Chapter 2
Cone and Partial Order Methods

2.1 Increasing Operators

Suppose that E is a real Banach space, θ is the zero element of E, $P \subset E$ is a cone. $A : D \to E$ is called an increasing operator, if $\forall x_1, x_2 \in D \subset E$, $x_1 \leq x_2 \Rightarrow Ax_1 \leq Ax_2$.

Theorem 2.1.1 (Jingxian Sun, see [170, 171]) *Suppose* $u_0, v_0 \in E$, $u_0 < v_0$, $A : [u_0, v_0] \to E$ *is an increasing operator satisfying*

$$u_0 \leq Au_0, \qquad Av_0 \leq v_0. \tag{2.1}$$

And if $A([u_0, v_0])$ *is relatively compact* (*sequentially compact*), *then* A *must have minimal fixed point* x_* *and maximal fixed point* x^*, *which satisfy*

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq x_* \leq \cdots \leq x^* \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0. \tag{2.2}$$

Proof (a) Firstly we prove A has a fixed point in $[u_0, v_0]$. Let $D = [u_0, v_0]$. By (2.1) and A is increasing, we know

$$A(D) \subset D. \tag{2.3}$$

Define $F = \{x \in D : Ax \geq x\}$. Because $u_0 \in F$, we have $F \neq \emptyset$. Suppose H is a totally ordered subset of F. Since A is increasing, $A(H)$ is also a totally ordered set. Noticing $A(D)$ is relatively compact thus separable and $A(H) \subset A(D)$, we conclude $A(H)$ is separable. So there exist a countable set $V = \{y_1, y_2, \ldots, y_n, \ldots\} \subset A(H)$ which is dense in $A(H)$. Since V is totally ordered, $z_n = \max\{y_1, y_2, \ldots, y_n\}$ exists (z_n is one of $y_1, y_2, \ldots, y_n$), and

$$z_n \in V \subset A(H) \quad (n = 1, 2, 3, \ldots), \tag{2.4}$$

$$z_1 \leq z_2 \leq \cdots \leq z_n \leq \cdots. \tag{2.5}$$

Since $A(D)$ is relatively compact, we have a subsequence $\{z_{n_i}\} \subset \{z_n\}$, such that

$$z_{n_i} \to z^* \in E \quad (i \to \infty). \tag{2.6}$$

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Through (2.3), (2.4), and (2.6), we know

$$z^* \in D. \tag{2.7}$$

Combining (2.5) and (2.6),

$$y_n \leq z_n \leq z^* \quad (n = 1, 2, 3, \ldots). \tag{2.8}$$

Also because V is dense in $A(H)$ we have $z \leq z^*$, $\forall z \in A(H)$. Thus

$$x \leq Ax \leq z^*, \quad \forall x \in H, \tag{2.9}$$

then

$$Ax \leq Az^* \quad \forall x \in H, \tag{2.10}$$

and

$$z_n \leq Az^* \quad (n = 1, 2, 3, \ldots). \tag{2.11}$$

By (2.6) and (2.11), we have

$$z^* \leq Az^*. \tag{2.12}$$

So $z^* \in F$. Also (2.9) implies z^* is an upper bound for H in F. So we can conclude F has a maximal element v^* by Zorn's Lemma. Since $v^* \in F$, $A(v^*) \geq v^*$. Also we have $A(A(v^*)) \geq A(v^*)$, so $A(v^*) \in F$. Because v^* is maximal, we have $A(v^*) = v^*$, which means v^* is a fixed point of A.

Similarly consider $G = \{s \in D : Ax \leq x\}$. We can obtain G has a minimal element u^* and $A(u^*) = u^*$.

(b) Secondly we prove A has minimal fixed point and maximal fixed point in D. Let $\mathrm{Fix}(A) = \{x \in D : Ax = x\}$. Then from the above proof, $\mathrm{Fix}(A) \neq \emptyset$. Let $S = \{I = [u, v] : u, v \in D, u \leq v, u \leq Au, Av \leq v, \mathrm{Fix}(A) \subset [u, v]\}$. Since $D \in S$, then $S \neq \emptyset$. S is a partial ordered set if we define $I_1 \leq I_2$ for $\forall I_1, I_2 \in S$ and $I_1 \subset I_2$. Suppose $S_1 = \{I_r : r \in \Lambda\}$ is a totally ordered subset of S, where $I_r = [u_r, v_r]$. Let $H_1 = \{u_r : r \in \Lambda\}$, then H_1 is a totally ordered subset of D and $Au_r \geq u_r$ $(r \in \Lambda)$. By substitute H by H_1 in the proof of part (a), we know there exist $z^* \in D$ and $\{z_n\} \subset A(H_1)$ such that $z_n \to z^*$ as $n \to \infty$ and

$$u_r \leq Au_r \leq z^*, \quad r \in \Lambda, \tag{2.13}$$

$$z^* \leq Az^*. \tag{2.14}$$

Since $u_r \leq x$, $\forall r \in \Lambda$, $x \in \mathrm{Fix}(A)$, then

$$u_r \leq Au_r \leq Ax = x, \quad r \in \Lambda, \; x \in \mathrm{Fix}(A),$$

so

$$z_n \leq x, \quad x \in \mathrm{Fix}(A) \quad (n = 1, 2, 3, \ldots). \tag{2.15}$$

Because $z_n \to z^*$, the above inequality implies that

$$z^* \leq x, \quad x \in \mathrm{Fix}(A). \tag{2.16}$$

Similarly considering $G_1 = \{v_r : r \in \Lambda\}$, we can obtain there exists $w^* \in D$ that satisfies

$$v_r \geq w^*, \quad r \in \Lambda, \tag{2.17}$$

$$w^* \geq Aw^*, \tag{2.18}$$

$$w^* \geq x, \quad x \in \mathrm{Fix}(A). \tag{2.19}$$

Combining (2.16), (2.19) and $\mathrm{Fix}(A) \neq \emptyset$, it has $z^* \leq w^*$. Let $I^* = [z^*, w^*]$. Then $I^* \in S$. Since (2.13) and (2.17), I^* is a lower bound for S_1. So by Zorn's Lemma, we find S has a minimal element $I_* = [x_*, x^*]$. It must have

$$x_* \leq x \leq x^*, \quad x \in \mathrm{Fix}(A). \tag{2.20}$$

By the definition of S and $I^* \in S$, it is easy to know

$$x_* \leq Ax_* \leq Ax^* \leq x^*, \tag{2.21}$$

$$Ax_* \leq Ax = x \leq Ax^*, \quad \forall x \in \mathrm{Fix}(A). \tag{2.22}$$

Since A is increasing,

$$x_* \leq Ax_* \leq A\big(Ax^*\big) \leq A(Ax_*) \leq Ax^* \leq x^*. \tag{2.23}$$

Through (2.22) and (2.23), we get $\bar{I} = \{Ax_*, Ax^*\} \in S$ and $\bar{I} \leq I_*$. Since I_* is a minimal element, it requires $\bar{I} = I_*$, which implies $Ax_* = x_*$ and $Ax^* = x^*$. (2.20) shows x_* and x^* is the minimal fixed point and maximal fixed point of A.

Finally, since A is increasing and $u_0 \leq x_* \leq x^* \leq v_0$, it is easy to obtain (2.2). □

Remark 2.1.1 From the above proof, the condition that $A([u_0, v_0])$ is relatively compact can be relaxed to $A(u_0, v_0)$ is relatively compact for every totally ordered subset. Additionally, by using the same method, the above theorem can be generalized to the following one: Suppose $u_0, v_0 \in E,\ u_0 < v_0,\ A : [u_0, v_0] \to E$ is an increasing operator satisfying (2.1). If $A = \sum_{i=1}^{m} C_i B_i$, where $B_i : [u_0, v_0] \to E_i$ (E_i is an ordered Banach space) and $C_i : [B_i u_0, B_i v_0] \to E$ is increasing operator and $B_i([u_0, v_0])$ is relatively compact for every totally ordered subset ($i = 1, 2, \ldots, m$), then the conclusion of Theorem 2.1.1 holds. (See Sun and Zhao [173].)

Definition 2.1.1 An operator $T : D(T) \subset E \to E$ is said to be convex if for x, $y \in D(T)$ with $x \leq y$ and every $t \in [0, 1]$, we have

$$T\big(tx + (1-t)y\big) \leq tT(x) + (1-t)T(y). \tag{2.24}$$

T is concave if $-T$ is convex.

Lemma 2.1.1 (Yihong Du, see [88, 89]) *Suppose P is a normal cone, $v > \theta$, $A : [\theta, v] \to E$ is a concave and increasing operator. If there exists $0 < \varepsilon < 1$ such that $A\theta \geq \varepsilon v$, $Av \leq v$, then A has minimal fixed point u^* in $[\theta, v]$, $u^* > \theta$. Let $u_0 = \theta$, $u_n = Au_{n-1}$ $(n = 1, 2, 3, \ldots)$, then we have*

$$\left\|u_n - u^*\right\| \leq N\|A\theta\|\varepsilon^{-2}(1-\varepsilon)^n \quad (n = 1, 2, 3, \ldots). \tag{2.25}$$

Here N is the normal constant of P.

Proof Let $\tau = 1 - \varepsilon (0 < \tau < 1)$, $B = \tau^{-1}A$. Then $B : [\theta, v] \to E$ is a concave and increasing operator. Also $B\theta \geq \tau^{-1}\varepsilon v$, $Bv \leq \tau^{-1}v$, $u_n = \tau Bu_{n-1}$ $(n = 1, 2, 3, \ldots)$. Obviously $u_1 = A\theta \geq \varepsilon v > \theta$ and $Av \leq v$, since A is increasing,

$$\theta = u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v, \tag{2.26}$$

$$B\theta \geq \tau^{-1}\varepsilon v \geq \varepsilon Bv \geq \varepsilon Bu_n = (1-\tau)Bu_n. \tag{2.27}$$

For $\theta \leq y \leq x \leq v$, $0 \leq t \leq 1$, we have

$$\theta \leq x - ty \leq (1-t)x + t(x-y) \leq x \leq v,$$

since B is concave,

$$B(x - ty) \geq (1-t)Bx + tB(x-y),$$

which means

$$Bx - B(x-ty) \leq t\big[Bx - B(x-y)\big], \quad \theta \leq y \leq x \leq v,\ 0 \leq t \leq 1. \tag{2.28}$$

Next we prove

$$u_{n+1} - u_n \leq \tau^n(Bu_n - B\theta) \quad (n = 1, 2, 3, \ldots). \tag{2.29}$$

Because $u_2 - u_1 = \tau(Bu_1 - B\theta)$, (2.29) holds when $n = 1$. Suppose it also holds when $n = k$, that is

$$u_{k+1} - u_k \leq \tau^k(Bu_k - B\theta). \tag{2.30}$$

Let $x = u_{k+1}$, $y = Bu_k - B\theta$. We have by (2.27), (2.28), and (2.30)

$$\theta \leq y \leq Bu_k - (1-\tau)Bu_k = \tau Bu_k = u_{k+1} = x \leq v, \tag{2.31}$$

$$u_k \geq x - \tau^k y, \tag{2.32}$$

$$u_{k+2} - u_{k+1} = \tau Bx - \tau Bu_k \leq \tau\big[Bx - B\big(x - \tau^k y\big)\big] \tag{2.33}$$

$$\leq \tau^{k+1}\big[Bx - B(x-y)\big] \leq \tau^{k+1}(Bx - B\theta), \tag{2.34}$$

so (2.29) holds when $n = k+1$. By induction, for every natural number n, (2.29) holds.

Since (2.29) and (2.27), we have

$$u_{n+1} - u_n \le \tau^n \left(\frac{B\theta}{1-\tau} - B\theta \right) = \frac{\tau^{n+1}}{1-\tau} B\theta,$$

so if $m > n$,

$$\begin{aligned} \theta \le u_m - u_n &\le (u_{n+1} - u_n) + \cdots + (u_m - u_{m-1}) \\ &\le \frac{B\theta}{1-\tau}\big(\tau^{n+1} + \cdots + \tau^m\big) \le \frac{\tau^{n+1} B\theta}{(1-\tau)^2}. \end{aligned} \tag{2.35}$$

Since P is normal, it must have $\{u_n\}$ is Cauchy sequence. So there exists $u^* \in E$ such that $u_n \to u^*$ as $n \to \infty$. Obviously $u_n \le u^* \le v$, then $\tau B u^* \ge \tau B u_n = u_{n+1}$. Let $n \to \infty$, we get

$$\tau B u^* \ge u^*. \tag{2.36}$$

In (2.35), let $m \to \infty$, we have

$$\theta \le u^* - u_n \le \frac{\tau^{n+1} B\theta}{(1-\tau)^2} \quad (n = 1, 2, 3, \ldots). \tag{2.37}$$

When n is large enough, we have $\tau^n (1-\tau)^{-2} < 1$. Combing (2.36), (2.37) with (2.28), it leads to

$$\begin{aligned} \theta \le \tau B u^* - u^* &\le \tau B u^* - u_{n+1} = \tau\big(B u^* - B u_n\big) \\ &\le \tau\left[B u^* - B\left(u^* - \frac{\tau^n}{(1-\tau)^2} \cdot \tau B\theta \right) \right] \\ &\le \frac{\tau^{n+1}}{(1-\tau)^2} \big[B u^* - B\big(u^* - \tau B\theta\big) \big] \\ &\le \frac{\tau^{n+1}}{(1-\tau)^2} \big(B u^* - B\theta \big) \to \theta \quad (n \to \infty), \end{aligned}$$

since P is normal, then $\tau B u^* - u^* = \theta$, that is $u^* = A u^*$. Also, (2.37) implies (2.25). Clearly $u^* > \theta$ for $A\theta \ge \varepsilon v > \theta$. Finally, u^* is the minimal fixed point of A in $[\theta, v]$. Suppose $\theta \le \bar{x} \le v$ such that $A\bar{x} = \bar{x}$, since A is increasing, by induction we get $u_n \le \bar{x}$ $(n = 0, 1, 2, \ldots)$. As $n \to \infty$, it shows $u^* \le \bar{x}$. □

Theorem 2.1.2 (Yihong Du [88]) *Suppose that the cone P is normal, $u_0, v_0 \in E$ and $u_0 < v_0$. Moreover, $A : [u_0, v_0] \to E$ is a increasing operator. Let $h_0 = v_0 - u_0$. If one of the following assumptions holds:*

(i) *A is a concave operator, $A u_0 \ge u_0 + \varepsilon h_0$, $A v_0 \le v_0$ where $\varepsilon \in (0, 1)$ is a constant;*

(ii) *A is a convex operator,* $Au_0 \geq u_0$, $Av_0 \leq v_0 - \varepsilon h_0$ *where* $\varepsilon \in (0, 1)$ *is a constant, then A has a unique fixed point* x^* *in* $[u_0, v_0]$. *Moreover, for any* $x_0 \in [u_0, v_0]$, *the iterative sequence* $\{x_n\}$ *given by* $x_n = Ax_{n-1}$ $(n = 1, 2, \ldots)$ *satisfying that*

$$\|x_n - x^*\| \to 0 \quad (n \to \infty),$$
$$\|x_n - x^*\| \leq M(1-\varepsilon)^n \quad (n = 1, 2, \ldots),$$

with M a positive constant independent of x_0.

Proof Firstly, we assume that (i) holds. Let $Bx = A(x + u_0) - u_0$. Clearly, $B : [\theta, h_0] \to E$ is concave and increasing. Moreover, $B\theta \geq \varepsilon h_0$ and $Bh_0 \leq h_0$. Lemma 2.1.1 implies that B has a minimal fixed point u^* in $[\theta, h_0]$ and

$$\|u_n - u^*\| \leq M_0(1-\varepsilon)^n \quad (n = 1, 2, \ldots),$$

where $u_0 = \theta$, $u_n = Bu_{n-1}$ $(n = 1, 2, \ldots)$, M_0 is a positive constant Let $h_n = Bh_{n-1}$ $(n = 1, 2, \ldots)$. Clearly, we have

$$h_0 \geq h_1 \geq \cdots \geq h_n \geq \cdots \geq u^*.$$

Let $t_n = \sup\{t > 0 : u^* \geq th_n\}$. Since $u^* = Bu^* \geq B\theta \geq \varepsilon h_0 \geq \varepsilon h_n$,

$$0 < \varepsilon \leq t_1 \leq \cdots \leq t_n \leq \cdots \leq 1, \qquad u^* \geq t_n h_n.$$

It follows that

$$\begin{aligned} u^* = Bu^* &\geq B(t_n h_n) \geq (1-t_n)B\theta + t_n bh_n \\ &\geq (1-t_n)\varepsilon h_0 + t_n h_{n+1} \geq \big[(1-t_n)\varepsilon + t_n\big]h_{n+1}. \end{aligned}$$

Thus, by the definition of t_{n+1}, we get $t_{n+1} \geq (1-t_n)\varepsilon + t_n$, and hence,

$$1 - t_{n+1} \leq (1-t_n)(1-\varepsilon) \quad (n = 1, 2, \ldots).$$

It follows that

$$1 - t_n \leq (1-t_1)(1-\varepsilon)^{n-1} \leq (1-\varepsilon)^n \quad (n = 1, 2, \ldots).$$

Notice that

$$\theta \leq h_n - u^* \leq h_n - t_n h_n \leq (1-t_n)h_0 \leq (1-\varepsilon)^n h_0.$$

This yields $\|h_n - u^*\| \leq N\|h_0\|(1-\varepsilon)^n$ $(n = 1, 2, \ldots)$.

For any $y_0 \in [\theta, h_0]$, let $y_n = By_{n-1}$ $(n = 1, 2, \ldots)$. Then $u_n \leq y_n \leq h_n$ $(n = 1, 2, \ldots)$. Therefore,

$$\begin{aligned} \|y_n - u^*\| &\leq \|y_n - u_n\| + \|u_n - u^*\| \\ &\leq N\|h_n - u_n\| + \|u_n - u^*\| \end{aligned}$$

$$\leq N\left\|h_n - u^*\right\| + (N+1)\left\|u_n - u^*\right\|$$
$$\leq M(1-\varepsilon)^n \quad (n = 1, 2, \ldots), \tag{2.38}$$

where $M = N^2\|h_0\| + (N+1)M_0$ is a constant. We can easily conclude from (2.38) that the fixed point of B in $[\theta, h_0]$ is unique. In fact, (2.38) yields

$$\left\|\bar{x} - u^*\right\| \leq M(1-\varepsilon)^n \to 0 \quad (n \to \infty),$$

and hence, $\bar{x} = u^*$. If we denote $x^* = u^* + u_0,\ x_n = y_n + u_0\ (n = 1, 2, \ldots)$, then we see that the conclusions of this theorem hold.

On the other hand, when (ii) holds. Let $Bx = v_0 - A(v_0 - x)$. It can be easily checked that $B : [\theta, h_0] \to E$ is a concave and increasing operator satisfying $B\theta \geq \varepsilon h_0$ and $Bh_0 \leq h_0$. By similar arguments as in the case of (i), we get the conclusions of this theorem. □

Corollary 2.1.1 (Yihong Du [88]) *Assume that P is a normal solid cone, $u_0, v_0 \in E,\ u_0 < v_0,\ A : [u_0, v_0] \to E$ is an increasing operator. Suppose one of the following assumptions holds*:

(i) *A is concave, $Au_0 \gg u_0,\ Av_0 \leq v_0$;*
(ii) *A is convex, $Au_0 \geq u_0,\ Av_0 \ll v_0$.*

Then A has a unique fixed point x_* in $[u_0, v_0]$. Moreover, for any $x_0 \in [u_0, v_0]$, the iterative sequence $\{x_n\}$ defined by $x_n = Ax_{n-1}\ (n = 1, 2, \ldots)$ satisfying that $\|x_n - x^*\| \to 0\ (n \to \infty)$ and

$$\left\|x_n - x^*\right\| \leq Mr^n \quad (n = 1, 2, \ldots), \tag{2.39}$$

where $r \in (0, 1),\ M > 0$ are constants.

Proof Let $h_0 = v_0 - u_0$. Since $Au_0 \gg u_0$ (or $Av_0 \ll v_0$), we can choose $\varepsilon \in (0, 1)$ small enough such that $Au_0 \geq u_0 + \varepsilon h_0$ (or $Av_0 \leq v_0 - \varepsilon h_0$). Then, we complete the proof by applying Theorem 2.1.2. □

Corollary 2.1.2 (Yihong Du [88]) *Suppose that P is a normal solid cone, $u_0, v_0 \in E,\ u_0 < v_0$ and $A : [u_0, v_0] \to E$ is a strongly positive operator (i.e., $x, y \in [u_0, v_0],\ x < y \Rightarrow Ax \ll Ay$). Suppose one of the following two conditions holds*:

(i) *A is concave, $Au_0 > u_0,\ Av_0 \leq v_0$;*
(ii) *A is convex, $au_0 \geq u_0,\ Av_0 < v_0$.*

Then A has a unique fixed point in $[u_0, v_0]$. Moreover, for any $x_0 \in [u_0, v_0]$ the iterative sequence $\{x_n\}$ given by $x_n = Ax_{n-1}\ (n = 1, 2, \ldots)$ satisfying that $\|x_n - x^*\| \to 0\ (n \to \infty)$ and

$$\left\|x_n - x^*\right\| \leq Mr^n \quad (n = 1, 2, \ldots),$$

where $r \in (0, 1),\ M > 0$ are constants.

Proof Assume that (i) holds (the proof is similar if condition (ii) holds). Let $u_1 = Au_0$, then $u_1 > u_0$. Since A is a strongly increasing operator, we have $Au_1 \gg Au_0 = u_1$. Applying Corollary 2.1.1 to $[u_1, v_0]$, we see that A has a unique fixed point x^* in $[u_1, v_0]$ and if we define $x_n = Ax_{n-1}$ then (2.39) holds. Suppose $\bar{x}$ is a fixed point of A in $[u_1, v_0]$, then $\bar{x} > u_0$. Hence, $\bar{x} = A\bar{x} \gg Au_0 = u_1$. Therefore, there is no fixed point for A in $[u_0, u_1]$. Moreover, for any $x_0 \in [u_0, v_0]$, let $x_n = Ax_{n-1}$ $(n = 1, 2, \ldots)$. It follows that $x_1 \in [u_1, v_0]$, and hence, (2.39) also holds. □

Example 2.1.1 Consider the following Hammerstein integral equation:

$$x(t) = \int_{\mathbb{R}^n} k(t,s) f\big(s, x(s)\big)\, ds,$$

where $k(t,s)$ is nonnegative and measurable in $\mathbb{R}^n \times \mathbb{R}^n$,

$$\lim_{t \to t_0} \int_{\mathbb{R}^n} \big|k(t,s) - k(t_0,s)\big|\, ds = 0, \quad \forall t_0 \in \mathbb{R}^n,$$

and there exist constants $M > m > 0$ such that

$$m \le \int_{\mathbb{R}^n} k(t,s)s \le M, \quad \forall t \in \mathbb{R}^n.$$

Moreover, for any $x \ge 0$, $f(\cdot, x)$ is measurable in $\mathbb{R}^n$; and for any $t \in \mathbb{R}^n$, $f(t, \cdot)$ is continuous in $[0, \infty)$. Furthermore, there exists $R > r > 0$ such that one of the following two conditions holds:

(i) for any $t \in \mathbb{R}^n$, $f(t, \cdot) : [r, R] \to \mathbb{R}$ is a concave increasing function satisfying that

$$f(t,r) \ge \left(\frac{1}{m} + \varepsilon_1\right) r, \quad f(r,R) \le \frac{1}{M} R, \quad \forall t \in \mathbb{R}^n,$$

with ε_1 is a positive constant;

(ii) for any $t \in \mathbb{R}^n$, $f(t, \cdot) : [r, R] \to \mathbb{R}$ is a convex increasing function satisfying that

$$f(t,r) \ge \frac{1}{m} r, \quad f(t,R) \le \left(\frac{1}{M} - \varepsilon_2\right) R, \quad \forall t \in \mathbb{R}^n,$$

with ε_2 is a positive constant.

We will show that the integral equation has a unique continuous solution $x^*(t)$ satisfying that $\forall t \in \mathbb{R}^n$, $r \le x^*(t) \le R$ and for any continuous function $x_0(t)$ with $r \le x_0(t) \le R$ the sequence $\{x_n(t)\}$ given by

$$x_n(t) = \int_{\mathbb{R}^n} k(t,s) f\big(s, x_{n-1}(s)\big)\, ds, \quad \forall t \in \mathbb{R}^n \ (n = 1, 2, \ldots),$$

has the following property:

$$\sup_{t \in \mathbb{R}^n} \big|x_n(t) - x^*(t)\big| \le M_0 \tau^n \to 0 \quad (n \to \infty),$$

where $M_0 > 0$ and $\tau \in (0, 1)$ are constants independent of $x_0(t)$.

