

Chapter 2

Stability Loss Problems Related to Solid and Hollow Circular Cylinders

In this chapter the Three-Dimensional Linearized Theory of Stability (TDLTS) is developed and applied to the investigation of global and local stability loss of circular solid and hollow cylinders fabricated from viscoelastic composite materials. The initial imperfection criterion is used for determination of the values of the critical parameters. The study is carried out within the scope of the 3D geometrically non-linear field equations of the theory for viscoelastic orthotropic bodies. The approach is developed and applied for a solution to the corresponding system of non-linear integro-differential equations. By the average-integrating procedure the corresponding approximate global (local) stability loss equations within the scope of the Bernoulli–Euler and the third order refined beam (of the Kirchhoff–Love and the third order shell) theories are derived from the corresponding TDLTS equations. The numerical results related to the critical forces and to the critical times which are obtained within the scope of the TDLTS and approximate approaches are presented and discussed.

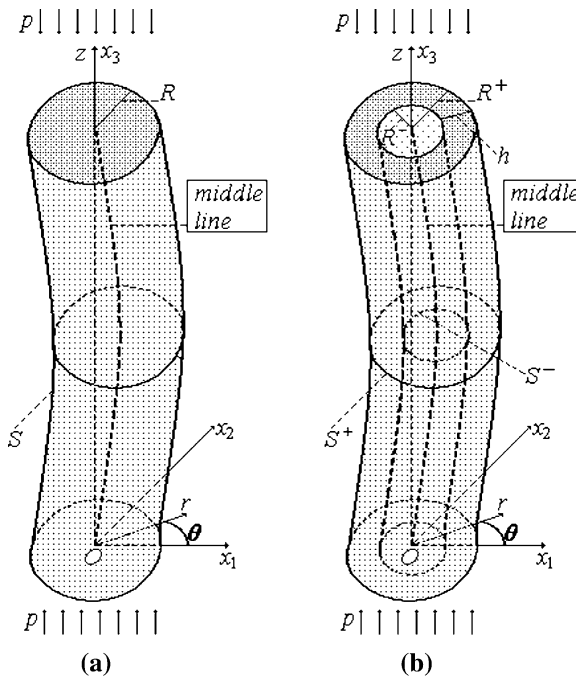
The results detailed in the present chapter are based on the investigations carried out in papers (Akbarov and Karakaya 2011, 2012a, b)

2.1 Formulation of the Problem Related to the Global Stability Loss

We consider a circular cylinder which has an initial global imperfection in the natural state and determine the position of the points of this cylinder with the Lagrange coordinates in the cylindrical $Or\theta z$ and the Cartesian system of coordinates $Ox_1x_2x_3$ (Fig. 2.1). We differ two cases: in Case 1 (Fig. 2.1a), in Case 2 (Fig. 2.1b) we assume that the cylinder is solid (hollow) one.

The noted initial global imperfection is given through the following equation of the “middle line” of the cylinders

Fig. 2.1 The geometry of the considered cylinders: **a** solid cylinder, **b** hollow cylinder



$$x_3 = t_3; \quad x_1 = L \sin\left(\frac{\pi}{\ell} t_3\right); \quad x_2 = 0 \quad (2.1)$$

where t_3 is a parameter and $t_3 \in (0, \ell)$, L is the amplitude of the initial imperfection form.

In Case 2 (in Case 1) we assume that the cylinders' cross section is on the plane which is perpendicular to its middle line tangent vector, is a closed circular ring of the constant inner radius R^- and outer radius R^+ (a circle of the constant radius R). Moreover, we assume that $L \ll l$ and introduce the small parameter

$$\varepsilon = \frac{L}{\ell}, \quad 0 \leq \varepsilon \ll 1. \quad (2.2)$$

We suppose that the material of the cylinders is viscoelastic transversal isotropic the symmetry axis of which coincides with the Ox_3 (Oz) axis. Within the foregoing assumptions, we investigate the evolution of the infinitesimal initial global imperfection of the cylinder with time in the case where the cylinder is loaded by uniformly distributed normal compressing forces with intensity p acting on the ends of the cylinder in the direction of the and Oz axis.

This investigation is made within the scope of the following field equations.

$$\begin{aligned}
\frac{\partial t_{rr}}{\partial r} + \frac{\partial t_{\theta r}}{r \partial \theta} + \frac{\partial t_{zr}}{\partial z} + \frac{1}{r}(t_{rr} - t_{\theta\theta}) &= 0, \\
\frac{\partial t_{r\theta}}{\partial r} + \frac{\partial t_{\theta\theta}}{r \partial \theta} + \frac{1}{r}(t_{\theta r} + t_{r\theta}) + \frac{\partial t_{z\theta}}{\partial z} &= 0, \\
\frac{\partial t_{rz}}{\partial r} + \frac{\partial t_{\theta z}}{r \partial \theta} + \frac{1}{r}t_{rz} + \frac{\partial t_{zz}}{\partial z} &= 0
\end{aligned} \tag{2.3a}$$

$$\begin{aligned}
t_{rr} &= \sigma_{rr} \left(1 + \frac{\partial u_r}{\partial r} \right) + \sigma_{r\theta} \left(\frac{\partial u_r}{r \partial \theta} - \frac{u_\theta}{r} \right) + \sigma_{rz} \frac{\partial u_r}{\partial z}, \\
t_{r\theta} &= \sigma_{rr} \frac{\partial u_\theta}{\partial r} + \sigma_{r\theta} \left(1 + \frac{\partial u_\theta}{r \partial \theta} + \frac{u_r}{r} \right) + \sigma_{rz} \frac{\partial u_\theta}{\partial z}, \\
t_{rz} &= \sigma_{rr} \frac{\partial u_z}{\partial r} + \sigma_{r\theta} \frac{\partial u_z}{r \partial \theta} + \sigma_{rz} \left(1 + \frac{\partial u_z}{\partial z} \right), \\
t_{\theta r} &= \sigma_{\theta r} \left(1 + \frac{\partial u_r}{\partial r} \right) + \sigma_{\theta\theta} \left(\frac{\partial u_r}{r \partial \theta} - \frac{u_\theta}{r} \right) + \sigma_{\theta z} \frac{\partial u_r}{\partial z}, \\
t_{\theta\theta} &= \sigma_{\theta r} \frac{\partial u_\theta}{\partial r} + \sigma_{\theta\theta} \left(1 + \frac{\partial u_\theta}{r \partial \theta} + \frac{u_r}{r} \right) + \sigma_{\theta z} \frac{\partial u_\theta}{\partial z}, \\
t_{\theta z} &= \sigma_{\theta r} \frac{\partial u_z}{\partial r} + \sigma_{\theta\theta} \frac{\partial u_z}{r \partial \theta} + \sigma_{\theta z} \left(1 + \frac{\partial u_z}{\partial z} \right), \\
t_{zr} &= \sigma_{zr} \left(1 + \frac{\partial u_r}{\partial r} \right) + \sigma_{z\theta} \left(\frac{\partial u_r}{r \partial \theta} - \frac{u_\theta}{r} \right) + \sigma_{zz} \frac{\partial u_r}{\partial z}, \\
t_{z\theta} &= \sigma_{zr} \frac{\partial u_\theta}{\partial r} + \sigma_{z\theta} \left(1 + \frac{\partial u_\theta}{r \partial \theta} + \frac{u_r}{r} \right) + \sigma_{zz} \frac{\partial u_\theta}{\partial z}, \\
t_{zz} &= \sigma_{zr} \frac{\partial u_z}{\partial r} + \sigma_{z\theta} \frac{\partial u_z}{r \partial \theta} + \sigma_{zz} \left(1 + \frac{\partial u_z}{\partial z} \right)
\end{aligned} \tag{2.3b}$$

$$\begin{aligned}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r} + \frac{1}{2} \left\{ \left(\frac{\partial u_r}{\partial r} \right)^2 + \left(\frac{\partial u_\theta}{\partial r} \right)^2 + \left(\frac{\partial u_z}{\partial r} \right)^2 \right\}, \\
\varepsilon_{r\theta} &= \frac{1}{2} \left(\frac{\partial u_\theta}{\partial r} + \frac{\partial u_r}{r \partial \theta} - \frac{u_\theta}{r} \right) + \frac{1}{2} \left\{ \frac{\partial u_r}{\partial r} \left(\frac{\partial u_r}{r \partial \theta} - \frac{u_\theta}{r} \right) + \frac{\partial u_\theta}{\partial r} \left(\frac{\partial u_\theta}{r \partial \theta} + \frac{u_r}{r} \right) + \frac{\partial u_z}{\partial r} \frac{\partial u_z}{r \partial \theta} \right\} \\
\varepsilon_{rz} &= \frac{1}{2} \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) + \frac{1}{2} \left\{ \frac{\partial u_r}{\partial r} \frac{\partial u_r}{\partial z} + \frac{\partial u_\theta}{\partial r} \frac{\partial u_\theta}{\partial z} + \frac{\partial u_z}{\partial r} \frac{\partial u_z}{\partial z} \right\} \\
\varepsilon_{\theta\theta} &= \frac{\partial u_\theta}{r \partial \theta} + \frac{u_r}{r} + \frac{1}{2} \left\{ \left(\frac{\partial u_r}{r \partial \theta} - \frac{u_\theta}{r} \right)^2 + \left(\frac{\partial u_\theta}{r \partial \theta} + \frac{u_r}{r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial u_z}{\partial \theta} \right)^2 \right\} \\
\varepsilon_{\theta z} &= \frac{1}{2} \left(\frac{\partial u_z}{r \partial \theta} + \frac{\partial u_\theta}{\partial z} \right) + \frac{1}{2} \left\{ \frac{\partial u_r}{\partial z} \left(\frac{\partial u_r}{r \partial \theta} - \frac{u_\theta}{r} \right) + \frac{\partial u_\theta}{\partial z} \left(\frac{\partial u_\theta}{r \partial \theta} + \frac{u_r}{r} \right) + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \frac{\partial u_z}{\partial z} \right\}, \\
\varepsilon_{zz} &= \frac{\partial u_z}{\partial z} + \frac{1}{2} \left\{ \left(\frac{\partial u_r}{\partial z} \right)^2 + \left(\frac{\partial u_\theta}{\partial z} \right)^2 + \left(\frac{\partial u_z}{\partial z} \right)^2 \right\}
\end{aligned} \tag{2.3c}$$

Here the Eq. (2.3a) is equilibrium equation in terms of the physical components t_{rr} , $t_{\theta\theta}$, t_{zz} , $t_{r\theta}$, $t_{\theta r}$, $t_{z\theta}$, $t_{\theta z}$, t_{zr} and t_{rz} of the non-symmetric Kirchhoff stress tensor, the Eq. (2.3b) shows the relation between the physical components of non-symmetric Kirchhoff stress tensor and the physical components of the ordinary stress tensor in the cylindrical system of coordinates, the non-linear relation between the physical components ε_{rr} , $\varepsilon_{\theta\theta}$, ε_{zz} , $\varepsilon_{r\theta}$, $\varepsilon_{\theta r}$, $\varepsilon_{z\theta}$, $\varepsilon_{\theta z}$, ε_{zr} and ε_{rz} of the Green strain tensor and the physical components u_r , u_θ and u_z of the displacement vector is illustrated by the Eq. (2.3c).

The constitutive relations for the cylinders material in the cylindrical system of coordinates are given as follows:

$$\begin{aligned}\sigma_{rr} &= A_{11}^* \varepsilon_{rr} + A_{12}^* \varepsilon_{\theta\theta} + A_{13}^* \varepsilon_{zz}; \quad \sigma_{\theta\theta} = A_{12}^* \varepsilon_{rr} + A_{11}^* \varepsilon_{\theta\theta} + A_{13}^* \varepsilon_{zz}; \\ \sigma_{zz} &= A_{13}^* \varepsilon_{rr} + A_{13}^* \varepsilon_{\theta\theta} + A_{13}^* \varepsilon_{zz}; \quad \sigma_{r\theta} = (A_{11}^* - A_{22}^*) \varepsilon_{r\theta}; \\ \sigma_{rz} &= 2G^* \varepsilon_{rz}; \quad \sigma_{\theta z} = 2G^* \varepsilon_{\theta z},\end{aligned}\quad (2.4)$$

where A_{ij}^* and G^* are following operators

$$\left\{ \begin{matrix} A_{ij}^* \\ G^* \end{matrix} \right\} \phi(t) = \left\{ \begin{matrix} A_{ij0} \\ G_0 \end{matrix} \right\} \phi(t) + \int_0^t \left\{ \begin{matrix} A_{ij1}(t-\tau) \\ G_1(t-\tau) \end{matrix} \right\} \phi(\tau) d\tau \quad (2.5)$$

Here, A_{ij0} and G_0 are the instantaneous values of elastic constants, $A_{ij1}(t)$ and $G_1(t)$ are the given functions which determine the hereditary properties of the cylinder material.

In the present section, all further discussions and considerations we will make for the hollow cylinder from which by obvious changing the corresponding ones can be obtained for the solid cylinder.

Assume that on the lateral surfaces $S^\pm$ of the hollow cylinder the following conditions are satisfied

$$\begin{aligned}t_{rr}|_{S^\pm} n_r^\pm + t_{r\theta}|_{S^\pm} n_\theta^\pm + t_{rz}|_{S^\pm} n_z^\pm &= 0, \quad t_{r\theta}|_{S^\pm} n_r^\pm + t_{\theta\theta}|_{S^\pm} n_\theta^\pm + t_{z\theta}|_{S^\pm} n_z^\pm = 0, \\ t_{rz}|_{S^\pm} n_r^\pm + t_{\theta z}|_{S^\pm} n_\theta^\pm + t_{zz}|_{S^\pm} n_z^\pm &= 0\end{aligned}\quad (2.6)$$

where n_r^+ , n_θ^+ and n_z^+ (n_r^- , n_θ^- and n_z^-) are the components of an unit normal vector to the outer (inner) lateral surface S^+ (S^-) of the hollow cylinder.

In the natural state, the upper and lower ends of the cylinder are on the inclined planes with the following unit normal vectors

$$\begin{aligned}\mathbf{n}_0 &= \frac{-\mathbf{k} - \varepsilon\pi\mathbf{i}}{\sqrt{1 + \varepsilon^2\pi^2}} \text{ (for the lower end plane);} \\ \mathbf{n}_\ell &= \frac{\mathbf{k} - \varepsilon\pi\mathbf{i}}{\sqrt{1 + \varepsilon^2\pi^2}} \text{ (for the upper end plane),}\end{aligned}\quad (2.7)$$

where $\mathbf{n}_0$ ($\mathbf{n}_\ell$) is an unit normal vector to the end plane of the cylinder at $z = 0$ ($z = \ell$), $\mathbf{i}$ and $\mathbf{k}$ ort-unit vectors in the Ox_3 and Ox_1 axes directions respectively.

Denote the upper (lower) end cross section of the cylinder through S_ℓ (S_0) and the conditions for the forces on these end cross sections we write as follows:

$$\begin{aligned} t_{rz}|_{S_0} n_{01} + t_{\theta z}|_{S_0} n_{02} + t_{zz}|_{S_0} n_{03} &= p, \\ t_{rz}|_{S_\ell} n_{\ell 1} + t_{\theta z}|_{S_\ell} n_{\ell 2} + t_{zz}|_{S_\ell} n_{\ell 3} &= -p, \end{aligned} \quad (2.8)$$

where n_{0j} ($n_{\ell j}$) is a component of the unit normal vector defined in (2.7). The end conditions for the displacements will be discussed below.

This completes the formulation of the considered problem. It follows from this formulation that the evolution of the infinitesimal initial global imperfection of the cylinder with time for the fixed value of the compressing force p (for the case where the material of the cylinder is viscoelastic) or with initial compressing force p (for the case where the material of the cylinder is pure elastic) will be investigated within the framework of the field equations (2.3), (2.4) and (2.5) and boundary condition (2.6) and (2.8).

2.2 Method of Solution for the Global Stability Loss Problem

Now we consider the method of solution of the problem formulated in the previous section. Note that the method employed below can be briefly summarized as follows:

- i. By employing the boundary-form perturbation techniques the considered boundary value problem for the non-linear integro-differential Eqs. (2.3)–(2.5) is reduced to the series boundary-value problems for the corresponding system of the linear integro-differential equations;
- ii. Owing to both the expressions of the operators (2.5) and the convolution theorem, by using the Laplace transformation with respect to time these series problems for the system of the linear integro-differential equations are reduced to the corresponding series boundary value problems for the linear system of differential equations in the Laplace transformation parameter space;
- iii. For each fixed value of the Laplace transformation parameter the linear problems are solved by employing the variable-separation method;
- iv. By applying the inverse transformation method (Schapery 1962, 1978) we determine the original of the sought values.

It should be noted that for the case where the material of the cylinder is pure elastic, the operators (2.5) are replaced by mechanical constants and therefore instead of the integro-differential equations we obtain differential equations and the corresponding problems for these equations are also investigated in the framework of the above procedure but without employing the Laplace transformation.

Since, according to the procedure summarized above and to the problem statement, first we derive the equation for the inner and outer lateral surfaces S^+ and S^- of the cylinder. According to the condition of the cylinder's cross section we can conclude that the coordinates of these surfaces must simultaneously satisfy the following equations.

$$\begin{aligned} \varepsilon f'(t_3)(x_{10}^\pm - \varepsilon f(t_3)) + x_{30}^\pm - t_3 &= 0, \\ (x_{20}^\pm)^2 + (x_{30}^\pm - t_3)^2 + (x_{10}^\pm - \varepsilon f(t_3))^2 &= (R^\pm)^2, \end{aligned} \quad (2.9)$$

where $f(t_3) = \ell \sin(\pi t_3 / \ell)$, $f'(t_3) = \pi \cos(\pi t_3 / \ell)$; $x_{10}^\pm$, $x_{20}^\pm$, $x_{30}^\pm$ are coordinates of the surface $S^\pm$.

Note that the first equation in (2.9) is an equation of the plane perpendicular to the vector which is the tangent vector to the middle line of the cylinder at the point that corresponds to the fixed value of the parameter t_3 ; but the second equation in (2.9) is an equation of the outer and inner circles with radiuses R^+ and R^- in turn which are counter to the cross section of the hollow cylinder which rises on the foregoing plane. It is also assumed that the thickness $h = R^+ - R^-$ is constant along the cylinder (Fig. 2.1).

Using the relations $x_{10}^\pm = r^\pm \cos \theta$ and $x_{20}^\pm = r^\pm \sin \theta$ we obtain the following equation for the surface $S^\pm$ in the cylindrical system of coordinates $Or\theta z$:

$$\begin{aligned} r^\pm &= r^\pm(\theta, t_3, \varepsilon) = \varepsilon f(t_3) \cos \theta \frac{1 + \varepsilon^2 (f'(t_3))^2}{1 + \varepsilon^2 (f'(t_3))^2 \cos^2 \theta} \\ &+ \left\{ \frac{(R^\pm)^2 - \varepsilon^2 (f(t_3))^2 (1 + \varepsilon^2 (f'(t_3))^2)}{1 + \varepsilon^2 (f'(t_3))^2 \cos^2 \theta} \right. \\ &\quad \left. + \varepsilon^2 (f(t_3))^2 \cos^2 \theta \frac{(1 + \varepsilon^2 (f'(t_3))^2)}{(1 + \varepsilon^2 (f'(t_3))^2 \cos^2 \theta)^2} \right\}^{\frac{1}{2}} \\ z^\pm &= t_3 - \varepsilon f'(t_3)(r^\pm(\theta, t_3, \varepsilon) - \varepsilon f(t_3)), \quad f'(t_3) = \frac{df(t_3)}{dt_3}. \end{aligned} \quad (2.10)$$

Using the assumption (2.2) and supposing that $(\varepsilon f'(t_3))^2 \ll 1$, after some mathematical manipulations, we obtain the following equations.

$$\begin{aligned} r^\pm &= R^\pm + \varepsilon f(t_3) \cos \theta + O(\varepsilon^2), \quad z^\pm = t_3 - \varepsilon R^\pm f'(t_3) \cos \theta + O(\varepsilon^2), \\ n_r^\pm &= \pm \left(1 - \varepsilon^2 \left(R^\pm f''(t_3) - \frac{f(t_3)}{R^\pm} \right)^2 + O(\varepsilon^2) \right), \\ n_\theta^\pm &= \pm \left(\varepsilon \frac{f(t_3)}{R^\pm} \sin \theta + O(\varepsilon^2) \right), \\ n_z^\pm &= \pm (-\varepsilon f'(t_3) \cos \theta + O(\varepsilon^2)) \end{aligned} \quad (2.11)$$

where $n_r^\pm$, $n_\theta^\pm$, $n_z^\pm$ are physical components of the unit normal vector to the surface $S^\pm$.

Now we write the equation of the planes on which lay the lower and upper inclined ends of the cylinder as follows.

$$\begin{aligned} x_3 &= -\varepsilon\pi x_1 \text{ (for the lower end),} \\ x_3 &= \varepsilon\pi x_1 + \ell \text{ (for the upper end).} \end{aligned} \quad (2.12)$$

According to the Eq. (2.7), we can also present the expression of the components of the normal vectors to these ends as follows

$$\begin{aligned} n_{01} &= n_{\ell 1} = -\varepsilon\pi \left(1 - \frac{1}{2}(\varepsilon\pi)^2 + O(\varepsilon\pi)^4 \right), \quad n_{03} = -\left(1 - \frac{1}{2}(\varepsilon\pi)^2 + O(\varepsilon\pi)^4 \right), \\ n_{\ell 3} &= \left(1 - \frac{1}{2}(\varepsilon\pi)^2 + O(\varepsilon\pi)^4 \right), \quad n_{02} = n_{\ell 2} = 0. \end{aligned} \quad (2.13)$$

According to the procedures of the boundary perturbation technique, we attempt to solve the considered problem by employing the boundary form perturbation method. For this purpose the unknowns are presented in series form in ε (2.2).

$$\{\sigma_{(ij)}; \varepsilon_{(ij)}; u_{(i)}\} = \sum_{q=0}^{\infty} \varepsilon^q \left\{ \sigma_{(ij)}^{(q)}; \varepsilon_{(ij)}^{(q)}; u_{(i)}^{(q)} \right\}, \quad (2.14)$$

where $(ij) = rr, \theta\theta, zz, r\theta, rz, \theta z$ and $(i) = r, \theta, z$.

Substituting Eq. (2.14) into Eq. (2.3a–2.3c), we obtain set equations for each approximation (2.14). Using Eq. (2.11) we expand the values of each approximation (2.14) in series form in the vicinity of the points $\{r = R^+; z_0 = t_3\}$ and $\{r = R^-; z_0 = t_3\}$. Substituting these last expressions in the boundary conditions in (2.6) and using the expressions of $n_r^\pm$, $n_\theta^\pm$ and $n_z^\pm$ given in (2.11), after some mathematical transformations we obtain boundary conditions which are satisfied at $\{r = R^+; z_0 = t_3\}$ (for outer surface S^+) and at $\{r = R^-; z_0 = t_3\}$ (for inner surface S^-) for each approximation in Eq. (2.14). It is evident that for the zeroth approximation, Eq. (2.3a–2.3c) is valid and condition (2.6) is replaced by the same one satisfied at points $\{r = R^+; z_0 = t_3\}$ and $\{r = R^-; z_0 = t_3\}$. We assume that the non-linear parts of the strain tensor components are very small and can be ignored with respect to their linear parts. According to this assumption, for the zeroth approximation, we obtain the following system of equations:

$$\begin{aligned}
\frac{\partial \sigma_{rr}^{(0)}}{\partial r} + \frac{\partial \sigma_{\theta r}^{(0)}}{r \partial \theta} + \frac{\partial \sigma_{zr}^{(0)}}{\partial z} + \frac{1}{r} (\sigma_{rr}^{(0)} - \sigma_{\theta\theta}^{(0)}) &= 0, \\
\frac{\partial \sigma_{r\theta}^{(0)}}{\partial r} + \frac{\partial \sigma_{\theta\theta}^{(0)}}{r \partial \theta} + \frac{2}{r} \sigma_{\theta r}^{(0)} + \frac{\partial \sigma_{z\theta}^{(0)}}{\partial z} &= 0, \\
\frac{\partial \sigma_{rz}^{(0)}}{\partial r} + \frac{\partial \sigma_{\theta z}^{(0)}}{r \partial \theta} + \frac{1}{r} \sigma_{rz}^{(0)} + \frac{\partial \sigma_{zz}^{(0)}}{\partial z} &= 0, \\
\varepsilon_{rr}^{(0)} = \frac{\partial u_r^{(0)}}{\partial r}, \quad \varepsilon_{r\theta}^{(0)} = \frac{1}{2} \left(\frac{\partial u_\theta^{(0)}}{\partial r} + \frac{\partial u_r^{(0)}}{r \partial \theta} - \frac{u_\theta^{(0)}}{r} \right), \quad \varepsilon_{rz}^{(0)} = \frac{1}{2} \left(\frac{\partial u_r^{(0)}}{\partial z} + \frac{\partial u_z^{(0)}}{\partial r} \right), \\
\varepsilon_{\theta\theta}^{(0)} = \frac{\partial u_\theta^{(0)}}{r \partial \theta} + \frac{u_r^{(0)}}{r}, \quad \varepsilon_{\theta z}^{(0)} = \frac{1}{2} \left(\frac{\partial u_z^{(0)}}{r \partial \theta} + \frac{\partial u_\theta^{(0)}}{\partial z} \right), \quad \varepsilon_{zz}^{(0)} = \frac{\partial u_z^{(0)}}{\partial z},
\end{aligned} \tag{2.15}$$

and boundary conditions

$$\sigma_{rr}^{(0)} \Big|_{r=R^\pm} = 0, \quad \sigma_{r\theta}^{(0)} \Big|_{r=R^\pm} = 0, \quad \sigma_{rz}^{(0)} \Big|_{r=R^\pm} = 0. \tag{2.16}$$

Moreover, we obtain the following end conditions for the zeroth approximation from (2.7), (2.8) and (2.13).

$$\sigma_{zz}^{(0)}(r, \theta, 0) = \sigma_{zz}^{(0)}(r, \theta, \ell) = -p. \tag{2.17}$$

Note that the mathematical procedure, according to which the end condition (2.17) is obtained, will be given below.

