{m_1} \qquad (j \in M_1), \tag{2.17}$$

and

$$y_j = (1-q)\bar{y}_j \qquad (j \in M_2). \tag{2.18}$$

Given the egalitarian nature of the chosen social welfare function W, all poor people, namely those belonging to set M_1, are treated equally, while all rich economic agents, those in set M_2, receive the same fraction, $1-q$, of their initial income.

The ideas here presented will be systematically explored and applied in the models proposed in Part II of this study.

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