Proof Let

$$E = C_B(\mathbb{R}^n) = \left\{x \in C(\mathbb{R}^n) : \sup_{t\in\mathbb{R}^n} |x(t)| < \infty\right\},$$

with its norm $\|x\|_{C_B} = \sup_{t\in\mathbb{R}^n} |x(t)|$, and $P = \{s \in C_B(\mathbb{R}^n) : x(t) \geq 0,\ \forall t \in \mathbb{R}^n\}$. Then P is a normal solid cone and $\mathring{P} = \{x \in C_B(\mathbb{R}^n) : \inf_{t\in\mathbb{R}^n} x(t) > 0\}$. Assume that condition (i) holds (the proof is much similar in the case of (ii)). Consider the following operator:

$$(Ax)(t) = \int_{\mathbb{R}^n} k(t, s) f\big(s, x(s)\big)\, ds,$$

and let $u_0(t) \equiv r\ (\forall t \in \mathbb{R}^n),\ v_0(t) \equiv R\ (\forall t \in \mathbb{R}^n)$. It can be easily checked that $A : [u_0, v_0] \to E$ is a concave increasing operator and $Au_0 \gg u_0,\ Av_0 \leq v_0$. Then the conclusions follows from Corollary 2.1.1. □

Note the results on concave increasing operators and the example above are obtained by Du Yihong, see [88, 89].

Theorem 2.1.3 (Dajun Guo [106]) *Suppose P is a solid and normal cone, $A : P \to P$ is an increasing operator. Assume there exist $v \in \mathring{P}$ and $c > 0$ such that $\theta < Av \leq v,\ A\theta \geq cAv$. Suppose for arbitrary $0 < a < b < 1$ and every bounded set $B \subset P$, there exists $\eta = \eta(a, b, B) > 0$ such that*

$$A(tx) \geq t(1 + \eta)Ax, \quad x \in B,\ t \in [a, b]. \tag{2.40}$$

Then A has a unique fixed point x^* in P and $x^* \in (\theta, v]$. Furthermore, for every initial point $x_0 \in P$, constructing a sequence as

$$x_n = Ax_{n-1} \quad (n = 1, 2, 3, \ldots), \tag{2.41}$$

we have

$$\|x_n - x^*\| \to \infty. \tag{2.42}$$

Proof Firstly we prove A has no more than one fixed point in P. Suppose $x^*,\ \bar{x} \in P$ such that $x^* = Ax^*,\ \bar{x} = A\bar{x}$. We have

$$x^* = Ax^* \geq A\theta \geq cAv > \theta, \qquad \bar{x} = A\bar{x} \geq A\theta \geq cAv > \theta. \tag{2.43}$$

Since $v \in \mathring{P}$, we can find $0 < t_0 < 1$ such that $t_0 x^* \leq v,\ t_0\bar{x} \leq v$, thus from (2.40), for certain $\eta_0 > 0$

$$Av \geq A\big(t_0 x^*\big) \geq t_0(1 + \eta_0)Ax^* = t_0(1 + \eta_0)x^*, \tag{2.44}$$

$$Av \geq A(t_0\bar{x}) \geq t_0(1 + \eta_0)A\bar{x} = t_0(1 + \eta_0)\bar{x}. \tag{2.45}$$

Recall (2.43), there exists $0 < c_1 < 1$ such that $x^* > c_1\bar{x}$ and $\bar{x} \geq c_1 x^*$. Define $t^* = \sup\{t > 0 : x^* \geq t\bar{x} \text{ and } \bar{x} \geq tx^*\}$. Then $c_1 \leq t \leq 1,\ x^* \geq t^*\bar{x},\ \bar{x} \geq t^* x^*$. So if $0 < t^* < 1$, from (2.40), for certain $\eta^* > 0$

$$x^* = Ax^* \geq A\big(t^*\bar{x}\big) \geq t^*\big(1+\eta^*\big)A\bar{x} = t^*\big(1+\eta^*\big)\bar{x},$$
$$\bar{x} = A\bar{x} \geq A\big(t^* x^*\big) \geq t^*\big(1+\eta^*\big)Ax^* = t^*\big(1+\eta^*\big)x^*,$$

this contradicts the definition of t^*. So $t^* = 1$, which means $x^* = \bar{x}$.

If $x^* \in P$ such that $x^* = Ax^*$, we can prove $x \in (\theta, v]$. In fact, $x^* > \theta$ is from (2.43). Let $t' = \sup\{t > 0 : v \geq tx^*\}$. Then $0 < t' < \infty,\ v \geq t' x^*$. If $0 < t' < 1$, from (2.40), for certain $\eta' > 0$ such that

$$v \geq Av \geq A\big(t' x^*\big) \geq t'\big(1+\eta'\big)Ax^* = t'\big(1+\eta'\big)x^*,$$

this contradicts the definition of t'. So $t' > 1$, which means $v \geq x^*$.

Finally, we prove A has a fixed point in P and (2.42) holds. Since $v \in \mathring{P}$, we can find $0 < t_* < 1$ such that

$$t_* x_0 \leq v. \tag{2.46}$$

Let $v_0 = t_*^{-1} v$. From (2.40), there exists $\eta_* > 0$ such that

$$Av = A(t_* v_0) \geq t_*(1+\eta_*)Av_0, \tag{2.47}$$

consequently

$$\theta < Av \leq Av_0 \leq \big[t_*(1+\eta_*)\big]^{-1} Av \leq t_*^{-1} v = v_0. \tag{2.48}$$

Also

$$A\theta \geq cAv \geq c_* Av_0, \tag{2.49}$$

where $c_* = ct_*(1+\eta_*) > 0$. Now let

$$u_0 = \theta, \qquad u_n = Au_{n-1}, \quad v_n = A(v_{n-1}) \quad (n = 1, 2, 3, \ldots). \tag{2.50}$$

Using (2.48) and the increase property of A, we get

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0. \tag{2.51}$$

(2.49) implies $u_1 \geq c_* v_1 > \theta$. Define $t_n = \sup\{t > 0 : u_n \geq tv_n\}$ $(n = 1, 2, 3, \ldots)$, then $u_n \geq t_n v_n$. From (2.51),

$$0 < c_* \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq \cdots \leq 1. \tag{2.52}$$

Suppose $t_n \to t^*$ $(n \to \infty)$, then $c_* < t^* \leq 1$. If $t^* < 1$, (2.40) implies that, for certain $\eta > 0$,

$$A(t_n v_n) \geq t_n(1+\eta)Av_n = t_n(1+\eta)v_{n+1} \quad (n = 1, 2, 3, \ldots),$$

so

$$u_{n+1} = Au_n \geq A(t_n v_n) \geq t_n(1+\eta)v_{n+1} \quad (n = 1, 2, 3, \ldots),$$

which means $t_{n+1} \geq t_n(1+\eta)$. So $t_{n+1} \geq t_1(1+\eta)^n \geq c_*(1+\eta)^n \to \infty$ $(n \to \infty)$. This contradicts (2.52). Thus $\lim_{n\to\infty} t_n = t^* = 1$.

From (2.51) and $u_n \geq t_n v_n$, we get

$$\theta \leq u_{n+p} - u_n \leq v_n - u_n \leq (1-t_n)v_n \leq (1-t_n)v_0, \tag{2.53}$$

since P is normal,

$$\|u_{n+p} - u_n\| \leq N(1-t_n)\|v_0\| \to 0, \quad n \to \infty,$$

where N is the normal constant in P. The above inequality means $\{u_n\}$ is a Cauchy sequence in E. Since E is complete, there exists u^* in P such that $\lim_{n\to\infty} u_n = u^*$.

Similarly by

$$\theta \leq v_n - v_{n+p} \leq v_n - u_n \leq (1-t_n)v_n \leq (1-t_n)v_0,$$

there exists v^* in P such that $\lim_{n\to\infty} v_n = v^*$. Obviously,

$$u_n \leq u^* \leq v^* \leq v_n, \tag{2.54}$$

and

$$\theta \leq v^* - u^* \leq v_n - u_n \leq (1-t_n)v_n \leq (1-t_n)v_0,$$

so $u^* = v^*$. Let $x^* = u^* = v^*$, (2.54) implies

$$u_{n+1} \leq Au_n \leq Ax^* \leq Av_n \leq v_{n+1}.$$

Let $n \to \infty$, we get $x^* \leq Ax^* \leq x^*$, so $x^* = Ax^*$. From (2.46), it can obtain $u_0 \leq x_0 \leq v_0$. Since A is increasing, by induction, $u_n \leq x_n \leq v_n$, where x_n is given by (2.41). Since $u_n \to x^*$ and $v_n \to x^*$ and P is normal, (2.42) must hold. $\square$

2.2 Decreasing Operators

Decreasing operators are a class of important operators, but it is difficult to obtain fixed point theorems (see [100, 106]). In this section, without assuming operators to be continuous or compact, we first study decreasing operators with convexity or concavity and give existence and uniqueness theorems, then we study eigenvalue problems and structure of solution set, finally apply them to nonlinear integral equations on unbounded regions and differential equations in Banach spaces.

Suppose that E is a real Banach space, θ is the zero element of E, $P \subset E$ is a cone. $A : D \to E$ is called a decreasing operator, if $\forall x_1, x_2 \in D$, $x_1 \leq x_2 \Rightarrow Ax_1 \geq Ax_2$. $A : D \to E$ is called a strictly decreasing operator, if $\forall x_1, x_2 \in D$, $x_1 < x_2 \Rightarrow Ax_1 > Ax_2$.

Theorem 2.2.1 (Dajun Guo, see [106]) *Assume that P is a normal cone and $A: P \to P$ is a completely continuous operator. Moreover, A is a decreasing operator satisfying that $A\theta > \theta$, $A^2\theta \geq \varepsilon_0 A\theta$, with $\varepsilon_0 > 0$, and for any $(x, t) \in (\theta, A\theta] \times (0, 1)$, there exists $\eta = \eta(x, t) > 0$, such that*

$$A(tx) \leq \big[t(1+\eta)\big]^{-1} Ax.$$

Then A has a unique fixed point $x^* > \theta$, and for any $x_0 \in P$ the iterative sequence $\{x_n\}$ given by $x_n = Ax_{n-1}$ $(n = 1, 2, \ldots)$ satisfying that

$$\|x_n - x^*\| \to 0.$$

Proof Let $u_0 = \theta,\ u_n = Au_{n-1}$ $(n = 1, 2, \ldots)$, then we have $u_0 \leq A^2 u_0,\ A^2 u_1 \leq u_1$. It follows from A is a continuous completely operator that $A^2([\theta, A\theta])$ is relatively compact. Theorem 2.1.1 implies that A^2 has a maximal fixed point u^* and a minimal fixed point u_*. Moreover,

$$\begin{aligned}
&\lim_{n\to\infty} u_{2n} = u_*, \qquad \lim_{n\to\infty} u_{2n+1} = u^*,\\
&\theta = u_0 \leq u_2 \leq \cdots \leq u_{2n} \leq \cdots \leq u_*\\
&\quad \leq u^* \leq \cdots \leq u_{2n+1} \leq \cdots \leq u_3 \leq u_1 = A\theta.
\end{aligned} \tag{2.55}$$

Since $u_2 = A^2\theta \geq \varepsilon_0 A\theta > \theta,\ u^* \geq u_* > \theta$. Letting $n \to \infty$ in the following equalities:

$$u_{2n} = Au_{2n-1}, \qquad u_{2n+1} = u_{2n},$$

we have $u_* = Au^*$ and $u^* = Au_*$, and hence, $u_* \geq u_2 \geq \varepsilon_0 u_1 \geq \varepsilon_0 u^*$. Let $t_0 = \sup\{t > 0 | u_* \geq tu^*\}$, then $\varepsilon_0 \leq t_0 \leq 1,\ u_* \geq t_0 u^*$. If $t_0 < 1$, then there exists $\eta_0 > 0$ such that

$$A\big(t_0 u^*\big) \leq \big[t_0(1+\eta_0)\big]^{-1} Au^* = \big[t_0(1+\eta_0)\big]^{-1} u_*.$$

Therefore, we have

$$u^* = Au_* \leq A\big(t_0 u^*\big) \leq \big[t_0(1+\eta_0)\big]^{-1} u_*,$$

that is $u_* \geq t_0(1+\eta_0)u^*$ in contradiction with the definition of t_0. Hence, $t_0 = 1$ and $u_* = u^*$. If we denote $x^* = u_* = u^*$, then $x^* > \theta$ and $x^* = Ax^*$, that is, x^* is a positive fixed point of A.

If $\bar{x} > \theta$ is a fixed point of A, then $\theta < \bar{x} \leq A\theta,\ \bar{x} = A^2\bar{x}$. In other words, A^2 has a fixed point in $[\theta, A\theta]$. By the minimal of u_* and the maximal of u^*, we conclude that $u_* \leq \bar{x} \leq u^*$, and therefore, $\bar{x} = x^*$. Thus, the positive fixed point of A is unique.

Finally, we show that $\|x_n - x^*\| \to 0$. We deduced from $x_0 \geq \theta$ that $\theta \leq Ax_0 \leq A\theta$, i.e., $u_0 \leq x_1 \leq u_1$. It follows from the decreasing of A that $u_2 \leq x_2 \leq u_1$. Moreover, we can get

$$u_{2n} \leq x_{2n} \leq u_{2n-1}, u_{2n} \leq x_{2n+1} \leq u_{2n+1} \quad (n = 1, 2, \ldots).$$

Noticing that $u_* = x^* = u^*$, we conclude from Theorem 1.3.1 and (2.55) that $\|x_n - x^*\| \to 0$. □

Theorem 2.2.2 (Zhitao Zhang, see [198, 202]) *Suppose that P is a normal cone of a real Banach space E, N is the normal constant of P, $A : P \to P$ is convex and decreasing, and $\exists \varepsilon > \frac{1}{2}$, such that*

$$A^2\theta \geq \varepsilon A\theta > \theta. \tag{2.56}$$

Then A has a unique fixed point x^* in P, and $\forall x_0 \in P$, constructing the sequence $x_n = Ax_{n-1}$, we have $\|x_n - x^*\| \to 0$ $(n \to +\infty)$, with the convergence rate $\|x_{2n} - x^*\| \leq 4N^2\|A\theta\| \cdot (\frac{1-\varepsilon}{\varepsilon})^{n-1}$, $\|x_{2n+1} - x^*\| \leq 2N^2\|A\theta\| \cdot (\frac{1-\varepsilon}{\varepsilon})^n$ $(n = 1, 2, \ldots)$.

Proof Let

$$u_0 = \theta, \qquad u_n = Au_{n-1} \quad (n = 1, 2, \ldots). \tag{2.57}$$

Since A is decreasing, we have

$$\theta = u_0 \leq u_2 \leq \cdots \leq u_{2n} \leq \cdots \leq u_{2n+1} \leq \cdots \leq u_3 \leq u_1 = A\theta, \tag{2.58}$$

and by (2.56) we get $u_2 \geq \varepsilon u_1 > \theta$, thus $u_{2n} \geq \varepsilon u_{2n+1}$ $(n = 1, 2, 3, \ldots)$. Let

$$t_n = \sup\{t > 0, u_{2n} \geq t u_{2n+1}\} \quad (n = 1, 2, 3, \ldots), \tag{2.59}$$

we have $u_{2n+2} \geq u_{2n} \geq t_n u_{2n+1} \geq t_n u_{2n+3}$, thus

$$0 < \varepsilon \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq \cdots \leq 1. \tag{2.60}$$

Now we prove $t_n \to 1$ $(n \to +\infty)$.

Noticing that A is convex decreasing, by (2.59) we have $u_{2n} \geq t_n u_{2n+1}$, and

$$\begin{aligned} u_{2n+1} = Au_{2n} \leq A(t_n u_{2n+1}) &= A\big(t_n u_{2n+1} + (1 - t_n)\theta\big) \\ &\leq t_n Au_{2n+1} + (1 - t_n)A\theta = t_n u_{2n+2} + (1 - t_n)A\theta. \end{aligned}$$

Noticing that $\varepsilon A\theta \leq A^2\theta = u_2 \leq u_{2n+2}$, $u_{2n+1} \geq u_{2n+3}$, we know that

$$u_{2n+3} \leq u_{2n+1} \leq t_n u_{2n+2} + \frac{1 - t_n}{\varepsilon} u_{2n+2} = \left(t_n + \frac{1 - t_n}{\varepsilon}\right) u_{2n+2},$$

i.e., $u_{2n+2} \geq (t_n + \frac{1-t_n}{\varepsilon})^{-1} u_{2n+3}$. Thus we get $t_{n+1} \geq (t_n + \frac{1-t_n}{\varepsilon})^{-1}$, and

$$1 - t_{n+1} \leq 1 - \frac{1}{t_n + \frac{1-t_n}{\varepsilon}} = \frac{(\frac{1}{\varepsilon} - 1)(1 - t_n)}{t_n + \frac{1-t_n}{\varepsilon}} \leq \left(\frac{1}{\varepsilon} - 1\right)(1 - t_n). \tag{2.61}$$

Noticing that $\varepsilon > \frac{1}{2}$, $\frac{1}{\varepsilon} - 1 < 1$,

$$1 - t_{n+1} \leq \left(\frac{1}{\varepsilon} - 1\right)(1 - t_n) \leq \left(\frac{1}{\varepsilon} - 1\right)^2 (1 - t_{n-1}) \leq \left(\frac{1}{\varepsilon} - 1\right)^n (1 - \varepsilon)$$

$$\leq \left(\frac{1-\varepsilon}{\varepsilon}\right)^{n+1}, \tag{2.62}$$

we have

$$t_n \to 1 \quad (n \to +\infty). \tag{2.63}$$

Since

$$\theta \leq u_{2n+2p} - u_{2n} \leq u_{2n+1} - u_{2n} \leq (1-t_n)u_{2n+1} \leq (1-t_n)A\theta, \tag{2.64}$$

noticing that P is normal, we know $\|u_{2n+2p} - u_{2n}\| \leq N \cdot (1-t_n) \cdot \|A\theta\|$. By (2.62) and (2.63), we find that $\exists u^*$ such that $\lim_{n\to\infty} u_{2n} = u^*$, and

$$\left\|u^* - u_{2n}\right\| \leq (1-t_n) \cdot N \cdot \|A\theta\| \leq \left(\frac{1-\varepsilon}{\varepsilon}\right)^n \cdot N \cdot \|A\theta\|.$$

Similarly, $\exists v^*$ such that $\lim_{n\to\infty} u_{2n+1} = v^*$, and

$$\left\|v^* - u_{2n+1}\right\| \leq (1-t_n) \cdot N \cdot \|A\theta\| \leq \left(\frac{1-\varepsilon}{\varepsilon}\right)^n \cdot N \cdot \|A\theta\|.$$

And it is clear that $\theta \leq v^* - u^* \leq u_{2n+1} - u_{2n} \leq (1-t_n)A\theta$, thus $\|v^* - u^*\| \leq N \cdot (1-t_n) \cdot \|A\theta\| \to 0$ $(n \to \infty)$, so we have $u^* = v^* := x^*$, and $u_{2n} \leq x^* \leq u_{2n+1},\ u_{2n+1} = Au_{2n} \geq Ax^* \geq Au_{2n+1} = u_{2n+2}$ $(n = 1, 2, 3, \ldots)$. Let $n \to +\infty$, we have $x^* \geq Ax^* \geq x^*$, thus $Ax^* = x^*$.

Now we prove the uniqueness. Suppose $\bar{x}$ is an arbitrary fixed point of A in P, then $u_0 = \theta \leq \bar{x} = A\bar{x} \leq A\theta = u_1$. As $u_{2n} \leq \bar{x} \leq u_{2n+1}$ $(n = 1, 2, \ldots)$, letting $n \to \infty$ we have $\bar{x} = x^*$.

At last, $\forall x_0 \in P$, we have $u_0 = \theta \leq x_1 = Ax_0 \leq A\theta = u_1,\ u_2 \leq x_2 = Ax_1 \leq u_1$. It is easy to get

$$u_{2n} \leq x_{2n} \leq u_{2n-1}, \quad u_{2n} \leq x_{2n+1} \leq u_{2n+1} \quad (n = 1, 2, \ldots), \tag{2.65}$$

thus $x_n \to x^*$. By (2.64) we get $\|u_{2n+1} - u_{2n}\| \leq N(1-t_n) \cdot \|A\theta\|$, thus

$$\begin{aligned}\left\|x^* - x_{2n}\right\| &\leq \left\|x^* - u_{2n}\right\| + \|u_{2n} - x_{2n}\| \leq 2N\|u_{2n-1} - u_{2n}\| \\ &\leq 2N\left(\left\|u_{2n-1} - x^*\right\| + \left\|x^* - u_{2n}\right\|\right) \leq 4N^2 \cdot \left(\frac{1-\varepsilon}{\varepsilon}\right)^{n-1} \cdot \|A\theta\|,\end{aligned}$$

$$\begin{aligned}\left\|x^* - x_{2n+1}\right\| &\leq \left\|x^* - u_{2n}\right\| + \|u_{2n} - x_{2n+1}\| \leq 2N\|u_{2n+1} - u_{2n}\| \\ &\leq 2N^2 \cdot \left(\frac{1-\varepsilon}{\varepsilon}\right)^n \cdot \|A\theta\|.\end{aligned}$$

□

Corollary 2.2.1 *Suppose that P is a normal cone of E, $A : [\theta, v] \to [\theta, v]$ is concave decreasing such that $A^2 v \leq (1-\varepsilon)v + \varepsilon A v$, where $v > \theta$, $\frac{1}{2} < \varepsilon \leq 1$. Then A has a unique fixed point $(v - x^*)$ in $[0, v]$; moreover, $\forall x_0 \in [\theta, v]$, constructing the sequence $x_n = v - A(v - x_{n-1})$, we have $x_n \to x^*$ $(n \to \infty)$.*

Proof Let $Bx = v - A(v-x)$, $\forall x \in [0, v]$, it is easy to get $B : [\theta, v] \to [\theta, v]$ is convex and decreasing, such that $B\theta = v - Av$, $B^2\theta = v - A(v - v + Av) = v - A^2 v \geq v - (1-\varepsilon)v - \varepsilon Av = \varepsilon(v - Av) = \varepsilon B\theta$. Thus by the proof of Theorem 2.2.2, we know B has a unique fixed point x^* such that $x^* = v - A(v - x^*)$, i.e., $v - x^*$ is a unique fixed point of A. Moreover, we can get x^* by iterative method. □

Corollary 2.2.2 *Suppose that* $A : [u, v] \to [u, v]$ *is concave and decreasing, and* $A^2 v \leq (1-\varepsilon)v + \varepsilon Av$, *where* $\frac{1}{2} < \varepsilon \leq 1$. *Then* A *has a unique fixed point in* $[u, v]$.

Proof Let $Bx = A(x+u) - u$, $\forall x \in [\theta, v-u]$, it is easy to know that $B : [\theta, v-u] \to [\theta, v-u]$ is concave decreasing, and $B(v-u) = Av - u$,

$$B^2(v-u) = A^2 v - u \leq (1-\varepsilon)v + \varepsilon Av - u = (1-\varepsilon)(v-u) + \varepsilon B(v-u).$$

Thus B satisfies the conditions of Corollary 2.2.1, and it is easy to get the conclusion. □

Theorem 2.2.3 (Zhitao Zhang, see [198]) *Suppose the hypotheses of Theorem 2.2.2 are satisfied. Let* $\lambda_0 = \frac{1}{2(1-\varepsilon)}$ (*as* $1/2 < \varepsilon < 1$), $\lambda_0 = +\infty$ (*as* $\varepsilon = 1$). *Then* $\forall \lambda \in [0, \lambda_0)$, *the operator equation* $\lambda Au = u$ *has a unique solution* $u(\lambda)$ *in* P, *and let* $u_0(\lambda) \equiv \theta$, $u_n(\lambda) = \lambda A u_{n-1}(\lambda)$, *we have* $u_n(\lambda) \to u(\lambda)$ $(n \to +\infty)$,

$$\left\| u_{2n}(\lambda) - u(\lambda) \right\| \leq 4N^2 \cdot \|A\theta\| \cdot \lambda \left(\frac{1-\varepsilon_\lambda}{\varepsilon_\lambda} \right)^{n-1}, \tag{2.66}$$

$$\left\| u_{2n+1}(\lambda) - u(\lambda) \right\| \leq 2N^2 \cdot \|A\theta\| \cdot \lambda \left(\frac{1-\varepsilon_\lambda}{\varepsilon_\lambda} \right)^{n}, \tag{2.67}$$

where $\varepsilon_\lambda = \min\{\varepsilon, 1 - (1-\varepsilon)\lambda\}$.