Taking the last assumption into account, for the subsequent approximations we obtain the following system of equations.

$$\begin{aligned}
&\frac{\partial \sigma_{rr}^{(q)}}{\partial r} + \frac{\partial \sigma_{\theta r}^{(q)}}{r \partial \theta} + \frac{\partial \sigma_{zr}^{(q)}}{\partial z} + \frac{1}{r} (\sigma_{rr}^{(q)} - \sigma_{\theta\theta}^{(q)}) + \sigma_{zz}^{(0)} \frac{\partial^2 u_r^{(q)}}{\partial z^2} \\
&= -\frac{\partial S_{rr}^{(q-1)}}{\partial r} - \frac{\partial S_{\theta r}^{(q-1)}}{r \partial \theta} - \frac{\partial S_{zr}^{(q-1)}}{\partial z} - \frac{1}{r} (S_{rr}^{(q-1)} - S_{\theta\theta}^{(q-1)}), \\
&\frac{\partial \sigma_{r\theta}^{(q)}}{\partial r} + \frac{\partial \sigma_{\theta\theta}^{(q)}}{r \partial \theta} + \frac{2}{r} \sigma_{\theta r}^{(q)} + \frac{\partial \sigma_{z\theta}^{(q)}}{\partial z} + \sigma_{zz}^{(0)} \frac{\partial^2 u_\theta^{(q)}}{\partial z^2} \\
&= -\frac{\partial S_{r\theta}^{(q-1)}}{\partial r} - \frac{\partial S_{\theta\theta}^{(q-1)}}{r \partial \theta} - \frac{2}{r} (S_{\theta r}^{(q-1)} + S_{r\theta}^{(q-1)}) - \frac{\partial S_{z\theta}^{(q-1)}}{\partial z}, \\
&\frac{\partial \sigma_{rz}^{(q)}}{\partial r} + \frac{\partial \sigma_{\theta z}^{(q)}}{r \partial \theta} + \frac{1}{r} \sigma_{rz}^{(q)} + \frac{\partial \sigma_{zz}^{(q)}}{\partial z} + \sigma_{zz}^{(0)} \frac{\partial^2 u_z^{(q)}}{\partial z^2} \\
&= -\frac{\partial S_{rz}^{(q-1)}}{\partial r} - \frac{\partial S_{\theta z}^{(q-1)}}{r \partial \theta} - \frac{1}{r} S_{rz}^{(q-1)} - \frac{\partial S_{zz}^{(q-1)}}{\partial z},
\end{aligned}$$

$$\begin{aligned}
S_{rr}^{(q-1)} &= \sum_{k=1}^{q-1} \left\{ \sigma_{rr}^{(k)} \frac{\partial u_r^{(q-k)}}{\partial r} + \sigma_{r\theta}^{(k)} \left(\frac{\partial u_r^{(q-k)}}{r \partial \theta} - \frac{u_\theta^{(q-k)}}{r} \right) + \sigma_{rz}^{(k)} \frac{\partial u_r^{(q-k)}}{\partial z} \right\}, \\
S_{r\theta}^{(q-1)} &= \sum_{k=1}^{q-1} \left\{ \sigma_{rr}^{(k)} \frac{\partial u_\theta^{(q-k)}}{\partial r} + \sigma_{r\theta}^{(k)} \left(\frac{\partial u_\theta^{(q-k)}}{r \partial \theta} + \frac{u_r^{(q-k)}}{r} \right) + \sigma_{rz}^{(k)} \frac{\partial u_\theta^{(q-k)}}{\partial z} \right\}, \\
S_{rz}^{(q-1)} &= \sum_{k=1}^{q-1} \left\{ \sigma_{rr}^{(k)} \frac{\partial u_z^{(q-k)}}{\partial r} + \sigma_{r\theta}^{(k)} \frac{\partial u_z^{(q-k)}}{r \partial \theta} + \sigma_{rz}^{(k)} \frac{\partial u_z^{(q-k)}}{\partial z} \right\}, \\
S_{\theta r}^{(q-1)} &= \sum_{k=1}^{q-1} \left\{ \sigma_{\theta r}^{(k)} \frac{\partial u_r^{(q-k)}}{\partial r} + \sigma_{\theta\theta}^{(k)} \left(\frac{\partial u_r^{(q-k)}}{r \partial \theta} - \frac{u_\theta^{(q-k)}}{r} \right) + \sigma_{\theta z}^{(k)} \frac{\partial u_r^{(q-k)}}{\partial z} \right\}, \\
S_{\theta\theta}^{(q-1)} &= \sum_{k=1}^{q-1} \left\{ \sigma_{\theta r}^{(k)} \frac{\partial u_\theta^{(q-k)}}{\partial r} + \sigma_{\theta\theta}^{(k)} \left(\frac{\partial u_\theta^{(q-k)}}{r \partial \theta} + \frac{u_r^{(q-k)}}{r} \right) + \sigma_{\theta z}^{(k)} \frac{\partial u_\theta^{(q-k)}}{\partial z} \right\}, \\
S_{\theta z}^{(q-1)} &= \sum_{k=1}^{q-1} \left\{ \sigma_{\theta r}^{(k)} \frac{\partial u_z^{(q-k)}}{\partial r} + \sigma_{\theta\theta}^{(k)} \frac{\partial u_z^{(q-k)}}{r \partial \theta} + \sigma_{\theta z}^{(k)} \frac{\partial u_z^{(q-k)}}{\partial z} \right\}, \\
S_{zz}^{(q-1)} &= \sum_{k=1}^{q-1} \left\{ \sigma_{rz}^{(k)} \frac{\partial u_z^{(q-k)}}{\partial r} + \sigma_{z\theta}^{(k)} \frac{\partial u_z^{(q-k)}}{r \partial \theta} + \sigma_{zz}^{(k)} \frac{\partial u_z^{(q-k)}}{\partial z} \right\}, \\
S_{zr}^{(q-1)} &= \sum_{k=1}^{q-1} \left\{ \sigma_{zr}^{(k)} \frac{\partial u_r^{(q-k)}}{\partial r} + \sigma_{z\theta}^{(k)} \left(\frac{\partial u_r^{(q-k)}}{r \partial \theta} - \frac{u_\theta^{(q-k)}}{r} \right) + \sigma_{zz}^{(k)} \frac{\partial u_r^{(q-k)}}{\partial z} \right\}, \\
S_{z\theta}^{(q-1)} &= \sum_{k=1}^{q-1} \left\{ \sigma_{zr}^{(k)} \frac{\partial u_\theta^{(q-k)}}{\partial r} + \sigma_{z\theta}^{(k)} \left(\frac{\partial u_\theta^{(q-k)}}{r \partial \theta} + \frac{u_r^{(q-k)}}{r} \right) + \sigma_{zz}^{(k)} \frac{\partial u_\theta^{(q-k)}}{\partial z} \right\}, \\
\varepsilon_{rr}^{(q)} &= \frac{\partial u_r^{(q)}}{\partial r} + \frac{1}{2} \sum_{k=1}^{q-1} \left\{ \left(\frac{\partial u_r^{(k)}}{\partial r} \frac{\partial u_r^{(q-k)}}{\partial r} \right) + \left(\frac{\partial u_\theta^{(k)}}{\partial r} \frac{\partial u_\theta^{(q-k)}}{\partial r} \right) + \left(\frac{\partial u_z^{(k)}}{\partial r} \frac{\partial u_z^{(q-k)}}{\partial r} \right) \right\}, \\
\varepsilon_{r\theta}^{(q)} &= \frac{1}{2} \left(\frac{\partial u_\theta^{(q)}}{\partial r} + \frac{\partial u_r^{(q)}}{r \partial \theta} - \frac{u_\theta^{(q)}}{r} \right) + \frac{1}{2} \sum_{k=1}^{q-1} \left\{ \frac{\partial u_r^{(k)}}{\partial r} \left(\frac{\partial u_r^{(q-k)}}{\partial \theta} - u_\theta^{(q-k)} \right), \right. \\
&\quad \left. \frac{\partial u_\theta^{(k)}}{\partial r} \left(\frac{\partial u_\theta^{(q-k)}}{r \partial \theta} + \frac{u_r^{(q-k)}}{r} \right) + \frac{\partial u_z^{(k)}}{\partial r} \frac{\partial u_z^{(q-k)}}{\partial \theta} \right\}, \varepsilon_{rz}^{(q)} = \frac{1}{2} \left(\frac{\partial u_r^{(q)}}{\partial z} + \frac{\partial u_z^{(q)}}{\partial r} \right) \\
&\quad + \frac{1}{2} \sum_{k=1}^{q-1} \left\{ \frac{\partial u_r^{(k)}}{\partial r} \frac{\partial u_r^{(q-k)}}{\partial z} + \frac{\partial u_\theta^{(k)}}{\partial r} \frac{\partial u_\theta^{(q-k)}}{\partial z} + \frac{\partial u_z^{(k)}}{\partial r} \frac{\partial u_z^{(q-k)}}{\partial z} \right\}, \\
\varepsilon_{\theta\theta}^{(q)} &= \frac{\partial u_\theta^{(q)}}{r \partial \theta} + \frac{u_r^{(q)}}{r} + \frac{1}{2} \sum_{k=1}^{q-1} \left\{ \frac{1}{r^2} \left(\frac{\partial u_r^{(k)}}{\partial \theta} - u_\theta^{(k)} \right) \left(\frac{\partial u_\theta^{(q-k)}}{\partial \theta} - u_\theta^{(q-k)} \right) \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{r^2} \left(\frac{\partial u_\theta^{(k)}}{\partial \theta} + u_r^{(k)} \right) \left(\frac{\partial u_\theta^{(q-k)}}{\partial \theta} + u_r^{(q-k)} \right) + \frac{1}{r^2} \left(\frac{\partial u_z^{(k)}}{\partial \theta} \right) \left(\frac{\partial u_z^{(q-k)}}{\partial \theta} \right) \Bigg\}, \\
& \underline{\varepsilon_{\theta z}^{(q)} = \frac{1}{2} \left(\frac{\partial u_z^{(q)}}{r \partial \theta} + \frac{\partial u_\theta^{(q)}}{\partial z} \right) + \frac{1}{2} \sum_{k=1}^{q-1} \left\{ \frac{1}{r} \frac{\partial u_r^{(k)}}{\partial z} \left(\frac{\partial u_r^{(q-k)}}{\partial \theta} - u_\theta^{(q-k)} \right) \right.} \\
& \left. + \frac{1}{r} \frac{\partial u_\theta^{(k)}}{\partial \theta} \left(\frac{\partial u_\theta^{(q-k)}}{\partial \theta} + u_r^{(q-k)} \right) + \frac{1}{r} \frac{\partial u_z^{(k)}}{\partial \theta} \frac{\partial u_z^{(q-k)}}{\partial z} \right\},} \\
& \underline{\varepsilon_{zz}^{(q)} = \frac{\partial u_z^{(q)}}{\partial z} + \frac{1}{2} \sum_{k=1}^{q-1} \left\{ \left(\frac{\partial u_r^{(k)}}{\partial z} \frac{\partial u_r^{(q-k)}}{\partial z} \right) \right.} \\
& \left. + \left(\frac{\partial u_\theta^{(k)}}{\partial z} \frac{\partial u_\theta^{(q-k)}}{\partial z} \right) + \left(\frac{\partial u_z^{(k)}}{\partial z} \frac{\partial u_z^{(q-k)}}{\partial z} \right) \right\}} \quad (2.18)
\end{aligned}$$

The underlined terms in the Eq. (2.18) are equal to zero for the first approximation. By direct verification it is proven that the Eq. (2.18) without the underlined terms coincide with the corresponding equations of the Three-Dimensional Linearized Theory of Stability (TDLTS) (Guz 1999). Due to linearity, the constitutive relations (2.4) are satisfied by each approximation separately, i.e.

$$\begin{aligned}
\sigma_{rr}^{(q)} &= A_{11}^* \varepsilon_{rr}^{(q)} + A_{12}^* \varepsilon_{\theta\theta}^{(q)} + A_{13}^* \varepsilon_{zz}^{(q)}; \quad \sigma_{\theta\theta}^{(q)} = A_{12}^* \varepsilon_{rr}^{(q)} + A_{11}^* \varepsilon_{\theta\theta}^{(q)} + A_{13}^* \varepsilon_{zz}^{(q)}; \\
\sigma_{zz}^{(q)} &= A_{13}^* \varepsilon_{rr}^{(q)} + A_{13}^* \varepsilon_{\theta\theta}^{(q)} + A_{13}^* \varepsilon_{zz}^{(q)}; \quad \sigma_{r\theta}^{(q)} = (A_{11}^* - A_{22}^*) \varepsilon_{r\theta}^{(q)}; \\
\sigma_{rz}^{(q)} &= 2G^* \varepsilon_{rz}^{(q)}; \quad \sigma_{\theta z}^{(q)} = 2G^* \varepsilon_{\theta z}^{(q)}.
\end{aligned} \quad (2.19)$$

Now we write the boundary conditions given on the lateral surface of the cylinder for the first approximation by the physical components of the stress tensor.

$$\begin{aligned}
& \sigma_{(ir)}^{(1)} \Big|_{(r=R^\pm, z=t_3)} + f_1 \frac{\partial \sigma_{(ir)}^{(0)}}{\partial r} \Big|_{(r=R^\pm, z=t_3)} + \varphi_1^\pm \frac{\partial \sigma_{(ir)}^{(0)}}{\partial z} \Big|_{(r=R^\pm, z=t_3)} \\
& + \gamma_\theta^\pm \sigma_{(i)\theta}^{(0)} \Big|_{(r=R^\pm, z=t_3)} + \gamma_z^\pm \sigma_{(i)z}^{(0)} \Big|_{(r=R^\pm, z=t_3)} = 0,
\end{aligned} \quad (2.20)$$

where $(i) = r, \theta, z$. In Eq. (2.20) replacing (i) with r, θ and z we obtain the explicit form of the corresponding boundary conditions in the considered approximation. Moreover, in Eq. (2.20) the following notation is used.

$$\begin{aligned}\gamma_z^\pm &= \pm(-f'(t_3) \cos(\theta)), \quad f_1 = f(t_3) \cos(\theta), \quad \varphi_1^\pm = -R^\pm f'(t_3) \cos(\theta), \\ \gamma_\theta^\pm &= \pm \frac{f(t_3)}{R^\pm} \sin(\theta), \quad f'(t_3) = \frac{df(t_3)}{dt_3}, \quad f''(t_3) = \frac{d^2f(t_3)}{dt_3^2}.\end{aligned}\quad (2.21)$$

Consider the satisfaction of the end conditions (2.8). To simplify the discussion we rewrite these conditions in the Cartesian system of coordinates $Ox_1x_2x_3$.

$$\sigma_{3n} \left(\delta_n^j + \frac{\partial u_j}{\partial x_n} \right) \Big|_{S_0} n_{0j} = p; \quad \sigma_{3n} \left(\delta_n^j + \frac{\partial u_j}{\partial x_n} \right) \Big|_{S_\ell} n_{\ell j} = -p \quad (2.22)$$

where δ_n^j is a Kronecker symbol, other notation used in (2.22) is conventional.

According to Eqs. (2.12) and (2.13), we can write the following expressions from the conditions (2.22).

$$\begin{aligned}\sigma_{3n} \left(\delta_n^j + \frac{\partial u_j}{\partial x_n} \right) \Big|_{S_0} n_{0j} &= \sigma_{3n}(x_1, x_2, -\varepsilon\pi x_1, t) \\ &\times \left(\delta_n^1 + \frac{\partial u_1(x_1, x_2, -\varepsilon\pi x_1, t)}{\partial x_n} \right) n_{01} \\ &+ \sigma_{3n}(x_1, x_2, -\varepsilon\pi x_1, t) \left(\delta_n^3 + \frac{\partial u_1(x_1, x_2, -\varepsilon\pi x_1, t)}{\partial x_n} \right) n_{03} = -p \\ \sigma_{3n} \left(\delta_n^j + \frac{\partial u_j}{\partial x_n} \right) \Big|_{S_\ell} n_{\ell j} &= \sigma_{3n}(x_1, x_2, \ell + \varepsilon\pi x_1, t) \\ &\times \left(\delta_n^1 + \frac{\partial u_1(x_1, x_2, \ell + \varepsilon\pi x_1, t)}{\partial x_n} \right) n_{\ell 1} \\ &+ \sigma_{3n}(x_1, x_2, \ell + \varepsilon\pi x_1, t) \left(\delta_n^3 + \frac{\partial u_3(x_1, x_2, \ell + \varepsilon\pi x_1, t)}{\partial x_n} \right) n_{\ell 3} = p\end{aligned}\quad (2.23)$$

Using the expansions

$$\begin{aligned}\sigma_{in}(x_1, x_2, -\varepsilon\pi x_1, t) &= \sum_{q=0}^{\infty} \varepsilon^q \sigma_{in}^{(q)}(x_1, x_2, -\varepsilon\pi x_1, t) = \sigma_{in}^{(0)}(x_1, x_2, 0, t) \\ &+ \varepsilon \left(\sigma_{in}^{(1)}(x_1, x_2, 0, t) + (-\pi x_1) \frac{\partial \sigma_{in}^{(0)}(x_1, x_2, 0, t)}{\partial x_3} \right) + O((\varepsilon\pi)^2); \\ \frac{\partial u_m(x_1, x_2, -\varepsilon\pi x_1, t)}{\partial x_j} &= \sum_{q=0}^{\infty} \varepsilon^q \frac{\partial u_m^{(q)}(x_1, x_2, -\varepsilon\pi x_1, t)}{\partial x_j} = \frac{\partial u_m^{(0)}(x_1, x_2, 0, t)}{\partial x_j} \\ &+ \varepsilon \left(\frac{\partial u_m^{(1)}(x_1, x_2, 0, t)}{\partial x_j} + (-\pi x_1) \frac{\partial^2 u_m^{(0)}(x_1, x_2, 0, t)}{\partial x_3 \partial x_j} \right) + O((\varepsilon\pi)^2);\end{aligned}$$

$$\begin{aligned}
\sigma_{in}(x_1, x_2, \ell + \varepsilon \pi x_1, t) &= \sum_{q=0}^{\infty} \varepsilon^q \sigma_{in}^{(q)}(x_1, x_2, \ell + \varepsilon \pi x_1, t) = \sigma_{in}^{(0)}(x_1, x_2, \ell, t) \\
&\quad + \varepsilon \left(\sigma_{in}^{(1)}(x_1, x_2, \ell, t) + (-\pi x_1) \frac{\partial \sigma_{in}^{(0)}(x_1, x_2, \ell, t)}{\partial x_3} \right) \\
&\quad + O((\varepsilon \pi)^2); \frac{\partial u_m(x_1, x_2, \ell + \varepsilon \pi x_1, t)}{\partial x_j} \\
&= \sum_{q=0}^{\infty} \varepsilon^q \frac{\partial u_m^{(q)}(x_1, x_2, \ell + \varepsilon \pi x_1, t)}{\partial x_j} = \frac{\partial u_m^{(0)}(x_1, x_2, \ell, t)}{\partial x_j} \\
&\quad + \varepsilon \left(\frac{\partial u_m^{(1)}(x_1, x_2, \ell, t)}{\partial x_j} + (-\pi x_1) \frac{\partial^2 u_m^{(0)}(x_1, x_2, \ell, t)}{\partial x_3 \partial x_j} \right) \\
&\quad + O((\varepsilon \pi)^2),
\end{aligned} \tag{2.24}$$

we obtain the following expression for the end conditions in (2.22).

$$\begin{aligned}
&\left\{ -\sigma_{3k}^{(0)} \left(\delta_k^3 + \frac{\partial u_3^{(0)}}{\partial x_k} \right) + \varepsilon \left[-\pi \sigma_{3k}^{(0)} \left(\delta_k^1 + \frac{\partial u_1^{(0)}}{\partial x_k} \right) - \sigma_{3k}^{(0)} \left(\frac{\partial u_3^{(0)}}{\partial x_k} - \pi x_1 \frac{\partial^2 u_3^{(0)}}{\partial x_3 \partial x_k} \right) \right. \right. \\
&\quad \left. \left. - \left(\sigma_{3k}^{(1)} - \pi x_1 \frac{\partial \sigma_{3k}^{(0)}}{\partial x_k} \right) \left(\delta_k^3 + \frac{\partial u_3^{(0)}}{\partial x_k} \right) \right] + O(\varepsilon^2) \right\}_{(x_1, x_2, 0)} = p, \\
&\left\{ \sigma_{3k}^{(0)} \left(\delta_k^3 + \frac{\partial u_3^{(0)}}{\partial x_k} \right) + \varepsilon \left[\pi \sigma_{3k}^{(0)} \left(\delta_k^1 + \frac{\partial u_1^{(0)}}{\partial x_k} \right) + \sigma_{3k}^{(0)} \left(\frac{\partial u_3^{(0)}}{\partial x_k} + \pi x_1 \frac{\partial^2 u_3^{(0)}}{\partial x_3 \partial x_k} \right) \right. \right. \\
&\quad \left. \left. + \left(\sigma_{3k}^{(1)} + \pi x_1 \frac{\partial \sigma_{3k}^{(0)}}{\partial x_k} \right) \left(\delta_k^3 + \frac{\partial u_3^{(0)}}{\partial x_k} \right) \right] + O(\varepsilon^2) \right\}_{(x_1, x_2, \ell)} = -p.
\end{aligned} \tag{2.25}$$

In a similar manner, we can write the following expansions for the physical components $u_{(i)}$ of the displacement vector at the ends of the cylinder

$$\begin{aligned}
&u_{(i)}^{(0)}(x_1, x_2, 0, t) + \varepsilon \left(u_{(i)}^{(1)}(x_1, x_2, 0, t) \right. \\
&\quad \left. + (-\pi x_1) \frac{\partial u_{(i)}^{(0)}(x_1, x_2, 0, t)}{\partial x_3} \right) + O((\varepsilon \pi)^2) = 0, \\
&u_{(i)}^{(0)}(x_1, x_2, \ell, t) + \varepsilon \left(u_{(i)}^{(1)}(x_1, x_2, \ell, t) + (-\pi x_1) \frac{\partial u_{(i)}^{(0)}(x_1, x_2, \ell, t)}{\partial x_3} \right) \\
&\quad + O((\varepsilon \pi)^2) = 0, \quad (i) = r, \theta, z.
\end{aligned} \tag{2.26}$$

We assume that the coefficient of ε^q in the expansion (2.26) for $(i) = r; \theta$ is equal to zero. Consequently, according to this assumption, we obtain the end conditions for the first and subsequent approximations for the displacements u_r and u_θ .

Taking the estimation $(\delta_k^3 + \partial u_3^{(0)} / \partial x_k) \approx \delta_k^3$, $(\delta_k^1 + \partial u_1^{(0)} / \partial x_k) \approx \delta_k^1$ and the expansions (2.23)–(2.26) into account we obtain the following end conditions for the stresses for the zeroth and first approximations from the condition (2.22).

For the zeroth approximation:

$$\sigma_{33}^{(0)}(x_1, x_2, 0) = \sigma_{33}^{(0)}(x_1, x_2, \ell) = -p. \quad (2.27)$$

For the first approximation

$$\begin{aligned} & \pi \sigma_{31}^{(0)}(x_1, x_2, 0, t) + \sigma_{3k}^{(0)}(x_1, x_2, 0, t) \\ & \left(\frac{\partial u_3^{(1)}(x_1, x_2, 0, t)}{k} - \pi x_1 \frac{\partial^2 u_3^{(0)}(x_1, x_2, 0, t)}{\partial x_3 \partial x_k} \right) \\ & + \left(\sigma_{33}^{(1)}(x_1, x_2, 0, t) - \pi x_1 \frac{\partial \sigma_{33}^{(0)}(x_1, x_2, 0, t)}{\partial x_3} \right) = 0, \\ & \left(\frac{\partial u_3^{(1)}(x_1, x_2, \ell, t)}{\partial x_k} - \pi x_1 \frac{\partial^2 u_3^{(0)}(x_1, x_2, \ell, t)}{\partial x_3 \partial x_k} \right) \\ & + \left(\sigma_{33}^{(1)}(x_1, x_2, \ell, t) - \pi x_1 \frac{\partial \sigma_{33}^{(0)}(x_1, x_2, \ell, t)}{\partial x_3} \right) = 0. \end{aligned} \quad (2.28)$$

Thus, rewriting the condition (2.27) in the cylindrical system of coordinates $Or\theta z$ we obtain the condition (2.17).

According to (2.15), (2.16) and (2.17), the values related to the zeroth approximation are determined as follows.

$$\sigma_{zz}^{(0)} = -p, \quad \sigma_{ij}^{(0)} = 0 \text{ for } (ij) \neq zz. \quad (2.29)$$

It follows from (2.29) that in the zeroth approximation the components of the displacement vector can be presented as follows

$$u_r^{(0)} = a(t)r + a_0, \quad u_\theta^{(0)} = b_0, \quad u_z^{(0)} = c(t)z + c_0 \quad (2.30)$$

where a_0 , b_0 and c_0 are constants, $a(t)$ and $c(t)$ are functions, t is a time. The functions $a(t)$ and $c(t)$ can be easily determined from Eqs. (2.19) and (2.29).

Now we consider determination of the values related to the first approximation. Taking the expression (2.29) into account the following field equations are obtained from Eq. (2.18) for this approximation.

$$\begin{aligned}
& \frac{\partial \sigma_{rr}^{(1)}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta r}^{(1)}}{\partial \theta} + \frac{\partial \sigma_{zr}^{(1)}}{\partial z} + \frac{1}{r} \left(\sigma_{rr}^{(1)} - \sigma_{\theta\theta}^{(1)} \right) + \sigma_{zz}^{(0)} \frac{\partial^2 u_r^{(1)}}{\partial z^2} = 0, \\
& \frac{\partial \sigma_{r\theta}^{(1)}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}^{(1)}}{\partial \theta} + \frac{\partial \sigma_{z\theta}^{(1)}}{\partial z} + \frac{2}{r} \sigma_{\theta r}^{(1)} + \sigma_{zz}^{(0)} \frac{\partial^2 u_\theta^{(1)}}{\partial z^2} = 0, \\
& \frac{\partial \sigma_{rz}^{(1)}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta z}^{(1)}}{\partial \theta} + \frac{\partial \sigma_{zz}^{(1)}}{\partial z} + \frac{1}{r} \sigma_{rz}^{(1)} + \sigma_{zz}^{(0)} \frac{\partial^2 u_z^{(1)}}{\partial z^2} = 0, \\
& \varepsilon_{rr}^{(1)} = \frac{\partial u_r^{(1)}}{\partial r}, \quad \varepsilon_{\theta\theta}^{(1)} = \frac{\partial u_\theta^{(1)}}{r \partial \theta} + \frac{u_r^{(1)}}{r}, \quad \varepsilon_{r\theta}^{(1)} = \frac{1}{2} \left(\frac{\partial u_r^{(1)}}{r \partial \theta} + \frac{\partial u_\theta^{(1)}}{\partial r} - \frac{u_\theta^{(1)}}{r} \right), \\
& \varepsilon_{zz}^{(1)} = \frac{\partial u_z^{(1)}}{\partial z}, \quad \varepsilon_{\theta z}^{(1)} = \frac{1}{2} \left(\frac{\partial u_\theta^{(1)}}{\partial z} + \frac{\partial u_z^{(1)}}{r \partial \theta} \right), \quad \varepsilon_{rz}^{(1)} = \frac{1}{2} \left(\frac{\partial u_r^{(1)}}{\partial z} + \frac{\partial u_z^{(1)}}{\partial r} \right).
\end{aligned} \tag{2.31}$$

The following conditions on the lateral surface of the cylinder are obtained from (2.20) and (2.29).

$$\begin{aligned}
& \sigma_{rr}^{(1)}(R^\pm, \theta, t_3, t) = 0, \quad \sigma_{r\theta}^{(1)}(R^\pm, \theta, t_3, t) = 0, \\
& \sigma_{rz}^{(1)}(R^\pm, \theta, t_3, t) = \pm \pi \sigma_{zz}^{(0)} \cos(\alpha_1 z) \cos \theta = 0, \quad \alpha_1 = \pi/\ell
\end{aligned} \tag{2.32}$$

According to Eqs. (2.26) and (2.28), the end conditions for the first approximation can be written as follows.

$$\begin{aligned}
& \sigma_{zz}^{(1)}(r, \theta, 0, t) + \sigma_{zz}^{(0)} \frac{\partial u_z^{(1)}(r, \theta, 0, t)}{\partial z} = 0, \quad u_r^{(1)}(r, \theta, 0, t) = 0, \quad u_\theta^{(1)}(r, \theta, 0, t) = 0, \\
& \sigma_{zz}^{(1)}(r, \theta, \ell, t) + \sigma_{zz}^{(0)} \frac{\partial u_z^{(1)}(r, \theta, \ell, t)}{\partial z} = 0, \\
& u_r^{(1)}(r, \theta, \ell, t) = 0, \quad u_\theta^{(1)}(r, \theta, \ell, t) = 0.
\end{aligned} \tag{2.33}$$

Thus, the Eq. (2.31), (2.19) and (2.5) and the conditions (2.32) and (2.33) complete the formulation of the problem for determination of the values of the first approximation. For solution to this problem we apply the Laplace transformation

$$\bar{\psi} = \int_0^\infty \psi(t) e^{-st} dt \tag{2.34}$$

with parameter $s > 0$, to all equations and relations related to the first approximation. After this application to Eq. (2.31), the boundary conditions (2.32) (in which $\sigma_{zz}^{(0)}$ must be replaced with $\sigma_{zz}^{(0)}/s$) and (2.33) are valid for the Laplace transformations of the corresponding sought-for quantities, whilst the constitutive relations (2.19) are transformed to the following ones:

$$\begin{aligned}
\bar{\sigma}_{rr}^{(1)} &= \bar{A}_{11}^* \bar{e}_{rr}^{(1)} + \bar{A}_{12}^* \bar{e}_{\theta\theta}^{(1)} + \bar{A}_{13}^* \bar{e}_{zz}^{(1)}, \quad \bar{\sigma}_{\theta\theta}^{(1)} = \bar{A}_{12}^* \bar{e}_{rr}^{(1)} + \bar{A}_{11}^* \bar{e}_{\theta\theta}^{(1)} + \bar{A}_{13}^* \bar{e}_{zz}^{(1)}, \\
\bar{\sigma}_{zz}^{(1)} &= \bar{A}_{13}^* \bar{e}_{rr}^{(1)} + \bar{A}_{13}^* \bar{e}_{\theta\theta}^{(1)} + \bar{A}_{13}^* \bar{e}_{zz}^{(1)}, \quad \bar{\sigma}_{r\theta}^{(1)} = (\bar{A}_{11}^* - \bar{A}_{22}^*) \bar{e}_{r\theta}^{(1)}, \\
\bar{\sigma}_{rz}^{(1)} &= 2\bar{G}^* \bar{e}_{rz}^{(1)}, \quad \bar{\sigma}_{\theta z}^{(1)} = 2\bar{G}^* \bar{e}_{\theta z}^{(1)}
\end{aligned} \tag{2.35}$$

where

$$\left\{ \begin{array}{c} \bar{A}_{ij}^* \\ \bar{G}^* \end{array} \right\} \bar{\phi}(s) = \left\{ \begin{array}{c} A_{ij0} \\ G_0 \end{array} \right\} \bar{\phi}(s) + \left\{ \begin{array}{c} \bar{A}_{ij1}(s) \\ G_1(s) \end{array} \right\} \bar{\phi}(s). \tag{2.36}$$

As has been noted above, Eqs. (2.31)–(2.36) coincide with the corresponding equations of the TDLTS. Therefore, to solve the obtained equation systems, according to the monograph (Guz 1999), in the cylindrical system of coordinates we can use the following representations.

$$\begin{aligned}
\bar{u}_r^{(1)} &= \frac{1}{r} \frac{\partial}{\partial \theta} \psi - \frac{\partial^2}{\partial r \partial z} \chi, \quad \bar{u}_\theta^{(1)} = -\frac{1}{r} \psi - \frac{1}{r} \frac{\partial^2}{\partial \theta \partial z} \chi, \\
\bar{u}_z^{(1)} &= (\bar{A}_{13}^* + \bar{G}^*)^{-1} \left(\bar{A}_{11}^* \Delta_1 + (\bar{G}^* + \sigma_{zz}^{(0)}) \frac{\partial^2}{\partial z^2} \right) \chi, \\
\Delta_1 &= \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial^2}{\partial \theta^2}.
\end{aligned} \tag{2.37}$$

The functions ψ and χ are determined from the equations.

$$\left(\Delta_1 + \zeta_1^2 \frac{\partial^2}{\partial z^2} \right) \psi = 0, \quad \left(\Delta_1 + \zeta_2^2 \frac{\partial^2}{\partial z^2} \right) \left(\Delta_1 + \zeta_3^2 \frac{\partial^2}{\partial z^2} \right) \chi = 0, \tag{2.38}$$

where

$$\begin{aligned}
\zeta_1^2 &= \frac{2\bar{G}^*}{\bar{A}_{11}^* - \bar{A}_{12}^*}, \quad \zeta_{2,3}^2 = c \pm \left(c^2 - \frac{(\bar{A}_{33}^* + \sigma_{zz}^{(0)}) (\bar{G}^* + \sigma_{zz}^{(0)})}{\bar{A}_{11}^* \bar{G}^*} \right)^{\frac{1}{2}}, \\
2\bar{A}_{11}^* \bar{G}^* c &= \bar{A}_{11}^* (\bar{A}_{33}^* + \sigma_{zz}^{(0)}) + \bar{G}^{*2} - (\bar{A}_{13}^* + \bar{G}^*)^2.
\end{aligned} \tag{2.39}$$

Taking the expressions of the right sides of the conditions (2.32) and (2.33) we find the solution to the Eq. (2.38) as follows.

$$\begin{aligned}
\psi &= (B_1 I_1(\xi_1 \alpha_1 r) + B_4 K_1(\xi_1 \alpha_1 r)) \sin(\alpha_1 z) \sin \theta, \\
\chi &= [B_2 I_1(\xi_2 \alpha_1 r) + B_3 I_1(\xi_3 \alpha_1 r) + B_5 K_1(\xi_2 \alpha_1 r) \\
&\quad + B_6 K_1(\xi_3 \alpha_1 r)] \cos(\alpha_1 z) \cos \theta,
\end{aligned} \tag{2.40}$$

where $\alpha_1 = \pi/\ell$, $I_1(x)$ is the first order Bessel function of a purely imaginary argument and $K_1(x)$ is the first order Macdonald Function, B_1 , B_2 , B_3 , B_4 , B_5 and B_6 are unknown constants.

Substituting these solutions into relations (2.37) and (2.35) we obtain the following expressions for Laplace transformation of the south values.

$$\begin{aligned}
\bar{u}_r^{(1)} &= \left[B_1 \frac{1}{r} I_1(\xi_2 \alpha_1 r) + B_2 \xi_2 \alpha_1^2 I_1'(\xi_2 \alpha_1 r) + B_3 \xi_3 \alpha_1^2 I_1'(\xi_3 \alpha_1 r) \right. \\
&\quad \left. + B_4 \frac{1}{r} K_1(\xi_1 \alpha_1 r) + B_5 \xi_2 \alpha_1^2 K_1'(\xi_2 \alpha_1 r) + B_6 \xi_3 \alpha_1^2 K_1'(\xi_3 \alpha_1 r) \right] \sin(\alpha_1 z) \cos \theta, \\
\bar{u}_\theta^{(1)} &= \left[-B_1 \xi_1 \alpha_1 I_1'(\xi_1 \alpha_1 r) - \frac{\alpha_1}{r} B_2 I_1(\xi_2 \alpha_1 r) - \frac{\alpha_1}{r} B_3 I_1(\xi_3 \alpha_1 r) \right. \\
&\quad \left. - B_4 \xi_1 \alpha_1 K_1'(\xi_1 \alpha_1 r) - \frac{\alpha_1}{r} B_5 K_1(\xi_2 \alpha_1 r) - \frac{\alpha_1}{r} B_6 \xi_3 \alpha_1^2 K_1(\xi_3 \alpha_1 r) \right] \sin(\alpha_1 z) \sin \theta, \\
\bar{u}_z^{(1)} &= [B_2 D_2 \alpha_1^2 I_1(\xi_2 \alpha_1 r) + B_3 D_2 \alpha_1^2 I_1(\xi_3 \alpha_1 r) \\
&\quad + B_5 D_2 \alpha_1^2 K_1(\xi_2 \alpha_1 r) + B_6 D_3 \alpha_1^2 K_1(\xi_3 \alpha_1 r)] \sin(\alpha_1 z) \cos \theta, \\
D_2 &= \frac{\bar{A}_{11}^* \xi_2^2 - \bar{G}^* - \sigma_{zz}^{(0)}}{\bar{A}_{13}^* + \bar{G}^*}, \quad D_3 = \frac{\bar{A}_{11}^* \xi_3^2 - \bar{G}^* - \sigma_{zz}^{(0)}}{\bar{A}_{13}^* + \bar{G}^*}, \\
\bar{\sigma}_{rr}^{(1)} &= \left\{ B_1 \left[-(\bar{A}_{11}^* - \bar{A}_{12}^*) \frac{1}{r^2} I_1(\xi_1 \alpha_1 r) + (\bar{A}_{11}^* - \bar{A}_{12}^*) \frac{\xi_1 \alpha_1}{r} I_1'(\xi_1 \alpha_1 r) \right] \right. \\
&\quad + B_2 [\bar{A}_{11}^* \alpha_1^3 \xi_2^2 I_1''(\xi_2 \alpha_1 r) + \bar{A}_{12}^* \left(\frac{-\alpha_1}{r^2} I_1(\xi_2 \alpha_1 r) + \frac{\xi_2 \alpha_1^2}{r} I_1'(\xi_2 \alpha_1 r) \right) - \bar{A}_{13}^* D_2 \alpha_1^3 I_1(\xi_2 \alpha_1 r)], \\
&\quad B_3 [\bar{A}_{11}^* \alpha_1^3 \xi_3^2 I_1''(\xi_3 \alpha_1 r) + \bar{A}_{12}^* \left(\frac{-\alpha_1}{r^2} I_1(\xi_3 \alpha_1 r) + \frac{\xi_3 \alpha_1^2}{r} I_1'(\xi_3 \alpha_1 r) \right) - \bar{A}_{13}^* D_3 \alpha_1^3 I_1(\xi_3 \alpha_1 r)], \\
&\quad B_4 \left[-(\bar{A}_{11}^* - \bar{A}_{12}^*) \frac{1}{r^2} K_1(\xi_1 \alpha_1 r) + (\bar{A}_{11}^* - \bar{A}_{12}^*) \frac{\xi_1 \alpha_1}{r} K_1'(\xi_1 \alpha_1 r) \right] \\
&\quad + B_5 [\bar{A}_{11}^* \alpha_1^3 \xi_2^2 K_1''(\xi_2 \alpha_1 r) + \bar{A}_{12}^* \left(\frac{-\alpha_1}{r^2} K_1(\xi_2 \alpha_1 r) + \frac{\xi_2 \alpha_1^2}{r} K_1'(\xi_2 \alpha_1 r) \right) \\
&\quad + \frac{\xi_2 \alpha_1^2}{r} K_1'(\xi_2 \alpha_1 r) - \bar{A}_{13}^* D_2 \alpha_1^3 K_1(\xi_2 \alpha_1 r)] \\
&\quad + B_6 [\bar{A}_{11}^* \alpha_1^3 \xi_3^2 K_1''(\xi_3 \alpha_1 r) + \bar{A}_{12}^* \left(\frac{-\alpha_1}{r^2} K_1(\xi_3 \alpha_1 r) + \frac{\xi_3 \alpha_1^2}{r} K_1'(\xi_3 \alpha_1 r) \right) \\
&\quad - \bar{A}_{13}^* D_3 \alpha_1^3 K_1(\xi_3 \alpha_1 r)] \} \sin(\alpha_1 z) \cos \theta, \bar{\sigma}_{r\theta}^{(1)} \\
&= \left\{ B_1 \left[\frac{1}{2} (\bar{A}_{11}^* - \bar{A}_{12}^*) \left(-\xi_1^2 \alpha_1^2 I_1''(\xi_1 \alpha_1 r) - \frac{1}{r^2} I_1(\xi_1 \alpha_1 r) + \frac{\xi_1 \alpha_1}{r} I_1'(\xi_1 \alpha_1 r) \right) \right] \right. \\
&\quad + B_2 \left[(\bar{A}_{11}^* - \bar{A}_{12}^*) \left(\frac{\alpha_1}{r^2} I_1(\xi_2 \alpha_1 r) - \xi_2 \frac{\alpha_1^2}{r} I_1'(\xi_2 \alpha_1 r) \right) \right] \\
&\quad + B_3 \left[(\bar{A}_{11}^* - \bar{A}_{12}^*) \left(\frac{\alpha_1}{r^2} I_1(\xi_3 \alpha_1 r) - \xi_3 \frac{\alpha_1^2}{r} I_1'(\xi_3 \alpha_1 r) \right) \right] \\
&\quad + B_4 \left[\frac{1}{2} (\bar{A}_{11}^* - \bar{A}_{12}^*) \left(-\xi_1^2 \alpha_1^2 K_1''(\xi_1 \alpha_1 r) - \frac{1}{r^2} K_1(\xi_1 \alpha_1 r) + \frac{\xi_1 \alpha_1}{r} K_1'(\xi_1 \alpha_1 r) \right) \right] \\
&\quad + B_5 \left[(\bar{A}_{11}^* - \bar{A}_{12}^*) \left(\frac{\alpha_1^2}{r^2} K_1(\xi_2 \alpha_1 r) - \xi_2 \frac{\alpha_1^2}{r} K_1'(\xi_2 \alpha_1 r) \right) \right] \\
&\quad + B_6 \left[(\bar{A}_{11}^* - \bar{A}_{12}^*) \left(\frac{\alpha_1}{r^2} K_1(\xi_3 \alpha_1 r) - \xi_3 \frac{\alpha_1^2}{r} K_1'(\xi_3 \alpha_1 r) \right) \right] \} \sin(\alpha_1 r) \sin \theta, \\
\bar{\sigma}_{rz}^{(1)} &= \left\{ B_1 \bar{G}^* \frac{\alpha_1}{r} I_1(\xi_1 \alpha_1 r) + B_2 \bar{G}^* \alpha_1^3 \xi_2 I_1'(\xi_2 \alpha_1 r) (1 + D_2) \right. \\
&\quad + B_3 \bar{G}^* \alpha_1^3 \xi_3 I_1'(\xi_3 \alpha_1 r) (1 + D_3) + B_4 \bar{G}^* \frac{\alpha_1}{r} K_1(\xi_1 \alpha_1 r) \\
&\quad \left. + B_5 \bar{G}^* \alpha_1^3 \xi_2 K_1'(\xi_2 \alpha_1 r) (1 + D_2) + B_6 \bar{G}^* \alpha_1^3 \xi_3 K_1'(\xi_3 \alpha_1 r) (1 + D_3) \right\} \cos(\alpha_1 z) \cos \theta,
\end{aligned}$$

$$\begin{aligned}
\bar{\sigma}_{zz}^{(1)} = & \left\{ B_2 \left[\bar{A}_{13}^* \left(\xi_2^2 \alpha_1^3 I_1'(\xi_2 \alpha_1 r) - \frac{\alpha_1}{r^2} I_1(\xi_2 \alpha_1 r) + \frac{\alpha_1^2 \xi_2}{r} I_1'(\xi_2 \alpha_1 r) \right) \right. \right. \\
& - \bar{A}_{33}^* D_2 \alpha_1^3 I_1(\xi_2 \alpha_1 r) \left. \right] + B_3 \left[\bar{A}_{13}^* (\xi_3^2 \alpha_1^3 I_1'(\xi_3 \alpha_1 r) \right. \\
& - \frac{\alpha_1}{r^2} I_1(\xi_3 \alpha_1 r) + \frac{\alpha_1^2 \xi_3}{r} I_1'(\xi_3 \alpha_1 r) \left. \right] - \bar{A}_{33}^* D_3 \alpha_1^3 I_1(\xi_3 \alpha_1 r) \left. \right] \\
& B_5 \left[\bar{A}_{13}^* \left(\xi_2^2 \alpha_1^3 K_1''(\xi_2 \alpha_1 r) - \frac{\alpha_1}{r^2} K_1(\xi_2 \alpha_1 r) + \frac{\alpha_1^2 \xi_2}{r} K_1'(\xi_2 \alpha_1 r) \right) \right. \\
& - \bar{A}_{33}^* D_2 \alpha_1^3 K_1(\xi_2 \alpha_1 r) \left. \right] + B_6 \left[\bar{A}_{13}^* (\xi_3^2 \alpha_1^3 K_1''(\xi_3 \alpha_1 r) - \frac{\alpha_1}{r^2} K_1(\xi_3 \alpha_1 r) \right. \\
& + \frac{\alpha_1^2 \xi_3}{r} K_1'(\xi_3 \alpha_1 r) \left. \right] - \bar{A}_{33}^* D_3 \alpha_1^3 K_1(\xi_3 \alpha_1 r) \left. \right] \Big\} \sin(\alpha_1 z) \cos \theta.
\end{aligned} \tag{2.41}$$

Here the notation $I_1'(x) = dI_1(x)/dx$, $I_1''(x) = d^2I_1(x)/dx^2$, $K_1'(x) = dK_1(x)/dx$ and $K_1''(x) = d^2K_1(x)/dx^2$ is used. The solution (2.41) to the problem under consideration satisfies automatically the end condition (2.33). Replacing the unknowns B_1 , B_2 , B_3 , B_4 , B_5 and B_6 with $\alpha_1^2 B_1 (= C_1)$, $\alpha_1^3 B_2 (= C_2)$, $\alpha_1^3 B_3 (= C_3)$, $\alpha_1^2 B_4 (= C_4)$, $\alpha_1^3 B_5 (= C_5)$ and $\alpha_1^3 B_6 (= C_6)$ respectively, we obtain the following algebraic equation from the boundary condition (2.32) for determination these unknowns.

$$\begin{aligned}
\bar{\sigma}_{rr}^{(1)}(R^+, \theta, t_3, t) = 0 & \Rightarrow C_1 a_{11}(\alpha_1 R^+) + C_2 a_{12}(\alpha_1 R^+) + C_3 a_{13}(\alpha_1 R^+) \\
& + C_4 a_{14}(\alpha_1 R^+) + C_5 a_{15}(\alpha_1 R^+) + C_6 a_{16}(\alpha_1 R^+) = 0, \\
\bar{\sigma}_{r\theta}^{(1)}(R^+, \theta, t_3, t) = 0 & \Rightarrow C_1 a_{21}(\alpha_1 R^+) + C_2 a_{22}(\alpha_1 R^+) + C_3 a_{23}(\alpha_1 R^+) \\
& + C_4 a_{24}(\alpha_1 R^+) + C_5 a_{25}(\alpha_1 R^+) + C_6 a_{26}(\alpha_1 R^+) = 0, \\
\bar{\sigma}_{rz}^{(1)}(R^+, \theta, t_3, t) = \pi \sigma_{zz}^{(0)} \frac{1}{s} & \Rightarrow C_1 a_{31}(\alpha_1 R^+) + C_2 a_{32}(\alpha_1 R^+) + C_3 a_{33}(\alpha_1 R^+) \\
& + C_4 a_{34}(\alpha_1 R^+) + C_5 a_{35}(\alpha_1 R^+) + C_6 a_{36}(\alpha_1 R^+) = \pi \sigma_{zz}^{(0)} \frac{1}{s}, \\
\bar{\sigma}_{rr}^{(1)}(R^-, \theta, t_3, t) = 0 & \Rightarrow C_1 a_{11}(\alpha_1 R^-) + C_2 a_{12}(\alpha_1 R^-) + C_3 a_{13}(\alpha_1 R^-) \\
& + C_4 a_{14}(\alpha_1 R^-) + C_5 a_{15}(\alpha_1 R^-) + C_6 a_{16}(\alpha_1 R^-) = 0, \\
\bar{\sigma}_{r\theta}^{(1)}(R^-, \theta, t_3, t) = 0 & \Rightarrow C_1 a_{21}(\alpha_1 R^-) + C_2 a_{22}(\alpha_1 R^-) + C_3 a_{23}(\alpha_1 R^-) \\
& + C_4 a_{24}(\alpha_1 R^-) + C_5 a_{25}(\alpha_1 R^-) + C_6 a_{26}(\alpha_1 R^-) = 0, \\
\bar{\sigma}_{rz}^{(1)}(R^-, \theta, t_3, t) = -\pi \sigma_{zz}^{(0)} \frac{1}{s} & \Rightarrow C_1 a_{31}(\alpha_1 R^-) + C_2 a_{32}(\alpha_1 R^-) + C_3 a_{33}(\alpha_1 R^-) \\
& + C_4 a_{34}(\alpha_1 R^-) + C_5 a_{35}(\alpha_1 R^-) + C_6 a_{36}(\alpha_1 R^-) = -\pi \sigma_{zz}^{(0)} \frac{1}{s}
\end{aligned} \tag{2.42}$$

Note that expressions for the coefficients a_{ij} ($i, j = 1, 2, 3, 4, 5, 6$) can be easily obtained from Eq. (2.41).

Thus, with the foregoing we determine completely the Laplace transformation of the values related to the first approximation. The Laplace transformation of the values of the second and subsequent approximations in (2.14) can also be determined as the values of the first approximation by taking the obvious changes into account. However, as follows from the structure of the Eq. (2.18), for stability loss problems, the consideration of only the zeroth and first approximation is sufficient, because accounting the second and subsequent approximation does not change the values of the critical parameters.

The original of the south values is determined by employing the method (Schapery 1962, 1978), according to which, for instance, the original of the displacement $u_r^{(1)}(r, \theta, t_3, t)$ is determined through the expression

$$u_r^{(1)}(r, \theta, t_3, t) \approx \left(s\bar{u}_r^{(1)}(r, \theta, t_3, t) \right) \Big|_{s=1/(2t)}. \quad (2.43)$$

Now we consider the selection of the stability loss criterion. In the present investigation, the case will be understood under stability loss, where

$$\max_{\substack{t_3 \in (0, \ell) \\ r \in (0, R) \\ \theta \in (0, 2\pi)}} |u_r^{(1)}(r, \theta, t_3, t)| \rightarrow \infty \text{ as } t \rightarrow t_{cr} \text{ (or as } p \rightarrow p_{cr} \text{ for the pure elastic case).} \quad (2.44)$$

Thus, the values of the critical time or the values of the critical force are determined from the initial imperfection criterion (2.44).

2.3 Approximate Equations for the Stability Loss of the Cylinder-Beam Obtained from Equations of the TDLTS by the Average-Integrating Procedure

2.3.1 Bernoulli Beam theory

For the case under consideration the approximate stability loss equations for the cylinder-beam can be derived from Eqs. (2.31)–(2.36) with the use of the Bernoulli hypothesis, according to which, the displacements of the cylinder are presented as follows.

$$\begin{aligned} u_r^{(1)} &= u_r^{(1)}(\theta, z) = w(z) \cos \theta, \quad u_\theta^{(1)} = u_\theta^{(1)}(\theta, z) = -w(z) \sin \theta, \\ u_z^{(1)} &= u_z^{(1)}(r, \theta, z) = -r \frac{dw(z)}{dz} \cos \theta. \end{aligned} \quad (2.45)$$

In writing the expression (2.45), it is assumed that the elongation of the middle line of the cylinder is very small with respect to the term $-rdw(z)/dz \cos \theta$ and is ignored. Thus, according to (2.31) and (2.45), we obtain that

$$\begin{aligned} \varepsilon_{zz}^{(1)} &= -r \cos \theta \frac{d^2 w(z)}{dz^2}, \quad \varepsilon_{(ij)}^{(1)} = 0 \text{ for } (ij) \neq zz, \quad \sigma_{rr}^{(1)} = \sigma_{\theta\theta}^{(1)} = \sigma_{r\theta}^{(1)} = 0, \\ \sigma_{zz}^{(1)} &= E_3^* \varepsilon_{zz}^{(1)} = -r \cos \theta E_3^* \frac{d^2 w(z)}{dz^2}, \end{aligned} \quad (2.46)$$

where E_3^* is an operator and $E_3^* \phi(t) = E_{30} \phi(t) + \int_0^t E_{31}(t - \tau) \phi(\tau) d\tau$, where E_{30} is the instantaneous value of the modulus of elasticity in the direction of the Oz axis of the cylinder material and $E_{31}(t)$ is a function which determine the relaxation of this material.

Assume that

$$\sigma_{rz}^{(1)} \neq 0, \quad \sigma_{\theta z}^{(1)} \neq 0. \quad (2.47)$$

Taking the relations (2.46) and (2.47) into account we obtain the following equation from (2.31).

$$\begin{aligned} \frac{\partial \sigma_{rz}^{(1)}}{\partial z} + \sigma_{zz}^{(0)} \frac{d^2 w}{dz^2} \cos \theta &= 0, \quad \frac{\partial \sigma_{\theta z}^{(1)}}{\partial z} - \sigma_{zz}^{(0)} \frac{d^2 w}{dz^2} \sin \theta = 0, \\ \frac{\partial \sigma_{rz}^{(1)}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta z}^{(1)}}{\partial \theta} + \frac{1}{r} \sigma_{rz}^{(1)} + \frac{\partial \sigma_{zz}^{(1)}}{\partial z} + \sigma_{zz}^{(0)} \frac{d^3 w}{dz^3} (-r \cos \theta) &= 0. \end{aligned} \quad (2.48)$$

Using the expressions of the equations in (2.48) we can write the following presentations.

$$\sigma_{rz}^{(1)} = s_{rz}^{(1)} \cos \theta, \quad \sigma_{\theta z}^{(1)} = s_{\theta z}^{(1)} \sin \theta, \quad \sigma_{zz}^{(1)} = s_{zz}^{(1)} \cos \theta. \quad (2.49)$$

According to the foregoing presentations, we obtain the following equations from (2.48) and (2.49).

$$\frac{ds_{rz}^{(1)}}{dz} + \sigma_{zz}^{(0)} \frac{d^2 w}{dz^2} = 0, \quad \frac{ds_{\theta z}^{(1)}}{dz} - \sigma_{zz}^{(0)} \frac{d^2 w}{dz^2} = 0. \quad (2.50)$$

Multiplying the last Eq. (2.48) with $(r^2 \cos \theta)$ and integrating with respect to r in the interval $[R^+, R^-]$ and with respect to θ in the interval $[0, 2\pi]$ we can write the following transformations.

$$\begin{aligned}
& \int_{R^-}^{R^+} \int_0^{2\pi} \frac{1}{r} \frac{\partial \sigma_{\theta z}^{(1)}}{\partial \theta} r^2 \cos \theta \, dr d\theta = \pi \int_{R^-}^{R^+} s_{rz}^{(1)} r dr; \\
& \int_{R^-}^{R^+} \int_0^{2\pi} \frac{1}{r} \sigma_{rz}^{(1)} r^2 \cos \theta \, dr d\theta = \pi \int_{R^-}^{R^+} s_{rz}^{(1)} r dr; \\
& \int_{R^-}^{R^+} \int_0^{2\pi} \frac{1}{r} \frac{\partial \sigma_{rz}^{(1)}}{\partial r} r^2 \cos \theta \, dr d\theta = \pi \int_{R^-}^{R^+} \frac{ds_{rz}^{(1)}}{dr} r^2 dr \\
& = \pi \left[\int_{R^-}^{R^+} \frac{d}{dr} \left(s_{rz}^{(1)} r^2 \right) dr - 2 \int_{R^-}^{R^+} s_{rz}^{(1)} r dr \right] \\
& = \pi s_{rz}^{(1)} \Big|_{r=R^+} (R^+)^2 - \pi s_{rz}^{(1)} \Big|_{r=R^-} (R^-)^2 - 2\pi \int_{R^-}^{R^+} s_{rz}^{(1)} r dr \\
& = \pi \sigma_{zz}^{(0)} \left((R^+)^2 + (R^-)^2 \right) \cos(\alpha_1 z) - 2\pi \int_{R^-}^{R^+} s_{rz}^{(1)} r dr, \\
& \int_{R^-}^{R^+} \int_0^{2\pi} \frac{\partial \sigma_{zz}^{(1)}}{\partial z} r^2 \cos \theta \, dr d\theta = \pi \int_{R^-}^{R^+} E_3^* \frac{d^3 w}{dz^3} (-r^3) dr \\
& = -\frac{\pi \left((R^+)^4 - (R^-)^4 \right)}{4} E_3^* \frac{d^3 w}{dz^3} = -J E_3^* \frac{d^3 w}{dz^3}, \\
& J = \frac{\pi \left((R^+)^4 - (R^-)^4 \right)}{4} \tag{2.51}
\end{aligned}$$

Here, the boundary conditions $\sigma_{rz}^{(1)}(R^+, \theta, t_3, t) = 2\pi \sigma_{zz}^{(0)} \cos(\alpha_1 z) \cos \theta, \Rightarrow s_{rz}^{(1)} \Big|_{r=R^+} = 2\pi \sigma_{zz}^{(0)} \cos(\alpha_1 z),$

$$\sigma_{rz}^{(1)}(R^-, \theta, t_3, t) = 2\pi \sigma_{zz}^{(0)} \cos(\alpha_1 z) \cos \theta, \Rightarrow s_{rz}^{(1)} \Big|_{r=R^-} = -2\pi \sigma_{zz}^{(0)} \cos(\alpha_1 z)$$

given in (2.32) are used. Thus, taking the transformation (2.51) and relation (2.50) into account we get the following equation for the displacement $w(z)$ from the last equation in (2.48).

$$\begin{aligned}
& - \left(E_3^* + \sigma_{zz}^{(0)} \right) J \frac{d^4 w}{dz^4} + \pi \left((R^+)^2 - (R^-)^2 \right) \sigma_{zz}^{(0)} \frac{d^2 w}{dz^2} \\
& = \left((R^+)^2 - (R^-)^2 \right) \alpha_1 \sigma_{zz}^{(0)} \sin(\alpha_1 z). \tag{2.52}
\end{aligned}$$

Applying the presentations (2.45) and (2.46) to the end conditions (2.33) and doing the integration over the area of the end cross sections we obtain the following ones for the approximate Bernoulli approach.

$$w|_{z=0;\ell} = 0, \quad \left. \frac{d^2 w}{dz^2} \right|_{z=0;\ell} = 0. \quad (2.53)$$

It is evident that for the real cases the inequality $|\sigma_{zz}^{(0)}| < |E_{31}^*|$ takes place and taking this inequality into account, the $\sigma_{zz}^{(0)}$ can be ignored in the first term of the Eq. (2.52). After this ignoring with the use of the notation $P = -\pi^2 \left((R^+)^2 - (R^-)^2 \right) \sigma_{zz}^{(0)}$ we obtain the classical stability loss equation of the Bernoulli beam from which the Euler critical force within the scope of the initial imperfection criterion is obtained.