Proof (i) As $\frac{1}{2} < \varepsilon < 1$, let $\lambda_0 = \frac{1}{2(1-\varepsilon)}$, then we know that $\lambda_0 > 1$. For any $\lambda \in [0, 1]$, by $\lambda A\theta \leq A\theta$, we get $A(\lambda A\theta) \geq A^2\theta \geq \varepsilon A\theta$, thus

$$\lambda A(\lambda A\theta) \geq \varepsilon(\lambda A\theta). \tag{2.68}$$

$\forall \lambda \in [1, \lambda_0)$, since $A^2\theta = A(\frac{1}{\lambda}\lambda A\theta) \leq \frac{1}{\lambda}A(\lambda A\theta) + (1 - \frac{1}{\lambda})A\theta$, we have

$$\lambda A(\lambda A\theta) \geq \lambda^2 \left[A^2\theta - \left(1 - \frac{1}{\lambda}\right) A\theta \right] \geq \lambda^2 \left[\varepsilon - 1 + \frac{1}{\lambda} \right] A\theta = \left[1 - (1-\varepsilon)\lambda\right](\lambda A\theta). \tag{2.69}$$

Since $\lambda_0 = \frac{1}{2(1-\varepsilon)}$, we know

$$1 - (1-\varepsilon)\lambda > 1 - (1-\varepsilon)\lambda_0 = 1 - (1-\varepsilon) \cdot \frac{1}{2(1-\varepsilon)} = \frac{1}{2}, \quad \forall \lambda < \lambda_0. \tag{2.70}$$

By (2.68)–(2.70), we see that $\forall \lambda \in [0, \lambda_0)$, λA satisfies all the conditions of Theorem 2.2.2, thus λA has a unique fixed point in P, i.e., $\lambda Au = u$ has a unique positive solution $u(\lambda)$, and let $u_0(\lambda) \equiv \theta,\ u_n(\lambda) = \lambda A u_{n-1}(\lambda)$, then $u_n(\lambda) \to u(\lambda)$ $(n \to +\infty)$; moreover,

$$\left\| u_{2n}(\lambda) - u(\lambda) \right\| \leq 4\lambda \cdot N^2 \cdot \|A\theta\| \cdot \left(\frac{1-\varepsilon_\lambda}{\varepsilon_\lambda} \right)^{n-1} \quad (n = 1, 2, \ldots),$$

$$\left\| u_{2n+1}(\lambda) - u(\lambda) \right\| \leq 2\lambda \cdot N^2 \cdot \|A\theta\| \cdot \left(\frac{1-\varepsilon_\lambda}{\varepsilon_\lambda} \right)^{n} \quad (n = 1, 2, \ldots),$$

where $\varepsilon_\lambda = \min\{\varepsilon, 1 - (1-\varepsilon)\lambda\}$.

(ii) As $\varepsilon = 1$, then $A\theta$ is the unique fixed point of A. Let $\lambda_0 = +\infty,\ \forall \lambda \in [0, 1]$, (2.68) is still valid, and we get $\lambda A(\lambda A\theta) \geq \lambda A\theta;\ \forall \lambda \in (1, +\infty)$, (2.69) is replaced by $\lambda A(\lambda A\theta) \geq \lambda A\theta$. Thus, $\forall \lambda \in [0, +\infty)$, by Theorem 2.2.2, we know that $\lambda Au = u$ has a unique positive solution $u(\lambda)$ such that $u(\lambda) = \lambda A\theta$, (2.66) and (2.67) are still valid. □

Theorem 2.2.4 (Zhitao Zhang, see [198]) *Suppose all the hypotheses of Theorem 2.2.3 are satisfied, Then $u(\cdot) : [0, \lambda_0) \to P$ is strictly increasing, i.e., for $0 \leq \lambda_1 < \lambda_2 < \lambda_0$, we have $u(\lambda_1) < u(\lambda_2)$.*

Proof If $\varepsilon = 1$, by the proof (ii) of Theorem 2.2.3, we know the conclusion is valid.

We may suppose that $\frac{1}{2} < \varepsilon < 1$. If $\lambda_2 > \lambda_1 = 0$, then by the proof of Theorem 2.2.3, we get $u(\lambda_2) > \theta = u(\lambda_1)$. We may suppose that $\lambda_1 > 0$.

Now using proof by contradiction, we first get

$$u(\lambda_2) \not\leq u(\lambda_1). \tag{2.71}$$

Suppose $u(\lambda_2) \leq u(\lambda_1)$, since A is decreasing, we get $\lambda_1 \cdot Au(\lambda_2) \geq \lambda_1 Au(\lambda_1) = u(\lambda_1)$, but $\lambda_1 \cdot Au(\lambda_2) = \frac{\lambda_1}{\lambda_2}\lambda_2 Au(\lambda_2) = \frac{\lambda_1}{\lambda_2}u(\lambda_2) < u(\lambda_2)$, thus $u(\lambda_1) < u(\lambda_2)$, which is a contradiction. Therefore, (2.71) is valid.

Now under the two conditions (i) $0 < \lambda_1 \leq 1,\ \frac{\lambda_1}{\lambda_2} \leq \frac{\varepsilon}{1-\varepsilon}$; (ii) $\lambda_1 > 1$ we prove that $\lambda_2 A(\lambda_2 Ax) \geq \lambda_1 A(\lambda_1 Ax),\ \forall x \in P$, respectively.

(i) As $0 < \lambda_1 \leq 1,\ \frac{\lambda_1}{\lambda_2} \leq \frac{\varepsilon}{1-\varepsilon}$, by $\lambda_2 > \lambda_1,\ \frac{1}{2} < \varepsilon \leq 1$, we get $1 < \frac{\lambda_2}{\lambda_1} \leq \frac{\varepsilon}{1-\varepsilon}$ (as $\lambda_1 = 1$, it is clear that $\lambda_2 < \frac{1}{2(1-\varepsilon)} < \frac{\varepsilon}{1-\varepsilon}$). Then we have $\lambda_1\varepsilon \geq \lambda_2(1-\varepsilon)$, thus $(\lambda_2 - \lambda_1)((\lambda_1 + \lambda_2)\varepsilon - \lambda_2) \geq 0$, i.e.,

$$\left(\lambda_2^2 - \lambda_1^2\right)\varepsilon - \lambda_2(\lambda_2 - \lambda_1) \geq 0. \tag{2.72}$$

Since $\lambda_1 \leq 1$, we get

$$A(\lambda_1 A\theta) \geq A^2\theta \geq \varepsilon A\theta. \tag{2.73}$$

Notice the following formula (2.74):

$$\forall x \in P, \quad \lambda_1 Ax \leq \lambda_1 A\theta, \quad A(\lambda_1 Ax) \geq A(\lambda_1 A\theta). \tag{2.74}$$

By (2.72)–(2.74), we have

$$\frac{\lambda_2^2-\lambda_1^2}{\lambda_1\lambda_2}A(\lambda_1 A\theta)-\frac{\lambda_2(\lambda_2-\lambda_1)}{\lambda_1\lambda_2}A(\theta)\geq\theta, \tag{2.75}$$

$$\frac{\lambda_2^2-\lambda_1^2}{\lambda_1\lambda_2}A(\lambda_1 Ax)-\frac{\lambda_2(\lambda_2-\lambda_1)}{\lambda_1\lambda_2}A(\theta)\geq\theta. \tag{2.76}$$

Since A is convex, we know $A(\frac{1}{\delta}\cdot\delta u)\leq\frac{1}{\delta}A(\delta u)+(1-\frac{1}{\delta})A\theta,\ \forall\delta\geq 1,\ \forall u\in P$. Thus $A(\delta u)\geq\delta A(u)-\delta(1-\frac{1}{\delta})A\theta$, therefore,

$$A(\lambda_2 Ax)=A\left(\frac{\lambda_2}{\lambda_1}\cdot\lambda_1 Ax\right)\geq\frac{\lambda_2}{\lambda_1}A(\lambda_1 Ax)-\frac{\lambda_2}{\lambda_1}\left(1-\frac{\lambda_1}{\lambda_2}\right)A\theta. \tag{2.77}$$

Noticing that

$$\frac{\lambda_2}{\lambda_1}A(\lambda_1 Ax)-\frac{\lambda_1}{\lambda_2}A(\lambda_1 Ax)=\frac{\lambda_2^2-\lambda_1^2}{\lambda_1\lambda_2}A(\lambda_1 Ax), \tag{2.78}$$

by (2.77), we have

$$A(\lambda_2 Ax)-\frac{\lambda_1}{\lambda_2}A(\lambda_1 Ax)\geq\frac{\lambda_2}{\lambda_1}A(\lambda_1 Ax)-\frac{\lambda_1}{\lambda_2}A(\lambda_1 Ax)-\left(\frac{\lambda_2}{\lambda_1}-1\right)A\theta. \tag{2.79}$$

By (2.76), (2.78), (2.79), we get $A(\lambda_2 Ax)-\frac{\lambda_1}{\lambda_2}A(\lambda_1 Ax)\geq\theta$ i.e.,

$$\lambda_2 A(\lambda_2 Ax)\geq\lambda_1 A(\lambda_1 Ax).$$

(ii) As $\lambda_1>1,\ \lambda_2>\lambda_1$, since $\lambda_0=\frac{1}{2(1-\varepsilon)}$, we know $\lambda_1+\lambda_2\leq\frac{1}{1-\varepsilon}$, thus $\frac{\lambda_1}{\lambda_2}\geq\frac{\lambda_1}{\frac{1}{1-\varepsilon}-\lambda_1}=\frac{(1-\varepsilon)\lambda_1}{1+(\varepsilon-1)\lambda_1}$, therefore, $(\frac{\lambda_1}{\lambda_2}+1)(\varepsilon\lambda_1-\lambda_1+1)\geq 1$. Moreover, $(\lambda_2^2-\lambda_1^2)(\varepsilon\lambda_1-\lambda_1+1)-\lambda_2(\lambda_2-\lambda_1)\geq 0$. By $\lambda_1>1$ and the convexity of A, we get

$$A(\lambda_1 A\theta)\geq\lambda_1 A^2\theta-\lambda_1\left(1-\frac{1}{\lambda_1}\right)A\theta\geq(\varepsilon\lambda_1+1-\lambda_1)A\theta.$$

Thus (2.75) is still valid. Notice that (2.74), (2.76) are still valid. So we can prove

$$\lambda_2 A(\lambda_2 Ax)\geq\lambda_1 A(\lambda_1 Ax).$$

By (i), (ii), $\forall x\in P$, as $0<\lambda_1\leq 1,\ 1<\frac{\lambda_2}{\lambda_1}\leq\frac{\varepsilon}{1-\varepsilon}$, or $1\leq\lambda_1<\lambda_2$, we have

$$(\lambda_2 A)^2x\geq(\lambda_1 A)^2x. \tag{2.80}$$

Noticing that $u_n(\lambda)=\lambda Au_{n-1}(\lambda),\ u_0(\lambda)=\theta,\ u_1(\lambda_1)=\lambda_1 A\theta<\lambda_2 A\theta=u_1(\lambda_2)$ and that $(\lambda A)^2$ is a increasing operator for fixed $\lambda\in[0,\lambda_0)$, by (2.80) and the proof of Theorem 2.2.2, we get $u(\lambda_1)\leq u(\lambda_2)$ as $0<\lambda_1\leq 1,\ 1<\frac{\lambda_2}{\lambda_1}\leq\frac{\varepsilon}{1-\varepsilon}$, or as $1 \le \lambda_1 < \lambda_2$. Since λ_1 is arbitrary and partial order has transitive relation, we get $u(\lambda_1) \le u(\lambda_2),\ \forall 0 < \lambda_1 < \lambda_2 < \lambda_0$. By (2.71) we have $u(\lambda_1) < u(\lambda_2),\ \forall \lambda_1, \lambda_2 \in [0, \lambda_0),\ 0 < \lambda_1 < \lambda_2$. □

Theorem 2.2.5 (Zhitao Zhang, see [198]) *Suppose all the hypotheses of Theorem 2.2.3 are satisfied. Then*

(i) $u(\lambda) : [0, \lambda_0) \to P$ *is continuous.*
(ii) $u(t\lambda) \ge tu(\lambda),\ \forall t \in [0, 1],\ \lambda \in [0, \lambda_0)$.
(iii) *If A is continuous, then there exists $u(\lambda_0)$ such that $\lambda_0 Au(\lambda_0) = u(\lambda_0)$.*

Proof (i) By Theorem 2.2.4 , we get $u(\lambda_1) < u(\lambda_2),\ \forall 0 \le \lambda_1 < \lambda_2$. Noticing $\lambda_1 Au(\lambda_1) = u(\lambda_1),\ \lambda_2 Au(\lambda_2) = u(\lambda_2)$, we have

$$\frac{u(\lambda_1)}{\lambda_1} = Au(\lambda_1) \ge Au(\lambda_2) = \frac{u(\lambda_2)}{\lambda_2}$$

i.e., $u(\lambda_2) \le \frac{\lambda_2}{\lambda_1} u(\lambda_1)$, thus

$$\theta \le u(\lambda_2) - u(\lambda_1) \le \left(\frac{\lambda_2}{\lambda_1} - 1\right) u(\lambda_1) \le \left(\frac{\lambda_2}{\lambda_1} - 1\right) \lambda_1 A\theta = (\lambda_2 - \lambda_1) A\theta.$$

Since P is normal, it is easy to know

$$\left\| u(\lambda_2) - u(\lambda_1) \right\| \le N \cdot |\lambda_2 - \lambda_1| \cdot \|A\theta\|. \tag{2.81}$$

Thus $u(\lambda) \to u(\lambda_1)$, as $\lambda \to \lambda_1^+$; $u(\lambda) \to u(\lambda_2)$, as $\lambda \to \lambda_2^-$. So we know $u(\lambda)$ is continuous on $[0, \lambda_0)$.

(ii) $\forall t \in [0, 1],\ \lambda \in [0, \lambda_0)$, we have $u(t\lambda) = t\lambda Au(t\lambda) \ge t\lambda Au(\lambda) = tu(\lambda)$.

(iii) By (2.81), we know $\lim_{\lambda \to \lambda_0^-} u(\lambda)$ exists; let $u(\lambda_0) = \lim_{\lambda \to \lambda_0^-} u(\lambda)$. Since $\lambda Au(\lambda) = u(\lambda),\ \lambda \in [0, \lambda_0)$ and A is continuous, we get $\lambda_0 Au(\lambda_0) = u(\lambda_0)$. □

Remark 2.2.1 It should be pointed out that A need not to be compact. In Theorems 2.2.2, 2.2.3, 2.2.4 and (i), (ii) of Theorem 2.2.5, Corollaries 2.2.1, 2.2.2, we do not assume A to be continuous. About convexity, we only use the following formula: $\forall x \in P,\ t \in [0, 1],\ A(tx) \le tAx + (1 - t)A\theta$.

Remark 2.2.2 By the proof of Theorem 2.2.2, for $u_n = Au_{n-1}$, we know if $\exists k_0 \ge 1$ such that $u_{2k_0} \ge \varepsilon A\theta$, the conclusion is still valid.

Remark 2.2.3 If P is normal, $A : P \to P$ is convex and decreasing, $A^2\theta \ge \varepsilon A\theta\ (\varepsilon > 0),\ A^3\theta \ge cA\theta\ (c > \frac{1}{2})$. Then $\forall \delta \in [0, \varepsilon],\ \delta Au = u$ has a unique solution in P.

Proof $\forall \delta \in [0, \varepsilon],\ A^2\theta \ge \varepsilon A\theta \ge \delta A\theta$, thus $A(\delta A\theta) \ge A^3\theta \ge cA\theta$, i.e., δA satisfies all the conditions of Theorem 2.2.2, δA has a unique fixed point in P, i.e., $\delta Au = u$ has a unique solution in P. □

Example 2.2.1 Consider nonlinear Hammerstein integral equations on $\mathbb{R}^N$:

$$x(t) = Ax(t) := \int_{\mathbb{R}^N} k(t,s)\frac{1}{1+x(s)}\,ds \tag{2.82}$$

Proposition 2.2.1 *Suppose that* $k : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ *is nonnegative and continuous,* $k \not\equiv 0$, *and* $\exists r > 0$, *such that* $\int_{\mathbb{R}^N} k(t,s)\,ds \le r < 1$. *Then* (2.82) *has a unique positive solution* $x^*(t)$, *and* $\forall x_0(t) \ge 0$, $x_n(t) = Ax_{n-1}(t)$, *we have* $x_n(t) \to x^*(t)(n \to \infty)$ (*uniformly for* $t \in \mathbb{R}^N$).

Proof Let $C_B(\mathbb{R}^N)$ denote the space of bounded continuous functions on $\mathbb{R}^N$, $\|x(\cdot)\| = \sup_{t\in\mathbb{R}^N} |x(t)|$, $P = C_B^+(\mathbb{R}^N)$ denote the nonnegative functions in $C_B(\mathbb{R}^N)$, then P is a normal and solid cone of $C_B(\mathbb{R}^N)$, and $A : P \to P$ is convex and decreasing. Noticing

$$A\theta = \int_{\mathbb{R}^N} k(t,s)\,ds > \theta, \quad A\theta \le r < 1,$$

$$A^2\theta \ge \left(\int_{\mathbb{R}^N} k(t,s)\,ds\right)\cdot\frac{1}{1+r} = \frac{1}{1+r}A\theta$$

and $\frac{1}{1+r} > \frac{1}{2}$, by Theorem 2.2.2 we get the conclusion. □

Example 2.2.2 Consider the following systems of nonlinear differential equations:

$$\begin{cases} -x_n'' = \dfrac{1}{n}\cdot(1+x_{n+2})^{-\frac{1}{2}}, \\ x_n(0) = x_n'(1) = 0, \quad n = 1, 2, \ldots. \end{cases} \tag{2.83}$$

Let $E = \{x | x = (x_1, x_2, \ldots), |x_i| < +\infty\}$, $\|x\| = \sup_i |x_i|$, $P = \{x \in E | x_i \ge 0\}$ is a normal and solid cone of E, (2.83) is equivalent to two-point boundary value problem in E.

Let $C[I, E] = \{x \mid x : I \to E$ is abstract continuous function$\}$, $I = [0, 1]$, $\bar{P} = \{x \in C[I, E] \mid x_i(t) \ge 0, i = 1, 2, \ldots, \forall t \in I\}$ is normal and solid cone of $C[I, E]$.

Proposition 2.2.2 *System of equations* (2.83) *has a unique positive solution in* $\bar{P}$, *and* $\forall x_0(t) \in \bar{P}$, *let* $x_n = Ax_{n-1}$, *we have* $x_n \to x^*$ $(n \to +\infty)$, *where* $Ax(t) = \int_0^1 G(t,s)f(x(s))\,ds$, $G(t,s) = \min\{t, s | (t,s) \in I \times I\}$, $f(x) = (f_1(x), f_2(x), \ldots)$, $f_i(x) = \frac{1}{i}(1+x_{i+2})^{-\frac{1}{2}}$, $i = 1, 2, \ldots$.

Proof It is easy to know $x(t) \in C^2[I, E] \cap \bar{P}$ is a solution of (2.83) if and only if $x \in \bar{P}$ is a solution of $Ax = x$. It is easy to know $A : \bar{P} \to \bar{P}$ is convex and decreasing, and

$$(A\theta)_i(t) = \frac{1}{i}\int_0^1 G(t,s)\,ds = \frac{1}{i}\left(t - \frac{t^2}{2}\right) \ge 0,$$

$$\left(A^2\theta\right)_i(t)=\int_0^1 G(t,s)f_i(A\theta)\,ds=\frac{1}{i}\int_0^1 G(t,s)\left(1+\frac{2s-s^2}{2(i+2)}\right)^{-\frac{1}{2}}ds,$$

since $s-\frac{s^2}{2}\geq 0,\ \forall s\in[0,1]$, we have $(1+\frac{s-\frac{s^2}{2}}{i+2})^{-\frac{1}{2}}\leq 1$. And by $1+\frac{2s-s^2}{2(i+2)}\leq 1+s-s^2/2\leq\frac{3}{2}$, we get $(1+\frac{2s-s^2}{2(i+2)})^{-\frac{1}{2}}\geq\sqrt{\frac{2}{3}}$, thus $A^2\theta\geq\sqrt{\frac{2}{3}}A\theta>\theta,\ \varepsilon=\sqrt{\frac{2}{3}}>\frac{1}{2}$. By Theorem 2.2.2, we get the conclusion. □

Example 2.2.3 Consider eigenvector problem:

$$\begin{cases}-x_n''=\dfrac{\lambda}{n}\cdot(1+x_{n+2})^{-\frac{1}{2}},\\ x_n(0)=x_n'(1)=0,\quad n=1,2,\ldots.\end{cases}\tag{2.84}$$

Proposition 2.2.3 *By Proposition 2.2.2 and Theorems 2.2.3, 2.2.4, we get* $\varepsilon=\sqrt{\frac{2}{3}}$, $\lambda_0=\frac{1}{2(1-\varepsilon)}=\frac{1}{2(1-\sqrt{\frac{2}{3}})}$, $\forall\lambda\in[0,\lambda_0)$, (2.84) *has a unique positive solution. Moreover,* $u(\lambda)$ *is continuous and increasing for* λ.

2.3 Mixed Monotone Operators

Let the real Banach space E be partially ordered by a cone P of E, i.e., $x\leq y$ if and only if $y-x\in P$. Let $D\subset E$, operator $A: D\times D\to E$ is said to be mixed monotone if $A(x,y)$ is non-decreasing in x and non-increasing in y. Point $(x^*,y^*)\in D\times D$ is called a coupled fixed point of A if $A(x^*,y^*)=x^*$ and $A(y^*,x^*)=y^*$. Element $x^*\in D$ is called a fixed point of A if $A(x^*,x^*)=x^*$.

Theorem 2.3.1 (Dajun Guo, see [101]) *Let the cone P be normal and $A:\mathring{P}\times\mathring{P}\to\mathring{P}$ be a mixed monotone operator. Suppose that there exists $0\leq a<1$ such that*

$$A\left(tx,t^{-1}y\right)\geq t^aA(x,y),\quad t\in(0,1),\ x,y\in\mathring{P}.\tag{2.85}$$

Then A has exactly one fixed point x^* in $\mathring{P}$ and constructing successively the sequences

$$x_n=A(x_{n-1},y_{n-1}),\quad y_n=A(y_{n-1},x_{n-1})\quad(n=1,2,3,\ldots)\tag{2.86}$$

for any initial $x_0,y_0\in[\mathring{P},\mathring{P}]$, we have

$$\|x_n-x^*\|\to 0,\quad \|y_n-x^*\|\to 0\quad(n\to\infty),\tag{2.87}$$

with the convergence rate

$$\|x_n-x^*\|=O\left(1-r^{a^n}\right),\qquad \|y_n-x^*\|=O\left(1-r^{a^n}\right),\tag{2.88}$$

where $0 < r < 1$ *and* r *depends on* (x_0, y_0). *Moreover, for any coupled fixed points* $(\bar{x}, \bar{y}) \in \mathring{P} \times \mathring{P}$ *of* A, *it must be* $\bar{x} = \bar{y} = x^*$.

Proof From hypothesis (2.85) we know first

$$A(x, y) = A\big(tt^{-1}x, t^{-1}ty\big) \geq t^a A\big(t^{-1}x, ty\big),$$

and so

$$A\big(t^{-1}x, ty\big) \leq t^{-a} A(x, y), \quad x, y \in \mathring{P},\ 0 < t < 1. \tag{2.89}$$

Let $z_0 \in \mathring{P}$ be arbitrarily given. Since $A(z_0, z_0) \in \mathring{P}$, we can choose $0 < t_0 < 1$ sufficiently small such that

$$t_0^{(1-a)/2} z_0 \leq A(z_0, z_0) \leq t_0^{-(1-a)/2} z_0. \tag{2.90}$$

Let $u_0 = t_0^{\frac{1}{2}} z_0,\ v_0 = t_0^{-\frac{1}{2}} z_0$ and

$$u_n = A(u_{n-1}, v_{n-1}), \quad v_n = A(v_{n-1}, u_{n-1}) \quad (n = 1, 2, \ldots). \tag{2.91}$$

Clearly,

$$u_0, v_0 \in \mathring{P}, \qquad u_0 < v_0, \qquad u_0 = t_0 v_0 \tag{2.92}$$

and by virtue of (2.85), (2.89) and the mixed monotone property of A, we have

$$\begin{aligned}
u_1 &= A(u_0, v_0) \leq A(v_0, u_0) = v_1 \\
u_1 &= A\big(t_0^{\frac{1}{2}} z_0, t_0^{-\frac{1}{2}} z_0\big) \geq t_0^{\frac{a}{2}} A(z_0, z_0) \geq t_0^{\frac{1}{2}} z_0 = u_0, \\
v_1 &= A\big(t_0^{-\frac{1}{2}} z_0, t_0^{\frac{1}{2}} z_0\big) \leq t_0^{-\frac{a}{2}} A(z_0, z_0) \leq t_0^{-\frac{1}{2}} z_0 = v_0.
\end{aligned}$$

Now, it is easy to show by induction that

$$u_0 \leq u_1 \leq \cdots \leq u_{n-1} \leq u_n \leq \cdots \leq v_n \leq v_{n-1} \leq \cdots \leq v_1 \leq v_0. \tag{2.93}$$

If $u_n \geq t_0^{a^n} v_n$, then $v_n \leq t_0^{-a^n} u_n$ and

$$\begin{aligned}
u_{n+1} = A(u_n, v_n) \geq A\big(t_0^{a^n} v_n, t_0^{-a^n} u_n\big) &\geq t_0^{a^{n+1}} A(v_n, u_n) \\
&= t_0^{a^{n+1}} v_n,
\end{aligned}$$

hence, by (2.92) and induction, we get

$$u_n \geq t_0^{a^n} v_n \quad (n = 0, 1, 2, \ldots). \tag{2.94}$$

From (2.93) and (2.94) we find

$$0 \leq u_{n+p} - u_n \leq v_n - u_n \leq \big(1 - t_0^{a^n}\big) v_n \leq \big(1 - t_0^{a^n}\big) v_0,$$

and consequently

$$\|u_{n+p} - u_n\| \leq N\big(1 - t_0^{a^n}\big)\|v_0\|,$$

which implies that $\{u_n\}$ converges (in norm) to some $u^* \in E$. Similarly, we can prove that $\{v_n\}$ also converges to some $v^* \in E$ and, by (2.93),

$$u_n \leq u^* \leq v^* \leq v_n \quad (n = 0, 1, \ldots). \tag{2.95}$$

Hence $u^*, v^* \in \mathring{P}$. Now, (2.95), (2.93) and (2.94) imply

$$0 \leq v^* - u^* \leq v_n - u_n \leq \big(1 - t_0^{a^n}\big)v_0 \quad (n = 1, 2, \ldots), \tag{2.96}$$

and therefore $u^* = v^*$. Let $v^* = u^* = x^*$. On account of (2.95),

$$A\big(x^*, x^*\big) \geq A(u_n, v_n) = u_{n+1} \quad (n = 0, 1, \ldots),$$
$$A\big(x^*, x^*\big) \leq A(v_n, u_n) = v_{n+1} \quad (n = 0, 1, \ldots).$$

Taking limit as $n \to \infty$, we get

$$x^* = u^* \leq A\big(x^*, x^*\big) \leq v^* = x^*,$$

hence, $A(x^*, x^*) = x^*$, i.e. x^* is a fixed point of A.

For any coupled fixed point $(\bar{x}, \bar{y}) \in \mathring{P} \times \mathring{P}$ of A, let $t_1 = \sup\{0 < t < 1 | tx^* \leq \bar{x} \leq t^{-1}x^*, tx^* \leq \bar{y} \leq t^{-1}x^*\}$. Clearly, $0 < t_1 \leq 1$ and $t_1x^* \leq \bar{x} \leq t_1^{-1}x^*$, $t_1x^* \leq \bar{y} \leq t_1^{-1}x^*$. If $0 < t_1 < 1$, then by virtue of (2.85) and (2.89), we have

$$\bar{x} = A(\bar{x}, \bar{y}) \geq A\big(t_1x^*, t_1^{-1}x^*\big) \geq t_1^a A\big(x^*, x^*\big) = t_1^a x^*,$$
$$\bar{x} = A(\bar{x}, \bar{y}) \leq A\big(t_1^{-1}x^*, t_1x^*\big) \leq t_1^{-a} A\big(x^*, x^*\big) = t_1^{-a} x^*$$

i.e.

$$t_1^a x^* \leq \bar{x} \leq t_1^{-a} x^*. \tag{2.97}$$

Similarly, we get

$$t_1^a x^* \leq \bar{y} \leq t_1^{-a} x^*. \tag{2.98}$$

(2.97) and (2.98) contradict the definition of t_1, since $t_1^a > t_1$. Hence $t_1 = 1$ and $\bar{x} = \bar{y} = x^*$. This at the same time proves the uniqueness of fixed point of A in $\mathring{P}$.

It remains to show that (2.87) and (2.88) hold. Let $(x_0, y_0) \in \mathring{P} \times \mathring{P}$ be given. We can choose t_0 $(0 < t_0 < 1)$ so small that (2.90) holds and $t_0^{\frac{1}{2}} z_0 \leq x_0 \leq t_0^{-\frac{1}{2}} z_0$, $t_0^{\frac{1}{2}} z_0 \leq y_0 \leq t_0^{-\frac{1}{2}} z_0$, i.e. $u_0 \leq x_0 \leq v_0$, $u_0 \leq y_0 \leq v_0$. Suppose $u_{n-1} \leq x_{n-1} \leq v_{n-1}$, then

$$x_n = A(x_{n-1}, y_{n-1}) \geq A(u_{n-1}, v_{n-1}) = u_n,$$
$$x_n = A(x_{n-1}, y_{n-1}) \leq A(v_{n-1}, u_{n-1}) = v_n,$$

and similarly, $y_n \geq u_n$, $y_n \leq v_n$. Hence, by induction,

$$u_n \leq x_n \leq v_n, \quad u_n \leq y_n \leq v_n \quad (n = 0, 1, \ldots). \tag{2.99}$$

Now, from

$$\begin{aligned}
&\|x_n - x^*\| \leq \|x_n - u_n\| + \|u_n - x^*\|, \\
&0 \leq x_n - u_n \leq v_n - u_n \leq \left(1 - t_0^{a^n}\right) v_0 \\
&0 \leq u^* - u_n \leq v_n - u_n \leq \left(1 - t_0^{a^n}\right) v_0,
\end{aligned}$$

it follows that

$$\|x_n - x^*\| \leq 2N\left(1 - t_0^{a^n}\right)\|v_0\| \quad (n = 0, 1, \ldots). \tag{2.100}$$

In the same way, we get

$$\|y_n - x^*\| \leq 2N\left(1 - t_0^{a^n}\right)\|v_0\| \quad (n = 0, 1, \ldots). \tag{2.101}$$

Finally, (2.100) and (2.101) imply (2.88) with $r = t_0$, and therefore (2.87) holds. The proof is complete. □

Theorem 2.3.2 (Dajun Guo, see [101]) *Let the cone P be normal and $A : \mathring{P} \times \mathring{P} \to \mathring{P}$ be a mixed monotone operator. Suppose that there exists $0 \leq a < 1$ such that* (2.85) *holds. Let x_t^* be the unique solution in $\mathring{P}$ of the equation*

$$A(x, x) = tx \quad (x > 0). \tag{2.102}$$

Then x_t^* is continuous with respect to t, i.e., $\|x_t^* - x_{t_0}^*\| \to 0$ as $t \to t_0$ $(t_0 > 0)$. If, in addition, $0 \leq a < \frac{1}{2}$, then x_t^* is strongly decreasing with respect to t, i.e.,

$$0 < t_1 < t_2 \quad \Longrightarrow \quad x_{t_1}^* \gg x_{t_2}^*, \tag{2.103}$$

and

$$\lim_{t \to \infty} \|x_t^*\| = 0, \qquad \lim_{t \to +0} \|x_t^*\| = +\infty. \tag{2.104}$$

Proof Since the operator $t^{-1}A$ satisfies all conditions of Theorem 2.3.1, (2.102) has exactly one solution $x_t^* \in \mathring{P}$. Given $t_2 > t_1 > 0$ arbitrarily and let $s_0 = \sup\{s > 0 | x_{t_1}^* \geq s x_{t_2}^*, x_{t_2}^* \geq s x_{t_1}^*\}$. Clearly, $0 < s_0 < +\infty$ and

$$x_{t_1}^* \geq s_0 x_{t_2}^*, \qquad x_{t_2}^* \geq s_0 x_{t_1}^*. \tag{2.105}$$

It is easy to see from (2.105) that $s_0 \geq 1$ is impossible. Hence $0 < s_0 < 1$. By (2.85) and (2.105), we find

$$t_1 x_{t_1}^* = A\left(x_{t_1}^*, x_{t_1}^*\right) \geq A\left(s_0 x_{t_2}^*, s_0^{-1} x_{t_2}^*\right) \geq s_0^a A\left(x_{t_2}^*, x_{t_2}^*\right) = t_2 s_0^a x_{t_2}^*,$$

$$t_2 x_{t_2}^* = A\left(x_{t_2}^*, x_{t_2}^*\right) \geq A\left(s_0 x_{t_1}^*, s_0^a x_{t_1}^*\right) \geq s_0^a A\left(x_{t_1}^*, x_{t_1}^*\right) = t_1 s_0^a x_{t_1}^*.$$

Consequently,

$$x_{t_1}^* \geq t_2 t_1^{-1} s_0^a x_{t_2}^*, \qquad x_{t_2}^* \geq t_1 t_2^{-1} s_0^a x_{t_1}^*. \tag{2.106}$$

Observing the definition of s_0 and $t_2 t_1^{-1} s_0^a > s_0$, we conclude $t_1 t_2^{-1} s_0^a \leq s_0$, and so

$$s_0 \geq (t_1/t_2)^{1/(1-a)}. \tag{2.107}$$

It follows from (2.105) and (2.107) that

$$(t_1/t_2)^{1/(1-a)} x_{t_2}^* \leq x_{t_1}^* \leq (t_2/t_1)^{1/(1-a)} x_{t_2}^*, \tag{2.108}$$

$$(t_1/t_2)^{1/(1-a)} x_{t_1}^* \leq x_{t_2}^* \leq (t_2/t_1)^{1/(1-a)} x_{t_1}^*. \tag{2.109}$$

Inequalities (2.108) and (2.109), together with the normality of cone P, imply that

$$\left\|x_{t_1}^* - x_{t_2}^*\right\| \to 0, \quad \text{as } t_1 \to t_2 - 0,$$

$$\left\|x_{t_2}^* - x_{t_1}^*\right\| \to 0, \quad \text{as } t_2 \to t_1 + 0.$$

Hence, the continuity of $x_{t_1}^*$ with respect to t $(t > 0)$ is proved.

Now, assume $0 \leq a < \frac{1}{2}$, by virtue of (2.106) and (2.107), we have

$$x_{t_1}^* \geq (t_2/t_1)^{(1-2a)/(1-a)} x_{t_2}^*, \tag{2.110}$$

which implies (2.103) since

$$(t_2/t_1)^{(1-2a)/(1-a)} > 1.$$

Letting $t_1 = 1$ and $t_2 = t$ in (2.110), we find

$$x_1^* \geq t^{(1-2a)/(1-a)} x_t^*,$$

and so

$$\left\|x_t^*\right\| \geq N t^{-(1-2a)/(1-a)} \left\|x_1^*\right\|, \quad t > 1$$

which implies $\|x_t^*\| \to 0$ as $t \to +\infty$. On the other hand, letting $t_1 = t$ and $t_2 = 1$ in (2.110), we get

$$x_t^* \geq t^{-(1-2a)/(1-a)} x_1^*,$$

and therefore

$$\left\|x_t^*\right\| \geq N t^{-(1-2a)/(1-a)} \left\|x_1^*\right\|, \quad 0 < t < 1,$$

which implies $\|x_t^*\| \to +\infty$ as $t \to +0$. Hence, (2.104) holds and our theorem is proved. $\square$

Remark 2.3.1 It should be pointed out that in Theorems 2.3.1 and 2.3.2 we do not require operator A to be continuous.

As above, an existence and uniqueness theorem was established for operator $A : \mathring{P} \times \mathring{P} \to \mathring{P}$ under the condition: there exists $0 \leq a < 1$ such that

$$A\big(tx, t^{-1}y\big) \geq t^a A(x, y), \quad x, y \in \mathring{P},\ 0 < t < 1. \tag{2.111}$$

This result was applied to the IVP of ordinary differential equations

$$\begin{cases} x' = \displaystyle\sum_{i=1}^{n} a_i(t) x^{r_i} + \left(\sum_{j=1}^{m} b_j(t) x^{s_j}\right)^{-1}, & t \in [0, T], \\ x(0) = x_0, \end{cases} \tag{2.112}$$

where $0 < r_i < 1,\ 0 < s_j < 1,\ i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m$.

Next we only assume that $A : P \times P \to P$ and replace (2.111) by, in some sense, weaker condition: for any $0 < a < b < 1$ and bounded $B \subset P$, there exists an $r = r(a, b, B) > 0$ such that

$$A\big(tx, t^{-1}y\big) \geq t(1 + r)A(x, y), \quad x, y \in B,\ t \in [a, b].$$

The result obtained can be applied to the IVP (2.112) and also the two-point BVP

$$\begin{cases} -x'' = \displaystyle\sum_{i=1}^{n} a_i(t) x^{r_i} + \left(\sum_{j=1}^{m} b_j(t) x^{s_j}\right)^{-1}, & t \in [0, 1], \\ ax(0) - bx'(0) = x_0, \qquad cx(1) + dx'(1) = x_1, \end{cases}$$

in the more general case that some one of the r_i and s_j may be equal to 1 and another one of the r_i and s_j may be equal to 0.

Theorem 2.3.3 (Dajun Guo, see [102]) *Let the cone P be normal and $A : P \times P \to P$ be a mixed monotone operator. Suppose that*

(a) *there exist $v > 0$ and $c > 0$ such that $0 < A(v, 0) \leq v$ and $A(0, v) \geq cA(v, 0)$,*
(b) *for any a, b satisfying $0 < a < b < 1$, there exists $r = r(a, b) > 0$ such that*

$$A\big(tx, t^{-1}y\big) \geq t(1 + r)A(x, y), \quad t \in [a, b],\ 0 \leq y \leq x \leq v.$$

Then A has exactly one fixed point x^* in $[0, v]$ and $x^* > 0$. Moreover, constructing successively the sequences

$$x_n = A(x_{n-1}, y_{n-1}), \quad y_n = A(y_{n-1}, x_{n-1}) \quad (n = 1, 2, 3, \ldots) \tag{2.113}$$

for any initial $x_0, y_0 \in [0, v]$, we have

$$\big\|x_n - x^*\big\| \to 0, \quad \big\|y_n - x^*\big\| \to 0 \quad (n \to \infty). \tag{2.114}$$

Proof Set $u_0 = 0$ and choose v and c to satisfy condition (a). Then $0 < A(v, 0) \le v$ and $A(0, v) \ge cA(v, 0)$. If we set $v_0 = v$ and define $u_1 = A(u_0, v_0),\ v_1 = A(v_0, u_0)$, then

$$\begin{aligned} &u_0 = 0 \le A(0, v) = u_1, \qquad v_1 = A(v, 0) \le v = v_0, \\ &u_1 = A(0, v) \le A(v, 0) = v_1 \quad \text{and} \quad u_1 \ge cv_1. \end{aligned} \tag{2.115}$$

Define $u_n = A(u_{n-1}, v_{n-1}),\ v_n = A(v_{n-1}, u_{n-1})$ for $n = 1, 2, 3, \ldots$ and assume that $u_{n-1} \le u_n \le v_n \le v_{n-1}$. Then

$$\begin{aligned} u_n &= A(u_{n-1}, v_{n-1}) \le A(u_n, v_n) = u_{n+1} \le A(v_n, u_n) = v_{n+1} \\ &\le A(v_{n-1}, u_{n-1}) = v_n. \end{aligned}$$

Hence, by induction,

$$0 = u_0 < u_1 \le \cdots \le u_{n-1} \le u_n \le \cdots \le v_n \le v_{n-1} \le \cdots \le v_1 \le v_0 = v. \tag{2.116}$$

It follows form (2.115) and (2.116) that

$$u_n \ge u_1 \ge cv_1 \ge cv_n \quad (n = 1, 2, 3, \ldots). \tag{2.117}$$

Let $t_n = \sup\{t > 0 : u_n \ge tv_n\}$. Then

$$u_n \ge t_n v_n \tag{2.118}$$

and, on account of (2.116) and (2.117) and the fact $u_{n+1} \ge u_n \ge t_n v_n \ge t_n v_{n+1}$, we have

$$0 < c \le t_1 \le t_2 \le \cdots \le t_n \le \cdots \le 1 \tag{2.119}$$

which implies that $\exists t^*$ such that

$$\lim_{n\to\infty} t_n = t^* \tag{2.120}$$

and $0 < t^* \le 1$. We check that

$$t^* = 1. \tag{2.121}$$

In fact, if $t^* < 1$, then $t_n \in [c, t^*]$ $(n = 1, 2, \ldots)$, and so, by virtue of (2.116) and condition (b), there exists $r > 0$ such that

$$A\left(t_n v_n, t_n^{-1} u_n\right) \ge t_n(1 + r)A(v_n, u_n) \quad (n = 1, 2, 3, \ldots). \tag{2.122}$$

Since $u_n \ge t_n u_n,\ v_n \le t_n^{-1} u_n$, we have by (2.122)

$$u_{n+1} = A(u_n, v_n) = A\left(t_n v_n, t_n^{-1} u_n\right) \ge t_n(1 + r)A(v_n, u_n) = t_n(1 + r)v_{n+1},$$

which implies that

$$t_{n+1} \ge t_n(1 + r) \quad (n = 1, 2, 3, \ldots)$$

and therefore

$$t_n \geq t_1(+r)^n \geq c(1+r)^n \quad (n = 1, 2, 3, \ldots).$$

Hence $t_n \to \infty$, which contradicts (2.119), and so (2.121) is true. Now, (2.116) and (2.118) imply

$$0 \leq u_{n+p} - u_n \leq v_n - u_n \leq (1 - t_n)v_n \leq (1 - t_n)v,$$

and so

$$\|u_{n+p} - u_n\| \leq N(1 - t_n)\|v\|, \tag{2.123}$$

where N is the normal constant of P. It follows from (2.123), (2.120) and (2.121) that $\lim_{n\to\infty} u_n = u^*$ exists. In the same way we can prove that $\lim_{n\to\infty} v_n = v^*$ also exists. Since

$$u_n \leq u^* \leq v^* \leq v_n,$$

we have

$$0 \leq v^* - u^* \leq v_n - u_n \leq (1 - t_n)v,$$

so

$$\left\|v^* - u^*\right\| \leq N(1 - t_n)\|v\| \to 0 \quad (n \to \infty).$$

Hence $u^* = v^*$. Let $x^* = u^* = v^*$, then $x^* \in [0, v],\ x^* > 0$. Since

$$u_n \leq x^* \leq v_n \quad (n = 1, 2, 3, \ldots),$$

we have

$$u_{n+1} = A(u_n, v_n) \leq A\left(x^*, x^*\right) \leq A(v_n, u_n) = v_{n+1},$$

and, after taking the limit,

$$x^* \leq A\left(x^*, x^*\right) \leq x^*.$$

Hence $A(x^*, x^*) = x^*$, i.e., x^* is a fixed point of A.

Let $\bar{x}$ be any fixed point of A in $[0, v]$. Then $u_0 = 0 \leq \bar{x} \leq v = v_0,\ A(\bar{x}, \bar{x}) = \bar{x}$, so

$$u_1 = A(u_0, v_0) \leq A(\bar{x}, \bar{x}) = \bar{x} \leq A(v_0, u_0) = v_1.$$

It is easy to see by induction that

$$u_n \leq \bar{x} \leq v_n \quad (n = 1, 2, 3, \ldots), \tag{2.124}$$

which implies by taking limit that $\bar{x} = x^*$.

Finally, we verify that (2.114) holds. Let $x_0, y_0 \in [0, v]$. Then, similar to (2.124), we get easily from (2.113) that

$$u_n \leq x_n \leq v_n, \quad u_n \leq y_n \leq v_n \quad (n = 1, 2, 3, \ldots). \tag{2.125}$$

Consequently, (2.114) follows from (2.125) and the fact that P is normal and $u_n \to x^*$ and $v_n \to x^*$. The proof is complete. □

Theorem 2.3.4 (Dajun Guo [102]) *Let the cone P be normal and solid and $A : P \times P \to P$ be a mixed monotone operator. Suppose that*

(a′) *there exist $v \in \mathring{P}$ and $c > 0$ such that $0 < A(v, 0) \le v$ and $A(0, v) \ge cA(v, 0)$,*
(b′) *for any a, b satisfying $0 < a < b < 1$, and any bounded set $B \subset P$, there exists an $r = r(a, b, B) > 0$ such that*

$$A\bigl(tx, t^{-1}y\bigr) \ge t(1 + r)A(x, y), \quad t \in [a, b],\ x, y \in B. \tag{2.126}$$

Then A has exactly one fixed point x^* in P and $0 < x^* \le v$. Moreover, constructing successively the sequences (2.113) for any initial $x_0, y_0 \in P$, we have (2.114) holds.

Proof By Theorem 2.3.3, A has a fixed point x^* in P and $0 < x^* \le v$. Let $\bar{x}$ be any fixed point of A in P and $x_0, y_0 \in P$ be given. Since $v \in \mathring{P}$, we can choose $0 < t_0 < 1$ sufficiently small such that

$$v \ge t_0\bar{x}, \qquad v \ge t_0x_0, \qquad v \ge t_0y_0. \tag{2.127}$$

Observing that $A(x, y) = A(tt^{-1}x, t^{-1}ty)$, it is easy to see from condition (b′) that for any $0 < a < b < 1$ and any bounded $B \subset P$, there exists $h = h(a, b, B) > 0$ such that

$$A\bigl(t^{-1}x, ty\bigr) \le \bigl(t(1 + h)\bigr)^{-1}A(x, y), \quad t \in [a, b],\ x, y \in B. \tag{2.128}$$

Now, (2.128) and conditions (a′) and (b′) imply that there exist $h_1 > 0$, $r_1 > 0$, $r_2 > 0$ such that

$$0 < A(v, 0) \le A\bigl(t_0^{-1}v, 0\bigr) = A\bigl(t_0^{-1}v, t_0 0\bigr) \le \bigl(t_0(1 + h_1)\bigr)^{-1}A(v, 0) \le t_0^{-1}v$$

and

$$\begin{aligned} A\bigl(0, t_0^{-1}v\bigr) &= A\bigl(t_0, t_0^{-1}v\bigr) \ge t_0(1 + r_1)A(0, v) \ge t_0(1 + r_1)cA(v, 0) \\ &= t_0(1 + r_1)cA\bigl(t_0t_0^{-1}v, t_0^{-1}0\bigr) \ge t_0(1 + r_1)ct_0(1 + r_2)A\bigl(t_0^{-1}v, 0\bigr) \\ &= c_1A\bigl(t_0^{-1}v, 0\bigr), \end{aligned}$$

where $c_1 = ct_0^2(1 + r_1)(1 + r_2) > 0$. Hence, conditions (a) and (b) of Theorem 2.3.3 are satisfied for $t_0^{-1}v$ instead of v. So, Theorem 2.3.3 implies that A has exactly one fixed point in $[0, t_0^{-1}v]$ and (2.114) holds for any initial $x_0, y_0 \in [0, t_0^{-1}v]$. Since, by (2.127), x^*, $\bar{x}$, x_0 and y_0 all belong to $[0, t_0^{-1}v]$, it must be $x^* = \bar{x}$ and (2.114) is true. The proof is complete. □

Remark 2.3.2 It should be pointed out that in Theorems 2.3.3 and 2.3.4 we do not assume operator A to be continuous or compact.

Convex (concave) operators are a class of important ones , which are extensively used in nonlinear differential and integral equations (see [8, 110]). We first study mixed monotone operators with convexity and concavity. Next, not assuming operators to be continuous or compact, we give existence and uniqueness theorems, then we study existence and uniqueness and continuity of eigenvectors, finally offering some applications to nonlinear integral equations on unbounded regions and differential equations in Banach spaces.