For the solution to the boundary value problem (2.52) and (2.53) we apply the Laplace transformation (2.34) to these equations. Using the convolution theorem for transformation of the term $E_3^* d^4 w / dz^4$ we obtain the same equation and end conditions written for $\bar{w}$. For instance, after this transformation we obtain the following equation instead of Eq. (2.52)

$$-\bar{E}_3^* J \frac{d^4 \bar{w}}{dz^4} - P \frac{d^2 \bar{w}}{dz^2} = -\frac{1}{s\pi} \alpha_1 P \sin(\alpha_1 z). \quad (2.54)$$

According to the end condition (2.53), the solution to the Eq. (2.54) is taken as follows

$$\bar{w} = \bar{A}(s) \sin \alpha_1 z. \quad (2.55)$$

Substituting (2.55) into (2.54) and doing some mathematical calculations we obtain the following expression for the unknown $\bar{A}(s)$.

$$\bar{A}(s) = -\frac{P\ell}{s\pi^2} \left[P - \frac{\pi^2 \bar{E}_3^* J}{\ell^2} \right]^{-1}. \quad (2.56)$$

Employing the method (Schapery 1962, 1978), the original of $\bar{A}(s)$ is determined as follows.

$$A(t) = (s\bar{A}(s))|_{s=(1/(2t))}. \quad (2.57)$$

For the cases where the cylinder material is pure elastic we obtain from (2.56) the expression

$$A = -\frac{P\ell}{\pi^2} \left[P - \frac{\pi^2 E_3 J}{\ell^2} \right]^{-1}. \quad (2.58)$$

According to (2.44), the stability loss criterion for the Bernoulli beam can be expressed as

$$\begin{aligned}
A(t) &\rightarrow \infty \text{ as } t \rightarrow t_{cr} \text{ (for the case where the cylinder material is viscoelastic),} \\
A &\rightarrow \infty \text{ as } P \rightarrow P_{cr} \text{ (for the case where the cylinder material is pure elastic)}
\end{aligned} \quad (2.59)$$

The following expression is obtained from (2.58) and (2.59) for the critical force which coincide with Euler critical force

$$P_{Eu.cr} = \frac{\pi^2 E_3 J}{\ell^2}. \quad (2.60)$$

2.4 The Third Order Refined Beam Theory

For the third order refined beam theory we use the modification of the theory (Kromm 1953, 1955) for the cylinder-beam, according to which, the displacements $u_r^{(1)}$ and $u_\theta^{(1)}$ are presented as in (2.45), but the displacement $u_z^{(1)}$ is presented as

$$\begin{aligned}
u_z^{(1)} = u_z^{(1)}(r, \theta, z) = & -r \frac{dw(z)}{dz} \cos \theta \\
& + \frac{1}{4} \left(\left((R^+)^2 + (R^-)^2 \right) r \cos \theta - \frac{1}{3} r^3 \cos^3 \theta \right) \varphi(z),
\end{aligned} \quad (2.61)$$

where $\varphi(z)$ is an average shear deformation of the cross section of the cylinder-beam.

Substituting the expressions for $u_r^{(1)}$, $u_\theta^{(1)}$ and $u_z^{(1)}$ into Eq. (2.31) we obtain the expression for strains and stresses given below.

$$\begin{aligned}
\varepsilon_{rz}^{(1)} &= \frac{1}{8} ((R^+)^2 + (R^-)^2) \cos \theta \left(1 - \frac{r^2 \cos^2 \theta}{(R^+)^2 + (R^-)^2} \right) \varphi(z), \\
\varepsilon_{\theta z}^{(1)} &= -\frac{1}{8} ((R^+)^2 + (R^-)^2) \sin \theta \left(1 - \frac{r^2 \cos^2 \theta}{(R^+)^2 + (R^-)^2} \right) \varphi(z), \\
\varepsilon_{rz}^{(1)} &= -r \cos \theta \frac{d^2 w(z)}{dz^2} + \frac{1}{4} (((R^+)^2 + (R^-)^2) r \cos \theta - \frac{r^3 \cos \theta}{3}) \frac{d\varphi(z)}{dz}, \\
\varepsilon_{rr}^{(1)} &= \varepsilon_{\theta\theta}^{(1)} = \varepsilon_{r\theta}^{(1)} = 0, \quad \sigma_{rr}^{(1)} = \sigma_{\theta\theta}^{(1)} = \sigma_{r\theta}^{(1)} = 0, \\
\sigma_{rz}^{(1)} &= 2G^* \varepsilon_{rz}^{(1)} = G^* \frac{1}{4} ((R^+)^2 + (R^-)^2) \cos \theta \\
&\quad \times \left(1 - \frac{r^2}{((R^+)^2 + (R^-)^2)} \cos^2 \theta \right) \varphi(z), \\
\sigma_{\theta z}^{(1)} &= 2G^* \varepsilon_{\theta z}^{(1)} = -G^* \frac{1}{4} ((R^+)^2 + (R^-)^2) \sin \theta \left(1 - \frac{r^2}{((R^+)^2 + (R^-)^2)} \cos^2 \theta \right) \varphi(z), \\
\sigma_{zz}^{(1)} &= E_3^* \varepsilon_{zz}^{(1)} = E_3^* \left(-r \cos \theta \frac{d^2 w(z)}{dz^2} + \frac{1}{4} ((R^+)^2 \right. \\
&\quad \left. + (R^-)^2) r \cos \theta - \frac{r^3 \cos \theta}{3} \right) \frac{d\varphi(z)}{dz}.
\end{aligned} \quad (2.62)$$

It follows from the expressions given in (2.62) that the Eqs. (2.48) and (2.49) hold also for the considered case. First, we apply the Laplace transformation to the Eqs. (2.48), (2.49) and (2.62) and doing the integration procedure which is similar to that made in the previous subsection we get from Eq. (2.48) the following equation for the function $\bar{w}(z, s)$.

$$E_3^* J \left[-1 - \frac{\sigma_{zz}^{(0)}}{E_3^*} - \frac{4}{3} \frac{\Re \sigma_{zz}^{(0)}}{\left((R^+)^2 + (R^-)^2 \right) G^*} \left(1 + \frac{\sigma_{zz}^{(0)}}{E_3^*} \right) \right] \frac{d^4 \bar{w}(z, s)}{dz^4} + \pi \left((R^+)^2 + (R^-)^2 \right) \sigma_{zz}^{(0)} \frac{d^2 \bar{w}(z, s)}{dz^2} = \frac{\alpha \pi \left((R^+)^2 + (R^-)^2 \right)}{s} \sigma_{zz}^{(0)} \sin \alpha_1 z, \quad (2.63)$$

where

$$\Re = \left((R^+)^2 + (R^-)^2 \right) - \frac{(R^+)^6 - (R^-)^6}{9 \left((R^+)^4 - (R^-)^4 \right)}. \quad (2.64)$$

Thus, using the presentation (2.55) and the estimation

$$-1 - \frac{\sigma_{zz}^{(0)}}{E_3^*} - \frac{4}{3} \frac{\Re \sigma_{zz}^{(0)}}{\left((R^+)^2 + (R^-)^2 \right) G^*} \left(1 + \frac{\sigma_{zz}^{(0)}}{E_3^*} \right) \approx 1 - \frac{4}{3} \frac{\Re \sigma_{zz}^{(0)}}{\left((R^+)^2 + (R^-)^2 \right) G^*}$$

we obtain from (2.63) the following expression for $\bar{A}(s)$

$$\bar{A}(s) = -\frac{P\ell}{s\pi^2} \left[P \left(1 + \frac{1}{3} \frac{\bar{E}_3^* \Re}{G^*} \left(\frac{\pi}{\ell} \right)^2 \right) - \frac{\pi^2 \bar{E}_3^* J}{\ell^2} \right]^{-1} \quad (2.65)$$

where, as in the previous case, the notation $P = -\pi^2 \left((R^+)^2 + (R^-)^2 \right) \sigma_{zz}^{(0)}$ is used.

By employing the method (2.57) we determine the function $A(t)$. For determination of the values of the critical parameters we can also use the expression (2.59). The following expression can be written for the unknown A from Eq. (2.65) for the case where the material of the cylinder is pure elastic.

$$A = -\frac{P\ell}{s\pi^2} \left[P \left(1 + \frac{1}{3} \frac{E_3 \Re}{G} \left(\frac{\pi}{\ell} \right)^2 \right) - \frac{\pi^2 E_3 J}{\ell^2} \right]^{-1} \quad (2.66)$$

From (2.66) the following expression for the critical force for the case where the cylinder's material is pure elastic one can be written.

$$P = -\frac{\pi^2 E_3 J}{\ell^2} \left(1 + \frac{1}{3} \frac{E_3 \Re}{G} \left(\frac{\pi}{\ell} \right)^2 \right)^{-1} \quad (2.67)$$

2.5 Numerical Results and Discussions

We assume that the cylinder is made from viscoelastic unidirectional fibrous composite material and its fibers lie along the Oz axis. In the discussions below, the values related to the matrix and the fibers will be denoted by upper indices (1) and (2), respectively. The material of the fibers is supposed to be pure elastic with Young's modulus $E^{(2)}$, Poisson coefficient $\nu^{(2)}$, shear modulus $\mu^{(2)}$, but the material of the matrix is supposed to be linearly viscoelastic with operators

$$\begin{aligned} E^*(1)\varphi &= E_0^{(1)}[\varphi(t) - \omega_0 \Pi_\alpha^*(-\omega_0 - \omega_\infty)\varphi], \\ \nu^{*(1)}\varphi &= \nu_0^{(1)}\left[\varphi(t) + \frac{1 - 2\nu_0^{(1)}}{2\nu_0^{(1)}}\omega_0 \Pi_\alpha^*(-\omega_0 - \omega_\infty)\varphi\right], \\ \mu^{*(1)}\varphi &= \mu_0^{(1)}\left[\varphi(t) - \frac{3}{2(1 + \nu_0^{(1)})}\omega_0 \Pi_\alpha^*\left(-\frac{1}{2(1 + \nu_0^{(1)})}\omega_0 - \omega_\infty\right)\varphi\right], \end{aligned} \quad (2.68)$$

where $E_0^{(1)}$, $\nu_0^{(1)}$ are the instantaneous values of Young's modulus and the Poisson coefficient, respectively, $\mu_0^{(1)}$ is the instantaneous value of shear modulus, α , ω_0 and ω_∞ are the rheological parameters of the matrix material, Π_α^* is the fractional exponential operator (Rabotnov 1977), and this operator is determined as

$$\Pi_\alpha^*(x)\varphi = \int_0^t \Pi_\alpha(x, t - \tau)\varphi(\tau)d\tau, \quad (2.69)$$

where

$$\Pi_\alpha(x, t) = t^\alpha \sum_{n=0}^{\infty} \frac{x^n t^{n(1+\alpha)}}{\Gamma((1+n)(1+\alpha))}, \quad -1 < \alpha < 0 \quad (2.70)$$

where $\Gamma(x)$ is the Gamma function.

Note that the fractional-exponential operators (Rabotnov 1977) allow us to describe with the accuracy required the initial parts of the experimentally and theoretically constructed creep and relaxation graphs. These operators also allow us to determine with very high accuracy, the asymptotic values of these graphs. Operators (2.69) and (2.70) are employed successfully to describe various polymer materials and epoxy-based composites with continuous fibers or layers. The values of the rheological parameters in relation (2.68) were determined for these materials in the monograph (Rabotnov 1977). Moreover, these operators have many simple rules for complicated mathematical transformations, for example, the Laplace transformation, the expression of which is given below.

We introduce the dimensionless rheological parameter $\omega(= \omega_\infty/\omega_0)$ and the dimensionless time $t' (= \omega_0^{1/(1+\alpha)} t)$ and assume that $\nu^{(2)} = \nu_0^{(1)} = 0.3$, $\eta^{(2)} = 0.5$ where $\eta^{(2)}$ is a fiber volume fraction in the composite under consideration.

It is known that, within the scope of the continuum approach this composite can be taken as homogeneous transversal isotropic one the isotropy axis of which coincide with the Oz axis. According to (Christensen 1979) and many others, by replacing the mechanical constants of the constituents of a composite with Laplace transformation of the corresponding operators in the expressions of the effective mechanical properties, we determine the Laplace transform of the effective operators. Therefore, in the Laplace transform of the constitutive relations (2.35) instead of $\bar{A}_{ij}^*$ and $\bar{G}^*$ we write these expressions. For the considered composite material the expressions for $\bar{A}_{ij}^*$ and $\bar{G}^*$ are determined as follows:

$$\begin{aligned}\bar{A}_{33}^* &= \bar{E}_3^* + 4(\bar{v}_{31}^*)^2 \bar{K}_{21}^*, \quad \bar{A}_{13}^* = 2\bar{v}_{31}^* \bar{K}_{21}^*, \quad \bar{A}_{11}^* = \bar{\mu}_{12}^* + \bar{K}_{21}^*, \quad \bar{A}_{12}^* = -\bar{\mu}_{12}^* + \bar{K}_{21}^*, \\ \bar{G}^* &= \bar{\mu}^{(1)} \frac{\bar{\mu}^{(2)}(1 + \eta^{(2)}) + \bar{\mu}^{*(1)}(1 - \eta^{(2)})}{\bar{\mu}^{(2)}(1 - \eta^{(2)}) + \bar{\mu}^*(1)(1 + \eta^{(2)})}\end{aligned}\quad (2.71)$$

where

$$\begin{aligned}\bar{K}_{12}^* &= \bar{K}^{*(1)} + \frac{1}{3}\bar{\mu}^{*(1)} + \eta^{(2)} \left[\left(\frac{1}{3}(\bar{\mu}^{(2)} - \bar{\mu}^{*(1)}) + \bar{K}^{(2)} - \bar{K}^{*(1)} \right)^{-1} \right. \\ &\quad \left. + \frac{(1 - \eta^{(2)})}{\bar{K}^{*(1)} + \frac{4}{3}\bar{\mu}^{*(1)}} \right]^{-1}, \quad \bar{E}_3^* = \eta^{(2)}\bar{E}^{(2)} + (1 - \eta^{(2)})\bar{E}^{*(1)} \\ &\quad + \frac{4\eta^{(2)}(1 - \eta^{(2)})(\bar{v}^{(2)} - \bar{v}^{*(1)})^2 \bar{\mu}^{*(1)}}{(1 - \eta^{(2)})\bar{\mu}^{*(1)}(\bar{K}^{(2)} + \bar{\mu}^{(2)}/3)^{-1} + \eta^{(2)}\bar{\mu}^{*(1)}(\bar{K}^{*(1)} + \bar{\mu}^{*(1)}/3)^{-1}}, \\ \bar{v}_{31}^* &= \eta^{(2)}\bar{v}^{(2)} + (1 - \eta^{(2)})\bar{v}^{*(1)} + 4\eta^{(2)}(1 - \eta^{(2)})\left(\bar{v}^{(2)} - \bar{v}^{*(1)}\right)^2 \\ &\quad \times \frac{\left[\bar{\mu}^{*(1)}(\bar{K}^{*(1)} + \bar{\mu}^{*(1)}/3)^{-1} - \bar{\mu}^{*(1)}(\bar{K}^{(2)} + \bar{\mu}^{(2)}/3)^{-1} \right]}{(1 - \eta^{(2)})\bar{\mu}^{*(1)}(\bar{K}^{(2)} + \bar{\mu}^{(2)}/3)^{-1} + \eta^{(2)}\bar{\mu}^{*(1)}(\bar{K}^{*(1)} + \bar{\mu}^{*(1)}/3)^{-1}}, \\ \bar{\mu}_{12}^* &= \bar{\mu}^{*(1)} \left\{ 1 + \eta^{(2)} \left[\frac{\bar{\mu}^{*(1)}}{\bar{\mu}^{(2)} - \bar{\mu}^{*(1)}} + \frac{\bar{K}^{*(1)} + 7\bar{\mu}^{*(1)}/3}{\bar{K}^{*(1)} + 8\bar{\mu}^{*(1)}/3} \right]^{-1} \right\}.\end{aligned}\quad (2.72)$$

Here the following notation is used:

$$\begin{aligned}\bar{K}^{*(1)} &= \frac{\bar{E}^{*(1)}}{3(1 - 2\bar{v}^{*(1)})}, \quad \bar{K}^{(2)} = \frac{\bar{E}^{(2)}}{3(1 - 2\bar{v}^{(2)})}, \\ \bar{\mu}^{(2)} &= \frac{\bar{E}^{(2)}}{3(1 + 2\bar{v}^{(2)})}, \quad \bar{E}^{(2)} = \frac{1}{s}E^{(2)}, \quad \bar{v}^{(2)} = \frac{1}{s}v^{(2)},\end{aligned}$$

$$\begin{aligned}
\bar{E}^{*(1)} &= E_1^{(0)} \left[\frac{1}{s} - \bar{\Pi}_x(-\omega) \right], \quad \bar{v}^{*(1)} = v_1^{(0)} \left[\frac{1}{s} - \frac{1 - 2v_0^{(1)}}{2v_0^{(1)}} \bar{\Pi}_x(-\omega) \right], \\
\bar{\mu}^{*(1)} &= \mu_0^{(1)} \left[\frac{1}{s} - \frac{3}{2(1+v_0^{(1)})} \bar{\Pi}_x \left(-\frac{3}{2(1+v_0^{(1)})} - \omega \right) \right], \\
\bar{\Pi}_x(-x) &= \frac{1}{s(s^{1+x}+x)}.
\end{aligned} \tag{2.73}$$

Thus, within the framework of the foregoing preparation we consider the numerical results and first examine the pure elastic stability loss under $t' = 0$ and $t' = \infty$, because in the viscoelastic stability loss of the cylinder the intensity of the external compressed force

$$p = \sigma_{zz}^{(0)} = P \left[\pi \left((R^+)^2 - (R^-)^2 \right) \right]^{-1} \tag{2.74}$$

must satisfy the inequality

$$p_{cr,\infty} < p < p_{cr,0} \tag{2.75}$$

where $p_{cr,0}$ ($p_{cr,\infty}$) is the critical force obtained at $t' = 0$ ($t' = \infty$).

We compare the results obtained within the scope of the TDLTS with the corresponding ones obtained within the scope of the Bernoulli beam theory and within the scope of the third order refined beam theory. Through $p'_{3D,c,0} (= p_{3D,c,0}/E_0^{(1)})$ and $p'_{3D,c,\infty} (= p_{3D,c,\infty}/E_0^{(1)})$ we denote the critical values of the intensity of the dimensionless forces obtained by employing the TDLTS. Here and below the sub-indices 0 and ∞ indicate that the critical values of the forces are calculated at $t' = 0$ and at $t' = \infty$ respectively. According to the Eqs. (2.60) and (2.67), the corresponding critical values of the intensity of the compressed forces obtained within the scope of the Bernoulli beam theory (denoted by $p'_{E,c,0}$ and $p'_{E,c,\infty}$) and within the scope of the third order refined beam theory (denoted by $p'_{R,c,0}$ and $p'_{R,c,\infty}$) can be calculated by the use of the following expressions.

$$\begin{aligned}
p'_{E,c,0} &= \frac{E_{30}}{4E_0^{(1)}} \left[\left(\frac{\pi R^+}{\ell} \right)^2 + \left(\frac{\pi R^-}{\ell} \right)^2 \right], \quad p'_{E,c,\infty} = \frac{E_{3\infty}}{4E_0^{(1)}} \left[\left(\frac{\pi R^+}{\ell} \right)^2 + \left(\frac{\pi R^-}{\ell} \right)^2 \right], \\
p'_{R,c,0} &= \frac{E_{30}}{4E_0^{(1)}} \left[\left(\frac{\pi R^+}{\ell} \right)^2 + \left(\frac{\pi R^-}{\ell} \right)^2 \right] \left(1 + \frac{1}{3} \frac{E_{30} \Re}{G_0} \left(\frac{\pi}{\ell} \right)^2 \right)^{-1}, \\
p'_{R,c,\infty} &= \frac{E_{3\infty}}{4E_0^{(1)}} \left[\left(\frac{\pi R^+}{\ell} \right)^2 + \left(\frac{\pi R^-}{\ell} \right)^2 \right] \left(1 + \frac{1}{3} \frac{E_{3\infty} \Re}{G_\infty} \left(\frac{\pi}{\ell} \right)^2 \right)^{-1}.
\end{aligned} \tag{2.76}$$

Substituting $R^+ = R$, $R^- = 0$ into expressions for J (2.51) and $\Re$ (2.64), we obtain from (2.76) the following expressions for the solid cylinder.

Table 2.1 The values of $p'_{3D.c.0}$, $p'_{R.c.0}$ and $p'_{E.c.0}$ obtained for various values of $E^{(2)}/E_0^{(1)}$ and ρ

$\frac{E^{(2)}}{E_0^{(1)}}$	$\rho = \frac{\pi R}{\ell}$								
	0.1			0.2			0.3		
	$p'_{3D.c.0}$	$p'_{R.c.0}$	$p'_{E.c.0}$	$p'_{3D.c.0}$	$p'_{R.c.0}$	$p'_{E.c.0}$	$p'_{3D.c.0}$	$p'_{R.c.0}$	$p'_{E.c.0}$
1	0.00247	0.00248	0.0025	0.0096	0.0097	0.010	0.0207	0.0210	0.0224
5	0.00740	0.00741	0.0075	0.0284	0.0286	0.030	0.0603	0.0611	0.0675
10	0.0134	0.01351	0.0137	0.0510	0.0513	0.055	0.1055	0.1066	0.1237

$$\begin{aligned}
p'_{E.c.0} &= \frac{E_{30}}{4E_0^{(1)}} \left(\frac{\pi R}{\ell} \right)^2, \quad p'_{R.c.0} = \frac{E_{30}}{4E_0^{(1)}} \left(\frac{\pi R}{\ell} \right)^2 \left(1 + \frac{8}{27} \frac{E_{30}}{G_0} \right)^{-1}, \\
p'_{E.c.\infty} &= \frac{E_{3\infty}}{4E_0^{(1)}} \left(\frac{\pi R}{\ell} \right)^2, \quad p'_{R.c.\infty} = \frac{E_{3\infty}}{4E_0^{(1)}} \left(\frac{\pi R}{\ell} \right)^2 \left(1 + \frac{8}{27} \frac{E_{3\infty}}{G_0} \right)^{-1}. \quad (2.77)
\end{aligned}$$

Now we analyze the numerical results obtained within the scope of the foregoing assumptions. The results related to the solid and hollow cylinders will be analyzed separately.

2.5.1 Solid Cylinder

Table 2.1 (Table 2.2) shows the values of $p'_{3D.c.0}$, $p'_{R.c.0}$ and $p'_{E.c.0}$ ($p'_{3D.c.\infty}$, $p'_{R.c.\infty}$ and $p'_{E.c.\infty}$) calculated for various values of $E^{(2)}/E_0^{(1)}$, ω and $\rho (= \pi R/\ell)$. Note that the values of $p'_{E.c.0}$, $p'_{R.c.0}$, $p'_{E.c.\infty}$ and $p'_{R.c.\infty}$ are calculated by the use of the expression (2.77). However, the values of $p'_{3D.c.0}$ and $p'_{3D.c.\infty}$ are calculated by the use of the criterion (2.44) and by the use of the solution of the system equation which is obtained from the first three equations of the system equations given in (2.42) by taking the equalities $C_4 = 0$, $C_5 = 0$ and $C_6 = 0$ into account and by replacing the notation R^+ with R .

It follows from these results that the values of $p'_{R.c.0}$ ($p'_{R.c.\infty}$) are very near to the corresponding values of $p'_{3D.c.0}$ ($p'_{3D.c.\infty}$) and the difference between them is insignificant. However the values of $p'_{E.c.0}$ ($p'_{E.c.\infty}$) are greater than the corresponding values of $p'_{3D.c.0}$ ($p'_{3D.c.\infty}$) and the difference between them becomes more significant with $E^{(2)}/E_0^{(1)}$ and ρ .

Consequently, it can be concluded from the foregoing results that for the pure elastic stability loss problems of the considered cylinder, the third order refined beam theory gives results which are sufficiently close to the corresponding ones obtained within the scope of the TDLTS.

Table 2.2 The values of $p'_{3D.c.\infty}$, $p'_{R.c.\infty}$ and $p'_{E.c.0}$ obtained for various values of $E^{(2)}/E_0^{(1)}$, $\rho = \pi R/\ell$ and ω

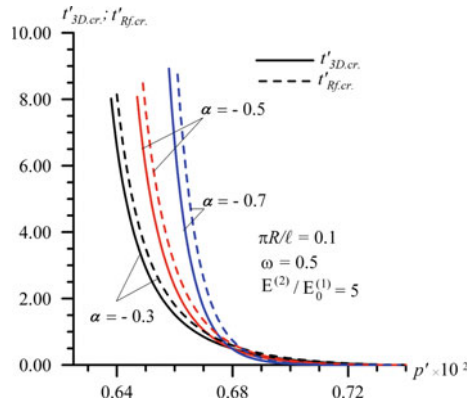
$\frac{E^{(2)}}{E_0^{(1)}}$	ρ								
	0.1			0.2			0.3		
	$p'_{3D.c.\infty}$	$p'_{R.c.0}$	$p'_{E.c.0}$	$p'_{3D.c.\infty}$	$p'_{R.c.0}$	$p'_{E.c.0}$	$p'_{3D.c.\infty}$	$p'_{R.c.0}$	$p'_{E.c.0}$
$\omega = 0.5$									
1	0.00163	0.00163	0.00166	0.00611	0.00614	0.0066	0.0124	0.0125	0.015
5	0.00621	0.00622	0.00666	0.02073	0.02073	0.0266	0.0366	0.0365	0.060
10	0.01138	0.01137	0.01291	0.03368	0.03353	0.0516	0.0530	0.0524	0.116
$\omega = 1.0$									
1	0.00184	0.00185	0.00187	0.00709	0.00713	0.0075	0.0149	0.0151	0.016
5	0.00664	0.00665	0.00687	0.02411	0.02420	0.0275	0.0471	0.0473	0.061
10	0.01235	0.01236	0.01312	0.04207	0.04212	0.0525	0.0760	0.0759	0.118
$\omega = 2.0$									
1	0.00206	0.00206	0.00208	0.00796	0.0080	0.0083	0.01698	0.0172	0.018
5	0.00692	0.00693	0.00708	0.02560	0.0261	0.0283	0.05315	0.0536	0.063
10	0.01285	0.01287	0.01333	0.04645	0.0466	0.0533	0.09021	0.0906	0.120

Now we consider the case where the cylinder material is viscoelastic and analyze the critical time t'_{cr} . To compare the values of t'_{cr} given by the TDLTS with those given by approximate beam theories, we must select the values of $p' (= p/E_0^{(1)})$ which fall in the intervals $[p'_{E.c.\infty}, p'_{E.c.0}]$, $[p'_{R.c.\infty}, p'_{R.c.0}]$ and $[p'_{3D.c.\infty}, p'_{3D.c.0}]$ simultaneously. However, as follows from the data given in Tables 2.1 and 2.2, in many cases such common values of $p' (= p/E_0^{(1)})$ exist only for the intervals $[p'_{R.c.\infty}, p'_{R.c.0}]$ and $[p'_{3D.c.\infty}, p'_{3D.c.0}]$. For this reason, we will compare only the critical time obtained by employing the TDLTS (denoted by $t'_{3D.cr.}$) with that obtained by employing the third order refined beam theory (denoted by $t'_{Rf.cr.}$).

Thus, consider the data given in Tables 2.3–2.5 which show the values of the $t'_{3D.cr.}$ and $t'_{Rf.cr.}$ obtained for the cases where $\rho = 0.1$, 0.2 and 0.3 respectively for various values of p' and α under $\omega = 0.5$, $E^{(2)}/E_0^{(1)} = 5$. Moreover, for a clearer illustration of the results related to the critical time in Fig. 2.2 the graphs of the dependencies among $t'_{3D.cr.}$, $t'_{Rf.cr.}$ and p' are given for various values of α in the case where $\rho = 0.1$, $\omega = 0.5$.

It follows from the results given in Fig. 2.2 and from other numerical results (which are not given here) that the graphs of the analyzed dependencies constructed for various values of α have a common intersection point. Before (after) this intersection point an increase in the absolute values of α causes an increase (decrease) in the values of the critical time. Figure 2.3 illustrates the parts of the graphs which appear after the aforementioned intersection point and it follows from this investigation that the position of the intersection point does not depend on the values of α . According to this statement and according to the expression of

Fig. 2.2 The graphs of the dependencies among $t'_{3D.cr.}$, $t'_{Rf.cr.}$, and dimensionless intensity of the compressed force p'



$\bar{\Pi}_\alpha$ given in (2.73), it can be concluded that the “intersection point” corresponds to the case where the Laplace transformation parameter s is equal to one, i.e. $s = 1$. In other words, according to the method (Schapery 1962, 1978), i.e. according to the expressions (2.43) or (2.57), the “intersection point” corresponds to the case where $t'_{cr} = 0.5$. Consequently, the foregoing conclusion on the influence of the rheological parameter α on the values of t'_{cr} can be rewritten as follows:

If $t'_{cr} > 0.5$ ($t'_{cr} < 0.5$) the increase in the absolute values of α causes an increase (a decrease) in the values of the t'_{cr} .

This conclusion holds for all problems considered and studied in the present book. From the viewpoint of the author, in the qualitative sense, the foregoing conclusion follows from the nature of the viscoelastic behavior of deformable solid bodies, but not from either special selection of the operator (2.69) or from the inverse transformation method (Schapery 1962, 1978). For example, it is evident that use of the more accurate inverse transformation method rather than Schapery’s method (1962, 1978), does not change significantly the value of t'_{cr} which corresponds to $s = 1$.

Note that the parameter α is one of the rheological parameters of the matrix material and characterizes the mechanical behavior of this viscoelastic material around the initial state of the deformation, i.e. in the near vicinity of $t' = 0$. However, the other dimensionless rheological parameter ω characterizes the mechanical properties of the matrix material around $t' = \infty$.

As the values of $E_\infty^{(1)}(v_\infty^{(1)})$ increase (decrease) with ω , therefore, it can be predicted that the values of $p'_{cr.\infty}$ must increase with ω . This prediction is proved by the data given in Table 2.2. Moreover, the influence of the parameter ω on the graphs of the dependencies among $t'_{3D.cr.}$, $t'_{Rf.cr.}$, and p' is illustrated by the graphs given in Fig. 2.4.

Analyses of the results given in Tables 2.3, 2.4, 2.5 show that in the case under consideration, i.e. in the case where $E^{(2)}/E_0^{(1)} = 5$, $\omega = 0.5$, $\rho = 0.1, 0.2$ and 0.3 the values of $t'_{Rf.cr.}$ are greater than the corresponding values of $t'_{3D.cr.}$. The

Fig. 2.3 The parts of the graphs given in the Fig. 2.2 which arise after the intersection point

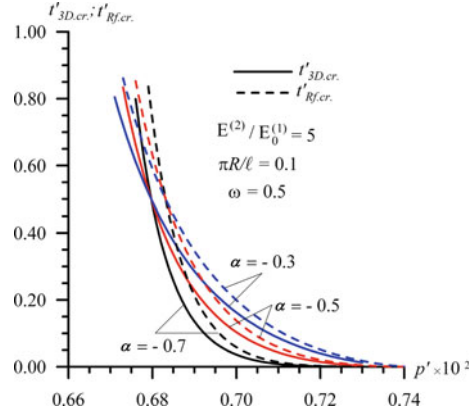
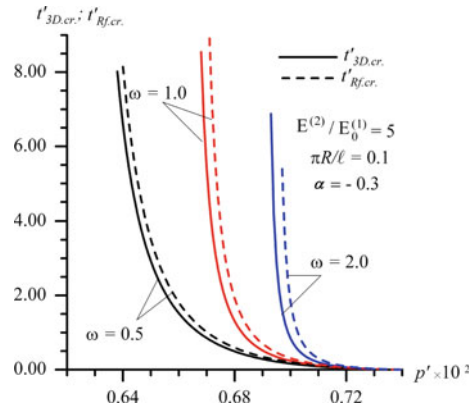


Fig. 2.4 The graphs of the dependencies among $t'_{3D.cr.}$, $t'_{Rf.cr.}$ and p' constructed for various values of rheological parameter ω



difference between $t'_{Rf.cr.}$ and $t'_{3D.cr.}$ depends not only on the rheological parameters ω and α , but also on the values of the intensity of the external compressive forces p' and, as can be predicted, there exist the following relations:

$$\begin{aligned} t'_{3D.cr.} &\rightarrow 0 \text{ as } p' \rightarrow p'_{3D.c.0}; \quad t'_{3D.cr.} \rightarrow \infty \text{ as } p' \rightarrow p'_{3D.c.\infty} \\ t'_{Rf.cr.} &\rightarrow 0 \text{ as } p' \rightarrow p'_{R.c.0}; \quad t'_{Rf.cr.} \rightarrow \infty \text{ as } p' \rightarrow p'_{R.c.\infty} \end{aligned} \quad (2.78)$$

The results given in Tables 2.3, 2.4, 2.5, and many other results (which are not given here), show that the insignificant difference between $p'_{3D.c.0}$ and $p'_{R.c.0}$, as well as the insignificant difference between $p'_{3D.c.\infty}$ and $p'_{R.c.\infty}$, causes the significant difference between the values of $t'_{3D.cr.}$ and $t'_{Rf.cr.}$. In this case, if $p'_{3D.c.0} > p'_{R.c.0}$ (or if $p'_{3D.c.0} < p'_{R.c.0}$) then the values of $t'_{3D.cr.}$ obtained for p' which are close to the $p'_{3D.c.0}$ or to the $p'_{R.c.0}$, are greater (less) than the corresponding values of $t'_{Rf.cr.}$. Moreover, if $p'_{3D.c.\infty} > p'_{R.c.\infty}$ (or if $p'_{3D.c.\infty} < p'_{R.c.\infty}$) then the values of $t'_{3D.cr.}$ obtained for the p' which is close to the $p'_{3D.c.\infty}$ or to the $p'_{R.c.\infty}$, are greater (less) than the corresponding

Table 2.3 The values of $t'_{3D.cr.}$ (upper number) and $t'_{Rf.cr.}$ (lower number) obtained for various values of p' and α under $\rho = 0.1$, $\omega = 0.5$ and $E^{(2)}/E_0^{(1)} = 5$

α	$p' \times 10^2$								
	0.637	0.650	0.660	0.670	0.680	0.690	0.700	0.710	0.720
-0.3	10.59 11.24	3.341 3.508	1.726 1.810	0.967 1.015	0.561 0.590	0.328 0.346	0.187 0.199	0.100 0.108	0.046 0.051
-0.5	35.92 39.06	7.143 7.648	2.835 3.030	1.259 1.347	0.588 0.631	0.277 0.299	0.187 0.199	0.052 0.058	0.018 0.020
-0.7	620.9 713.9	42.05 47.13	9.015 10.07	2.331 2.609	0.655 0.738	0.187 0.212	0.050 0.058	0.011 0.014	0.0019 0.0025

Table 2.4 The values of $t'_{3D.cr.}$ (upper number) and $t'_{Rf.cr.}$ (lower number) obtained for various values of p' and α under $\rho = 0.2$, $\omega = 0.5$ and $E^{(2)}/E_0^{(1)} = 5$

α	$p' \times 10^2$								
	2.250	2.300	2.350	2.400	2.450	2.500	2.550	2.600	2.650
-0.3	5.704 5.998	3.465 3.660	2.227 2.366	1.475 1.578	0.990 1.068	0.664 0.724	0.439 0.486	0.282 0.318	0.173 0.200
-0.5	15.10 16.20	7.517 8.117	4.049 4.408	2.275 2.501	1.301 1.447	0.743 0.840	0.417 0.480	0.225 0.266	0.113 0.139
-0.7	146.4 165.7	45.79 52.04	16.33 18.81	6.247 7.315	2.462 2.941	0.969 1.187	0.370 0.468	0.132 0.174	0.042 0.059

Table 2.5 The values of $t'_{3D.cr.}$ (upper number) and $t'_{Rf.cr.}$ (lower number) obtained for various values of p' and α under $\rho = 0.3$, $\omega = 0.5$ and $E^{(2)}/E_0^{(1)} = 5$

α	$p' \times 10^2$								
	4.100	4.200	4.300	4.400	4.500	4.600	4.700	4.800	4.900
-0.3	8.151 8.208	5.634 5.730	4.074 4.178	3.035 3.137	2.308 2.403	1.779 1.866	1.383 1.462	1.080 1.151	0.843 0.908
-0.5	24.89 25.13	14.84 15.20	9.428 9.769	6.246 6.541	4.257 4.504	2.957 3.161	2.078 2.247	1.469 1.608	1.040 1.153
-0.7	336.9 342.4	142.3 148.0	66.80 70.87	33.62 36.31	17.74 19.50	9.670 10.81	5.373 6.119	3.015 3.504	1.695 2.013

values of $t'_{Rf.cr.}$. The results illustrated in Tables 2.3, 2.4, 2.5 correspond to the case where $p'_{3D.c.0} < p'_{R.c.0}$ and $p'_{3D.c.\infty} < p'_{R.c.\infty}$. Therefore in the all cases considered in these tables the values of $t'_{3D.cr.}$ are less than the corresponding values of $t'_{Rf.cr.}$.

Consider the numerical results related to the other aforementioned cases, namely the case where $p'_{3D.c.0} < p'_{R.c.0}$ and $p'_{3D.c.\infty} > p'_{R.c.\infty}$. The results given in Tables 2.1 and 2.2 show that the values of $p'_{3D.c.0}$, $p'_{R.c.0}$, $p'_{3D.c.\infty}$ and $p'_{R.c.\infty}$ obtained, for example for the case where $\rho = 0.1$, $E^{(2)}/E_0^{(1)} = 10$ and $\omega = 0.5$ satisfy these inequalities. The corresponding values of $t'_{3D.cr.}$ and $t'_{Rf.cr.}$ obtained for the case mentioned are given in Table 2.6. Consequently, according to the foregoing discussion, the following relations must occur:

Table 2.6 The values of $t'_{3D.cr.}$ (upper number) and $t'_{Rf.cr.}$ (lower number) obtained for various values of p' and α under $\rho = 0.1$, $\omega = 0.5$ and $E^{(2)}/E_0^{(1)} = 10$

α	$p' \times 10^2$								
	1.15	1.16	1.17	1.18	1.19	1.20	1.21	1.22	1.23
-0.3	73.51 68.95	27.39 26.73	14.47 14.35	8.865 8.869	5.881 5.920	4.094 4.141	2.937 2.984	2.148 2.191	1.588 1.627
-0.5	541.1 494.7	135.8 131.3	55.61 54.96	28.00 28.02	15.76 15.91	9.493 9.648	5.964 6.098	3.849 3.958	2.522 2.608

Table 2.7 The values of $t'_{cdm.cr.}$ (upper number) and $t'_{3D.cr.}$ (lower number) obtained for various values of p' and α under $\rho = 0.1$, $\omega = 0.5$ and $E^{(2)}/E_0^{(1)} = 5$

α	$p' \times 10^2$						
	0.660	0.670	0.680	0.690	0.700	0.710	0.720
-0.3	52.77 1.726	52.77 0.967	1.179 0.561	0.532 0.328	0.262 0.187	0.127 0.100	0.055 0.046
-0.5	340.2 2.835	7.577 1.259	1.662 0.588	0.545 0.277	0.202 0.126	0.074 0.052	0.023 0.018
-0.7	26326 9.015	46.40 2.331	3.701 0.655	0.578 0.187	0.110 0.050	0.020 0.011	0.0029 0.0019

$$\begin{aligned}
 t'_{3D.cr.} &> t'_{Rf.cr.} \text{ for } p' \text{ which is close to the values of } p'_{3D.c.0} \text{ or } p'_{R.c.0} \\
 t'_{3D.cr.} &< t'_{Rf.cr.} \text{ for } p' \text{ which is close to the values of } p'_{3D.c.\infty} \text{ or } p'_{R.c.\infty}
 \end{aligned}
 \tag{2.79}$$

The analysis of the results given in Table 2.6 proves the reliability of the relation (2.79).

Compare the values of $t'_{3D.cr.}$ with the corresponding ones calculated by employing the critical deformation method (Gerard and Gilbert 1958). According to this method, it is assumed that the critical deformation of the viscoelastic cylinder is equal to the critical deformation of the corresponding elastic cylinder. Consequently, using this assumption the critical deformation for the pure elastic cylinder is determined within the scope of the TDLTS. Note that in the considered case the critical deformation mentioned corresponds to $p'_{3D.c.0}$. According to this determination, using the relation $p'_{3D.c.0} = p/E_3^*$ the critical time is determined for the selected values of p .

The values of the dimensionless critical time (denoted by $t'_{cdm.cr.}$) determined by employing the critical deformation method are given in Table 2.7. These values are calculated for the case where $\rho = 0.1$, $E^{(2)}/E_0^{(1)} = 5$ and $\omega = 0.5$. At the same time, in this table the corresponding values of $t'_{3D.cr.}$ are also illustrated. Comparison of the $t'_{cdm.cr.}$ with the $t'_{3D.cr.}$ shows that the critical deformation method is not acceptable for determination of the critical time for the stability loss of the cylinder made from viscoelastic composite material.

With this we restrict to consideration of the numerical results related to the solid cylinder.

Table 2.8 The values of $p'_{3D.c.0}$, $p'_{R.c.0}$ and $p'_{E.c.0}$ obtained for various values of $E^{(2)}/E_0^{(1)}$, h/R^+ and $\rho = \pi R/\ell$

$\frac{h}{R^+}$	$\frac{E^{(2)}}{E^{(1)}}$	ρ					
		0.2		0.3			
		$p'_{3D.c.0}$	$p'_{R.c.0}$	$p'_{E.c.0}$	$p'_{3D.c.0}$	$p'_{R.c.0}$	$p'_{E.c.0}$
0.3	1	0.0138	0.0142	0.0148	0.028	0.030	0.033
	5	0.0400	0.0417	0.0446	0.079	0.086	0.100
	10	0.0697	0.0739	0.0819	0.132	0.142	0.184
0.5	1	0.0117	0.0120	0.0125	0.024	0.025	0.028
	5	0.0344	0.0354	0.0374	0.070	0.074	0.108
	10	0.0606	0.0630	0.0687	0.119	0.128	0.154
0.8	1	0.0099	0.0100	0.0103	0.021	0.021	0.023
	5	0.0294	0.0297	0.0311	0.061	0.063	0.070
	10	0.0526	0.0532	0.0571	0.107	0.110	0.128

2.5.2 Hollow Cylinder

The values of $p'_{3D.c.0}$, $p'_{R.c.0}$ and $p'_{E.c.0}$ are given in Table 2.8 for various values of $E^{(2)}/E_0^{(1)}$, $\rho (= \pi R^+/\ell)$ and h/R^+ . The corresponding values of $p'_{3D.c.\infty}$, $p'_{R.c.\infty}$ and $p'_{E.c.\infty}$ obtained in case where $E^{(2)}/E_0^{(1)} = 5$, $\omega = 0.5$ are tabulated in Table 2.9. Note that the values of $p'_{E.c.0}$, $p'_{R.c.0}$, $p'_{E.c.\infty}$ and $p'_{R.c.\infty}$ are calculated by the use of the expression (2.76). However, the values of $p'_{3D.c.0}$ and $p'_{3D.c.\infty}$ are calculated by the use of the criterion (2.44) and the solution of the system of Eq. (2.42).

It follows from the foregoing results that for the considered all cases the following relations occur.

$$\begin{aligned}
 p'_{3D.c.0} &< p'_{R.c.0} < p'_{E.c.0}, \quad p'_{3D.c.\infty} < p'_{R.c.\infty} < p'_{E.c.\infty}, \\
 |p'_{E.c.0} - p'_{3D.c.0}| &> |p'_{E.c.0} - p'_{R.c.0}|, \\
 |p'_{E.c.\infty} - p'_{3D.c.\infty}| &> |p'_{E.c.\infty} - p'_{R.c.\infty}|.
 \end{aligned} \tag{2.80}$$

Moreover, it follows from these results that the values of differences given in (2.80) increase with $E^{(2)}/E_0^{(1)}$ and ρ . But, decrease in the values of h/R^+ causes to increase of the foregoing dimensionless intensity of the critical forces. If we consider the total dimensionless force

$$P' = \frac{P}{E_0^{(1)}} \left(\frac{\pi}{\ell}\right)^2 = p' \pi \left(\frac{\pi}{\ell}\right)^2 \left((R^+)^2 - (R^-)^2\right) \tag{2.81}$$

instead of the p' , then decrease in the values of h/R^+ must cause to decrease in the critical values of P' . This affirmation is proven by data given in Table 2.10 which shows the values of $P'_{3D.c.0}$ for the case where $E^{(2)}/E_0^{(1)} = 5$.

Table 2.9 The values of $p'_{3D.c.0}$, $p'_{R.c.0}$ and $p'_{E.c.0}$ obtained for various values of $E^{(2)}/E_0^{(1)}$, h/R^+ and $\rho = \pi R/\ell$ under $\omega = 0.5$

$\frac{h}{R^+}$	$\frac{E^{(1)}}{E_0^{(1)}}$	ρ					
		0.2			0.3		
		$p'_{3D.c.0}$	$p'_{R.c.0}$	$p'_{E.c.0}$	$p'_{3D.c.0}$	$p'_{R.c.0}$	$p'_{E.c.0}$
0.3	1	0.0082	0.0087	0.0099	0.015	0.017	0.022
	5	0.0236	0.0276	0.0397	0.035	0.045	0.089
	10	0.0337	0.0421	0.0769	0.045	0.060	0.173
0.5	1	0.0072	0.0075	0.0083	0.013	0.015	0.018
	5	0.0220	0.0244	0.0333	0.035	0.041	0.075
	10	0.0331	0.0382	0.0645	0.047	0.056	0.145
0.8	1	0.0062	0.0063	0.0069	0.012	0.013	0.015
	5	0.0209	0.0213	0.0277	0.036	0.037	0.062
	10	0.0334	0.0343	0.0537	0.051	0.053	0.120

Table 2.10 The values of $P'_{3D.c.0} \times 10^2$ obtained in the case where $E^{(2)}/E_0^{(1)} = 5$ where $P'_{3D.c.0}$ (2.81) is a dimensionless total critical force

$\rho = \frac{\pi R^+}{\ell}$	h/R^+						
	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.2	0.256	0.297	0.324	0.341	0.350	0.355	0.357
0.3	1.152	1.354	1.494	1.587	1.647	1.682	1.701

Table 2.11 The influence of the rheological parameter ω on the values of $p'_{3D.c.0}$, $p'_{R.c.0}$ and $p'_{E.c.0}$ in the case where $E^{(2)}/E_0^{(1)} = 5$ and $h/R^+ = 0.5$

ω	$\rho = \pi R^+/\ell$					
	0.2			0.3		
	$p'_{3D.c.0}$	$p'_{R.c.0}$	$p'_{E.c.0}$	$p'_{3D.c.0}$	$p'_{R.c.0}$	$p'_{E.c.0}$
0.5	0.02208	0.02442	0.03334	0.03524	0.04118	0.07501
1	0.02756	0.02928	0.03438	0.04991	0.05559	0.07736
2	0.03067	0.03199	0.03542	0.05927	0.06423	0.07970

The influence of the rheological parameter ω on the values of the critical forces are illustrated by the data given in Table 2.11 which are obtained for various ρ under $E^{(2)}/E_0^{(1)} = 5$ and $h/R^+ = 0.5$.

It follows from the foregoing results that for pure elastic stability loss problems in the cases where the relations $E^{(2)}/E_0^{(1)} \leq 5$, $h/R^+ \geq 0.5$ and $\rho = 0.2$ satisfy simultaneously the difference between the results obtained in the framework of the TDLTS and in the framework of the third order refined beam theory are not more than 5-6 %.

The comparison of the results given in Tables 2.8 and 2.9 with the results given in Tables 2.1 and 2.2 shows that the values of the critical forces obtained for the

Table 2.12 The values of $t'_{3D.cr.}$ (upper number) and $t'_{Rf.cr.}$ (lower number) obtained for various values of the rheological parameter α in the case where $h/R^+ = 0.3$ under $E^{(2)}/E_0^{(1)} = 5$, $\rho = 0.2$ and $\omega = 0.5$

α	$p' \times 10^2$						
	2.80	2.90	3.10	3.20	3.30	3.40	3.60
-0.3	4.088 179.1	2.688 24.79	1.277 5.306	0.895 3.153	0.625 1.988	0.430 1.289	0.182 0.546
-0.5	9.475 1883.	5.269 118.1	1.859 13.64	1.130 6.587	0.684 3.453	0.405 1.883	0.122 0.566
-0.7	67.35 $\sim O(10^6)$	25.33 4516	4.463 123.7	1.948 36.74	0.843 12.52	0.351 4.561	0.047 0.614

Table 2.13 The values of $t'_{3D.cr.}$ (upper number) and $t'_{Rf.cr.}$ (lower number) obtained for various values of the rheological parameter α in the case where $h/R^+ = 0.5$ under $E^{(2)}/E_0^{(1)} = 5$, $\rho = 0.2$ and $\omega = 0.5$

α	$p' \times 10^2$						
	2.50	2.60	2.80	2.90	3.00	3.20	3.30
-0.3	5.348 65.03	2.969 13.23	1.092 2.827	0.673 1.579	0.405 0.907	0.117 0.280	0.046 0.135
-0.5	13.80 455.8	6.056 49.07	1.492 5.653	0.759 2.501	0.373 1.152	0.065 0.222	0.017 0.080
-0.7	126.0 42853	31.94 1044	3.095 28.48	1.003 7.316	0.307 2.009	0.016 0.129	0.0019 0.0239

hollow cylinder approach to the corresponding ones obtained for the solid cylinder with h/R^+ .

Now we consider the case where the cylinder material is viscoelastic and analyze the critical time. As in the previous section, to compare the values of the critical time given by the TDLTS with those given by approximate beam theories we must select the values of $p' (= p/E_0^{(1)})$ which fall in intervals $[p'_{E.c.\infty}, p'_{E.c.0}]$, $[p'_{R.c.\infty}, p'_{R.c.0}]$ and $[p'_{3D.c.\infty}, p'_{3D.c.0}]$ simultaneously. It follows from the data given in Tables 2.8 and 2.9, in many cases such common values of $p' (= p/E_0^{(1)})$ exist only for the intervals $[p'_{R.c.\infty}, p'_{R.c.0}]$ and $[p'_{3D.c.\infty}, p'_{3D.c.0}]$. For this reason, we will compare only the critical time obtained by employing the TDLTS (denoted by $t'_{3D.cr.}$) with that obtained by employing the third order refined beam theory (denoted by $t'_{Rf.cr.}$).

Thus, we consider data given in Tables 2.12, 2.13 and 2.14 which show the values of $t'_{3D.cr.}$ (upper number) and $t'_{Rf.cr.}$ (lower number) in the cases where $h/R^+ = 0.3, 0.5$ and 0.8 respectively under $E^{(2)}/E_0^{(1)} = 5$, $\rho = 0.2$ and $\omega = 0.5$ for various values of rheological parameter α .

The foregoing results show that for the cases under considerations the values of $t'_{Rf.cr.}$ are greater than $t'_{3D.cr.}$. In the cases where p' is close to the values of $p'_{3D.c.\infty}$ (or to the values of $p'_{R.c.\infty}$) the difference between $t'_{3D.cr.}$ and $t'_{Rf.cr.}$ becomes more non-negligible. The results given above also show the influence of the rheological parameter α on the values of $t'_{3D.cr.}$ and $t'_{Rf.cr.}$.

Table 2.14 The values of $t'_{3D.cr.}$ (upper number) and $t'_{Rf.cr.}$ (lower number) obtained for various values of the rheological parameter α in the case where $h/R^+ = 0.8$ under $E^{(2)}/E_0^{(1)} = 5$, $\rho = 0.2$ and $\omega = 0.5$

α	$p' \times 10^2$						
	2.50	3.00	4.00	4.50	5.00	5.50	6.50
-0.3	8.358 14.29	5.023 7.609	2.192 2.984	1.513 2.010	1.056 1.382	0.738 0.959	0.347 0.454
-0.5	25.78 54.64	12.64 22.61	3.959 6.098	2.357 3.507	1.425 2.076	0.862 1.244	0.299 0.437
-0.7	357.2 1249.	108.8 287.0	15.73 32.31	6.631 12.85	2.866 5.366	1.240 2.286	0.213 0.400

For more clear illustration this influence in Fig. 2.5 the graphs of the dependencies among $t'_{3D.cr.}$, $t'_{Rf.cr.}$ and p' are given for various values of α for the cases where $h/R^+ = 0.3$ (Fig. 2.5a) and $h/R^+ = 0.5$ (Fig. 2.5b) under $E^{(2)}/E_0^{(1)} = 5$, $\omega = 0.5$. It follows from the results given in Fig. 2.5 that the graphs of the analyzed dependencies constructed for various α have a common intersection point. Before (after) this intersection point the increase in the absolute values of α causes to increase (to decrease) the values of the critical time. Note that the parts of the graphs which appear after the intersection point are illustrated in Fig. 2.6 in the cases where $h/R^+ = 0.3$ (Fig. 2.6a) and $h/R^+ = 0.5$ (Fig. 2.6b). We remind that the related explanation on the mentioned “intersection point” is given in the previous subsection.

The influence of the dimensionless rheological parameter ω on the graphs of the dependencies among $t'_{3D.cr.}$, $t'_{Rf.cr.}$ and p' are given in Fig. 2.7 in the cases where $h/R^+ = 0.3$ (Fig. 2.7a) and $h/R^+ = 0.5$ (Fig. 2.7b) under $E^{(2)}/E_0^{(1)} = 5$, $\rho = 0.1$ and $\alpha = -0.3$. The character of these graphs can be explained by the increasing of the values of $p'_{3D.c.\infty}$ and $p'_{R.c.\infty}$ with ω .

Therefore for common values of p' the values of $t'_{3D.cr.}$ and $t'_{Rf.cr.}$ increase with ω . At the same time, it should be noted that the values of the critical time depend not only on the values of the rheological parameters, but also on the values of the intensity of the external compressive force p' . In this case, as in the previous section, there exist the relations (2.78).

Compare the values of $t'_{3D.cr.}$ with the corresponding ones calculated with employing critical deformation method (Gerard and Gilbert 1958). Remind that the critical deformation method has been explained in the previous subsection.

The values of the dimensionless critical time (denoted by $t'_{cdm.cr.}$) determined by employing the critical deformation method are given in Table 2.15. These values are calculated for the various α in the case where $\rho = 0.1$, $E^{(2)}/E_0^{(1)} = 5$, $h/R^+ = 0.5$ and $\omega = 0.5$. At the same time, in this table the corresponding values of $t'_{3D.cr.}$ (upper number) are also illustrated. The comparison the values of $t'_{cdm.cr.}$ with corresponding values of $t'_{3D.cr.}$ shows that the critical deformation method does not acceptable for determination of the critical time for the stability loss of the hollow cylinder made from viscoelastic composite material.

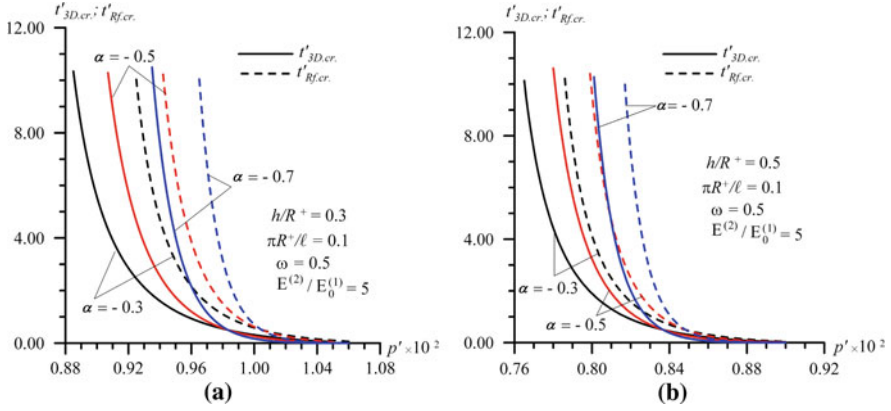


Fig. 2.5 The graphs of the dependencies among $t'_{3D.cr.}$, $t'_{Rf.cr.}$ and p' constructed for various values of rheological parameter α in the cases where $h/R^+ = 0.3$ (a) and $h/R^+ = 0.5$ (b)

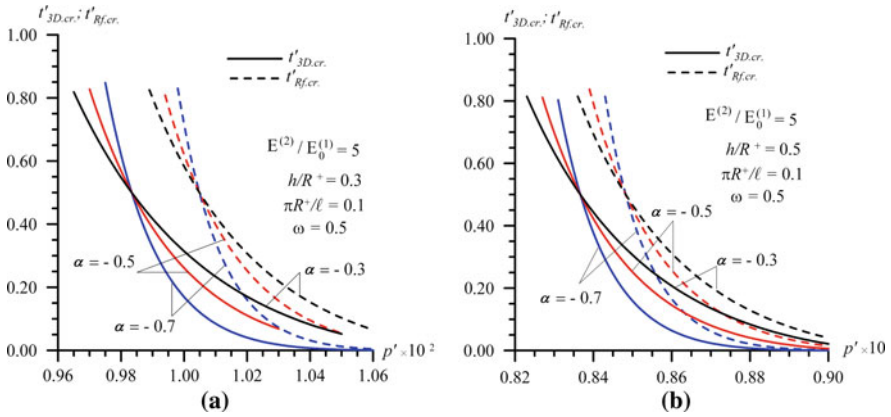


Fig. 2.6 The graphs of the dependencies among $t'_{3D.cr.}$, $t'_{Rf.cr.}$ and p' constructed for various values of rheological parameter ω in the cases where $h/R^+ = 0.3$ (a) and $h/R^+ = 0.5$ (b)

2.6 Formulation of the Problem Related to the Rotationally Symmetric Local Stability Loss

We consider a hollow cylinder which has an initial rotationally symmetric infinitesimal imperfection in the natural state and determine the position of the points of this cylinder with the Lagrange coordinates in the cylindrical $Or\theta z$ and the Cartesian $Ox_1x_2x_3$ system of coordinates (Fig. 2.8). The noted initial imperfection is given through the following equation of the middle surface of the hollow cylinder:

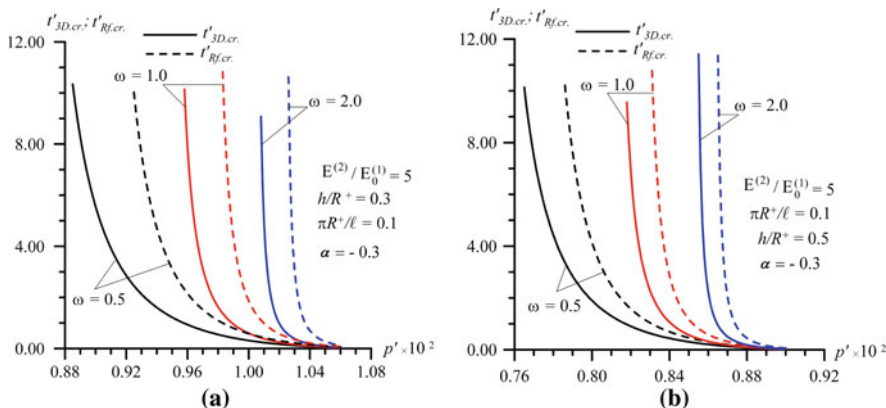


Fig. 2.7 The graphs of the dependencies among $t'_{3D.cr.}$, $t'_{Rf.cr.}$ and p' constructed for various values of rheological parameter ω in the case where $h/R^+ = 0.3$ (a) and $h/R^+ = 0.5$ (b)

Table 2.15 The values of $t'_{3D.cr.}$ (upper number) and $t'_{cdm.cr.}$ (lower number) obtained for various values of the rheological parameter α in the case where $\rho = 0.1$ under $E^{(2)}/E_0^{(1)} = 5$, $h/R^+ = 0.8$ and $\omega = 0.5$

α	$p' \times 10^2$				
	0.820	0.830	0.840	0.850	0.850
-0.3	0.906 21.44	0.634 3.587	0.443 1.434	0.306 0.722	0.207 0.399
-0.5	1.150 96.45	0.698 7.891	0.4227 2.186	0.251 0.837	0.145 0.364
-0.7	2.006 3220.	0.872 49.65	0.377 5.845	0.159 1.181	0.0641 0.2955

$$r = r_0(t_3) = R + L \sin(\alpha_1 m t_3); \quad \alpha_1 = \pi/\ell \quad (2.82)$$

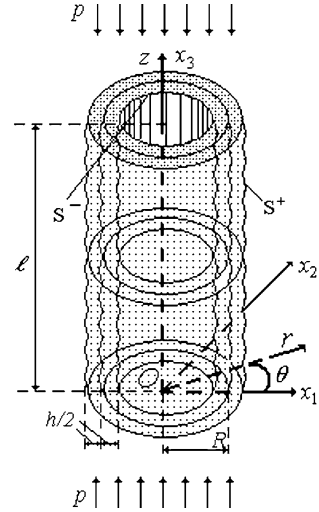
where t_3 is a parameter and $t_3 \in (0, \ell)$, L is the amplitude of the initial imperfection form, and m is the number of the half-length of the period of the initial imperfection.