Theorem 2.3.5 (Zhitao Zhang [196]) *Let P be a normal cone of E, $A : P \times P \to P$ be a mixed monotone operator, suppose that*

(i) *for fixed y, $A(\cdot, y) : P \to P$ is concave; for fixed x, $A(x, \cdot) : P \to P$ is convex;*
(ii) *$\exists v > \theta,\ c > \frac{1}{2}$ such that $\theta < A(v, \theta) \le v$ and*

$$A(\theta, v) \ge cA(v, \theta). \tag{2.129}$$

Then A has exactly one fixed point $x^* \in [\theta, v]$, and constructing successively sequences

$$x_n = A(x_{n-1}, y_{n-1}), \quad y_n = A(y_{n-1}, x_{n-1}) \quad (n = 1, 2, \ldots), \tag{2.130}$$

for any initial $(x_0, y_0) \in [\theta, v] \times [\theta, v]$, we have

$$\|x_n - x^*\| \to 0, \quad \|y_n - x^*\| \to 0 \quad (n \to \infty),$$

with convergence rate

$$\|x_n - x^*\| \le N^2\left(\frac{1-c}{c}\right)^n \cdot \|v\|, \qquad \|y_n - x^*\| \le N^2\left(\frac{1-c}{c}\right)^n \cdot \|v\|. \tag{2.131}$$

Proof Let $u_0 = \theta$, $v_0 = v$, then

$$u_0 < v_0. \tag{2.132}$$

Let

$$u_n = A(u_{n-1}, v_{n-1}), \quad v_n = A(v_{n-1}, u_{n-1}) \quad (n = 1, 2, \ldots). \tag{2.133}$$

It is easy to show

$$\theta = u_0 < u_1 \le u_2 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_2 \le v_1 \le v_0 = v, \tag{2.134}$$

hence, by (2.129), (2.134), we get

$$u_n \ge u_1 \ge cv_1 \ge cv_n. \tag{2.135}$$

Let

$$t_n = \sup\{t > 0, u_n \ge tv_n\} \quad (n = 1, 2, \ldots), \tag{2.136}$$

then

$$u_n \geq t_n v_n, \tag{2.137}$$

and on account of (2.135) and the fact $u_{n+1} \geq u_n \geq t_n v_n \geq t_n v_{n+1}$, we have

$$0 < c \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq \cdots \leq 1, \tag{2.138}$$

which implies that $\lim_{n\to\infty} t_n = t^*$ exists and $0 < t^* \leq 1$, we check that

$$t^* = 1. \tag{2.139}$$

From the hypothesis (i), we get the following (2.140)–(2.142), $\forall x_1 \leq x_2,\ y_1 \leq y_2,\ t \in [0, 1]$:

$$A\big(tx_1 + (1-t)x_2, y\big) \geq tA(x_1, y) + (1-t)A(x_2, y), \tag{2.140}$$

$$A\big(x, ty_1 + (1-t)y_2\big) \leq tA(x, y_1) + (1-t)A(x, y_2), \tag{2.141}$$

$$A(x, y) = A\big(x, t\cdot t^{-1}y\big) \leq tA\big(x, t^{-1}y\big) + (1-t)A(x, \theta), \quad \forall t \in (0, 1]. \tag{2.142}$$

Then by (2.142), we get

$$A\big(x, t^{-1}y\big) \geq t^{-1}\big[A(x, y) - (1-t)A(x, \theta)\big], \quad \forall t \in (0, 1]. \tag{2.143}$$

By (2.133), (2.134), (2.137), (2.140)–(2.143), and the fact that A is a mixed monotone operator, we have

$$\begin{aligned}
u_{n+1} &= A(u_n, v_n) \geq A(t_n v_n, v_n) \geq t_n A(v_n, v_n) + (1-t_n)A(\theta, v_n) \\
&\geq t_n A\big(v_n, t_n^{-1}u_n\big) + (1-t_n)A(\theta, v) \\
&\geq t_n \cdot \big[t_n^{-1}\cdot A(v_n, u_n) - t_n^{-1}\cdot(1-t_n)A(v_n, \theta)\big] + (1-t_n)A(\theta, v) \\
&= A(v_n, u_n) + (1-t_n)\big[A(\theta, v) - A(v_n, \theta)\big] \\
&\geq v_{n+1} + (1-t_n)[u_1 - v_1] \\
&\geq v_{n+1} + (1-t_n)\left(1 - \frac{1}{c}\right)u_1 \\
&\geq v_{n+1} + (1-t_n)\left(1 - \frac{1}{c}\right)v_{n+1} \\
&= \left[1 + (1-t_n)\left(1 - \frac{1}{c}\right)\right]v_{n+1}
\end{aligned} \tag{2.144}$$

which implies that

$$t_{n+1} \geq 1 + (1-t_n)\left(1 - \frac{1}{c}\right), \tag{2.145}$$

therefore

$$1 - t_{n+1} \leq (1 - t_n)\left(\frac{1}{c} - 1\right). \tag{2.146}$$

By the hypothesis (ii), we get $\frac{1}{2} < c \leq 1$ and $\frac{1}{c} - 1 < 1$. Thus (2.146) implies that

$$1 - t_{n+1} \leq (1 - t_n)\left(\frac{1}{c} - 1\right) \leq \left(\frac{1}{c} - 1\right)^n (1 - t_1) \leq \left(\frac{1-c}{c}\right)^{n+1}. \tag{2.147}$$

Hence

$$t_n \to 1 \quad (n \to \infty). \tag{2.148}$$

Now from (2.134) and (2.148), we have

$$\theta \leq u_{n+p} - u_n \leq v_n - u_n \leq (1 - t_n)v_n \leq (1 - t_n)v. \tag{2.149}$$

Since P is normal, we get

$$\|u_{n+p} - u_n\| \leq N(1 - t_n) \cdot \|v\| \leq N \cdot \left(\frac{1-c}{c}\right)^n \cdot \|v\|, \tag{2.150}$$

$$\|v_n - u_n\| \leq N(1 - t_n) \cdot \|v\| \leq N \cdot \left(\frac{1-c}{c}\right)^n \cdot \|v\|, \tag{2.151}$$

where N is the normal constant of P. So by (2.150), we know that $\lim_{n\to\infty} u_n = u^*$ exists. In the same way we can prove that $\lim_{n\to} v_n = v^*$ exists. By (2.150), we get

$$\|u_n - u^*\| \leq N \cdot \left(\frac{1-c}{c}\right)^n \cdot \|v\|. \tag{2.152}$$

Similarly,

$$\|v_n - v^*\| \leq N \cdot \left(\frac{1-c}{c}\right)^n \cdot \|v\|. \tag{2.153}$$

Since $u_n \leq u^* \leq v^* \leq v_n$, we have

$$\theta \leq v^* - u^* \leq v_n - u_n \leq (1 - t_n)v. \tag{2.154}$$

It follows that $\|v^* - u^*\| \leq N \cdot (1 - t_n)\|v\| \to 0$ $(n \to \infty)$, hence $u^* = v^*$. Let $x^* = u^* = v^*$, then $x^* \in [\theta, v]$, $x^* > \theta$. Since

$$u_n \leq x^* \leq v_n, \qquad u_{n+1} = A(u_n, v_n) \leq A(x^*, x^*) \leq A(v_n, v_n) = v_{n+1}, \tag{2.155}$$

after taking the limit, we have

$$x^* \leq A(x^*, x^*) \leq x^*. \tag{2.156}$$

Hence $A(x^*, x^*) = x^*$, i.e., x^* is a fixed point of A. Let $\bar{x}$ be any fixed point of A in $[\theta, v]$, then

$$u_0 = \theta \le \bar{x} \le v = v_0, \qquad A(\bar{x}, \bar{x}) = \bar{x},$$

so

$$u_1 = A(u_0, v_0) \le A(\bar{x}, \bar{x}) = \bar{x} \le A(v_0, u_0) = v_1.$$

It is easy to see by induction that

$$u_n \le \bar{x} \le v_n \quad (n = 1, 2, \ldots), \tag{2.157}$$

which implies by taking limit that $\bar{x} = x^*$. Finally, $\forall x_0, y_0 \in [\theta, v]$, then, similar to (2.157), we get

$$u_n \le x_n \le v_n, \quad u_n \le y_n \le v_n \quad (n = 1, 2, \ldots). \tag{2.158}$$

Consequently, by (2.151) we get

$$\|x_n - x^*\| \le N\|v_n - u_n\| \le N^2 \cdot \left(\frac{1-c}{c}\right)^n \|v\|, \tag{2.159}$$

$$\|y_n - x^*\| \le N\|v_n - u_n\| \le N^2 \cdot \left(\frac{1-c}{c}\right)^n \|v\|, \tag{2.160}$$

therefore $x_n \to x^*,\ y_n \to x^*\ (n \to \infty)$. □

Theorem 2.3.6 (Zhitao Zhang [196]) *Let the cone P be normal and $A : [\theta, v] \times [\theta, v] \to [\theta, v]$ be a mixed monotone operator. Suppose that*

(i) *for fixed $y \in [\theta, v]$, $A(\cdot, y) : [\theta, v] \to [\theta, v]$ is convex; for fixed $x \in [\theta, v]$, $A(x, \cdot) : [\theta, v] \to [\theta, v]$ is concave;*
(ii) *there is a constant c, $\frac{1}{2} < c \le 1$ such that*

$$A(v, \theta) \le cA(\theta, v) + (1 - c)v. \tag{2.161}$$

Then A has exactly one fixed point $x^* \in [\theta, v]$. Moreover, constructing successively the sequences

$$x_n = v - A(v - x_{n-1}, v - y_{n-1}), \quad y_n = v - A(v - y_{n-1}, v - x_{n-1}) \quad (n = 1, 2, \ldots) \tag{2.162}$$

for any initial $x_0, y_0 \in [\theta, v]$, we have

$$v - x_n \to x^*, \quad v - y_n \to x^* \quad (n \to \infty). \tag{2.163}$$

Proof Let

$$B(x, y) = v - A(v - x, v - y), \quad \forall x, y \in [\theta, v - u] \tag{2.164}$$

then B is a mixed monotone operator. Moreover, for fixed $y \in [\theta, v]$, $B(\cdot, y) : [\theta, v] \to [\theta, v]$ is concave, and for fixed $x \in [\theta, v]$, $B(x, \cdot) : [\theta, v] \to [\theta, v]$ is convex. If $A(\theta, v) = v$, then $A(x, y) = v$, $\forall x, y \in [\theta, v]$, and $A(v, v) = v$, all the results are obvious. If $A(\theta, v) < v$, then $\theta < B(v, \theta) = v - A(\theta, v) \le v$. By (2.161), we know that

$$A(v, \theta) < cv + (1 - c)v = v, \tag{2.165}$$

and

$$v - A(v, \theta) \ge v - \big[cA(\theta, v) + (1 - c)v\big] = c\big(v - A(\theta, v)\big), \tag{2.166}$$

i.e.,

$$B(\theta, v) \ge cB(v, \theta). \tag{2.167}$$

So B satisfies all the conditions of Theorem 2.3.5 and B has exactly one fixed point $y^* > \theta$, i.e.

$$y^* = B\big(y^*, y^*\big) = v - A\big(v - y^*, v - y^*\big), \tag{2.168}$$

thus $A(v - y^*, v - y^*) = v - y^*$. Let $x^* = v - y^*$, then $\forall x_0, y_0 \in [\theta, v]$ by (2.162), (2.164) and Theorem 2.3.5, we get

$$x_n \to y^*, \quad y_n \to y^* \quad (n \to \infty). \tag{2.169}$$

Hence

$$v - x^n \to x^*, \quad v - y_n \to x^* \quad (n \to \infty). \tag{2.170}$$

□

Theorem 2.3.7 (Zhitao Zhang [196]) *Suppose all the conditions of Theorem 2.3.5 are satisfied, then* $\exists \lambda_0 \ge 1$, *such that* $\lambda_0 A(v, \theta) \le v$, *and* $\forall \lambda \in [0, \lambda_0]$, *the equation*

$$u = \lambda A(u, u) \tag{2.171}$$

has exactly one solution $u(\lambda)$. *Let* $u_0(\lambda) = \theta$, $v_0(\lambda) = v$, $u_n(\lambda) = \lambda A(u_{n-1}(\lambda), v_{n-1}(\lambda))$, $v_n(\lambda) = \lambda A(v_{n-1}(\lambda), u_{n-1}(\lambda))$, *we have*

$$\big\|u_n(\lambda) - u(\lambda)\big\| \le N \cdot \left(\frac{1-c}{c}\right)^n \|v\| \to 0 \quad (n \to \infty), \tag{2.172}$$

$$\big\|v_n(\lambda) - u(\lambda)\big\| \le N \cdot \left(\frac{1-c}{c}\right)^n \|v\| \to 0 \quad (n \to \infty). \tag{2.173}$$

Proof If $\lambda = 0$, then the conclusion is obvious and $u(0) = \theta$.

Suppose $\lambda \in (0, \lambda_0]$, where $\lambda_0 = \sup\{t > 0, tA(v, \theta) \le v)\}$. From $A(v, \theta) \le v$, we get $\lambda_0 \ge 1$, and from $\theta < \lambda A(v, \theta) \le \lambda_0 A(v, \theta) \le v$, $\lambda A(\theta, v) \ge c\lambda A(v, \theta)$, we see that λA satisfies all conditions of Theorem 2.3.5, so λA has exactly one fixed point $u(\lambda) \in [\theta, v]$, and $u(\lambda) > \theta$; obviously, other results are valid. □

Lemma 2.3.1 (See [88]) *Let E be an ordered Banach space with positive cone P such that $\mathring{P} \neq \emptyset$. Let X be an ordered Banach space with positive cone K, K is normal. Suppose $A : D(A) \subset E \to X$ is a concave or convex operator, $x_0 \in \mathring{D}(A)$, the interior of $D(A)$ in E. Then A is continuous at x_0 if and only if A is locally bounded at x_0, i.e., there is a $\delta > 0$ such that A is bounded on the δ-neighborhood $N_\delta(x_0)$ of x_0.*

Theorem 2.3.8 (Zhitao Zhang [196]) *Let P be a normal and solid cone of E, $A : P \times P \to P$ is a mixed monotone operator, $A(v, \theta) \gg \theta$, and the hypotheses* (i), (ii) *of Theorem 2.3.5 are satisfied. Then the equation*

$$\lambda A(u, u) = u, \quad \lambda \in [0, \lambda_0], \tag{2.174}$$

where $\lambda_0 = \sup\{t > 0, tA(v, \theta) \leq v\}$, has exactly one solution $u(\lambda)$ satisfying

(i) $u(\cdot) : [0, \lambda_0] \to [\theta, v]$ *is continuous*;
(ii) $\forall 0 < \lambda_1 < \lambda_2 \leq \lambda_0$, *we have*

$$u(\lambda_2) \geq \frac{\lambda_2}{\lambda_1} c \cdot u(\lambda_1), \tag{2.175}$$

$$u(\lambda_1) \geq \frac{\lambda_1}{\lambda_2} c \cdot u(\lambda_2). \tag{2.176}$$

Proof (i) Set $u_0(\lambda_1) = \theta, v_0(\lambda) = v$,

$$u_n(\lambda) = \lambda A\big(u_{n-1}(\lambda), v_{n-1}(\lambda)\big), \quad v_n(\lambda) = \lambda A\big(v_{n-1}(\lambda), u_{n-1}(\lambda)\big) \quad (n = 1, 2, \ldots) \tag{2.177}$$

by Theorem 2.3.7, we know that the convergences of $u_n(\lambda) \to u(\lambda)$, $v_n(\lambda) \to v(\lambda)$ $(n \to \infty)$ are both uniform for $\lambda \in [0, \lambda_0]$. Hence $u(\lambda)$ is continuous on $[0, \lambda_0]$ if each $u_n(\lambda)$ $v_n(\lambda)$ is.

In fact, $\forall x_0, y_0 \in \mathring{P} \cap [\theta, v]$, $x, y \in [\theta, v]$, then

$$\big\|A(x, y) - A(x_0, y_0)\big\| \leq \big\|A(x, y) - A(x_0, y)\big\| + \big\|A(x_0, y) - A(x_0, y_0)\big\|, \tag{2.178}$$

and for fixed y, $A(\cdot, y)$ is bounded on $[\theta, v]$, so $A(\cdot, y)$ is continuous at x_0, similarly $A(x_0, \cdot)$ is continuous at y_0. By Lemma 2.3.1 and (2.178), we get A is continuous at (x_0, y_0). since x_0, y_0 are arbitrary, A is continuous in $\mathring{P} \cap [\theta, v]$. Obviously,

$$\lim_{\lambda \to 0} u(\lambda) = \lim_{\lambda \to 0} \lambda A\big(u(\lambda), u(\lambda)\big) = \theta = u(0) \tag{2.179}$$

by $A(v, \theta) \gg \theta$ and (2.129), (2.177), we get $\forall \lambda \in (0, \lambda_0]$,

$$u_1(\lambda) = \lambda A(\theta, v) \gg \theta, \qquad v_1(\lambda) = \lambda A(v, \theta) \gg \theta \tag{2.180}$$

and $u_1(\lambda)$, $v_1(\lambda)$ are continuous, hence we can easily prove by induction that $u_n(\lambda)$, $v_n(\lambda)$ are continuous on $[0, \lambda_0]$, therefore, $u(\lambda)$ is continuous on $[0, \lambda_0]$.

(ii) Since $u(\lambda) \in [\theta, v]$, by (2.129) we know that

$$u(\lambda_1) = \lambda_1 A(u(\lambda_1), u(\lambda_1)) \geq \lambda_1 A(\theta, v) \geq \lambda_1 c \cdot A(v, \theta)$$
$$\geq \frac{\lambda_1}{\lambda_2} c \cdot \lambda_2 \cdot A(u(\lambda_2), u(\lambda_2)) = \frac{\lambda_1}{\lambda_2} c \cdot u(\lambda_2). \tag{2.181}$$

Similarly

$$u(\lambda_2) \geq \frac{\lambda_2}{\lambda_1} c \cdot u(\lambda_1). \tag{2.182}$$

□

Corollary 2.3.1 *Let P be a normal cone of E, $A : [u, v] \times [u, v] \to [u, v]$ is a mixed monotone operator, suppose that the hypothesis* (i) *of Theorem* 2.3.5 *is satisfied, and* $\exists c$ *such that* $\frac{1}{2} < c \leq 1$,

$$A(u, v) \geq cA(v, u) + (1 - c)u. \tag{2.183}$$

Then A has exactly one fixed point $\bar{x} \in [u, v]$.

Proof Let

$$B(x, y) = A(x + u, y + u) - u, \quad \forall x, y \in [\theta, v - u] \tag{2.184}$$

then $B : [\theta, v - u] \times [\theta, v - u]$ is a mixed monotone operator satisfying the hypothesis (i) of Theorem 2.3.5. Moreover,

$$B(v - u, \theta] = A(v, u) - u, \tag{2.185}$$
$$B(\theta, v - u) = A(u, v) - u. \tag{2.186}$$

By (5.56), (5.58) and (5.59), we get

$$B(v - u, \theta) \leq v,\, A(u, v) - u \geq cA(v, u) - cu, \tag{2.187}$$

i.e.,

$$B(\theta, v - u) \geq cB(v - u, \theta). \tag{2.188}$$

We may suppose that $B(v - u) \geq \theta$, (since if $B(v - u, \theta) = \theta$ then $A(v, u) = u$, therefore, $\forall x, y \in [u, v]$, $A(x, y) = u$, and u is the unique fixed point of A). Thus the hypothesis (ii) of Theorem 2.3.5 is satisfied, so B has exactly one fixed point $x^* \in [\theta, v - u]$, i.e.,

$$A\big(x^* + u, x^* + u\big) - u = x^*. \tag{2.189}$$

Obviously, A has the unique fixed point $\bar{x} = u + x^* \in [u, v]$. □

Corollary 2.3.2 *Let P be a normal cone of E, $A : [u, v] \times [u, v] \to [u, v]$ is a mixed monotone operator. Suppose that the hypothesis* (i) *of Theorem* 2 *is satisfied, and* $\exists c$ *such that*

$$\frac{1}{2} < c \leq 1, \qquad A(v, u) \leq cA(u, v) + (1 - c)v. \tag{2.190}$$

Then A has exactly one fixed point $x^* \in [u, v]$.

Proof Let $B(x, y) = A(x + u, y + u) - u$, $\forall x, y \in [\theta, v - u]$. Similarly to the proof of Corollary 2.3.1, we can verify that $B : [\theta, v - u] \times [\theta, v - u] \to [\theta, v - u]$ satisfies all the hypotheses of Theorem 2.3.6 and B has exactly one fixed point $x^* \in [\theta, v - u]$, i.e.,

$$x^* = B\big(x^*, x^*\big) = A\big(x^* + u, x^* + u\big) - u, \tag{2.191}$$

thus A has the unique fixed point $\bar{x} = x^* + u \in [u, v]$. □

Remark 2.3.3 It should be pointed out that we do not assume operator A to be continuous or compact in Theorems 2.3.5–2.3.8 too.

Definition 2.3.1 Operator $A : \mathring{P} \to \mathring{P}$. If there exists $0 \leq \alpha < 1$, such that

$$A(tx) \geq t^{\alpha} A(x), \quad \text{or} \quad A(tx) \leq t^{-\alpha} Ax, \quad \forall x \in \mathring{P},\ 0 < t < 1, \tag{2.192}$$

then A is called α concave or $-\alpha$ convex, respectively.

We usually assume $A(x, y)$ has the same type of convex-concave property, namely $A(x, y)$ is convex in x, concave in y (refer to [196]), or α concave in x, $-\alpha$ convex in y (refer to [101]). Here $A(x, y)$ has different convex-concave type, such as A is concave in x and $-\alpha$ convex in y, or α concave in x and convex in y.

Theorem 2.3.9 (Zhitao Zhang [197]) *Let the cone P be normal and solid and $A : \mathring{P} \times \mathring{P} \to \mathring{P}$ be a mixed monotone operator.*

(i) *for fixed y, $A(\cdot, y) : P \to \mathring{P}$ is concave; for fixed x, $A(x, \cdot) : \mathring{P} \to \mathring{P}$ is $-\alpha$ convex.*
(ii) $\exists u_0,\ v_0 \in P,\ \varepsilon > 0,\ \varepsilon \geq \alpha$ *such that such that*

$$0 \ll u_0 \leq v_0, \qquad u_0 \leq A(u_0, v_0), \qquad A(v_0, u_0) \leq v_0, \tag{2.193}$$

and

$$A(\theta, v_0) \geq \varepsilon A(v_0, u_0). \tag{2.194}$$

Then A has a unique fixed point x^* in $[u_0, v_0]$. For any $(x_0, y_0) \in [u_0, v_0] \times [u_0, v_0]$, constructing successively the sequences

$$x_n = A(x_{n-1}, y_{n-1}), \quad y_n = A(y_{n-1}, x_{n-1}) \quad (n = 1, 2, 3, \ldots) \tag{2.195}$$

we have

$$\|x_n - x^*\| \to 0, \quad \|y_n - x^*\| \to 0 \quad (n \to \infty). \tag{2.196}$$

Proof Define

$$u_n = A(u_{n-1}, v_{n-1}), \quad v_n = A(v_{n-1}, u_{n-1}), \quad n = 1, 2, \ldots. \tag{2.197}$$

It is easy to show

$$0 \ll u_0 \le u_1 \le \cdots \le u_{n-1} \le u_n \le \cdots \le v_n \le v_{n-1} \le \cdots \le v_1 \le v_0 \le v. \tag{2.198}$$

By (2.194),

$$u_n \ge u_1 \ge \varepsilon v_1 \ge \varepsilon v_n. \tag{2.199}$$

Let

$$t_n = \sup\{t > 0 | u_n \ge t v_n\}, \quad n = 1, 2, \ldots \tag{2.200}$$

then $u_n \ge t_n v_n$.

From $u_{n+1} \ge u_n \ge t_n v_n \ge t_n v_{n+1}$, we have

$$0 < \varepsilon \le t_1 \le t_2 \le \cdots \le t_n \le \cdots \le 1 \tag{2.201}$$

which implies $\lim_{n\to\infty} t_n = t^*$ exists and $\varepsilon \le t^* \le 1$. We check that $t^* = 1$.