We assume that the thickness of the cylinder's wall is constant along the cylinder and is equal to h . Moreover, we assume that $L \ll l$ and introduce the small parameter:

$$\varepsilon = \frac{L}{\ell}, \quad 0 \leq \varepsilon \ll \left(\frac{1}{m\pi} \right). \quad (2.83)$$

We suppose that the material of the cylinder is viscoelastic transversal isotropic, the symmetry axis of which coincides with the Ox_3 (Oz) axis. Within the foregoing assumptions, we investigate the evolution of the infinitesimal rotationally symmetric initial imperfection of the cylinder with time for the case where the cylinder is loaded by uniformly distributed normal compressed forces with intensity p acting on the ends of the cylinder in the direction of the Oz axis.

Fig. 2.8 The geometry of the considered hollow cylinder with rotationally symmetric initial imperfection



The investigation is also made within the scope of the field Eqs. (2.3a), (2.3b), (2.3c), (2.4) and (2.5). For both completeness and clarity of the case under consideration we rewrite these equations for the rotationally symmetric case:

$$\frac{\partial t_{rr}}{\partial r} + \frac{\partial t_{rz}}{\partial z} + \frac{1}{r}(t_{rr} - t_{\theta\theta}) = 0, \quad \frac{\partial t_{rz}}{\partial r} + \frac{1}{r}t_{rz} + \frac{\partial t_{zz}}{\partial z} = 0, \quad (2.84a)$$

$$\begin{aligned} t_{rr} &= \sigma_{rr} \left(1 + \frac{\partial u_r}{\partial r} \right) + \sigma_{rz} \frac{\partial u_r}{\partial z}, \quad t_{rz} = \sigma_{rr} \frac{\partial u_z}{\partial r} + \sigma_{rz} \left(1 + \frac{\partial u_z}{\partial z} \right), \\ t_{\theta\theta} &= \sigma_{\theta r} \frac{\partial u_\theta}{\partial r} + \sigma_{\theta\theta} \left(1 + \frac{u_r}{r} \right), \quad t_{zr} = \sigma_{zr} \left(1 + \frac{\partial u_r}{\partial r} \right) + \sigma_{zz} \frac{\partial u_r}{\partial z}, \\ t_{zz} &= \sigma_{zr} \frac{\partial u_z}{\partial r} + \sigma_{zz} \left(1 + \frac{\partial u_z}{\partial z} \right), \end{aligned} \quad (2.84b)$$

$$\begin{aligned} \varepsilon_{rr} &= \frac{\partial u_r}{\partial r} + \frac{1}{2} \left\{ \left(\frac{\partial u_r}{\partial r} \right)^2 + \left(\frac{\partial u_z}{\partial r} \right)^2 \right\}, \\ \varepsilon_{rz} &= \frac{1}{2} \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) + \frac{1}{2} \left\{ \frac{\partial u_r}{\partial r} \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \frac{\partial u_z}{\partial z} \right\} \\ \varepsilon_{\theta\theta} &= \frac{u_r}{r} + \frac{1}{2} \left(\frac{u_r}{r} \right)^2, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} + \frac{1}{2} \left\{ \left(\frac{\partial u_r}{\partial z} \right)^2 + \left(\frac{\partial u_z}{\partial z} \right)^2 \right\}. \end{aligned} \quad (2.84c)$$

$$\begin{aligned} \sigma_{rr} &= A_{11}^* \varepsilon_{rr} + A_{12}^* \varepsilon_{\theta\theta} + A_{13}^* \varepsilon_{zz}; \quad \sigma_{\theta\theta} = A_{12}^* \varepsilon_{rr} + A_{11}^* \varepsilon_{\theta\theta} + A_{13}^* \varepsilon_{zz}; \\ \sigma_{zz} &= A_{13}^* \varepsilon_{rr} + A_{13}^* \varepsilon_{\theta\theta} + A_{33}^* \varepsilon_{zz}; \quad \sigma_{rz} = 2G^* \varepsilon_{rz}; \quad \sigma_{\theta z} = 2G^* \varepsilon_{\theta z} \end{aligned} \quad (2.85)$$

where A_{ij}^* and G^* are the operators determined by expression (2.5).

The meaning of the notation used in Eqs. (2.84a–2.84c) and (2.85) is explained in Sect. 2.1. Denoting the inner and outer surface of the hollow cylinder by S^- and S^+ (Fig. 2.8), respectively we may write the free-of-traction conditions on these surfaces as follows:

$$t_{rr}|_{S^\pm} n_r^\pm + t_{zr}|_{S^\pm} n_z^\pm = 0, \quad t_{rz}|_{S^\pm} n_r^\pm + t_{zz}|_{S^\pm} n_z^\pm = 0 \quad (2.86)$$

where n_r^+ and n_z^+ (n_r^- and n_z^-) are the components of the unit normal vector to the outer (inner) lateral surface S^+ (S^-) of the hollow cylinder.

Moreover, we assume that at the ends of the cylinder the conditions written below are satisfied:

$$t_{zz}|_{z=0} = -p, \quad t_{zz}|_{z=\ell} = -p. \quad (2.87)$$

The end conditions for the displacements will be discussed below.

This completes the formulation of the considered problem and it follows from this formulation that the evolution of the infinitesimal initial rotationally symmetric imperfection of the cylinder with time for the fixed value of the external compressed force p (for the case where the material of the cylinder is viscoelastic) or with initial compressing force p (for the case where the material of the cylinder is pure elastic) will be investigated within the framework of the field Eqs. (2.84) and (2.85) and boundary conditions (2.86) and (2.87).

2.7 Method of Solution for the Rotationally Symmetric Problem

We now develop the method summarized and proposed in Sect. 2.2 with respect to the solution to the rotationally symmetric stability loss problem of the hollow cylinder formulated in the previous section.

According to the procedure summarized in Sect. 2.2 and to the problem statement, first we derive the equation for the inner and outer lateral surfaces S^+ and S^- of the cylinder shown in Fig. 2.8. Using the condition of the constant thickness of the cylinder's wall we can conclude that the coordinates of these surfaces must simultaneously satisfy the following equations:

$$z^\pm - t_3 = -\frac{dr_0}{dt_3}(r^\pm - r_0(t_3)), \quad (z^\pm - t_3)^2 + (r^\pm - r_0(t_3))^2 = \left(\frac{h}{2}\right)^2 \quad (2.88)$$

where $z^\pm$ and $r^\pm$ are the coordinates of the surface $S^\pm$. Note that the first equation in (2.88) is an equation of the straight line (plane) perpendicular to the vector which is the tangent vector to the middle surface of the cylinder at the point that corresponds to the fixed value of the parameter t_3 , but the second equation in (2.88) shows that the thickness of the cylinder's wall remains constant along the cylinder. We can write the expressions

$$r^\pm = r_0 \pm \frac{h}{2} \left(1 + \left(\frac{dr_0}{dt_3} \right)^2 \right)^{-\frac{1}{2}}, \quad z^\pm = t_3 \mp \frac{dr_0}{dt_3} \frac{h}{2} \left(1 + \left(\frac{dr_0}{dt_3} \right)^2 \right)^{-\frac{1}{2}} \quad (2.89)$$

from the Eq. (2.88).

Using the assumption (2.83) and the condition $(dr_0/dt_3)^2 \ll 1$, after some mathematical manipulations, we obtain the following expansions from Eq. (2.89):

$$\begin{aligned} r^\pm &= R \pm \frac{h}{2} + \varepsilon \ell \sin(\alpha_1 m t_3) \mp \frac{h}{4} \varepsilon^2 m^2 \pi^2 \cos^2(\alpha_1 m t_3) + O(\varepsilon^4), \\ z^\pm &= t_3 \mp \varepsilon \frac{h}{2} \pi m \cos(\alpha_1 m t_3) \pm \varepsilon^3 \frac{h}{4} (m \pi \cos^2(\alpha_1 m t_3))^3 + O(\varepsilon^5), \\ n_r^\pm &= \pm \frac{dz^\pm}{dt_3} (V(t_3))^{-1}, \quad n_z^\pm = \mp \frac{dr^\pm}{dt_3} (V(t_3))^{-1}, \quad V(t_3) = \left(\left(\frac{dz^\pm}{dt_3} \right)^2 + \left(\frac{dr^\pm}{dt_3} \right)^2 \right)^{\frac{1}{2}} \\ &\Rightarrow n_r^\pm = \pm 1 + O(\varepsilon^2), \quad n_z^\pm = \mp \varepsilon m \pi \cos(\alpha_1 m t_3) + O(\varepsilon^3), \end{aligned} \quad (2.90)$$

where $n_r^\pm$ and $n_z^\pm$ are the physical components of the unit normal vector to the surface $S^\pm$ (Fig. 2.8).

As in Sect. 2.2, we attempt to solve the considered problem by employing the boundary form perturbation method. For this purpose, the unknowns are presented in series form in ε (2.83):

$$\{\sigma_{(ij)}; \varepsilon_{(ij)}; u_{(i)}\} = \sum_{q=0}^{\infty} \varepsilon^q \left\{ \sigma_{(ij)}^{(q)}; \varepsilon_{(ij)}^{(q)}; u_{(i)}^{(q)} \right\}, \quad (2.91)$$

where $(ij) = rr, \theta\theta, zz, rz$ and $(i) = r, z$.

Substituting Eq. (2.91) into Eq. (2.84a–c), we obtain set equations for each approximation (2.91). Using relation (2.90) we expand the values of each approximation (2.91) in series form in the vicinity of the points $\{r = R + h/2; z_0 = t_3\}$ and $\{r = R - h/2; z_0 = t_3\}$. Substituting these last expressions in the boundary conditions in (2.86) and using the expressions of $n_r^\pm$ and $n_z^\pm$ given in (2.90), after some mathematical transformations we obtain boundary conditions which are satisfied at $\{r = R + h/2; z = t_3\}$ (for outer surface S^+) and $\{r = R - h/2; z = t_3\}$ (for inner surface S^-) for each approximation in Eq. (2.91). It is evident that for the zeroth approximation, equations in (2.84a–2.84c) are valid and condition (2.86) is replaced by the same one satisfied at points $\{r = R + h/2; z = t_3\}$ and $\{r = R - h/2; z = t_3\}$. We assume that in the zeroth approximation the non-linear parts of the strain tensor components are very small and can be ignored with respect to their linear parts. According to this assumption, for the zeroth approximation, we obtain the following system of equations:

$$\begin{aligned}
\frac{\partial \sigma_{rr}^{(0)}}{\partial r} + \frac{\partial \sigma_{zr}^{(0)}}{\partial z} + \frac{1}{r} (\sigma_{rr}^{(0)} - \sigma_{\theta\theta}^{(0)}) &= 0, \quad \frac{\partial \sigma_{rz}^{(0)}}{\partial r} + \frac{1}{r} \sigma_{rz}^{(0)} + \frac{\partial \sigma_{zz}^{(0)}}{\partial z} = 0, \\
\varepsilon_{rr}^{(0)} = \frac{\partial u_r^{(0)}}{\partial r}, \quad \varepsilon_{rz}^{(0)} &= \frac{1}{2} \left(\frac{\partial u_r^{(0)}}{\partial z} + \frac{\partial u_z^{(0)}}{\partial r} \right), \quad \varepsilon_{\theta\theta}^{(0)} = \frac{u_r^{(0)}}{r}, \quad \varepsilon_{zz}^{(0)} = \frac{\partial u_z^{(0)}}{\partial z}
\end{aligned} \tag{2.92}$$

and boundary conditions:

$$\sigma_{rr}^{(0)}|_{r=R\pm h/2} = 0, \quad \sigma_{rz}^{(0)}|_{r=R\pm h/2} = 0. \tag{2.93}$$

Moreover, we obtain the following end conditions for the zeroth approximation from (2.87):

$$\sigma_{zz}^{(0)}(r, \theta, 0) = \sigma_{zz}^{(0)}(r, \theta, \ell) = -p. \tag{2.94}$$

Taking the assumption $1 + \partial u_z^{(0)}/\partial z \approx 1$, $1 + u_r^{(0)}/r \approx 1$ and $1 + \partial u_z^{(0)}/\partial z \approx 1$ into account, for the subsequent approximations we obtain the following system of equations:

$$\begin{aligned}
&\frac{\partial \sigma_{rr}^{(q)}}{\partial r} + \frac{\partial \sigma_{zr}^{(q)}}{\partial z} + \frac{1}{r} (\sigma_{rr}^{(q)} - \sigma_{\theta\theta}^{(q)}) + \sigma_{zz}^{(0)} \frac{\partial^2 u_r^{(q)}}{\partial z^2} \\
&= -\frac{\partial S_{rr}^{(q-1)}}{\partial r} - \frac{\partial S_{zr}^{(q-1)}}{\partial z} - \frac{1}{r} (S_{rr}^{(q-1)} - S_{\theta\theta}^{(q-1)}), \\
&\frac{\partial \sigma_{rz}^{(q)}}{\partial r} + \frac{1}{r} \sigma_{rz}^{(q)} + \frac{\partial \sigma_{zz}^{(q)}}{\partial z} + \sigma_{zz}^{(0)} \frac{\partial^2 u_z^{(q)}}{\partial z^2} = -\frac{\partial S_{rz}^{(q-1)}}{\partial r} - \frac{1}{r} S_{rz}^{(q-1)} - \frac{\partial S_{zz}^{(q-1)}}{\partial z}, \\
&S_{rr}^{(q-1)} = \sum_{k=1}^{q-1} \left\{ \sigma_{rr}^{(k)} \frac{\partial u_r^{(q-k)}}{\partial r} + \sigma_{rz}^{(k)} \frac{\partial u_r^{(q-k)}}{\partial z} \right\}, \\
&S_{rz}^{(q-1)} = \sum_{k=1}^{q-1} \left\{ \sigma_{rr}^{(k)} \frac{\partial u_z^{(q-k)}}{\partial r} + \sigma_{rz}^{(k)} \frac{\partial u_z^{(q-k)}}{\partial z} \right\}, \\
&S_{\theta\theta}^{(q-1)} = \sum_{k=1}^{q-1} \left\{ \sigma_{\theta\theta}^{(k)} \frac{u_r^{(q-k)}}{r} \right\}, \quad S_{zz}^{(q-1)} = \sum_{k=1}^{q-1} \left\{ \sigma_{rz}^{(k)} \frac{\partial u_z^{(q-k)}}{\partial r} + \sigma_{zz}^{(k)} \frac{\partial u_z^{(q-k)}}{\partial z} \right\}, \\
&S_{zr}^{(q-1)} = \sum_{k=1}^{q-1} \left\{ \sigma_{zr}^{(k)} \frac{\partial u_r^{(q-k)}}{\partial r} + \sigma_{zz}^{(k)} \frac{\partial u_z^{(q-k)}}{\partial z} \right\}, \\
&\varepsilon_{rr}^{(q)} = \frac{\partial u_r^{(q)}}{\partial r} + \frac{1}{2} \sum_{k=1}^{q-1} \left\{ \left(\frac{\partial u_r^{(k)}}{\partial r} \frac{\partial u_r^{(q-k)}}{\partial r} \right) + \left(\frac{\partial u_z^{(k)}}{\partial r} \frac{\partial u_z^{(q-k)}}{\partial r} \right) \right\}, \\
&\varepsilon_{rz}^{(q)} = \frac{1}{2} \left(\frac{\partial u_r^{(q)}}{\partial z} + \frac{\partial u_z^{(q)}}{\partial r} \right) + \frac{1}{2} \sum_{k=1}^{q-1} \left\{ \frac{\partial u_r^{(k)}}{\partial r} \frac{\partial u_r^{(q-k)}}{\partial z} + \frac{\partial u_z^{(k)}}{\partial r} \frac{\partial u_z^{(q-k)}}{\partial z} \right\},
\end{aligned}$$

$$\begin{aligned} \varepsilon_{\theta\theta}^{(q)} &= \frac{u_r^{(q)}}{r} + \frac{1}{2} \sum_{k=1}^{q-1} \left\{ \frac{1}{r^2} u_r^{(k)} u_r^{(q-k)} \right\}, \\ \varepsilon_{zz}^{(q)} &= \frac{\partial u_z^{(q)}}{\partial z} + \frac{1}{2} \sum_{k=1}^{q-1} \left\{ \left(\frac{\partial u_r^{(k)}}{\partial z} \frac{\partial u_r^{(q-k)}}{\partial z} \right) + \left(\frac{\partial u_z^{(k)}}{\partial z} \frac{\partial u_z^{(q-k)}}{\partial z} \right) \right\}. \end{aligned} \quad (2.95)$$

For the problem under consideration the underlined terms in Eq. (2.95) are equal to zero for the first approximation. By direct verification it is proven that Eq. (2.95) without the underlined terms coincides with the corresponding equations of the TDLTS (Guz 1999) related to the rotationally symmetric case. Due to linearity, the constitutive relations in (2.85) are satisfied by each approximation separately, i.e.

$$\begin{aligned} \sigma_{rr}^{(q)} &= A_{11}^* \varepsilon_{rr}^{(q)} + A_{12}^* \varepsilon_{\theta\theta}^{(q)} + A_{13}^* \varepsilon_{zz}^{(q)}; \quad \sigma_{\theta\theta}^{(q)} = A_{12}^* \varepsilon_{rr}^{(q)} + A_{11}^* \varepsilon_{\theta\theta}^{(q)} + A_{13}^* \varepsilon_{zz}^{(q)}; \\ \sigma_{zz}^{(q)} &= A_{13}^* \varepsilon_{rr}^{(q)} + A_{13}^* \varepsilon_{\theta\theta}^{(q)} + A_{33}^* \varepsilon_{zz}^{(q)}; \quad \sigma_{rz}^{(q)} = 2G^* \varepsilon_{rz}^{(q)}. \end{aligned} \quad (2.96)$$

Now we write the boundary conditions given on the lateral surface of the cylinder for the first approximation by the physical components of the stress tensor:

$$\sigma_{rr}^{(1)}|_{r=R \pm h/2} = 0, \quad \sigma_{rz}^{(1)}|_{r=R \pm h/2} = \sigma_{zz}^{(0)} m \pi \cos(\alpha_1 m t_3). \quad (2.97)$$

According to the expressions (2.90) and (2.91), we obtain from Eq. (2.87) the following end conditions for the first approximation:

$$\sigma_{zz}^{(1)}|_{t_3=0;\ell} + \sigma_{zz}^{(0)} \frac{\partial u_z^{(1)}}{\partial z} \Big|_{t_3=0;\ell} = 0. \quad (2.98)$$

Moreover, we assume that:

$$u_r^{(1)}|_{t_3=0;\ell} = 0. \quad (2.99)$$

According to (2.92), (2.93) and (2.94), the values related to the zeroth approximation are determined as follows:

$$\sigma_{zz}^{(0)} = -p, \quad \sigma_{ij}^{(0)} = 0 \text{ for } (ij) \neq zz. \quad (2.100)$$

It follows from (2.100) that in the zeroth approximation the components of the displacement vector can be presented as follows:

$$u_r^{(0)} = a(t)r + a_0, \quad u_z^{(0)} = c(t)z + c_0 \quad (2.101)$$

where a_0 and c_0 are constants, $a(t)$ and $c(t)$ are functions and t is a time. The functions $a(t)$ and $c(t)$ can be easily determined from Eqs. (2.92), (2.96) and (2.101).

Now we consider determination of the values related to the first approximation. Taking the expression (2.100) into account the following field equations are obtained from the Eq. (2.95) for this approximation:

$$\begin{aligned} \frac{\partial \sigma_{rr}^{(1)}}{\partial r} + \frac{\partial \sigma_{zr}^{(1)}}{\partial z} + \frac{1}{r} \left(\sigma_{rr}^{(1)} - \sigma_{\theta\theta}^{(1)} \right) + \sigma_{zz}^{(0)} \frac{\partial^2 u_r^{(1)}}{\partial z^2} &= 0, \\ \frac{\partial \sigma_{rz}^{(1)}}{\partial r} + \frac{\partial \sigma_{zz}^{(1)}}{\partial z} + \frac{1}{r} \sigma_{rz}^{(1)} + \sigma_{zz}^{(0)} \frac{\partial^2 u_z^{(1)}}{\partial z^2} &= 0, \\ \varepsilon_{rr}^{(1)} = \frac{\partial u_r^{(1)}}{\partial r}, \quad \varepsilon_{\theta\theta}^{(1)} = \frac{u_r^{(1)}}{r}, \quad \varepsilon_{zz}^{(1)} = \frac{\partial u_z^{(1)}}{\partial z}, \quad \varepsilon_{zr}^{(1)} = \frac{1}{2} \left(\frac{\partial u_z^{(1)}}{\partial r} + \frac{\partial u_r^{(1)}}{\partial z} \right). \end{aligned} \quad (2.102)$$

We attempt to solve these Eqs. (2.96) and (2.102) within the scope of the conditions (2.97), (2.98) and (2.99). For this purpose we apply the Laplace transformation (2.34) to all equations and relations related to the first approximation. After this application, the Eq. (2.96) and the boundary conditions (2.97) (in which $\sigma_{zz}^{(0)}$ must be replaced with $\sigma_{zz}^{(0)}/s$), (2.98) and (2.99) are valid for the Laplace transformation of the corresponding sought-for quantities, whereas the constitutive relations in (2.96) are transformed to the following ones:

$$\begin{aligned} \bar{\sigma}_{rr}^{(1)} &= \bar{A}_{11}^* \bar{\varepsilon}_{rr}^{(1)} + \bar{A}_{12}^* \bar{\varepsilon}_{\theta\theta}^{(1)} + \bar{A}_{13}^* \bar{\varepsilon}_{zz}^{(1)}; \quad \bar{\sigma}_{\theta\theta}^{(1)} = \bar{A}_{12}^* \bar{\varepsilon}_{rr}^{(1)} + \bar{A}_{11}^* \bar{\varepsilon}_{\theta\theta}^{(1)} + \bar{A}_{13}^* \bar{\varepsilon}_{zz}^{(1)}; \\ \bar{\sigma}_{zz}^{(1)} &= \bar{A}_{13}^* \bar{\varepsilon}_{rr}^{(1)} + \bar{A}_{13}^* \bar{\varepsilon}_{\theta\theta}^{(1)} + \bar{A}_{33}^* \bar{\varepsilon}_{zz}^{(1)}; \quad \bar{\sigma}_{r\theta}^{(1)} = (\bar{A}_{11}^* - \bar{A}_{22}^*) \bar{\varepsilon}_{r\theta}^{(1)}; \\ \bar{\sigma}_{rz}^{(1)} &= 2\bar{G}^* \bar{\varepsilon}_{rz}^{(1)}; \quad \bar{\sigma}_{\theta z}^{(1)} = 2\bar{G}^* \bar{\varepsilon}_{\theta z}^{(1)} \end{aligned} \quad (2.103)$$

where

$$\left\{ \begin{matrix} \bar{A}_{ij}^* \\ \bar{G}^* \end{matrix} \right\} \bar{\phi}(s) = \left\{ \begin{matrix} A_{ij0} \\ G_0 \end{matrix} \right\} \bar{\phi}(s) + \left\{ \begin{matrix} \bar{A}_{ij1} \\ \bar{G}_1 \end{matrix} \right\} \bar{\phi}(s). \quad (2.104)$$

As has been noted above, Eqs. (2.102), (2.103) and (2.104) coincide with the corresponding equations of the TDLTS related to the rotationally symmetric case. Therefore to solve the obtained equation systems, according to the monograph (Guz 1999), in the cylindrical system of coordinates we can use the following representations:

$$\begin{aligned} \bar{u}_r^{(1)} &= -\frac{\partial^2}{\partial r \partial z} \chi, \quad \Delta_1 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}, \\ \bar{u}_z^{(1)} &= (\bar{A}_{13}^* + \bar{G}^*) \left(\bar{A}_{11}^* \Delta_1 + (\bar{G}^* + \sigma_{zz}^{(0)}) \frac{\partial^2}{\partial z^2} \right) \chi. \end{aligned} \quad (2.105)$$

The function χ is determined from the equation:

$$\left(\Delta_1 + \xi_2^2 \frac{\partial^2}{\partial z^2} \right) \left(\Delta_1 + \xi_3^2 \frac{\partial^2}{\partial z^2} \right) \chi = 0 \quad (2.106)$$

where

$$\begin{aligned}\xi_{2,3}^2 &= c \pm \left(c^2 - \frac{(\bar{A}_{33}^* + \sigma_{zz}^{(0)})(\bar{G}^* + \sigma_{zz}^{(0)})}{\bar{A}_{11}^* \bar{G}^*} \right)^{\frac{1}{2}}, \\ 2\bar{A}_{11}^* \bar{G}^* c &= \bar{A}_{11}^* (\bar{A}_{33}^* + \sigma_{zz}^{(0)}) + \bar{G}^{*2} - (\bar{A}_{13}^* + \bar{G}^*)^2.\end{aligned}\quad (2.107)$$

Taking the expressions of the right sides of the conditions (2.97) and (2.98) into account we find the solution to the Eq. (2.106) as follows:

$$\begin{aligned}\chi &= [B_1 I_0(\xi_2 \alpha_1 m r) + B_2 K_0(\xi_2 \alpha_1 m r) \\ &\quad + B_3 I_0(\xi_3 \alpha_1 m r) + B_4 K_0(\xi_3 \alpha_1 m r)] \cos(\alpha_1 m z)\end{aligned}\quad (2.108)$$

where $I_0(x)$ is the zeroth order Bessel function of a purely imaginary argument, $K_0(x)$ is the zeroth order Macdonald function and B_1 , B_2 , B_3 and B_4 are unknown constants, the values of which also depend on the transformation parameter s .

Substituting this solution into relations (2.102), (2.105) and (2.96) we obtain the following expressions for Laplace transformation of the sought functions:

$$\begin{aligned}\bar{u}_r^{(1)} &= [B_1 \xi_2 (\alpha_1 m)^2 I_0'(\xi_2 \alpha_1 m r) + B_3 \xi_3 (\alpha_1 m)^2 I_0'(\xi_3 \alpha_1 m r) \\ &\quad + B_2 \xi_2 (\alpha m)^2 K_0'(\xi_2 \alpha_1 m r) + B_4 \xi_3 (\alpha_1 m)^2 K_0'(\xi_3 \alpha_1 m r)] \sin(\alpha_1 m z), \\ \bar{u}_z^{(1)} &= [B_1 D_2 (\alpha_1 m)^2 I_0(\xi_2 \alpha_1 m r) + B_3 D_3 (\alpha_1 m)^2 I_0(\xi_3 \alpha_1 m r) \\ &\quad + B_2 D_2 (\alpha m)^2 K_0(\xi_2 \alpha_1 m r) + B_4 D_3 (\alpha_1 m)^2 K_0(\xi_3 \alpha_1 m r)] \sin(\alpha_1 m z), \\ D_2 &= \frac{\bar{A}_{11}^* \xi_2^2 - \bar{G}^* - \sigma_{zz}^{(0)}}{\bar{A}_{13}^* + \bar{G}^*}, \quad D_3 = \frac{\bar{A}_{11}^* \xi_3^2 - \bar{G}^* - \sigma_{zz}^{(0)}}{\bar{A}_{13}^* + \bar{G}^*}, \\ \bar{\sigma}_{rr}^{(1)} &= \left\{ B_1 \left[\bar{A}_{11}^* (\alpha_1 m)^3 \xi_2^2 I_0''(\xi_2 \alpha_1 m r) + \bar{A}_{12}^* \frac{\xi_2 (\alpha_1 m)^2}{r} I_0'(\xi_2 \alpha_1 m r) \right. \right. \\ &\quad \left. \left. + \bar{A}_{13}^* (\alpha_1 m)^3 I_0(\xi_2 \alpha_1 m r) \right] + B_3 \left[\bar{A}_{11}^* (\alpha_1 m)^3 \xi_3^2 I_0''(\xi_3 \alpha_1 m r) \right. \right. \\ &\quad \left. \left. + \bar{A}_{12}^* \frac{\xi_3 (\alpha_1 m)^2}{r} I_0'(\xi_3 \alpha_1 m r) + \bar{A}_{13}^* D_3 (\alpha_1 m)^3 I_0(\xi_3 \alpha_1 m r) \right] \right. \\ &\quad \left. + B_2 \left[\bar{A}_{11}^* (\alpha_1 m)^3 \xi_2^2 K_0''(\xi_2 \alpha_1 m r) + \bar{A}_{12}^* \frac{\xi_2 (\alpha_1 m)^2}{r} K_0'(\xi_2 \alpha_1 m r) \right. \right. \\ &\quad \left. \left. + \bar{A}_{13}^* D_2 (\alpha_1 m)^3 K_0(\xi_2 \alpha_1 m r) \right] + B_4 \left[\bar{A}_{11}^* (\alpha_1 m)^3 \xi_3^2 K_0''(\xi_3 \alpha_1 m r) \right. \right. \\ &\quad \left. \left. + \bar{A}_{12}^* \frac{\xi_3 (\alpha_1 m)^2}{r} K_0'(\xi_3 \alpha_1 m r) + \bar{A}_{13}^* D_3 (\alpha_1 m)^3 K_0(\xi_3 \alpha_1 m r) \right] \right\} \sin(\alpha_1 m z), \\ \bar{\sigma}_{rz}^{(1)} &= \left\{ B_1 \bar{G}^* (\alpha_1 m)^3 \xi_2^2 I_0'(\xi_2 \alpha_1 m r) (1 + D_2) \right. \\ &\quad \left. + B_3 \bar{G}^* (\alpha_1 m)^3 \xi_3^2 I_0'(\xi_3 \alpha_1 m r) (1 + D_3) + B_2 \bar{G}^* (\alpha_1 m)^3 \xi_2^2 K_0'(\xi_2 \alpha_1 m r) (1 + D_2) \right. \\ &\quad \left. + B_4 \bar{G}^* (\alpha_1 m)^3 \xi_3^2 K_0'(\xi_3 \alpha_1 m r) (1 + D_3) \right\} \cos(\alpha_1 m z), \quad \bar{\sigma}_{zz}^{(1)} = \\ &\left\{ B_1 \left[\bar{A}_{13}^* \left(\xi_2^2 (\alpha_1 m)^3 I_0''(\xi_2 \alpha_1 m r) + \frac{(\alpha_1 m)^2 \xi_2}{r} I_0'(\xi_2 \alpha_1 m r) \right) - \bar{A}_{33}^* D_2 (\alpha_1 m)^3 I_0(\xi_2 \alpha_1 m r) \right] \right.\end{aligned}$$

$$\begin{aligned}
& +B_3 \left[\bar{A}_{13}^* \left(\zeta_3^2 (\alpha_1 m)^3 I_0''(\zeta_3 \alpha_1 m r) + \frac{(\alpha_1 m)^2 \zeta_3}{r} I_0'(\zeta_3 \alpha_1 m r) \right) \right. \\
& - \bar{A}_{33}^* D(\alpha m)^3 I_0(\zeta_3 \alpha_1 m r) \left. \right] + B_2 \left[\bar{A}_{13}^* \left(\zeta_2^2 (\alpha_1 m)^3 K_0''(\zeta_2 \alpha_1 m r) \right. \right. \\
& + \frac{(\alpha_1 m)^2 \zeta_2}{r} K_0'(\zeta_2 \alpha_1 m r) \left. \left. \right) - \bar{A}_{33}^* D_2(\alpha_1 m)^3 K_0(\zeta_2 \alpha_1 m r) \right] \\
& + B_4 \left[\bar{A}_{13}^* \left(\zeta_3^2 (\alpha_1 m)^3 K_0''(\zeta_3 \alpha_1 m r) \right. \right. \\
& - \frac{(\alpha_1 m)^2 \zeta_3}{r} K_0'(\zeta_3 \alpha_1 m r) \left. \left. \right) - \bar{A}_{33}^* D(\alpha_1 m)^3 K_0(\zeta_3 \alpha_1 m r) \right] \Big\} \sin(\alpha_1 m z) \quad (2.109)
\end{aligned}$$

Here the notation $I_0'(x) = dI_0(x)/dx$, $I_0''(x) = d^2I_0(x)/dx^2$, $K_0'(x) = dK_0(x)/dx$ and $K_0''(x) = d^2K_0(x)/dx^2$ is used. The solution (2.109) to the problem under consideration satisfies automatically the end conditions (2.98) and (2.99). Replacing the unknowns B_1 , B_2 , B_3 and B_4 with C_1 , C_2 , C_3 and C_4 , respectively, where $C_i = (\alpha_1 m)^3 B_i$ ($i = 1, 2, 3, 4$), we obtain the following algebraic equation from the boundary condition (2.97) for determination of these unknowns:

$$\begin{aligned}
\bar{\sigma}_{rr}^{(1)}(R^+, t_3, t) = 0 & \Rightarrow C_1 a_{11}(\alpha_1 m R^+) + C_2 a_{12}(\alpha_1 m R^+) \\
& + C_3 a_{13}(\alpha_1 m R^+) + C_4 a_{14}(\alpha_1 R^+) = 0, \\
\bar{\sigma}_{rz}^{(1)}(R^+, t_3, t) = 2\pi m \sigma_{zz}^{(0)} \frac{1}{s} & \Rightarrow C_1 a_{31}(\alpha_1 m R^+) \\
& + C_3 a_{33}(\alpha_1 m R^+) + C_4 a_{34}(\alpha_1 R^+) = 2\pi m \sigma_{zz}^{(0)} \frac{1}{s}, \\
\bar{\sigma}_{rr}^{(1)}(R^-, t_3, t) = 0 & \Rightarrow C_1 a_{11}(\alpha_1 m R^-) + C_2 a_{12}(\alpha_1 m R^-) + \\
& C_3 a_{13}(\alpha_1 m R^-) + C_4 a_{14}(\alpha_1 R^-) = 0, \\
\bar{\sigma}_{rz}^{(1)}(R^-, t_3, t) = 2\pi m \sigma_{zz}^{(0)} \frac{1}{s} & \Rightarrow C_1 a_{31}(\alpha_1 m R^-) \\
& + C_3 a_{33}(\alpha_1 m R^-) + C_4 a_{34}(\alpha_1 R^-) = 2\pi m \sigma_{zz}^{(0)} \frac{1}{s}
\end{aligned} \quad (2.110)$$

where $R^+ = R + h/2$, $R^- = R - h/2$. The expressions for the coefficients in Eq. (2.110) can be written easily from the relation (2.109).

Thus, with the foregoing, we determine completely the Laplace transformation of the values related to the first approximation. The Laplace transformation of the values of the second and subsequent approximations in (2.91) can also be determined as the values of the first approximation by taking the obvious changes into account. However, as it has been noted in Sect. 2.2 for stability loss problems, consideration of only the zeroth and the first approximations is sufficient, because, according to the structure of the field equations (2.95) and (2.96), accounting for the second and subsequent approximations does not change the values of the critical parameters.

The original of the sought values is determined by employing the method (Schapery 1962, 1978), according to which, for instance, the original of the displacement $u_r^{(1)}(r, t_3, t)$ is determined through the expression:

$$u_r^{(1)}(r, t_3, t) \approx \left(s \bar{u}_r^{(1)}(r, t_3, s) \right) \Big|_{s=1/(2t)}. \quad (2.111)$$

Now we consider the selection of the stability loss criterion. In the present investigation, as in Sect. 2.2, the case will be understood under stability loss, where:

$$\max_{\substack{t_3(0, \ell) \\ r \in (0, R)}} |u_r^{(1)}(r, t_3, t)| \rightarrow \infty \text{ as } t \rightarrow t_{cr}. \text{ (or as } p \rightarrow p_{cr} \text{ for the pure elastic case).} \quad (2.112)$$

Thus, the values of the critical time or the values of the critical force are determined from the initial imperfection criterion (2.112).

2.8 Approximate Equations of the Stability Loss of the Cylinder-Shell Obtained from Equations of the TDLTS by the Average-Integrating Procedure

2.8.1 Kirchhoff–Love Shell Theory

For the case under consideration, the approximate stability loss equations for the hollow cylinder-shell can be derived from Eqs. (2.97)–(2.103) by using the Kirchhoff–Love hypothesis, according to which, the displacements of the cylinder are presented as follows:

$$u_r^{(1)} = w(z), \quad u_z^{(1)} = r' \frac{dw(z)}{dz}, \quad r' = R - r. \quad (2.113)$$

Underwriting the expression (2.113) is the assumption that the displacement of the middle surface of the cylinder, in the direction of the Oz axis, is very small with respect to the term $-r' dw(z)/dz$ and is ignored. Thus, according to (2.92), (2.96) and (2.113), we obtain:

$$\begin{aligned} \varepsilon_{\theta\theta}^{(1)} &= \frac{w(z)}{r}, \quad \varepsilon_{zz}^{(1)} = r' \frac{d^2 w(z)}{dz^2}, \quad \varepsilon_{rr}^{(1)} = \varepsilon_{rz}^{(1)} = 0, \\ \sigma_{\theta\theta}^{(1)} &= A_{11}^* \varepsilon_{\theta\theta}^{(1)} + A_{13}^* \varepsilon_{zz}^{(1)}, \quad \sigma_{zz}^{(1)} = A_{13}^* \varepsilon_{\theta\theta}^{(1)} + A_{33}^* \varepsilon_{zz}^{(1)}. \end{aligned} \quad (2.114)$$

Assume that:

$$\sigma_{rz}^{(1)} \neq 0. \quad (2.115)$$

Taking the relations (2.114) and (2.115) into account we obtain the following equation from (2.102)

$$\begin{aligned} \frac{\partial \sigma_{rz}^{(1)}}{\partial z} - \frac{1}{r} \sigma_{\theta\theta} + \sigma_{zz}^{(0)} \frac{d^2 w}{dz^2} &= 0, \\ \frac{\partial \sigma_{rz}^{(1)}}{\partial r} + \frac{1}{r} \sigma_{rz}^{(1)} + \frac{\partial \sigma_{zz}^{(1)}}{\partial z} + \sigma_{zz}^{(0)} \frac{d^2 u_z^{(1)}}{dz^2} &= 0. \end{aligned} \quad (2.116)$$

We can write the following presentations from Eq. (2.114):

$$\sigma_{\theta\theta}^{(1)} = A_{11}^* \frac{w(z)}{r} + A_{13}^* r' \frac{d^2 w(z)}{dz^2}, \quad \sigma_{zz}^{(1)} = A_{13}^* \frac{w(z)}{r} + A_{33}^* r' \frac{d^2 w(z)}{dz^2}. \quad (2.117)$$

Multiplying the first equation in (2.116) by $(1 - r'/R)dr'$ and integrating in the interval $[-h/2, +h/2]$ we obtain the following relation:

$$\int_{-h/2}^{+h/2} \frac{\partial \sigma_{rz}^{(1)}}{\partial z} \left(1 - \frac{r'}{R}\right) dr' = \frac{h}{R^2} A_{11}^* w - \sigma_{zz}^{(0)} h \frac{d^2 w}{dz^2}. \quad (2.118)$$

Multiplying the second equation in (2.116) by $(1 - r'/R)r'dr'$, also integrating in the interval $[-h/2, +h/2]$ and taking the relation (2.117) into account, we derive the following equation:

$$\begin{aligned} \int_{-h/2}^{+h/2} \frac{\partial \sigma_{rz}^{(1)}}{\partial z} \left(1 - \frac{r'}{R}\right) r' dr' + \int_{-h/2}^{+h/2} \frac{1}{R - r'} \sigma_{rz}^{(1)} \left(1 - \frac{r'}{R}\right) r' dr' + \\ \frac{A_{33}^* h^3}{12} \frac{d^3 w}{dz^3} + \frac{\sigma_{zz}^{(0)} h^3}{12} \frac{d^3 w}{dz^3} = 0. \end{aligned} \quad (2.119)$$

We make the following mathematical manipulations:

$$\begin{aligned} \int_{-h/2}^{+h/2} \frac{\partial \sigma_{rz}^{(1)}}{\partial r} \left(1 - \frac{r'}{R}\right) r' dr' + \int_{-h/2}^{+h/2} \frac{1}{R - r'} \sigma_{rz}^{(1)} \left(1 - \frac{r'}{R}\right) r' dr' \\ = - \int_{-h/2}^{+h/2} \frac{\partial (\sigma_{rz}^{(1)} r')}{\partial r} \left(1 - \frac{r'}{R}\right) r' dr' + \int_{-h/2}^{+h/2} \sigma_{rz}^{(1)} \left(1 - \frac{r'}{R}\right) dr' + \frac{1}{R} \int_{-h/2}^{+h/2} \sigma_{rz}^{(1)} r' dr' \\ = - \int_{-h/2}^{+h/2} \frac{\partial (\sigma_{rz}^{(1)} r')}{\partial r} dr' + \frac{1}{R} \int_{-h/2}^{+h/2} \frac{\partial (\sigma_{rz}^{(1)} r')}{\partial r} r' dr' \end{aligned}$$

$$\begin{aligned}
& + \int_{-h/2}^{+h/2} \sigma_{rz}^{(1)} \left(1 - \frac{r'}{R}\right) dr' + \frac{1}{R} \int_{-h/2}^{+h/2} \sigma_{rz}^{(1)} r' dr' \\
& = \left(\sigma_{rz}^{(1)} r' \right) \Big|_{-h/2}^{+h/2} + \frac{1}{R} \left(\sigma_{rz}^{(1)} r'^2 \right) \Big|_{-h/2}^{+h/2} + \int_{-h/2}^{+h/2} \sigma_{rz}^{(1)} \left(1 - \frac{r'}{R}\right) dr' \\
& = \sigma_{rz}^{(0)} h \left(\frac{h}{2R} - 1 \right) m\pi \cos(\alpha_1 m z) + \int_{-h/2}^{+h/2} \sigma_{rz}^{(1)} \left(1 - \frac{r'}{R}\right) dr' \quad (2.120)
\end{aligned}$$

Substituting (2.120) into (2.119), taking the relation (2.118) and $\sigma_{zz}^{(0)} = -p$ into account and assuming that $A_{33}^* 1 - p \approx A_{33}^* 1$, we obtain the following equation for the stability loss of the cylinder within the scope of the Kirchhoff-Love hypothesis:

$$\frac{h^2}{12} A_{33}^* \frac{d^4 w}{dz^4} + p \frac{d^2 w}{dz^2} + \frac{A_{11}^*}{R^2} w = -\alpha_1 m p \left(\frac{h}{2R} - 1 \right) \sin(\alpha_1 m z). \quad (2.121)$$

According to (2.113) and (2.114), we obtain from (2.98) and (2.99) the following end conditions for the displacement $w(z)$:

$$w(z)|_{z=0;\ell} = 0, \quad \frac{d^2 w(z)}{dz^2} \Big|_{z=0;\ell} = 0. \quad (2.122)$$

If we assume that the cylinder material is an isotropic one, then it must be written $E^* / (1 - (\nu^*)^2)$ instead of A_{33}^* and A_{11}^* . For the case where the material of the cylinder is isotropic and pure elastic, the Eq. (2.121) is transformed to the following one:

$$\frac{T}{h} \frac{d^4 w}{dz^4} + p \frac{d^2 w}{dz^2} + \frac{E}{R^2(1 - \nu^2)} w = -\alpha_1 m p \left(\frac{h}{2R} - 1 \right) \sin(\alpha_1 m z), \quad (2.123)$$

where $T = h^3 E / (12(1 - \nu^2))$, E and ν are the modulus of elasticity and Poisson's coefficient of the cylinder material and E^* and ν^* are the corresponding operators.

If we replace the term $E / (R^2(1 - \nu^2))$ in Eq. (2.123) with E / R^2 , then the left side of this equation coincides with the well-known classical one given in almost all books related to stability loss of the shells (see, for instance the book (Volmir 1967)). Consequently, obtaining the stability loss equation of the cylindrical shell within the scope of the Kirchhoff-Love hypothesis from the TDLTS Eqs. (2.97)–(2.103) of the circular hollow cylinder by the average-integrating procedure described above gives the foregoing type corrections in the expressions of the coefficients of this equation.

Now we turn to the solution of the Eq. (2.121). This solution is found by employing Laplace transformation (2.34) to this equation. According to the

convolution theorem for the Laplace transformation, the Eq. (2.121) is transformed to the following one:

$$\frac{h^2}{12} \bar{A}_{33}^* \frac{d^4 \bar{w}}{dz^4} + p \frac{d^2 \bar{w}}{dz^2} + \frac{\bar{A}_{11}^*}{R^2} \bar{w} = -\frac{1}{s} \alpha_1 m p \left(\frac{h}{2R} - 1 \right) \sin(\alpha_1 m z). \quad (2.124)$$

Taking the end condition (2.122) and the expression of the right hand-side of the Eq. (2.124), we select the solution to Eq. (2.124) as

$$\bar{w} = \bar{A}(s) \sin(\alpha_1 z). \quad (2.125)$$

Substituting (2.125) into (2.124) and doing some mathematical calculations we obtain the following expression for the unknown $\bar{A}(s)$:

$$\bar{A}(s) = \frac{-\ell p \left(\frac{h}{2R} - 1 \right)}{m \pi s \left[\frac{\bar{A}_{33}^*}{12} \left(\frac{h}{R} \right)^2 \lambda^4 - p \lambda^2 + \bar{A}_{11}^* \right]}, \quad (2.126)$$

where $\lambda = \pi m R / \ell$.

Employing the method (Schapery 1962, 1978), the original of $\bar{A}(s)$ is determined as follows:

$$A(t) = (s \bar{A}(s))|_{s=(1/(2t))}. \quad (2.127)$$

For the cases where the cylinder material is pure elastic we obtain from (2.126) the following expression for determination of the values of the critical force:

$$p_{cr.} = \frac{A_{11}}{\lambda^2} + \frac{A_{33}}{12} \left(\frac{h}{R} \right) \lambda^2. \quad (2.128)$$

It follows from the relation $dp_{cr.}/d(\lambda^2) = 0$ that the minimum critical force is obtained under:

$$\lambda \approx \sqrt[4]{12 \frac{A_{11}}{A_{33}}} \sqrt{\frac{R}{h}} \Rightarrow m \approx \frac{\ell}{\pi \sqrt{R h}} \sqrt[4]{12 \frac{A_{11}}{A_{33}}}. \quad (2.129)$$

For the case where the material of the cylinder is isotropic, we can write the relations given below from the Eq. (2.129):

$$\lambda \approx \sqrt[4]{12} \sqrt{\frac{R}{h}} \Rightarrow m \approx \frac{\ell}{\pi \sqrt{R h}} \sqrt[4]{12}. \quad (2.130)$$

By replacing $\sqrt[4]{12}$ with $\sqrt[4]{12(1-\nu^2)}$ we obtain the relations from (2.130) which are given in related papers and books (see, for instance the book (Volmir 1967)).

In the general case, i.e. in the case where the cylinder material is viscoelastic transversal isotropic, the values of the critical parameters, according to (2.112), are defined from the following equation:

$A(t) \rightarrow \infty$ as $t \rightarrow t_{cr}$. (for the case where the cylinder material is viscoelastic),
 $A \rightarrow \infty$ as $p \rightarrow p_{cr}$. (for the case where the cylinder material is pure elastic).
(2.131)

The values of the parameter λ under which the critical time or critical force becomes minimal will be determined by direct verification of the corresponding numerical results.

2.9 The Third Order Refined Shell Theory

As for the third order refined shell theory, we use some modification of the theory by (Kromm 1953, 1955) for the cylindrical shell, according to which, the displacement $u_r^{(1)}$ is presented as in (2.113), but the displacement $u_z^{(1)}$ is presented as

$$u_z^{(1)} = r' \frac{dw(z)}{dz} + \left(\frac{h^2}{4} r' - \frac{1}{3} r'^3 \right) \varphi(z), \quad r' = R - r, \quad (2.132)$$

where $\varphi(z)$ is an average shear deformation of the cross-section of the cylindrical shell.

Substituting the expressions for $u_r^{(1)}$ and $u_z^{(1)}$ into Eqs. (2.101) and (2.96) we obtain the expression for strains and stresses given below:

$$\begin{aligned} \varepsilon_{rr}^{(1)} &= 0, \quad \varepsilon_{\theta\theta}^{(1)} = \frac{w(z)}{r}, \quad \varepsilon_{zz}^{(1)} = r' \frac{d^2 w(z)}{dz^2} + \left(\frac{h^2}{4} r' - \frac{1}{3} r'^3 \right) \frac{d\varphi}{dz}, \\ \varepsilon_{rz}^{(1)} &= -\frac{1}{2} \left(\frac{h^2}{4} - r'^2 \right) \varphi(z), \\ \sigma_{rr}^{(1)} &= 0, \quad \sigma_{\theta\theta}^{(1)} = A_{11}^* \frac{w}{r} + A_{13}^* \left(r' \frac{d^2 w(z)}{dz^2} + \left(\frac{h^2}{4} r' - \frac{1}{3} r'^3 \right) \frac{d\varphi}{dz} \right), \\ \sigma_{zz}^{(1)} &= A_{13}^* \frac{w}{r} + A_{33}^* \left(r' \frac{d^2 w(z)}{dz^2} + \left(\frac{h^2}{4} r' - \frac{1}{3} r'^3 \right) \frac{d\varphi}{dz} \right), \\ \sigma_{rz}^{(1)} &= -G^* \left(\frac{h^2}{4} - r'^2 \right) \varphi(z). \end{aligned} \quad (2.133)$$

It follows from the expressions given in (2.133) that the equations in (2.116) hold also for the case considered.

As in the previous subsection, multiplying the first equation in (2.116) by $(1 - r'/R)dr'$, integrating in the interval $[-h/2, +h/2]$ and taking the relation (2.133) into account we obtain the following expressions:

$$\begin{aligned}
& \int_{-h/2}^{+h/2} \frac{\partial \sigma_{rz}^{(1)}}{\partial z} \left(1 - \frac{r'}{R}\right) dr' = -G^* \frac{h^3}{6} \frac{d\varphi(z)}{dz}, \\
& - \int_{-h/2}^{+h/2} \frac{1}{r} \sigma_{\theta\theta}^{(1)} \left(1 - \frac{r'}{R}\right) dr' = -\frac{h}{R^2} A_{11} w(z) \\
& \int_{-h/2}^{+h/2} \sigma_{zz}^{(1)} \frac{\partial^2 w}{\partial z^2} \left(1 - \frac{r'}{R}\right) dr' = -\sigma_{zz}^{(0)} h \frac{d^2 w}{dz^2}
\end{aligned} \tag{2.134}$$

Thus, after integrating (2.134), the first equation in (2.126) is transformed into the following one:

$$-G^* \frac{h^3}{6} \frac{d\varphi(z)}{dz} - \frac{h}{R^2} A_{11} w(z) + \sigma_{zz}^{(0)} h \frac{d^2 w}{dz^2} = 0. \tag{2.135}$$

Multiplying the second equation in (2.126) by $(1 - r'/R)r'dr'$, also integrating in the interval $[-h/2, +h/2]$ and taking into account the relations

$$\begin{aligned}
& \int_{-h/2}^{+h/2} \frac{\partial \sigma_{rz}^{(1)}}{\partial z} \left(1 - \frac{r'}{R}\right) r' dr' + \int_{-h/2}^{+h/2} \frac{1}{R - r'} \sigma_{rz}^{(1)} \left(1 - \frac{r'}{R}\right) r' dr' \\
& = \sigma_{zz}^{(0)} h \left(\frac{h}{2R} - 1\right) m\pi \cos(\alpha_1 m z) - \frac{h^3}{6} G^* \varphi(z), \\
& \int_{-h/2}^{+h/2} \frac{\partial \sigma_{zz}^{(1)}}{\partial z} \left(1 - \frac{r'}{R}\right) r' dr' = A_{33}^* \frac{h^3}{12} \frac{d^3 w}{dz^3} + A_{33}^* \frac{h^5}{60} \frac{d^2 \varphi}{dz^2}, \\
& \int_{-h/2}^{+h/2} \sigma_{zz}^{(1)} \left[r' \frac{d^3 w}{dz^3} + \left(\frac{h^2}{4} r' - \frac{r'^3}{3}\right) \frac{d\varphi}{dz} \right] r' \left(1 - \frac{r'}{R}\right) dr' \\
& = \sigma_{zz}^{(0)} \left[\frac{h^3}{12} \frac{d^3 w}{dz^3} + \frac{h^5}{60} \frac{d^2 \varphi}{dz^2} \right]
\end{aligned} \tag{2.136}$$

the mentioned second equation in (2.126) is transformed into the following one:

$$\begin{aligned}
& -\frac{h^3}{6} G^* \varphi + \left(A_{33}^* + \sigma_{zz}^{(0)}\right) \frac{h^3}{12} \frac{d^3 w}{dz^3} + \left(A_{33}^* + \sigma_{zz}^{(0)}\right) \frac{h^5}{60} \frac{d^2 \varphi}{dz^2} \\
& = -h \left(\frac{h}{2R} - 1\right) m\pi \sigma_{zz}^{(0)} \cos(\alpha_1 m z).
\end{aligned} \tag{2.137}$$

Thus, from (2.136) and (2.137) we obtain the following equation:

$$\begin{aligned} & \left[\left(A_{33}^* + \sigma_{zz}^{(0)} \right) \frac{h^2}{12} + \left(\frac{A_{33}^*}{G^*} + \frac{\sigma_{zz}^{(0)}}{G^*} \right) \frac{\sigma_{zz}^{(0)} h^2}{10} \right] \frac{d^4 w}{dz^4} \\ & + \left[-\sigma_{zz}^{(0)} - \left(\frac{A_{33}^*}{G^*} + \frac{\sigma_{zz}^{(0)}}{G^*} \right) \frac{A_{11}^* h^2}{10R^2} \right] \frac{d^2 w}{dz^2} + \frac{A_{11}^*}{R} w \\ & = \pi \alpha_1 m^2 \left(\frac{h}{2R} - 1 \right) \sigma_{zz}^{(0)} \sin(\alpha_1 m z). \end{aligned} \quad (2.138)$$

The end condition (2.122) holds also for the Eq. (2.138). As in the previous subsection, employing the Laplace transformation (2.34) to the Eq. (2.138), using the presentation (2.125) and the equality $\sigma_{zz}^{(0)} = -p$ we obtain the following expression for the unknown $\bar{A}(s)$:

$$\begin{aligned} \bar{A}(s) &= \frac{-\ell p \left(\frac{h}{2R} - 1 \right)}{\Theta(s, \lambda, h/R)}, \\ \Theta(s, \lambda, h/R) &= m\pi s \left[A_{11}^* - \left[p - \left(\frac{A_{33}^*}{G^*} - \frac{p}{G^*} \right) \frac{A_{11}^*}{10} \left(\frac{h}{R} \right)^2 \right] \lambda^2 \right. \\ & \quad \left. + \left[\frac{A_{33}^*}{12} - \left(\frac{A_{33}^*}{G^*} - \frac{p}{G^*} \right) \frac{p}{10} \right] \left(\frac{h}{R} \right)^2 \lambda^4 \right]. \end{aligned} \quad (2.139)$$

For the case where the material of the cylinder is a pure elastic one, within the scope of the assumption $(A_{33}/G - p/G)p/10 \approx (A_{33}/G)p/10$, we determine the following expression from the equation for the calculation of the critical values of the external compressed force:

$$p_{cr} = \frac{A_{11} + A_{11} \frac{A_{33}}{10G} \left(\frac{h}{R} \right)^2 \lambda^2 + \frac{A_{33}}{12} \lambda^4}{\left[1 + \frac{A_{11}}{10G} \left(\frac{h}{R} \right)^2 \right] \lambda^2 + \frac{A_{33}}{10G} \left(\frac{h}{R} \right)^2 \lambda^4}. \quad (2.140)$$

2.10 Numerical Results Related to the Rotationally Symmetric Stability Loss Problem

As in Sect. 2.5, we assume that the cylinder is made from viscoelastic unidirectional fibrous composite materials consisting of a viscoelastic matrix and pure elastic fibers lying along the Oz axis. Moreover, we assume that the notation and expressions (2.68)–(2.73) given in Sect. 2.5 hold also for the case under consideration.