From (i) we know

$$A\big(x, t^{-1}y\big) \ge t^\alpha A(x, y), \quad \forall x \in P,\ y \in \mathring{P},\ 0 < t < 1. \tag{2.202}$$

Therefore, through (2.194), we have

$$\begin{aligned} u_{n+1} = A(u_n, v_n) &\ge A(t_n v_n, v_n) \ge t_n A(v_n, v_n) + (1 - t_n)A(\theta, v_n) \\ &\ge t_n A\big(v_n, t^{-1}u_n\big) + (1 - t_n)A(\theta, v_n) \\ &\ge t_n t_n^\alpha A(v_n, u_n) + (1 - t_n)A(\theta, v_0) \\ &\ge \big(t_n^{1+\alpha} + (1 - t_n)\varepsilon\big)v_{n+1}. \end{aligned} \tag{2.203}$$

So

$$t_{n+1} \ge t_n^{1+\alpha} + (1 - t_n)\varepsilon, \tag{2.204}$$

let $n \to \infty$, then t^* satisfies

$$t^* \ge \big(t^*\big)^{1+\alpha} + \big(1 - t^*\big)\varepsilon. \tag{2.205}$$

Observing

$$f(t) = t^{1+\alpha} - (1 + \varepsilon)t + \varepsilon, \quad t \in [0, \infty) \tag{2.206}$$

it is easy to know

$$f(0) = \varepsilon, \qquad f(1) = 0, \tag{2.207}$$

$$f'(t) = (1+\alpha)t^{\alpha} - (1+\varepsilon). \tag{2.208}$$

We divide the problem into two cases:

(i) $\alpha = 0$. $\forall t \in (0,1]$, $f'(t) = -\varepsilon < 0$. So $f(t)$ is strictly decreasing in $[\varepsilon, 1]$, from (2.207),

$$f(t) > 0, \quad \forall t \in [\varepsilon, 1).$$

(ii) $0 < \alpha < 1$. $f''(t) = (1+\alpha)\alpha t^{\alpha-1} > 0$, $\forall t \in (0, \infty)$, also

$$f'(0) = -(1+\varepsilon) < 0, \qquad f'\left(\left(\frac{1+\varepsilon}{1+\alpha}\right)^{\frac{1}{\alpha}}\right) = 0.$$

Thus $f'(t) < 0$, $\forall t \in [0, (\frac{1+\varepsilon}{1+\alpha})^{\frac{1}{\alpha}})$.

Since $\varepsilon \geq \alpha$, $(\frac{1+\varepsilon}{1+\alpha})^{\frac{1}{\alpha}} \geq 1$, we get $f'(t) < 0$, $\forall t \in [0,1)$, which means $f(t)$ is strictly decreasing in $[0,1)$, by (2.207), $f(t) > 0$, $\forall t \in [\varepsilon, 1)$.

From the above discussion (i) and (ii), if $t^*[\varepsilon, 1)$, then $f(t^*) > 0$, which is

$$\left(t^*\right)^{1+\alpha} + \left(1 - t^*\right)\varepsilon > t^*. \tag{2.209}$$

This contradicts (2.205). So $t^* = 1$. The rest of proof is routine. Readers can refer to Theorem 2.3.5 or [196]. □

Theorem 2.3.10 (Zhitao Zhang [197]) *Let the cone P be normal and solid and $A : \mathring{P} \times \mathring{P} \to \mathring{P}$ be a mixed monotone operator. Assume that*

(i) *for fixed y, $A(\cdot, y) : \mathring{P} \to \mathring{P}$ is α concave; for fixed x, $A(x, \cdot) : P \to \mathring{P}$ is convex;*
(ii) *$\exists u_0, v_0 \in P$, $\varepsilon \geq \frac{1}{2-\alpha}$ such that*

$$0 \ll u_0 \leq v_0, \qquad u_0 \leq A(u_0, v_0), \qquad A(v_0, u_0) \leq v_0;$$

and

$$A(u_0, v_0) \geq \varepsilon A(v_0, \theta). \tag{2.210}$$

Then A has a unique fixed point x^* in $[u_0, v_0]$. For any $(x_0, y_0) \in [u_0, v_0] \times [u_0, v_0]$, (2.195) and (2.196) still hold.

Proof From the proof of Theorem 2.3.9, we only need to prove $t^* = 1$ when $\frac{1}{2-a} \leq \varepsilon < 1$ (the notations follow that of Theorem 2.3.9).

By condition (i), for fixed x, we have

$$A\left(x, t \cdot t^{-1} y\right) \leq t A\left(x, t^{-1} y\right) + (1-t) A(x, \theta), \quad \forall t \in [0,1].$$

Therefore,

$$A\big(x, t^{-1}y\big) \geq t^{-1}\big[A(x, y) - (1-t)A(x, \theta)\big], \tag{2.211}$$

consequently,

$$\begin{aligned} u_{n+1} &= A(u_n, v_n) \geq A(t_n v_n, v_n) \geq t_n^{\alpha} A\big(v_n, t_n^{-1} u_n\big) \\ &\geq t_n^{\alpha} t_n^{-1}\big[A(v_n, u_n) - (1-t_n)A(v_n, \theta)\big] \\ &\geq t_n^{\alpha-1}\big[v_{n+1} - (1-t_n)A(v_0, \theta)\big] \\ &\geq t_n^{\alpha-1}\left[v_{n+1} - (1-t_n)\frac{1}{\varepsilon}u_1\right] \\ &\geq t_n^{\alpha-1}\left[v_{n+1} - (1-t_n)\frac{1}{\varepsilon}u_{n+1}\right], \end{aligned}$$

equivalently,

$$u_{n+1} \geq t_n^{\alpha-1}\left[1 + (1-t_n)t_n^{\alpha-1} \cdot \frac{1}{\varepsilon}\right]^{-1} v_{n+1}. \tag{2.212}$$

So

$$t_{n+1} \geq t_n^{\alpha-1}\left[1 + (1-t_n)t_n^{\alpha-1} \cdot \frac{1}{\varepsilon}\right]^{-1}. \tag{2.213}$$

Let $n \to \infty$, then

$$t^* \geq \big(t_n^*\big)^{\alpha-1}\left[1 + \big(1-t^*\big)\big(t_n^*\big)^{\alpha-1} \cdot \frac{1}{\varepsilon}\right]^{-1}. \tag{2.214}$$

Consider the function

$$g(t) = t^{2-\alpha} + t \cdot \frac{1-t}{\varepsilon} - 1, \quad t \in [\varepsilon, 1]. \tag{2.215}$$

Then it is easy to know

$$g(1) = 0, \qquad g'(t) = (2-\alpha)t^{1-\alpha} + \frac{1-t}{\varepsilon} - \frac{t}{\varepsilon}, \tag{2.216}$$

$$g''(t) = (2-\alpha)(1-\alpha)t^{-\alpha} - \frac{2}{\varepsilon}. \tag{2.217}$$

Since $\varepsilon \geq \frac{1}{2-\alpha}$, $g'(1) = 2 - \alpha - \frac{1}{\varepsilon} \geq 0$. At the same time, from $\varepsilon < 1$, $0 \leq \alpha < 1$,

$$\varepsilon^{1-\alpha} < \frac{2}{(2-\alpha)(1-\alpha)}, \tag{2.218}$$

which means $\varepsilon < \frac{2\varepsilon^{\alpha}}{(2-\alpha)(1-\alpha)}$. If $t \in [\varepsilon, 1]$, $\varepsilon < \frac{2t^{\alpha}}{(2-\alpha)(1-\alpha)}$, that is

$$(2-\alpha)(1-\alpha) \cdot t^{-\alpha} < \frac{2}{\varepsilon}. \tag{2.219}$$

From (2.217), we get $g''(t) < 0$, $\forall t \in [\varepsilon, 1]$. Also by $g'(1) \geq 0$, we get $g'(t) > 0$, $\forall t \in [\varepsilon, 1]$. Therefore $g(1) = 0$ leads to

$$g(t) < 0, \quad t \in [\varepsilon, 1). \tag{2.220}$$

This means $t^{2-\alpha} + t \cdot \frac{1-t}{\varepsilon} < 1$, $t \in [\varepsilon, 1)$ in (2.215). This means

$$t^2 + t^{1+\alpha}(1-t)\frac{1}{\varepsilon} < t^{\alpha}, \qquad t + t^{\alpha}(1-t)\frac{1}{\varepsilon} < t^{\alpha-1},$$

$$t < t^{\alpha-1}\left[1 + t^{\alpha-1}(1-t)\frac{1}{\varepsilon}\right]^{-1}, \quad \forall t \in [\varepsilon, 1). \tag{2.221}$$

If $t^* \in [\varepsilon, 1)$, then t^* satisfies (2.221), which contradicts (2.214). Thus $t^* = 1$. The rest of the proof is routine. □

2.4 Applications of Mixed Monotone Operators

We give applications of Theorem 2.3.1 and 2.3.2 to the following initial value problem:

$$\begin{cases} x' = \displaystyle\sum_{i=1}^{n} a_i(t)x^{r_i} + \left(\sum_{j=1}^{m} b_j(t)x^{s_j}\right)^{-1}, & \text{a.e. on } J \\ x(0) = x_0, \end{cases} \tag{2.222}$$

where $J = [0, T]$ $(T > 0)$, $0 < r_i < 1$, $0 < s_j < 1$ $(i = 0, 1, \ldots, n;\ j = 0, 1, \ldots, m)$, $x_0 > 0$, $a_i(t)$ are nonnegative bounded measurable functions (on J) and $b_j(t)$ are nonnegative measurable functions such that

$$\inf_{t \in J} \sum_{j=1}^{m} b_j(t) > 0.$$

The set of all absolutely continuous functions from J into $\mathbb{R}$ is denoted by $AC[J, \mathbb{R}]$. A function $x(t)$ on J is said to be a solution of the initial value problem (2.222) if $x(t) \in AC[J, \mathbb{R}]$ and satisfies (2.222).

Theorem 2.4.1 (Dajun Guo [101]) *Under conditions mentioned above, initial value problem* (2.222) *has exactly one positive solution* $x^*(t)$. *Moreover, constructing a* *successive sequence of functions*

$$x_n(t) = x_0 + \int_0^T \left\{ \sum_{i=1}^{n} a_i(s)\big(x_{n-1}(s)\big)^{r_i} \right\} ds + \int_0^T \left\{ \sum_{j=1}^{m} b_j(s)\big(x_{n-1}(s)\big)^{s_j} \right\}^{-1} ds, \tag{2.223}$$

$(n = 1, 2, \ldots)$ *for any initial positive function* $x_0(t) \in AC[J, \mathbb{R}]$, *the sequence of functions* $\{x_n(t)\}$ *converges to* $x^*(t)$ *uniformly on* J.

Proof It is clear, $x(t) \in AC[J, \mathbb{R}]$ is a positive solution of (2.222) if and only if $x(t) \in C[J, \mathbb{R}]$ is a positive solution of the following integral equation:

$$x(t) = x_0 + \int_0^T \left\{ \sum_{i=1}^{n} a_i(s)\big(x(s)\big)^{r_i} \right\} ds + \int_0^T \left\{ \sum_{j=1}^{m} b_j(s)\big(x(s)\big)^{s_j} \right\}^{-1} ds, \quad t \in J. \tag{2.224}$$

Let $E = C[J, \mathbb{R}]$ and $P = \{x \in C[J, \mathbb{R}] : x(t) \ge 0 \text{ for } t \in J\}$, then P is a normal and solid cone in E and (2.224) can be written in the form

$$x = A(x, x), \tag{2.225}$$

where

$$\begin{aligned} A(x, y) &= A_1(x) + A_2(y), \\ A_1(x) &= x_0 + \int_0^T \left\{ \sum_{i=1}^{n} a_i(s)\big(x(s)\big)^{r_i} \right\} ds, \\ A_2(y) &= \int_0^T \left\{ \sum_{j=1}^{m} b_j(s)\big(x(s)\big)^{s_j} \right\}^{-1} ds. \end{aligned}$$

It is clear that $A_1 : \mathring{P} \to \mathring{P}$ is non-decreasing and $A_2 : \mathring{P} \to \mathring{P}$ is non-increasing, so $A : \mathring{P} \times \mathring{P} \to \mathring{P}$ is a mixed monotone operator. We now check that A satisfies all of the conditions of Theorem 2.3.1. For $x, y \in \mathring{P}$ and $0 < t < 1$, it is easy to see

$$\begin{aligned} A_1(tx) &\ge t^{r_0} A_1(x), \\ A_2\big(t^{-1} y\big) &\ge t^{s_0} A_2(y), \end{aligned}$$

where $r_0 = \max\{r_1, \ldots, r_n\}$, $s_0 = \max\{s_1, \ldots, s_m\}$, $0 < r_0 < 1$, $0 < s_0 < 1$. Therefore

$$A\big(tx, t^{-1} y\big) \ge t^r A(x, y), \quad x, y \in \mathring{P},\ 0 < t < 1,$$

where $r = \max\{r_0, s_0\},\ 0 < r < 1$. Hence, by Theorem 2.3.1, we conclude that A has exactly one fixed point x^* in $\mathring{P}$ and, for any initial $x_0 \in \mathring{P}$,

$$\|x_n - x^*\| = \max_{t\in J}|x_n(t) - x^*(t)| \to 0 \quad (n \to \infty),$$

where

$$x_n = A(x_{n-1}, x_{n-1}) \quad (n = 1, 2, \ldots).$$

The proof is complete. □

Using Theorem 2.3.2, we get similarly the following.

Theorem 2.4.2 (Dajun Guo [101]) *Let the hypotheses of Theorem 2.4.1 be satisfied. Denote by $x^*(t)$ the unique positive solution of the initial value problem*

$$\begin{cases} rx' = \sum_{i=1}^{n} a_i(t)x^{r_i} + \left(\sum_{j=1}^{m} b_j(t)x^{s_j}\right)^{-1}, & a.e.\ on J \\ rx(0) = x_0. \end{cases} \tag{2.226}$$

Then $x_r^*(t)$ converges to $x_{r_0(t)}^*$ uniformly on $t \in J$ as $r \to r_0$ $(r_0 > 0)$. If, in addition, $0 < r_i < \frac{1}{2},\ 0 < s_j < \frac{1}{2}$ $(i = 0, 1, \ldots, n;\ j = 0, 1, \ldots, m)$, then

$$0 < r < r' \implies x_r^*(t) > x_{r'}^*(t), \quad t \in J,$$

and

$$\max_{t\in J} x_r^*(t) \to 0 \quad as\ r \to +\infty, \qquad \max_{t\in J} x_r^*(t) \to +\infty \quad as\ r \to +0.$$

We apply Theorem 2.3.4 to the IVP

$$\begin{cases} x' = \sum_{i=0}^{n} a_i(t)x^{r_i} + \left(\sum_{j=0}^{m} b_j(t)x^{s_j}\right)^{-1}, & t \in [0, T], \\ x(0) = x_0, \end{cases} \tag{2.227}$$

and the two-point BVP

$$\begin{cases} -x'' = \sum_{i=0}^{n} a_i(t)x^{r_i} + \left(\sum_{j=0}^{m} b_j(t)x^{s_j}\right)^{-1}, & t \in [0, 1], \\ ax(0) - bx'(0) = x_0, \qquad cx(1) + dx'(1) = x_1, \end{cases} \tag{2.228}$$

where

$$0 = r_0 < r_1 < \cdots < r_{n-1} < r_n = 1, \qquad 0 = s_0 < s_1 < \cdots < s_{m-1} < s_n = 1,$$

$$T > 0, \quad x_0 \geq 0, \quad x_1 \geq 0, \quad a \geq 0, \quad b \geq 0, \quad c \geq 0, \quad d \geq 0,$$
$$Q = ac + ad + bc > 0$$

and $a_i(t), b_j(t)$ $(i = 0, 1, \ldots, n;\ j = 0, 1, \ldots, m)$ are nonnegative continuous functions on $J = [0, T]$ (for problem (2.227)) or $I = [0, 1]$ (for problem (2.228)).

Theorem 2.4.3 (Dajun Guo [102]) *Suppose that $a_0(t) > 0,\ b_0(t) > 0$ for $t \in J$ and*

$$\int_0^T a_n(t)\,dt < 1. \tag{2.229}$$

Then IVP (2.227) has exactly one nonnegative nontrivial C^1 solution $x^*(t)$ on J. Moreover, constructing a successive sequence of functions

$$x_n(t) = x_0 + \int_0^T \left\{\sum_{i=0}^n a_i(s)\big(x_{n-1}(s)\big)^{r_i}\right\} ds + \int_0^T \left\{\sum_{j=0}^m b_j(s)\big(x_{n-1}(s)\big)^{s_j}\right\}^{-1} ds$$
$$(n = 1, 2, \ldots)$$

for any initial nonnegative function $x_0(t) \in C(J, \mathbb{R})$, the sequence $\{x_n(t)\}$ converges to $x^*(t)$ uniformly on J.

Proof It is clear that $x(t) \in C^1[J, \mathbb{R}]$ is a nonnegative solution of (2.227) if and only if $x(t) \in C[J, \mathbb{R}]$ is a nonnegative solution of the following integral equation:

$$x(t) = x_0 + \int_0^T \left\{\sum_{i=0}^n a_i(s)\big(x(s)\big)^{r_i}\right\} ds + \int_0^T \left\{\sum_{j=0}^m b_j(s)\big(x(s)\big)^{s_j}\right\}^{-1} ds,$$
$$t \in J. \tag{2.230}$$

Let $E = C[J, \mathbb{R}]$ and $P = \{x \in C[J, \mathbb{R}] : x(t) \geq 0 \text{ for } t \in J\}$, then P is a normal and solid cone in E and (2.230) can be written in the form

$$x = A(x, x), \tag{2.231}$$

where

$$A(x, y) = A_1(x) + A_2(y),$$
$$A_1(x) = x_0 + \int_0^T \left\{\sum_{i=0}^n a_i(s)\big(x(s)\big)^{r_i}\right\} ds,$$
$$A_2(y) = \int_0^T \left\{\sum_{j=0}^m b_j(s)\big(x(s)\big)^{s_j}\right\}^{-1} ds.$$

Evidently, $A_1 : P \to P$ is non-decreasing and $A_2 : P \to P$ is non-increasing, so $A : P \times P \to P$ is a mixed monotone operator. We now check that A satisfies all of the conditions of Theorem 2.3.4.

By virtue of (2.229), we can choose a constant $R > 0$ sufficiently large such that

$$x_0 + \sum_{i=0}^{n} R^{r_i} \int_0^T a_i(s)\,ds + \int_0^T b_0(s)^{-1}\,ds < R. \tag{2.232}$$

Hence, putting $v(t) \equiv R$ $(t \in J)$, we have $v \in \mathring{P}$ and $0 < A(v, 0) \le v$. On the other hand, since, by hypothesis,

$$\min_{t\in J} a_0(t) = \bar{a}_0 > 0, \qquad \min_{t\in J} b_0(t) = \bar{b}_0 > 0,$$
$$\max_{t\in J} a_i(t) = a_i^* \ge 0, \qquad \max_{t\in J} b_j(t) = b_j^* \ge 0,$$
$$(i = 0, 1, 2, \ldots, n;\ \ j = 0, 1, 2, \ldots, m),$$

we can choose $0 < c_0 < 1$ sufficiently small such that

$$c_0\left(\frac{1}{\bar{b}_0} + \sum_{i=0}^{n} a_i^* R^{r_i}\right) < \bar{a}_0. \tag{2.233}$$

Let

$$F(t) = x_0 + \int_0^t a_0(s)\,ds + \int_0^t \left(\sum_{j=0}^{m} b_j(s) R^{s_j}\right)^{-1} ds$$
$$- c_0\left(x_0 + \int_0^t a_i(s) R^{r_i}\,ds + \int_0^t \big(b_0(s)\big)^{-1}\,ds\right), \quad t \in J.$$

Then, (2.233) implies

$$F(t) \ge (1 - c_0)x_0 + \int_0^t \left\{\bar{a}_0 - c_0\left(\frac{1}{\bar{b}_0} + \sum_{i=0}^{n} a_i^* R^{r_i}\right)\right\} ds \ge 0, \quad t \in J$$

i.e., $A(0, v) \ge c_0 A(v, 0)$. Thus, condition (a′) of Theorem 2.3.4 is satisfied.

Let $0 < a < b < 1$ and a bounded $B \subset P$ be given. So, $\|x\| \le R_1$ for all $x \in B$, where $R_1 > 0$ is a constant. We now define

$$r = \min\left\{\bar{a}_0(1-b)\left(\bar{a}_0 b + b\sum_{i=1} n a_i^* R_1^{r_i}\right)^{-1}, \bar{b}_0(1-b)\left(\bar{b}_0 b + \sum_{j=1}^{m} b_j^* R_1^{s_j}\right)^{-1}\right\} > 0, \tag{2.234}$$

and check that it satisfies (2.126). For any $t_1 \in [a, b],\ x, y \in B$, we have

$$A_1(t_1 x) = x_0 + \int_0^t \left\{\sum_{i=0}^{n} a_i(s) t_1^{r_i} \big(x(s)\big)^{r_i}\right\} ds$$

$$\geq t_1\left(x_0 b^{-1} + \int_0^t \left\{a_0(s)b^{-1} + \sum_{i=1}^n a_i(s)\big(x(s)\big)^{r_i}\right\} ds\right).$$

To show that $A_1(t_1 x) \geq t_1(1 + rA_1(x))$, it is sufficient to show that

$$a_0(s)b^{-1} + \sum_{i=1}^n a_i(s)\big(x(s)\big)^{r_i} \geq (1+r)\left\{a_0(s) + \sum_{i=1}^n a_i(s)\big(x(s)\big)^{r_i}\right\}, \quad s \in J$$

since $b^{-1} \geq 1 + r$ by (2.234). This inequality can be written in the form

$$a_0(s)\big(b^{-1} - 1 - r\big) \geq r\sum_{i=1}^n a_i(s)\big(x(s)\big)^{r_i},$$

which will certainly be true if

$$\bar{a}_0\big(1 - b(1+r)\big) \geq br\sum_{i=1}^n a_i^* R_1^{r_i}.$$

This last inequality follows from (2.234). Similarly, we have

$$\begin{aligned}
A_2\big(t_1^{-1}y\big) &\geq t_1\int_0^t \left\{bb_0(s) + \sum_{j=1}^m b_j(s)\big(y(s)\big)^{s_j}\right\}^{-1} ds \\
&\geq t_1(1+r)\int_0^t \left\{b_0(s) + \sum_{j=1}^m b_j(s)\big(y(s)\big)^{s_j}\right\}^{-1} ds \\
&= t_1(1+r)A_2(y).
\end{aligned}$$

Hence, $A(t_1 x, t_1^{-1} y) \geq t_1(1+r)A(x, y)$, and (2.126) is true.

Finally, our conclusions follow from Theorem 2.3.4 and the proof is complete. □

Theorem 2.4.4 (Dajun Guo [102]) *Suppose that $a_0(t) > 0,\ b_0(t) > 0$ for $t \in I$ and*

$$\int_0^1 a_n(s)\, ds < \frac{Q}{(a+b)(c+d)}.$$

Then BVP (2.228) has exactly one nonnegative nontrivial C^2 solution $x^*(t)$ on I. Moreover, constructing successively the sequence of functions

$$x_k(t) = z_0(t) + \int_0^1 G(t,s)\left(\sum_{i=0}^n a_i(s)\big(x_{k-1}(s)\big)^{r_i}\left\{\sum_{j=0}^m b_j(s)\big(x_{k-1}(s)\big)^{s_j}\right\}^{-1}\right) ds$$

$(k = 1, 2, \ldots)$

for any initial nonnegative function $x_0(t) \in C[I, \mathbb{R}]$, *the sequence* $\{x_k(t)\}$ *converges to* $x^*(t)$ *uniformly on* I, *where*

$$G(t,s) = \begin{cases} Q^{-1}(at+b)(c(1-s)+d), & t \le s, \\ Q^{-1}(as+b)(c(1-t)+d), & t > s \end{cases}$$

and

$$z_0(t) = Q^{-1}\big\{\big(c(1-t)+d\big)x_0 + (at+b)x_1\big\}.$$

Proof It is well known that $x(t) \in C^2(I, \mathbb{R})$ is a nonnegative solution of (2.228) if and only if $x(t) \in C(I, \mathbb{R})$ is a nonnegative solution of the following integral equation:

$$x(t) = z_0(t) + \int_0^1 G(t,s)\left(\sum_{i=0}^{n} a_i(s)\big(x(s)\big)^{r_i} + \left\{\sum_{j=1}^{m} b_j(s)\big(x(s)\big)^{s_j}\right\}^{-1}\right) ds.$$

This integral equation can be regarded as an equation of the form (2.231), where $A(x,y) = A_1(x) + A_2(y)$ and

$$A_1(x) = z_0(t) + \int_0^1 G(t,s)\left\{\sum_{i=0}^{n} a_i(s)\big(x(s)\big)^{r_i}\right\} ds,$$

$$A_2(y) = \int_0^1 G(t,s)\left\{\sum_{j=0}^{m} b_j(s)\big(x(s)\big)^{s_j}\right\}^{-1} ds.$$

Let $E = C(I, \mathbb{R})$ and $P = \{x \in C(I, \mathbb{R}) : x(t) \ge 0 \text{ for } t \in I\}$. Then $A : P \times P \to P$ is a mixed monotone operator. In the same way as in the proof of Theorem 2.4.3, we can show the operator A satisfies condition (a′) and (b′) of Theorem 2.3.4. Hence our conclusion follow from Theorem 2.3.4. □

Next we use some of our results to several existence theorems for nonlinear integral equations on unbounded region and nonlinear differential equations in Banach spaces.