Thus, within the foregoing assumptions we consider the numerical results and, according to the relation (2.75), first examine the pure elastic stability loss under

$t' = 0$ and $t' = \infty$. We compare the results obtained within the scope of the TDLTS with the corresponding ones obtained within the scope of the Kirchhoff–Love shell theory and within the scope of the third order refined shell theory. Through $p'_{3D.c.0} (= p_{3D.c.0}/E_0^{(1)})$ and $p'_{3D.c.\infty} (= p_{3D.c.\infty}/E_0^{(1)})$ we denote the critical values of the intensity of the dimensionless forces obtained by employing the TDLTS. Here and below the sub-indices 0 and ∞ indicate that the critical values of the forces are calculated at $t' = 0$ and $t' = \infty$ respectively. According to the Eqs. (2.128) and (2.140), the corresponding critical values of the intensity of the compressing forces obtained within the scope of the Kirchhoff–Love shell theory (denoted by $p'_{K.c.0}$ and $p'_{K.c.\infty}$) and within the scope of the third order refined shell theory (denoted by $p'_{R.c.0}$ and $p'_{R.c.\infty}$) can be calculated by the use of the following expressions:

$$\begin{aligned}
 p'_{K.c.0} &= \frac{p_{K.c.0}}{E_0^{(1)}} = \frac{A_{110}}{\lambda^2 E_0^{(1)}} + \frac{A_{330}}{12E_0^{(1)}} \left(\frac{h}{R}\right)^2 \lambda^2, \\
 p'_{K.c.\infty} &= \frac{p_{K.c.\infty}}{E_0^{(1)}} = \frac{A_{11\infty}}{\lambda^2 E_0^{(1)}} + \frac{A_{33\infty}}{12E_0^{(1)}} \left(\frac{h}{R}\right)^2 \lambda^2, \\
 p'_{R.c.0} &= \frac{p_{R.c.0}}{E_0^{(1)}} = \frac{A_{110} + A_{110} \frac{A_{330}}{12G_0} \left(\frac{h}{R}\right)^2 \lambda^2 + \frac{A_{330}}{12} \lambda^4}{E_0^{(1)} \left[1 + \frac{A_{110}}{10G_0} \left(\frac{h}{R}\right)^2\right] \lambda^2 + \frac{A_{330}}{10G_0} \left(\frac{h}{R}\right)^2 \lambda^4}, \\
 p'_{R.c.\infty} &= \frac{p_{R.c.\infty}}{E_0^{(1)}} = \frac{A_{11\infty} + A_{11\infty} \frac{A_{33\infty}}{12G_\infty} \left(\frac{h}{R}\right)^2 \lambda^2 + \frac{A_{33\infty}}{12} \lambda^4}{E_0^{(1)} \left[1 + \frac{A_{11\infty}}{10G_\infty} \left(\frac{h}{R}\right)^2\right] \lambda^2 + \frac{A_{33\infty}}{10G_\infty} \left(\frac{h}{R}\right)^2 \lambda^4},
 \end{aligned} \tag{2.141}$$

where $A_{110} = \bar{A}_{11}^*|_{s=\infty}$, $A_{330} = \bar{A}_{33}^*|_{s=\infty}$, $A_{11\infty} = \bar{A}_{11}^*|_{s=0}$, $A_{33\infty} = \bar{A}_{33}^*|_{s=0}$, $G_0 = \bar{G}^*|_{s=\infty}$ and $G_\infty = \bar{G}^*|_{s=0}$.

Thus, for the problem under consideration we have the dimensionless parameters h/R , $\lambda = \pi m R / \ell$, $E^{(2)}/E_0^{(1)}$, ω and α . Under numerical investigation, the critical values of the forces are minimized with respect to the parameter λ . Table 2.16 shows the minimized values of $p'_{3D.c.0}$, $p'_{R.c.0}$ and $p'_{K.c.0}$ for various $E^{(2)}/E_0^{(1)}$ and h/R . The corresponding values of $p'_{3D.c.\infty}$, $p'_{R.c.\infty}$ and $p'_{K.c.\infty}$ are given in Table 2.17 which are obtained for the case where $\omega = 1.0$. In the foregoing tables the values of the parameter λ under which the critical forces have their minimum are also shown.

Note that the parameter ω characterizes the mechanical behavior of the viscoelastic matrix material in the vicinity of $t' = \infty$. Consequently, an increase in the values of ω corresponds to an increase (a decrease) in the values of $E_\infty^{(1)}$ ($v_\infty^{(1)}$). Therefore the values of $p'_{3D.c.\infty}$, $p'_{R.c.\infty}$ and $p'_{K.c.\infty}$ must be increase with ω . This consideration is confirmed by the data given in Table 2.18 which show the values

Table 2.16 The values of $p'_{3D.c.0}$, $p'_{R.c.0}$ and $p'_{K.c.0}$ obtained for various values of $E^{(2)}/E_0^{(1)}$ and h/R

$\frac{E^{(2)}}{E_0^{(1)}}$	$p'_{3D.c.0} \times 10^2$	$p'_{R.c.0} \times 10^2$ $h/R = 0.01$	$p'_{E.c.0} \times 10^2$
1	0.6000($\lambda = 18.2$)	0.7678($\lambda = 18.7$)	0.7772($\lambda = 18.6$)
5	1.2816($\lambda = 15.8$)	1.6471($\lambda = 16.7$)	1.6688($\lambda = 16.6$)
10	1.7868($\lambda = 13.9$)	2.2674($\lambda = 15.0$)	2.3019($\lambda = 14.9$)
$h/R = 0.05$			
1	2.8996($\lambda = 8.40$)	3.6540($\lambda = 8.60$)	3.8860($\lambda = 8.30$)
5	6.2111($\lambda = 7.20$)	7.8134($\lambda = 7.70$)	8.3442($\lambda = 7.40$)
10	8.6517($\lambda = 6.40$)	10.660($\lambda = 6.90$)	11.510($\lambda = 6.70$)
$h/R = 0.10$			
1	5.5644($\lambda = 6.10$)	6.8662($\lambda = 6.30$)	7.7721($\lambda = 5.90$)
5	11.958($\lambda = 5.30$)	14.618($\lambda = 5.60$)	16.691($\lambda = 5.30$)
10	16.634($\lambda = 4.70$)	19.765($\lambda = 5.10$)	23.019($\lambda = 4.70$)
$h/R = 0.20$			
1	10.286($\lambda = 4.60$)	12.095($\lambda = 4.70$)	15.546($\lambda = 4.20$)
5	22.226($\lambda = 4.00$)	25.571($\lambda = 4.20$)	33.377($\lambda = 3.70$)
10	30.798($\lambda = 3.60$)	33.950($\lambda = 3.90$)	46.047($\lambda = 3.30$)

Table 2.17 The values of $p'_{3D.c.\infty}$, $p'_{R.c.\infty}$ and $p'_{K.c.\infty}$ obtained for various values of $E^{(2)}/E_0^{(1)}$ and h/R under $\omega = 0.1$

$\frac{E^{(2)}}{E_0^{(1)}}$	$p'_{3D.c.\infty} \times 10^2$	$p'_{R.c.\infty} \times 10^2$ $h/R = 0.01$	$p'_{E.c.\infty} \times 10^2$
1	0.3279($\lambda = 15.8$)	0.6277($\lambda = 18.2$)	0.6437($\lambda = 18.6$)
5	0.6627($\lambda = 12.1$)	1.3315($\lambda = 15.7$)	1.3786($\lambda = 15.4$)
10	0.9202($\lambda = 10.4$)	1.8103($\lambda = 14.0$)	1.8938($\lambda = 13.7$)
$h/R = 0.05$			
1	1.5824($\lambda = 7.30$)	2.8637($\lambda = 8.50$)	3.2106($\lambda = 8.00$)
5	3.1809($\lambda = 5.60$)	5.7933($\lambda = 7.50$)	6.8929($\lambda = 6.90$)
10	4.3762($\lambda = 4.90$)	7.5643($\lambda = 6.90$)	9.4694($\lambda = 6.10$)
$h/R = 0.10$			
1	3.0307($\lambda = 5.30$)	5.1058($\lambda = 6.30$)	6.4214($\lambda = 5.70$)
5	6.0469($\lambda = 4.20$)	9.7688($\lambda = 5.80$)	13.786($\lambda = 4.90$)
10	8.2102($\lambda = 3.70$)	12.191($\lambda = 5.40$)	18.940($\lambda = 4.30$)
$h/R = 0.20$			
1	5.5785($\lambda = 4.00$)	8.1647($\lambda = 5.00$)	12.842($\lambda = 4.00$)
5	10.914($\lambda = 3.30$)	14.270($\lambda = 4.70$)	27.582($\lambda = 3.40$)
10	14.368($\lambda = 3.10$)	16.706($\lambda = 4.60$)	37.885($\lambda = 3.10$)

of $p'_{3D.c.\infty}$, $p'_{R.c.\infty}$ and $p'_{K.c.\infty}$ obtained for various ω under $E^{(2)}/E_0^{(1)} = 5$. It follows from the foregoing numerical results that the relations

$$p'_{3D.c.0} < p'_{R.c.0} < p'_{K.c.0}, \quad p'_{3D.c.\infty} < p'_{R.c.\infty} < p'_{K.c.\infty} \quad (2.142)$$

Table 2.18 The values of $p'_{3D.c.\infty}$, $p'_{R.c.\infty}$ and $p'_{K.c.\infty}$ obtained for various values of ω and h/R under $E^{(2)}/E_0^{(1)} = 5$

ω	$p'_{3D.c.\infty} \times 10^2$	$p'_{R.c.\infty} \times 10^2$ $h/R = 0.01$	$p'_{E.c.\infty} \times 10^2$
0.5	0.4394($\lambda = 10.1$)	1.2282($\lambda = 15.7$)	1.3196($\lambda = 15.2$)
1.0	0.6627($\lambda = 12.1$)	1.3315($\lambda = 15.7$)	1.3786($\lambda = 15.4$)
2.0	0.8792($\lambda = 13.6$)	1.4268($\lambda = 15.9$)	1.4584($\lambda = 15.8$)
$h/R = 0.05$			
0.5	2.0815($\lambda = 4.80$)	4.6327($\lambda = 8.10$)	6.5982($\lambda = 6.80$)
1.0	3.1809($\lambda = 5.60$)	5.7933($\lambda = 7.50$)	6.8929($\lambda = 6.90$)
2.0	4.2441($\lambda = 6.30$)	6.5321($\lambda = 7.40$)	7.2930($\lambda = 7.00$)
$h/R = 0.10$			
0.5	3.8823($\lambda = 5.60$)	6.7360($\lambda = 6.70$)	13.196($\lambda = 4.80$)
1.0	6.0469($\lambda = 4.20$)	9.7688($\lambda = 5.80$)	13.786($\lambda = 4.90$)
2.0	8.2169($\lambda = 4.60$)	11.699($\lambda = 5.60$)	14.585($\lambda = 5.00$)
$h/R = 0.20$			
0.5	6.7019($\lambda = 3.10$)	8.2085($\lambda = 5.90$)	26.393($\lambda = 3.40$)
1.0	10.914($\lambda = 3.30$)	14.270($\lambda = 4.70$)	27.582($\lambda = 3.40$)
2.0	14.928($\lambda = 3.50$)	18.875($\lambda = 4.30$)	29.172($\lambda = 3.50$)

occur for all the considered values of the problem parameters. The difference between the values of $p'_{3D.c.0}$ ($p'_{3D.c.\infty}$) and of $p'_{R.c.0}$ ($p'_{R.c.\infty}$) as well as the difference between $p'_{R.c.0}$ ($p'_{R.c.\infty}$) and $p'_{K.c.0}$ ($p'_{K.c.\infty}$) depends not only on the parameters $E^{(2)}/E_0^{(1)}$ and h/R , but also on the values of the rheological parameter ω .

Now we consider the case where the cylinder material is viscoelastic and analyze the critical time. To compare the values of the critical time given by the TDLTS with those given by approximate shell theories we must select the values of p' ($= p/E_0^{(1)}$) which fall in the intervals $[p'_{K.c.\infty}, p'_{K.c.0}]$, $[p'_{R.c.\infty}, p'_{R.c.0}]$ and $[p'_{3D.c.\infty}, p'_{3D.c.0}]$ simultaneously. However, as follows from the data given in Tables 2.16, 2.17 and 2.18, in many cases such common values of p' ($= p/E_0^{(1)}$), exist only for the intervals $[p'_{R.c.\infty}, p'_{R.c.0}]$ and $[p'_{3D.c.\infty}, p'_{3D.c.0}]$. For this reason, we will compare only the critical time obtained by employing the TDLTS (denoted by $t'_{3D.cr.}$) with that obtained by employing the third order refined shell theory (denoted by $t'_{Rf.cr.}$). For consideration we select the case where $E^{(2)}/E_0^{(1)} = 5$, $\omega = 1$ and $\alpha = -0.3$. Tables 2.19 and 2.20 show the values of $t'_{3D.cr.}$ (upper number) and $t'_{Rf.cr.}$ (lower number) for the cases where $R/h = 0.1$ and 0.2 respectively, for various values of λ . Note that the selected values of λ are in the interval $[\lambda_{m.0}, \lambda_{m.\infty}]$, where $\lambda_{m.0}$ ($\lambda_{m.\infty}$) is the value of λ under which the values of $p'_{3D.c.0}$ ($p'_{R.c.\infty}$) have their minimum.

Table 2.19 The values of $t'_{3D.cr.}$ (upper number) and $t'_{Rf.cr.}$ (lower number) obtained for various values of the parameter λ in the case where $h/R = 0.1$ under $\alpha = -0.3$ and $\omega = 1.0$

λ	$p' \times 10^2$						
	11.5	11.7	11.8	11.9	12.0	12.2	12.4
4.2	0.027 64.33	0.021 13.08	0.018 8.427	0.016 5.936	0.014 4.426	0.010 2.736	0.008 2.230
5.3	0.006 0.688	0.002 0.525	0.001 0.461	$\frac{O(10^{-4})}{0.405}$	$\frac{O(10^{-4})}{0.375}$	$\frac{O(10^{-5})}{0.278}$	$\frac{O(10^{-6})}{0.245}$
5.6	0.008 0.649	0.004 0.499	0.002 0.439	$\frac{0.001}{0.387}$	$\frac{O(10^{-5})}{0.341}$	$\frac{O(10^{-6})}{0.266}$	$\frac{O(10^{-6})}{0.235}$

Table 2.20 The values of $t'_{3D.cr.}$ (upper number) and $t'_{Rf.cr.}$ (lower number) obtained for various values of the parameter λ in the case where $h/R = 0.2$ under $\alpha = -0.3$ and $\omega = 1.0$

λ	$p' \times 10^2$						
	15.0	16.0	16.5	17.0	18.0	18.5	19.0
4.0	0.561 44.65	0.320 4.895	0.244 2.847	0.187 1.847	0.109 0.918	0.082 0.678	0.060 0.510
4.2	0.651 20.54	0.365 3.902	0.278 2.406	0.212 1.615	0.123 0.834	0.092 0.623	0.068 0.473
4.7	1.075 13.87	0.578 3.532	0.436 2.284	0.333 1.583	0.196 0.855	0.150 0.650	0.114 0.501

For estimation of the difference between the values of the critical forces obtained within the scope of the TDLST and within the scope of the refined shell theory we introduce the parameters $d_{p,0} = p'_{R,c,0}/p'_{3D,c,0}$ and $d_{p,\infty} = p'_{R,c,\infty}/p'_{3D,c,\infty}$. Furthermore, to estimate the difference between the critical times obtained within the scope of the TDLST and within the scope of the refined shell theory we introduce the parameter $d_t = t'_{Rf.cr.}/t'_{3D.cr.}$. It follows from the numerical results given in Tables 2.16 and 2.17 that $d_{p,0} \approx 1.22$ and 1.15 ($d_{p,\infty} \approx 1.61$ and 1.30) in the cases where $h/R = 0.1$ and 0.2 respectively under $E^{(2)}/E_0^{(1)} = 5$, $\omega = 1$. However, Tables 2.19 and 2.20 show that the values of d_t may be greater than 10^5 . Consequently, in many cases, especially in the cases where the values of p' ($= p/E_0^{(1)}$) are near the values of $p'_{3D,c,0}$ or $p'_{3D,c,\infty}$, to investigate the stability loss of the considered cylinder it is necessary to use the TDLTS developed in the present book.

For a clearer illustration of the difference between the $t'_{3D.cr.}$ and $t'_{Rf.cr.}$ in Fig. 2.9 the graphs of the dependencies among $t'_{3D.cr.}$, $t'_{Rf.cr.}$ and p' are given for the various values of the parameter λ in the cases where $h/R = 0.1$ (Fig. 2.9a) and 0.2 (Fig. 2.9b) under $\omega = 1$ and $\alpha = -0.3$. It follows from these graphs that, as can be predicted, the following relations occur:

$$\begin{aligned} t'_{3D.cr.} &\rightarrow 0 \text{ as } p' \rightarrow p'_{3D,c,0}, \quad t'_{3D.cr.} \rightarrow \infty \text{ as } p' \rightarrow p'_{3D,c,\infty}, \\ t'_{Rf.cr.} &\rightarrow 0 \text{ as } p' \rightarrow p'_{R,c,0}, \quad t'_{Rf.cr.} \rightarrow \infty \text{ as } p' \rightarrow p'_{R,c,\infty}. \end{aligned} \quad (2.143)$$

Table 2.21 The values of $t'_{3D.cr.}$ (upper number) and $t'_{Rf.cr.}$ (lower number) obtained for various values of p' in the case where $\lambda = 4.0$, $\alpha = -0.3$, $E^{(2)}/E_0^{(1)} = 5$, $h/R = 0.2$ and $\omega = 1.0$

$p' \times 10^2$						
20.4	20.5	20.6	20.7	20.8	20.9	21.0
$t'_{3D.cr.}/t'_{cdm.cr.}$						
0.0210	0.0190	0.0172	0.0155	0.0138	0.0122	0.0108
118.48	8.4127	3.2494	1.7368	1.0685	0.7095	0.4931

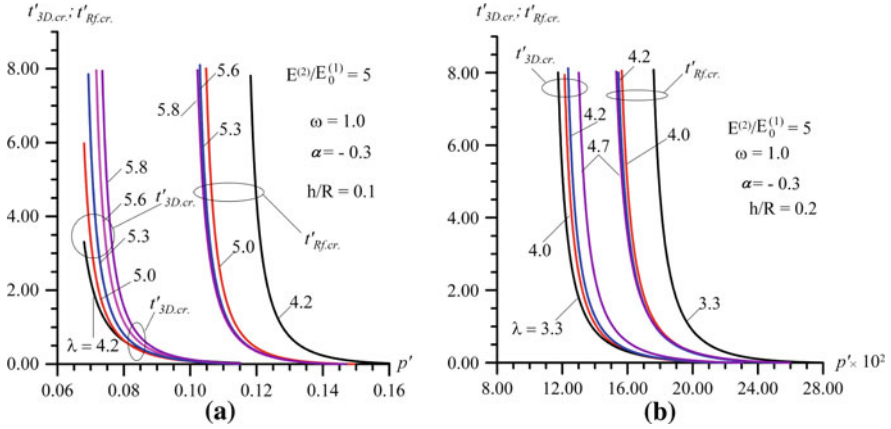


Fig. 2.9 The graphs of the dependencies among $t'_{3D.cr.}$, $t'_{Rf.cr.}$ and dimensionless intensity of the compressed force p' constructed for various values of the parameter λ in the cases where $h/R = 0.1$ (a) and $h/R = 0.2$ (b)

Consider the influence of the rheological parameter α on the values of $t'_{3D.cr.}$. The graphs of the dependencies between $t'_{3D.cr.}$ and p' are given in Fig. 2.10 which are constructed for the cases where $h/R = 0.1$ (Fig. 2.10a) and 0.2 (Fig. 2.10b) under $\omega = 1$ and $\lambda = 4$. It follows from the results given in Fig. 2.10 (and from other numerical results which are not given here) that the graphs of the analyzed dependencies constructed for various α have a common intersection point.

Before (after) this intersection point the increase in the absolute values of α causes an increase (a decrease) in the values of $t'_{3D.cr.}$. Note that the parts of the graphs which appear after the intersection point are illustrated in Fig. 2.11 for the cases where $h/R = 0.1$ (Fig. 2.11a) and $h/R = 0.2$ (Fig. 2.11b) under $\omega = 1$ and $\lambda = 4$. Remind that the explanation of the character of the influence of the parameter α on the values of the critical time is given in Subsection 2.5.1.

Now we consider the question of how the rheological parameter ω influences the behavior of the graphs related to the dependence between $t'_{3D.cr.}$ and p' . The corresponding graphs are given in Fig. 2.12 for various values of the parameter ω in the case where $E^{(2)}/E_0^{(1)} = 5$, $h/R = 0.2$ and $\alpha = -0.5$. The character of these

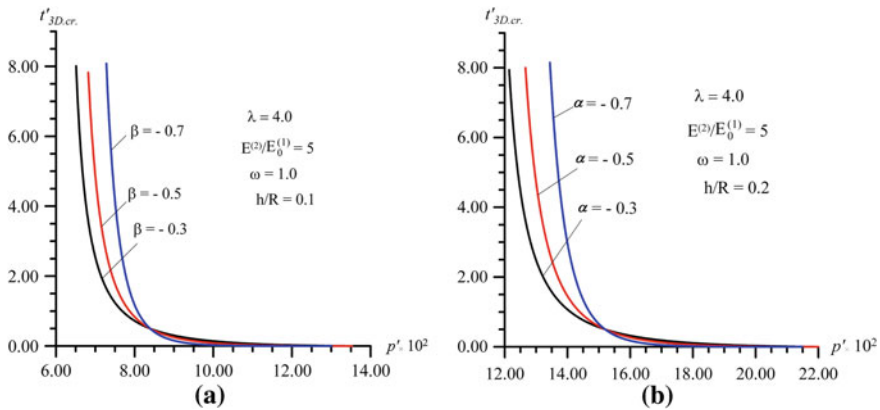


Fig. 2.10 The influence of the rheological parameter α on the dependencies between $t'_{3D.cr.}$ and compressed force p' in the cases where $h/R = 0.1$ (a) and $h/R = 0.2$ (b)

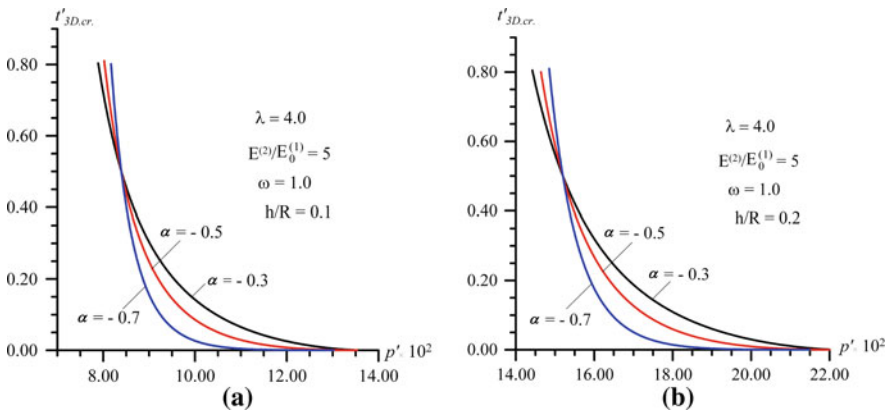
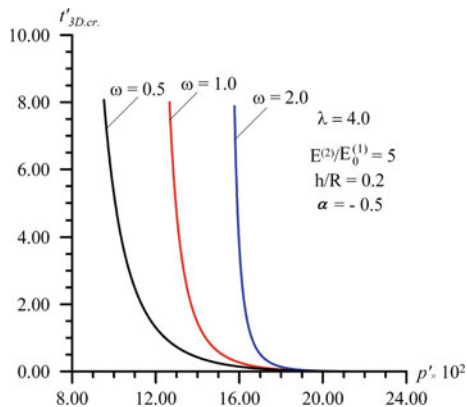


Fig. 2.11 The parts of the graphs given in Fig. 2.10 which arise after the intersection point in the cases where $h/R = 0.1$ (a) and $h/R = 0.2$ (b)

graphs can be explained by the increase in the values of $p'_{3D.c.\infty}$ with ω . Therefore for common p' (with respect to the selected values of ω), the values of $t'_{3D.cr.}$ increase with ω .

As in the previous sections, compare the values of $t'_{3D.cr.}$ with the corresponding ones calculated by employing the critical deformation method (Gerard and Gilbert 1958). According to this method, it is assumed that the critical deformation of the viscoelastic hollow cylinder is equal to the critical deformation of the corresponding elastic cylinder. Using this assumption, the critical deformation of the pure elastic cylinder is determined within the scope of the TDLTS. Note that in the considered case the mentioned critical deformation corresponds to the $p'_{3D.c.0}$.

Fig. 2.12 The graphs of the dependencies between $t'_{3D.cr.}$ and dimensionless intensity of the compressed force p' constructed for various values of the rheological parameter ω



According to this determination, using the relation $p'_{3D.c.0} = p/E_3^*$, the critical time is determined for selected values of p . The values of the dimensionless critical time (denoted by $t'_{cdm.cr.}$), which are determined by employing the critical deformation method, are given in Table 2.21. These values are calculated for the various p' for the case where $\lambda = 4.0$, $\alpha = -0.3$, $E^{(2)}/E_0^{(1)} = 5$, $h/R = 0.2$ and $\omega = 1.0$. At the same time, in this table the corresponding values of $t'_{3D.cr.}$ (upper number) are also illustrated. The comparison of $t'_{cdm.cr.}$ with $t'_{3D.cr.}$ shows that the critical deformation method is not acceptable for determination of the critical time for the local stability loss of the hollow cylinder made from viscoelastic composite material.

2.11 Conclusions

Thus, in the present chapter within the scope of the three-dimensional geometrically non-linear field equations of the theory for viscoelastic orthotropic bodies an approach has been developed for 3D global and local stability loss analyses of solid and hollow cylinders made from viscoelastic composite materials. In studying the global (local) stability loss of the cylinder it is supposed that it has an initial infinitesimal global (local) imperfection and the values of a critical time as well as the values of a critical force are determined from the initial imperfection criterion. According to this criterion, the moment at which the mentioned initial imperfections start to increase and grow indefinitely is taken as a critical time (or critical force). To study the corresponding non-linear boundary value problems the boundary-form disturbance method and small parameter method are employed simultaneously. From the results of these applications, the solution to the aforementioned non-linear problem is reduced to the solution for the corresponding series of linear problems. By direct verification it is proven that the equations and

relations related to the first and subsequent approximations coincide with the equations and relations of the well known TDLTS. It is established that for investigation of the stability loss problem it is enough to use only the zeroth and first approximations. By the average-integrating procedure the corresponding approximate global stability loss equations within the scope of the Bernoulli and third order refined beam theories are derived from the equations and relations of the aforementioned first approximations. In addition, by the average-integrating procedure the corresponding approximate local stability loss equations within the scope of the Kirchhoff–Love and third order shell theories are obtained from the equations and relations related to the rotationally symmetric 3D local stability loss of the hollow cylinder. Final numerical results are obtained by employing Laplace transformation and variation separation methods for the case where the cylinder is made from the viscoelastic composite consisting of the viscoelastic matrix and uni-directional elastic fibers lying along the cylinder. First, the numerical results related to the pure elastic stability loss of the cylinder at $t = 0$ and $t = \infty$ are considered. It is established that the critical forces obtained within the scope of the third order refined beam and shell theories are very close to the corresponding results obtained within the scope of the 3D approach. However, the critical time obtained within the scope of the 3D approach can be significantly less than the corresponding values of the critical time obtained within the scope of the third order refined beam and shell theories. The difference between the critical times obtained by employing 3D and third order refined beam and shell theories becomes increasingly non-negligible if the values of the external compressed force is close to the critical compressed force obtained at $t = \infty$. Consequently, for investigation of the global and local stability loss of the cylinders made from viscoelastic composites, in many cases it is necessary to use the TDLTS developed in the present chapter. This approach can be easily extended for investigation of the more complicated stability loss problems, for instance, for non-symmetric local stability loss of the hollow cylinders (shells) made from viscoelastic materials.

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