We first consider the following nonlinear integral equation on $\mathbb{R}^N$:

$$x(t) = Ax(t) = \int_{\mathbb{R}^N} K(t,s)\left[\frac{x^2(s) + x(s)}{2} + \sqrt{1 - x^2(s)}\right] ds. \tag{2.235}$$

Proposition 2.4.1 (Zhitao Zhang [196]) *Suppose that* $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ *is continuous and* $K(t,s) \ge 0,\ K \not\equiv 0$, *moreover,* $\exists a > 0$ *such that*

$$\int_{\mathbb{R}^N} K(t,s)\, ds \le a < \frac{1}{4}. \tag{2.236}$$

Then (2.235) *has unique one solution* $x^*(t)$ *satisfying* $0 \le x^*(t) < 1$, *and* $x^*(t) \not\equiv 0$.

Proof We use Theorem 2.3.6 to prove it. Let $C_B(\mathbb{R}^N)$ denote the set of all bounded continuous functions, we define $\|x\| = \sup_{t\in\mathbb{R}^N} |x(t)|$, then $C_B(\mathbb{R}^N)$ is a real Banach space. Let $P = C_B^+(\mathbb{R}^N)$ denote the set of nonnegative functions of $C_B(\mathbb{R}^N)$, then P is a normal and solid cone of $C_B(\mathbb{R}^N)$. (2.235) can be written in the form

$$x = A(x, x); \tag{2.237}$$

where

$$A(x, y) = A_1(x) + A_2(y); \tag{2.238}$$

$$A_1(x) = \int_{\mathbb{R}^N} K(t, s) \cdot \frac{x^2(s) + x(s)}{2}\, ds; \tag{2.239}$$

$$A_2(y) = \int_{\mathbb{R}^N} K(t, s) \cdot \sqrt{1 - y^2(s)}\, ds. \tag{2.240}$$

Let $v \equiv 1$, then

$$A(v, 0) = A_1(v) + A_2(0) = 2\int_{\mathbb{R}^N} K(t, s)\, ds \le 2a < \frac{1}{2}, \tag{2.241}$$

and

$$A(x, y) \le A(v, 0), \quad \forall x, y \in [0, v]. \tag{2.242}$$

So $A : [0, v] \times [0, v] \to [0, v]$ is a mixed monotone operator, and $A(\cdot, y)$ is a convex operator for fixed y, $A(x, \cdot)$ is a concave operator for fixed x, thus the hypothesis (i) of Theorem 2.3.6 is satisfied. Moreover,

$$A(0, v) = A_1(0) + A_2(1) = 0. \tag{2.243}$$

Let $c = 1 - 2a > \frac{1}{2}$, then

$$A(v, 0) \le 2a \le cA(0, v) + (1 - c)v, \tag{2.244}$$

thus the hypothesis (ii) of Theorem 2.3.6 is satisfied. So A has exactly one fixed point $x^* \in [0, 1]$, moreover, $\forall x_0, y_0 \in [0, v]$, $x_n = 1 - A(1 - x_{n-1}, 1 - y_{n-1})$, $y_n = 1 - A(1 - y_{n-1}, 1 - x_{n-1})$, we have

$$1 - x_n \to x^*, \quad 1 - y_n \to x^* \quad (n \to \infty). \qquad \square$$

Now we consider the system of equations:

$$-x_n'' = \frac{1}{n} \cdot \left[\frac{1}{3} x_{2n}^{\frac{1}{2}} + (1 + x_{n+2})^{-\frac{1}{2}}\right]; \tag{2.245}$$

$$x_n(0) = x_n'(1) = 0, \quad n = 1, 2, \ldots. \tag{2.246}$$

Let $E = \{x | x = (x_1, x_2, \ldots), \sup_i |x_i| < \infty\}$, $\|x\| = \sup_i |x_i|$. $P = \{x \mid x \in E, x_i \ge 0\} \subset E$ is a normal and solid cone. Let $I = [0, 1]$, $C[I, E] = \{x \mid x(\cdot) : I \to$

E is continuous}, $\|x\| = \max_{t\in I} \|x(t)\|$. $\bar{P} = \{x \in C[I, E] | x(t) \in P,\ \forall t \in I\} \subset C[I, E]$ is a normal and solid cone. Then (2.245)–(2.246) is equivalent to the two-point Boundary Value Problem in E. □

Proposition 2.4.2 (Zhitao Zhang [196]) *The system* (2.245)–(2.246) *has a unique positive solution* $x^*(t) \in [0, v]$, *where* $v = (1, 1, \dots, 1, \dots)$. *Moreover,* $\forall x_0, y_0 \in [0, v]$, $x_n = A(x_{n-1}, y_{n-1}), y_n = A(y_{n-1}, x_{n-1})$, *we have* $x_n \to x^*$, $y_n \to x^*$ $(n \to \infty)$,

$$\|x_n - x^*\| \le \left(\frac{4\sqrt{2}}{3} - 1\right)^n, \qquad \|y_n - x^*\| \le \left(\frac{4\sqrt{2}}{3} - 1\right)^n, \tag{2.247}$$

where

$$A(x, y) = A_1(x) + A_2(y) \tag{2.248}$$

(*in the following* $i = 1, 2, \dots$)

$$\begin{aligned}
\big(A_1(x)\big)_i &= \frac{1}{3i}\int_0^1 G(t,s)\cdot x_{2i}^{\frac{1}{2}}(s)\,ds;\\
\big(A_2(y)\big)_i &= \frac{1}{i}\int_0^1 G(t,s)\cdot \big(1 + x_{i+2}(s)\big)^{-\frac{1}{2}}\,ds;\\
G(t,s) &= \begin{cases} t, & 0 \le t \le s \le 1,\\ s, & 1 \ge t > s \ge 0.\end{cases}
\end{aligned} \tag{2.249}$$

Proof It is easy to know $x(t) \in C^2[I, E] \cap \bar{P}$ is a solution of (2.245)–(2.246) iff $x \in \bar{P}$ is a solution of $A(x, x) = x$. We shall prove A has exactly one fixed point in $[0, v]$. Obviously, $A : \bar{P} \times \bar{P} \to \bar{P}$ is a mixed monotone operator, and $A(\cdot, y)$ is a concave operator for fixed y, $A(x, \cdot)$ is a convex one for fixed x.

$$\begin{aligned}
\big(A(v,0)\big)_i &= \big(A_1(v)\big)_i + \big(A_2(0)\big)_i = \frac{4}{3i}\int_0^1 G(t,s)\,ds = \frac{4}{3i}\big(t - t^2\big)\\
&\le \frac{4}{3i}\cdot\frac{1}{2} \le 1,
\end{aligned} \tag{2.250}$$

and since $\forall x, y \in [0, v]$, $A(x, y) \le A(v, 0)$, we get $A : [0, v] \times [0, v] \to [0, v]$. Moreover,

$$\begin{aligned}
\big(A(0,v)\big)_i &= \big(A_1(0)\big)_i + \big(A_2(v)\big)_i = 0 + \frac{1}{i}\int_0^1 G(t,s)(1+1)^{-\frac{1}{2}}\,ds\\
&= \frac{1}{i}\cdot 2^{-\frac{1}{2}}\int_0^1 G(t,s)\,ds
\end{aligned} \tag{2.251}$$

by (2.250) and (2.251), we get $A(v,0) > 0$,

$$\begin{aligned}\big(A(0,v)\big)_i &= \frac{1}{i}\cdot 2^{-\frac{1}{2}}\cdot\frac{3i}{4}\big(A(v,0)\big)_i\\ &= \frac{3}{4\sqrt{2}}\big(A(v,0)\big)_i. \end{aligned}\tag{2.252}$$

Since $c = \frac{3}{4\sqrt{2}} > \frac{1}{2}$ such that $A(0,v) \geq cA(v,0)$, we know that all the hypotheses of Theorem 2.3.5 are satisfied and A has a unique fixed point $x^* \in [0,v]$, i.e., (2.245)–(2.246) has a unique solution in $[0,v]$. Moreover, $\forall x_0, y_0 \in [0,v]$, let

$$(x_n)_i = \frac{1}{i}\int_0^1 G(t,s)\left[\frac{1}{3}\big(x_{n-1}(s)\big)_{2i}^{\frac{1}{2}} + \big(1+y_{n-1}(s)\big)_{i+2}{}^{-\frac{1}{2}}\right]ds, \tag{2.253}$$

$$(y_n)_i = \frac{1}{i}\int_0^1 G(t,s)\left[\frac{1}{3}\big(y_{n-1}(s)\big)_{2i}^{\frac{1}{2}} + \big(1+x_{n-1}(s)\big)_{i+2}{}^{-\frac{1}{2}}\right]ds, \tag{2.254}$$

we have

$$(x_n)_i \to x_i^*, \quad (y_n)_i \to x_i^* \quad (n\to\infty)$$

uniformly convergent for $i = 1,2,\ldots$. Since the normal constant of $\bar{P}$ is 1, we get

$$\big\|x_n - x^*\big\| \leq \left(\frac{1-\frac{3}{4\sqrt{2}}}{\frac{3}{4\sqrt{2}}}\right)^n\cdot\|v\| = \left(\frac{4\sqrt{2}}{3}-1\right)^n;$$

$$\big\|y_n - x^*\big\| \leq \left(\frac{1-\frac{3}{4\sqrt{2}}}{\frac{3}{4\sqrt{2}}}\right)^n\cdot\|v\| = \left(\frac{4\sqrt{2}}{3}-1\right)^n.$$

□

Consider the Hammerstein type integral equation on $\mathbb{R}^N$

$$x(t) = (Ax)(t) = \int_{\mathbb{R}^N} K(t,s)\big[\big(1+x(s)\big) + x^{-\frac{1}{3}}(s)\big]ds. \tag{2.255}$$

Proposition 2.4.3 (Zhitao Zhang [197]) *Suppose* $K: \mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}$ *is continuous,* $K(t,s)\geq 0$, *and*

$$\frac{1}{21} \leq \int_{\mathbb{R}^N} K(t,s)\,ds \leq \frac{1}{5} \tag{2.256}$$

then (2.255) *has a unique positive solution* $x^*(t)$, *satisfying* $10^{-1}\leq x^*(t)\leq 1$, *and* $\forall (x_0(t), y_0(t)) \in [10^{-1},1]\times[10^{-1},1]$, (2.195) *and* (2.196) *hold.*

Proof Let $E = C_B(\mathbb{R}^N)$ is the space of bounded continuous functions on $\mathbb{R}^N$, define norm $\|x\| = \sup_{t\in R^N}|x(t)|$, then E is a Banach space.

Let $P = C_B^+(\mathbb{R}^N)$ denotes the set of all nonnegative functions in E, then P is a normal and solid cone. Equation (2.255) can be read as $x = A(x, x)$, where

$$A(x, y) = A_1(x) + A_2(y),$$
$$A_1(x) = \int_{R^N} K(t, s)\big(1 + x(s)\big)\,ds,$$
$$A_2(y) = \int_{R^N} K(t, s) y^{-\frac{1}{3}}(s)\,ds.$$

We aim to apply Theorem 2.3.9. Let $u_0 = 10^{-1},\ v_0 = 1$, and it is easy to know $A : P \times \mathring{P} \to \mathring{P}$ is mixed monotone operator. For fixed $y,\ A(\cdot, y) : P \to \mathring{P}$ is concave; for fixed $x,\ A(x, \cdot) : \mathring{P} \to \mathring{P}$ is $-\frac{1}{3}$ convex.

Obviously $\theta \ll u_0 \le v_0$, and by (2.256),

$$A(u_0, v_0) = \int_{\mathbb{R}^N} K(t, s)\big[\big(10^{-1} + 1\big)\big]\,ds \ge 10^{-1}, \tag{2.257}$$

$$A(v_0, u_0) = \int_{\mathbb{R}^N} K(t, s)\big[(1 + 1) + 10^{\frac{1}{3}}\big]\,ds \le 1, \tag{2.258}$$

$$A(\theta, v_0) = \int_{\mathbb{R}^N} K(t, s)\big[(1 + 0) + 1\big]\,ds \ge \frac{2}{5} A(v_0, u_0). \tag{2.259}$$

Let $\varepsilon = \frac{2}{5}$, then $\varepsilon > 0,\ \varepsilon > \frac{1}{3}$. Then by Theorem 2.3.9, this proposition is proved. □

2.5 Further Results on Cones and Partial Order Methods

Let E be a real Banach space, P be a cone in E. Let $P^* = \{f \in E^* | f(x) \ge 0,\ \forall x \in P\}$, if $\overline{P - P} = E$, then $P^* \subset E^*$ is a cone; by Theorem 1 of [13], we know P is normal $\Leftrightarrow$ P^* is generating.

Theorem 2.5.1 (Zhitao Zhang [199]) *Let P be a cone in E, then the following assertions are equivalent*:

(i) *P is normal*;
(ii) *$x_n \le z_n \le y_n$ $(n = 1, 2, \ldots)$, $x_n \to x$ weakly and $y_n \to x$ weakly imply $z_n \to x$ weakly.*

Proof (ii) $\Rightarrow$ (i) Suppose (i) is not true, then $\exists \theta \le x_n \le y_n$ such that $\|x_n\| > n^2 \|y_n\|,\ n = 1, 2, \ldots$. Let $z_n = \frac{x_n}{n\|y_n\|},\ y_n' = \frac{y_n}{n\|y_n\|}$, then $\theta \le z_n \le y_n'$, and $\|y_n'\| \to 0$, thus $y_n' \to \theta$ weakly, but we know that $\|z_n\| > n$, so $\{z_n\}$ is unbounded, $z_n \to \theta$ weakly is impossible, which contradicts (ii).

(i) $\Rightarrow$ (ii) $\forall x_n \le z_n \le y_n,\ x_n \to x$ weakly, $y_n \to x$ weakly. By (i), we know P^* is generating, i.e., $\forall f \in E^*,\ \exists f_1, f_2 \in P^*$ such that $f = f_1 - f_2$. We have

$$f_i(x_n) \le f_i(z_n) \le f_i(y_n), \quad f_i(x_n) \to f_i(x), \quad f_i(y_n) \to f_i(x)$$
$$(i = 1, 2,\ n \to +\infty)$$

thus we get $f_i(z_n) \to f_i(x)$ and $f(z_n) = f_1(z_n) - f_2(z_n) \to f_1(x) - f_2(x) = f(x)$, i.e., $z_n \to x$ weakly. □

Theorem 2.5.2 (Zhitao Zhang [199]) *Let P be a strongly minihedral and solid cone, then every bounded (in norm) set D has a least upper bound* $\sup D$.

Proof We know $\exists M > 0$ such that $\forall x \in D,\ \|x\| < M$. Since P is a solid cone, we find that $\exists u_0 \in \mathring{P},\ \delta > 0$ such that $u_0 - \delta \frac{x}{\|x\|} \in P$. So $x \le \frac{\|x\|}{\delta} u_0 < \frac{M}{\delta} u_0$, thus D is a bounded set (in order), by the definition of strongly minihedral cone, we get D has a least upper bound $\sup M$. □

Next we consider two operators form equations:

$$A(x, x) + Bx = x. \tag{2.260}$$

Definition 2.5.1 $B : D \to E$ satisfies

$$B\big(tx + (1-t)y\big) = tBx + (1-t)By$$

for $x, y \in D$, and $t \in [0, 1]$, then B is said to be affine.

Theorem 2.5.3 (Li, Liang and Xiao [129]) *Let P be normal, N be the normal constant of P. Let $u, v \in P \cap \mathcal{D}(B)$, $u < v$, operator $A : [u, v] \times [u, v] \to E$ be mixed monotone and $B : \mathcal{D}(B) \to E$ be affine on $[u, v]$, where $[u, v] = \{x \in E | u \le x \le v\}$. Assume that*:

(i) *for fixed y, $A(\cdot, y) : [u, v] \to E$ is concave; for fixed x, $A(x, \cdot) : [u, v] \to E$ is convex*;
(ii) *$(I - B)^{-1} : E \to \mathcal{D}(B)$ exists and is an increasing operator on $[u - Bu, v - Bv]$, i.e., for $x, y \in [u - Bu, v - Bv]$, $x \ge y$ implies that $(I - B)^{-1}x \ge (I - B)^{-1}y$, where I is the identity operator on E*;
(iii) *$A(u, v) \ge u$, $A(v, u) \le v$, $Bu \ge \theta$ and $Bv \le \theta$*;
(iv) *there exists some $m_0 \in \mathbb{N} \cup \{0\}$ such that*

$$u_{m_0+1} \ge \frac{1}{2}(v_{m_0+1} + u_{m_0}), \tag{2.261}$$

where

$$u_0 = u, \qquad v_0 = v,$$
$$u_n = (I - B)^{-1} A(u_{n+1}, v_{n+1}) \quad (n = 1, 2, \ldots), \tag{2.262}$$
$$v_n = (I - B)^{-1} A(v_{n+1}, u_{n+1}) \quad (n = 1, 2, \ldots). \tag{2.263}$$

Then (2.260) *has a unique positive solution* x^* *in* $[u, v]$. *Moreover, constructing successively the sequences*

$$x_n = (I - B)^{-1} A(x_{n-1}, y_{n-1}) \quad (n = 1, 2, 3, \ldots), \tag{2.264}$$

$$y_n = (I - B)^{-1} A(y_{n-1}, x_{n-1}) \quad (n = 1, 2, 3, \ldots) \tag{2.265}$$

for any initial $(x_0, y_0) \in [0, v] \times [u, v]$, *we have*

$$x_n \to x^*, \quad y_n \to x^* \quad (n \to \infty),$$

and the convergence rates are

$$\left\| x_{m_0+n} - x^* \right\| \le \frac{2N^2}{n+1} \|v - u\| \quad (n \to \infty), \tag{2.266}$$

$$\left\| y_{m_0+n} - x^* \right\| \le \frac{2N^2}{n+1} \|v - u\| \quad (n \to \infty). \tag{2.267}$$

Proof For convenience of presentation, we denote $C = (I - B)^{-1} A$. From (ii) and (iii) we have

$$C(u, v) = (I - B)^{-1} A(u, v) \ge (I - B)^{-1} u \ge u, \tag{2.268}$$

and

$$C(v, u) = (I - B)^{-1} A(v, u) \ge (I - B)^{-1} v \ge v. \tag{2.269}$$

Hence $\mathcal{R}(C|_{[u,v]\times[u,v]}) \subset [u, v]$. Since B is affine on $[u, v]$, if follows that $(I - B)^{-1}$ is affine on $[u - Bu, v - Bv]$. So by (ii) we know that C is mixed monotone, and for fixed y, $C(\cdot, y) : [u, v] \to E$ is concave, for fixed x, $C(x, \cdot) : [u, v] \to E$ is convex.

By (2.262), (2.263), (2.268) and (2.269), we show easily that for $n \in \mathbb{N}$,

$$\begin{aligned} u = u_0 &\le u_1 \le \cdots \le u_{m_0} \le u_{m_0+1} \le \cdots \le u_{m_0+n} \le u_{u+m_0+1} \le \cdots \\ &\le v_{m_0+n+1} \le v_{m_0+n} \le \cdots \le v_{m_0+1} \le v_{m_0} \le \cdots \le v_1 \\ &\le v_0 = v. \end{aligned} \tag{2.270}$$

Thus, it follows from (2.261) and (2.270) that

$$\theta \le \frac{1}{2}(v_{m_0+n} - u_{m_0}) \le \cdots \le \frac{1}{2}(v_{m_0+1} - u_{m_0}) \le u_{m_0+1} - u_{m_0} \le \cdots \le u_{m_0+n} - u_{m_0}.$$

We set

$$t_n = \sup\left\{t > 0 : u_{m_0+n} \ge t v_{m_0+n} + (1 - t) u_{m_0}\right\} \quad (n = 1, 2, \ldots).$$

Obviously,

$$u_{m_0+n} \ge t_n v_{m_0+n} + (1 - t_n) u_{m_0}, \tag{2.271}$$

and

$$\frac{1}{2} \le t_1 \le t_2 \le \cdots \le t_n \le \cdots \le 1.$$

Next, we prove $t_n \to 1$ as $n \to \infty$. For $n \in \mathbb{N}$, making use of (2.271) we obtain

$$\begin{aligned}
u_{m_0+n} &\ge t_n u_{m_0+n} + (1-t_n)u_{m_0} \\
&\ge t_n v_{m_0+n} + (1-t_n)\frac{2u_{m_0} - t_n v_{m_0+n}}{2-t_n} \\
&\ge \frac{t_n}{2-t_n} v_{m_0+n} + \frac{2(1-t_n)}{2-t_n} u_{m_0}.
\end{aligned}$$

So,

$$\begin{aligned}
u_{m_0+n+1} &= C(u_{m_0+n}, v_{m_0+n}) \\
&\ge \frac{t_n}{2-t_n} C(v_{m_0+n}, v_{m_0+n}) + \frac{2(1-t_n)}{2-t_n} C(u_{m_0}, v_{m_0+n}), \\
&\ge \frac{t_n}{2-t_n} C(v_{m_0+n}, v_{m_0+n}) + \frac{2(1-t_n)}{2-t_n} C(u_{m_0}, v_{m_0+1}),
\end{aligned}$$

i.e.,

$$t_n C(v_{m_0+n}, v_{m_0+n}) + 2(1-t_n) C(u_{m_0}, v_{m_0+1}) \le (2-t_n) u_{m_0+n+1}. \tag{2.272}$$

From (2.261), (2.271) and (2.272), we have

$$\begin{aligned}
v_{m_0+n+1} &= C(v_{m_0+n}, u_{m_0+n}) \\
&\le t_n C(v_{m_0+n}, v_{m_0+n}) + (1-t_n) C(v_{m_0+n}, u_{m_0}) \\
&\le t_n C(v_{m_0+n}, v_{m_0+n}) + (1-t_n) v_{m_0+1} \\
&\le t_n C(v_{m_0+n}, v_{m_0+n}) + (1-t_n)(2u_{m_0+1-u_{m_0}}) \\
&= t_n C(v_{m_0+n}, v_{m_0+n}) + 2(1-t_n) u_{m_0+1} - (1-t_n) u_{m_0} \\
&\le t_n C(v_{m_0+n}, v_{m_0+n}) + 2(1-t_n) C(u_{m_0}, v_{m_0+1}) - (1-t_n) u_{m_0} \\
&\le (2-t_n) u_{m_0+n+1} - (1-t_n) u_{m_0}.
\end{aligned}$$

Consequently,

$$u_{m_0+n+1} \ge \frac{1}{2-t_n} v_{m_0+n+1} - (1-t_n) u_{m_0}.$$

This means that

$$t_{n+1} \ge \frac{1}{2-t_n},$$

i.e.,

$$1 - t_{n+1} \leq \frac{1 - t_n}{2 - t_n}. \tag{2.273}$$

For convenience, we set $s_n = 1 - t_n \neq 0$. Then (2.273) can be rewritten as follows:

$$s_{n+1} \leq \frac{s_n}{1 + s_n} = \frac{1}{1 + \frac{1}{s_n}}. \tag{2.274}$$

Therefore,

$$\frac{1}{s_{n+1}} \geq 1 + \frac{1}{s_n}$$

and

$$\frac{1}{s_n} \geq 1 + \frac{1}{s_n}. \tag{2.275}$$

Combining (2.274) and (2.275) gives

$$1 - t_{n+1} = s_{n+1} \leq \frac{1}{2 + \frac{1}{s_{n-1}}} \leq \cdots \leq \frac{1}{n + \frac{1}{s_1}} \leq n + 2. \tag{2.276}$$

Hence, $s_n \to 0$ $(n \to \infty)$, i.e., $t_n \to 1$ as $n \to \infty$. In addition, from (2.270) and (2.271) we obtain, for $n, p \in \mathbb{N}$,

$$\begin{aligned} \theta \leq u_{m_0+n+p} - u_{m_0+n} &\leq v_{m_0+n} - u_{m_0+n} \\ &\leq (1 - t_n)(v_{m_0+n} - u_{m_0}) \leq (1 - t_n)(v - u). \end{aligned} \tag{2.277}$$

Since P is normal we get

$$\|u_{m_0+n+p} - u_{m_0+n}\| \leq N(1 - t_n)\|v - u\| \tag{2.278}$$

and

$$\|v_{m_0+n} - u_{m_0+n}\| \leq N(1 - t_n)\|v - u\|. \tag{2.279}$$

From (2.278) we know that $\{u_{m_0+n}\}_{n=1}^{\infty}$ is a Cauchy sequence. Hence there exists $x^* \in [u, v]$ such that $u_n \to x^*$ $(n \to \infty)$. In combination with (2.270), (2.277), and (2.279), this implies

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = x^*, \tag{2.280}$$

and

$$u_{m_0+n} \leq x^* \leq v_{m_0+n}. \tag{2.281}$$

Therefore,

$$u_{m_0+n+1} = C(u_{m_0+n}, v_{m_0+n}) \leq C\left(x^*, x^*\right) \leq C(v_{m_0+n}, u_{m_0+n}) = v_{m_0+n+1}.$$

Taking limit and using (2.280), we conclude that

$$C\bigl(x^*, x^*\bigr) = x^*,$$

i.e.,

$$(I - B)^{-1} A\bigl(x^*, x^*\bigr) = x^*.$$

So,

$$A\bigl(x^*, x^*\bigr) + Bx^* = x^*,$$

that is, (2.260) has a positive solution x^*.

Now for each $(x_0, y_0) \in [0, v] \times [u, v]$, considering the sequences (2.264) and (2.265) we have

$$u_n \le x_n \le v_n \quad (n = 0, 1, \ldots), \tag{2.282}$$

and

$$u_n \le y_n \le v_n \quad (n = 0, 1, \ldots). \tag{2.283}$$

Hence by (2.280) we get

$$\bigl\|x_n - x^*\bigr\| \to 0 \quad (n \to \infty), \tag{2.284}$$

and

$$\bigl\|y_n - x^*\bigr\| \to 0 \quad (n \to \infty). \tag{2.285}$$

Moreover, using (2.276), (2.279), (2.281), and (2.282) we obtain the following inequality:

$$\begin{aligned}
\bigl\|x_{m_0+n} - x^*\bigr\| &\le \|x_{m_0+n} - u_{m_0+n}\| + \bigl\|x^* - u_{m_0+n}\bigr\| \\
&\le 2N\|v_{m_0+n} - u_{m_0+n}\| \\
&\le 2N^2(1 - t_n)\|v - u\| \\
&\le \frac{2N^2}{n+1}\|v - u\| \quad (n = 1, 2, \ldots).
\end{aligned}$$

In the same way we can get (2.267).

Finally, let $y^* \in [u, v]$, $y^* \ne x^*$ and $C(y^*, y^*) = y^*$. We take initial value $(x_0, y_0) = (y^*, y^*)$. Then,

$$x_1 = C(x_0, y_0) = C\bigl(y^*, y^*\bigr) = y^*$$

and

$$y_1 = C(y_0, x_0) = C\bigl(y^*, y^*\bigr) = y^*.$$

By induction we have $x_n = y_n = y^*$ $(n = 1, 2, \ldots)$. Noting that (2.284) and (2.285), we get $y^* = x^*$. This implies the uniqueness of positive solution of (2.260). $\square$

Remark 2.5.1 In Theorem 2.5.3 they do not require operators A and B to be compact or continuous.

In the special case $B = 0$, we have the following corollary, which improved Corollary 2.3.1.

Corollary 2.5.1 (Li, Liang and Xiao [129]) *Let P be normal, $u, v \in P$, $u < v$, operator $A : [u, v] \times [u, v] \to E$ is mixed monotone and satisfy the following condition* (i) *of Theorem* 2.5.3. *Suppose that*

$$A(u, v) \geq u, \quad A(v, u) \leq v \quad \textit{and} \quad A(u, v) \geq \frac{1}{2}\big[A(v, u) + u\big].$$

Then A has a unique fixed point x^* in $[u, v]$.

Remark 2.5.2 In Corollary 2.3.1, the constant c must be strictly larger than $\frac{1}{2}$. But Corollary 2.5.1 works even for $c = \frac{1}{2}$.

Example 2.5.1 (Zhitao Zhang and Liming Sun) *An example to show that constant $c = \frac{1}{2}$ is the best one*:

According to the Theorem 2.3.5 and Corollary 2.5.1, it requires that the constant $c \in [\frac{1}{2}, 1]$. Then a natural question comes out that whether this result holds when $c < \frac{1}{2}$. The answer is negative. We will give an example to show mixed monotone operator A may has no fixed point when $c < \frac{1}{2}$.

We consider Banach space

$$l^{\infty} = \big\{x = (x_1, x_2, \ldots) \mid x_i \in \mathbb{R}, i = 1, 2, \ldots \text{ and } |x_i| \text{ is uniformly bounded}\big\}$$

with norm $\|x\| = \sup_{i \in \mathbb{N}} |x_i|$. Define subspace

$$V = \Big\{x \in l^{\infty} \mid \text{there exists } \bar{x} \in \mathbb{R}, \text{ s.t. } \lim_{i \to \infty} x_i = \bar{x} < \infty\Big\}.$$

Then V is a Banach space. Let $P = \{x \in V \mid x_i \geq 0, \forall i \geq 1\}$, then it is easy to verify that P is a normal cone of V.

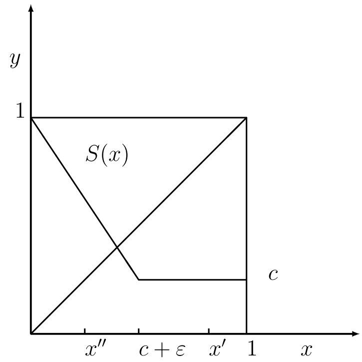
First, we need a function $S : [0, 1] \to [0, 1]$ which is convex, decreasing and continuous in $[0, 1]$, but $S^n(0)$ and $S^n(1)$ do not converge to the fixed point of S. For example, suppose $c < \frac{1}{2}$, choose $\varepsilon > 0$ small enough such that $c + \varepsilon < \frac{1}{2}$, let

$$S(x) = \begin{cases} -\frac{1-c}{c+\varepsilon}x + 1, & x \in [0, c+\varepsilon], \\ c, & x \in (c+\varepsilon, 1]. \end{cases} \tag{2.286}$$

The graph of S is presented in Fig. 2.1. It is easy to know that S has a unique fixed point $x = \frac{c+\varepsilon}{1+\varepsilon}$ and a couple fixed points $x' = \frac{c^2+\varepsilon}{c+\varepsilon}$, $x'' = c$, which means $S(x') = x''$ and $S(x'') = x'$. We can verify that

$$S(0) = 1, \qquad S(1) = c,$$

Fig. 2.1 The graph of S



$$S^{2n}(0) = S^{2n+1}(1) = x'', \quad n \geq 1,$$
$$S^{2n+1}(0) = S^{2n}(1) = x', \quad n \geq 1.$$

Let $\bar{v} = (1, 1, 1, \ldots)$, $\theta = (0, 0, \ldots)$,

$$A : P \to P$$
$$x \mapsto Ax = \big(1, S(x_1), S(x_2), \ldots\big)$$

A maps P to P, in fact suppose $x \in P$ with $\lim_{i\to\infty} x_i = \bar{x}$, then $\lim_{i\to\infty} S(x_i) = S(\bar{x})$, since S is continuous.

Then it is easy to verify A is decreasing and convex,

$$A(\bar{v}) = \big(1, S(1), S(1), \ldots\big),$$
$$A(\theta) = \big(1, S(0), S(0), \ldots\big) = \bar{v}.$$

So if there exists k such that $A(\bar{v}) \geq kA(\theta)$, then $k \leq \frac{S(1)}{S(0)} = c < \frac{1}{2}$, we can just let $k = c < \frac{1}{2}$.

We next show A has no fixed point in P. If there exists $x \in V$ such that $Ax = x$ then

$$x_1 = 1,$$
$$x_2 = S(x_1),$$
$$x_3 = S(x_2),$$
$$\vdots$$
$$x_n = S(x_{n-1}),$$
$$\vdots\ .$$

So $x_i = S^{i-1}(1)$, but we know that this consequence diverges, thus A has no fixed point in P.

Operator A is a special mixed monotone operator, A only has decreasing part. □

Definition 2.5.2 Let P be a cone of Banach space E, $A : P \to P$, $u_0 > \theta$ (i.e., $u_0 \in P$, $u \neq \theta$), A is called a u_0-concave operator, if

(i) $\forall x > \theta$ there exist $\alpha = \alpha(x) > 0$, $\beta = \beta(x) > 0$ such that

$$\alpha x \leq Ax \leq \beta x; \tag{2.287}$$

(ii) $\forall x \in P$ such that $\alpha_1 u_0 \leq x \leq \beta_1 u_0$ for some $\alpha_1 = \alpha_1(x) > 0$, $\beta_1 = \beta_1(x) > 0$, and $0 < t < 1$, there exists $\eta = \eta(x, t) > 0$ such that

$$A(tx) \geq (1 + \eta)tAx.$$

Definition 2.5.3 Let P be a cone of Banach space E, $A : P \to P$, $u_0 > \theta$ (i.e. $u_0 \in P, u \neq \theta$), A is called a u_0-convex operator, if

(i) $\forall x > \theta$ there exist $\alpha = \alpha(x) > 0$, $\beta = \beta(x) > 0$ such that (2.287) is satisfied;
(ii) $\forall x \in P$ such that $\alpha_1 u_0 \leq x \leq \beta_1 u_0$ for some $\alpha_1 = \alpha_1(x) > 0$, $\beta_1 = \beta_1(x) > 0$, and $0 < t < 1$, there exists $\eta = \eta(x, t) > 0$ such that

$$A(tx) \leq (1 - \eta)tAx.$$

Theorem 2.5.4 (Zhitao Zhang [199]) *Let $P \subset E$ be a normal and solid cone, $A : P \to P$ is a condensing map, $A^2\theta > \theta$, A is strongly decreasing, i.e., $\theta \leq x < y$ implies $Ax \gg Ay$, and $A(tx) < t^{-1}Ax$ for $x > \theta$, $t \in (0, 1)$. Then A has a unique fixed point $x^* > \theta$, and $\forall x_0 \in P$, let $x_n = Ax_{n-1}$ $(n = 1, 2, \ldots)$, we have $\|x_n - x^*\| \to 0 (n \to \infty)$.*

Proof Since P is normal we know $[\theta, A\theta]$ is bounded. Moreover, $[\theta, A\theta]$ is convex closed subset. Since A is decreasing, we get $A([\theta, A\theta]) \subset [\theta, A\theta]$. By the Sadovskii Theorem, we see that A has at least one positive fixed point. Obviously, $A^2 : P \to P$ is a condensing and increasing map. Let $u_0 = \theta$, $u_n = Au_{n-1}$, it is easy to know A^2 has a minimal fixed point u_* and a maximal fixed point u^* in $[\theta, A\theta]$. Moreover,

$$\begin{aligned} \theta = u_0 \leq u_2 \leq \cdots \leq u_{2n} \leq \cdots \leq u_* \leq \cdots \leq u^* \leq \cdots \\ \leq u_{2n+1} \leq \cdots \leq u_3 \leq u_1 = A\theta, \end{aligned} \tag{2.288}$$

$$\lim_{n\to\infty} u_{2n} = u_*, \qquad \lim_{n\to\infty} u_{2n+1} = u^*.$$

Since $A(tx) < t^{-1}Ax$, $\forall t \in (0, 1)$, we get

$$A\big(t \cdot t^{-1}x\big) < t^{-1} \cdot A\big(t^{-1}x\big),$$

i.e., $A(t^{-1}x) > t \cdot Ax$. Since A is strongly decreasing, we get

$$A^2(tx) \gg A\big(t^{-1}Ax\big) > tA^2x.$$

Thus A^2 is strongly sublinear, and $\exists e \in \mathring{P}$ such that A^2 is e-concave (see [110]). Therefore, by Theorem 2.2.2 of [110], we see that A^2 has at most one positive (i.e., $>\theta$) fixed point. Let $u_* = u^* = x^*$, then $A^2x^* = x^*$, and $A(A^2x^*) = A^2(Ax^*) = Ax^*$, thus $Ax^* = x^*$. Moreover, $\forall x_0 \in P,\ \theta \le Ax_0 \le A\theta,\ A^2\theta \le A^2x_0 \le A\theta$, i.e., $u_0 \le x_1 \le u_1,\ u_2 \le x_2 \le u_1$, by induction, we get

$$u_{2n} \le x_{2n+1} \le u_{2n+1}, \qquad u_{2n} \le x_{2n} \le u_{2n-1}.$$

By taking the limit, we get $x_n \to x^*$ $(n \to \infty)$. □

Theorem 2.5.5 (Zhitao Zhang [199]) *Suppose $P \subset E$ is a normal and solid cone, $A : P \times P \to P$ is completely continuous, and $A(x, y)$ is strongly increasing in x, $A(x, y)$ is decreasing in y. Moreover, $\exists \theta < u_0 < v_0$ such that $A([u_0, v_0] \times [u_0, v_0]) \subset [u_0, v_0]$; $A(tx, y) \gg tA(x, y)$, for $x, y \in P,\ t \in (0, 1)$. Then A has a unique fixed point $x^* \in [u_0, v_0]$ and*

$$\forall x_0, y_0 \in [u_0, v_0], \quad x_n = A(x_{n-1}, y_{n-1}), \quad y_n = A(y_{n-1}, x_{n-1}),$$

we have $x_n \to x^*,\ y_n \to x^*$ $(n \to \infty)$.

Proof By Theorem 2.1.7 of [110], we know A has a couple quasi-fixed point $(x^*, y^*) \in [u_0, v_0] \times [u_0, v_0]$, i.e., $x^* = A(x^*, y^*),\ y^* = A(y^*, x^*)$, which is minimal and maximal in the sense that $x^* \le \bar{x} \le y^*$ and $x^* \le \bar{y} \le y^*$ for any coupled quasi-fixed point $(\bar{x}, \bar{y}) \in [u_0, v_0] \times [u_0, v_0]$. Moreover, we have $x^* = \lim_{n\to\infty} u_n, y^* = \lim_{n\to\infty} v_n$, where $u_n = A(u_{n-1}, v_{n-1}),\ v_n = A(v_{n-1}, u_{n-1})$ $(n = 1, 2, \ldots)$ which satisfy

$$u_0 \le u_1 \le \cdots \le u_n \le x^* \le \cdots \le y^* \le v_n \le \cdots \le v_1 \le v_0. \tag{2.289}$$

Now we prove $x^* = y^*$.

If $x^* \ne y^*$ then $x^* < y^*$. Let $t_0 = \inf\{t | x^* \le ty^*\}$, then since A is strongly increasing in x, we get

$$x^* = A\big(x^*, y^*\big) \ll A\big(y^*, y^*\big) \le A\big(y^*, x^*\big) = y^*, \tag{2.290}$$

thus by $x^* \ge u_0 > \theta$ and (2.290), we have $0 < t_0 < 1$ and

$$x^* = A\big(x^*, y^*\big) \le A\big(t_0 y^*, x^*\big) \ll t_0 A\big(y^*, x^*\big) = t_0 y^*, \tag{2.291}$$

which contradicts the definition of t_0.

Thus we have $x^* = y^*$ and

$$x^* = A\big(x^*, x^*\big), \qquad u_n \to x^*, \quad v_n \to x^* \quad (n \to \infty).$$

Moreover, $\forall x_0, y_0 \in [u_0, v_0]$, let

$$x_n = A(x_{n-1}, y_{n-1}), \qquad y_n = A(y_{n-1}, x_{n-1}).$$

It is obvious that $u_n \le x_n \le v_n,\ u_n \le y_n \le v_n$. Following the normality of P, we get $x_n \to x^*,\ y_n \to x^*$ $(n \to \infty)$. □

Proposition 2.5.1 *There is no operator $A : P \to P$ which is decreasing and e-convex, where $e > \theta$.*

Proof If $A : P \to P$ is decreasing and e-convex, then for $x \in P$ such that $\alpha_1(x)e \leq x \leq \beta_1(x)e$ $(\alpha_1(x) > 0,\ \beta_1(x) > 0)$, and $\forall t \in (0, 1),\ \exists \eta = \eta(x, t) > 0$ (see [110]) such that

$$Ax \leq A(tx) \leq (1 - \eta)tAx,$$

so we have $(1 - (1 - \eta)t)Ax \leq \theta$, thus $Ax \leq \theta$, but by the definition of e-convex, we have $\exists \alpha = \alpha(x) > 0$ such that $Ax \geq \alpha e$. So we know $e \leq \theta$, which contradicts $e > \theta$. □

Proposition 2.5.2 *Let E be weakly complete and P be a normal and solid cone in E, $A : P \to P$ is continuous and strongly decreasing, $A^2\theta > \theta$; $A(tx) < t^{-1}Ax$, for $t \in (0, 1)$, $x \in P$. Then A has a unique fixed point $x^* \in P$, and $\forall x_0 \in P$, let $x_n = Ax_{n-1}$, we have $x_n \to x^*$ $(n \to \infty)$.*

Proof Let

$$u_0 = \theta, \qquad u_n = Au_{n-1}.$$

We know $A^2 : [\theta, A\theta] \to [\theta, A\theta]$ is continuous and strongly increasing. Moreover,

$$\theta \leq u_0 \leq u_2 \leq \cdots \leq u_{2n} \leq \cdots \leq u_{2n+1} \leq \cdots \leq u_1 = A\theta. \tag{2.292}$$

Since E is weakly complete and P is normal, by Theorem 2.2 of [89], we get P is regular. By Theorem 2.1.1 of [110], we know A^2 has a minimal fixed point u_* and a maximal fixed point u^*, and $u_{2n} \to u_*$, $u_{2n+1} \to u^*$. Similarly to the proof of Theorem 2.5.4, we can prove $u^* = u_*$, and let $x^* = u_* = u^*$, then $Ax^* = x^*$, and $\forall x_0 \in P$, $x_n = Ax_{n-1}$, we have $x_n \to x^*$ $(n \to \infty)$. □

On Differentiable Maps Let (E, P) be an OBS with open unit ball B, a subset $D \subset E$ is called a right (or left) neighborhood of point $x \in E$ if there exists a positive number ε such that $x + \varepsilon B^+ \subset D$ (or $x - \varepsilon B^+ \subset D$). The set D is called P-open (or $-P$-open), if D is a right (or left) neighborhood of each of its points (see [8]).

Let D be a right neighborhood of some $x \in E$ and let F be an arbitrary Banach Space. A map $A : D \to F$ is said to be right differentiable in x, if there exists a bounded linear operator $T \in L(\overline{P - P}, F)$ such that

$$\lim_{h \to \theta, h \in P} \frac{\|A(x + h) - A(x) - Th\|}{\|h\|} = 0. \tag{2.293}$$

T is denoted by $A'_+(x)$ and called the right derivative of A at x. Let $D \subset E$ be P-open, then $A : D \to F$ is called right differentiable if A has the right derivative at every $x \in D$. In this case, the map $A'_+ : D \to L(\overline{P - P}, F)$ is called the right derivative of A. If A'_+ maps D continuously into the Banach space $L(\overline{P - P}, F)$ then A is said to be continuously right differentiable. Left derivatives and left differentiable maps are defined in the obvious way (see [8]).

Theorem 2.5.6 (Zhitao Zhang, see [199]) *Suppose that $P \subset E$ is a normal cone, $A : P \times P \to P$ is a mixed monotone operator, and $A(x, y) = A_1(x) + A_2(y)$, where A_1 is increasing and A_2 is decreasing. Moreover, $\exists v \in P$ such that $A : [\theta, v] \times [\theta, v] \to [\theta, v]$; $A_1 : P \to P$ and $A_2 : P \to P$ are both continuously right differentiable, and $\sup_{x\in[\theta,v]} \|A'_{1+}(x)\| + \sup_{y\in[\theta,v]} \|A'_{2+}(y)\| = \delta < 1$. Then A has a unique fixed point $x^* \in [\theta, v]$.*

Proof Let

$$u_0 = \theta, \qquad v_0 = v, \qquad u_n = A(u_{n-1}, v_{n-1}), \qquad v_n = A(v_{n-1}, u_{n-1}).$$

Since A is mixed monotone, we know easily

$$\theta = u_0 \le u_1 \le u_2 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_3 \le v_1 \le v_0 = v. \tag{2.294}$$

Moreover, we know that $x + t(y - x) \in P$, $\forall t \in [0, 1]$, $\forall \theta \le x \le y \le v$. So $A_i(x + t(y - x)) : [0, 1] \to P$, $i = 1, 2$ are right differentiable on [0, 1], and by mean value theorem (see [11]), we get $A_i(y) - A_i(x) \in (1-0)\overline{co}\{A'_{i+}(x+t(y-x))(y-x), t \in [0, 1]\} = \overline{co}\{A'_{i+}(x + t(y - x))(y - x), t \in [0, 1]\}$ $(i = 1, 2)$. So we have

$$\begin{aligned}
\big\|A(y, x) - A(x, y)\big\| &= \big\|A_1(y) + A_2(x) - A_1(x) - A_2(y)\big\| \\
&\le \big\|A_1(y) - A_1(x)\big\| + \big\|A_2(x) - A_2(y)\big\| \\
&\le \sup_{x\in[\theta,v]} \big\|A'_{1+}(x)\big\| \cdot \|y - x\| + \sup_{y\in[\theta,v]} \big\|A'_{2+}(y)\big\| \cdot \|y - x\| \\
&= \delta\|y - x\|.
\end{aligned} \tag{2.295}$$

So by standard argument, we know that $\exists u^*$ such that $u_n \to u^*$, $v_n \to u^*$, and $A(u^*, u^*) = u^*$. □

Theorem 2.5.7 (Zhitao Zhang [199]) *Let $P \subset E$ be a normal cone, $A : P \to P$ be a continuous, continuously right differentiable, twice right differentiable decreasing map. Suppose that one of the following hypotheses is satisfied:*

(i) *A is order convex, i.e., $\forall \theta \le x \le y$, $t \in [0, 1]$, $A(tx + (1 - t)y) \le tAx + (1 - t)Ay$, and $A'_+(0)u \ge -Nu$, $\forall u \in P$;*
(ii) *A is order concave on $[\theta, A\theta]$, i.e., $\forall x, y \in [\theta, A\theta]$, $x \le y$, $t \in [0, 1]$, $A(tx + (1 - t)y) \ge tAx + (1 - t)Ay$, and $A'_+(A\theta)u \ge -Nu$, $\forall u \in P$, where N is a positive constant.*

Then A has a unique fixed point $x^* \in P$.

Proof (i) Since A is order convex, in virtue of Theorem 23.1, Theorem 23.3 in [8] and Proposition 3.2 in [7], we know

$$\forall \theta \le x \le y, \quad A'_+(y)h \ge A'_+(x)h, \ \forall h \in P \tag{2.296}$$

and

$$Ay - Ax \geq A'_+(x)(y - x) \geq A'_+(0)(y - x) \geq -N(y - x). \tag{2.297}$$

So we get $(A + N)y - (A + N)x \geq \theta$. Let

$$Bx = \frac{Ax + Nx}{1 + N},$$

then $B : P \to P$ is increasing and $\forall x \in [\theta, A\theta]$,

$$\theta \leq B\theta = \frac{A\theta}{1 + N} \leq A\theta, \qquad B(A\theta) = \frac{A^2\theta + NA\theta}{1 + N} \leq \frac{A\theta + NA\theta}{1 + N} \leq A\theta. \tag{2.298}$$

Let

$$u_0 = \theta, \qquad v_0 = A\theta, \qquad u_n = Bu_{n-1}, \qquad v_n = Bv_{n-1},$$

we have

$$\theta = u_0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_2 \leq v_1 \leq v_0 = A\theta, \tag{2.299}$$

and

$$\begin{aligned} \theta \leq v_n - u_n &= \frac{Av_{n-1} + Nv_{n-1}}{1 + N} - \frac{Au_{n-1} + Nu_{n-1}}{1 + N} \\ &= \frac{Av_{n-1} - Au_{n-1} + N(v_{n-1} - u_{n-1})}{1 + N} \leq \frac{N}{1 + N}(v_{n-1} - u_{n-1}) \\ &\leq \left(\frac{N}{1 + N}\right)^n (v_0 - u_0) = \left(\frac{N}{1 + N}\right)^n \cdot A\theta. \end{aligned} \tag{2.300}$$

Following the normality of P, there exists a unique $u^* \in P$ such that $u_n \to u^*,\ v_n \to u^*\ (n \to \infty)$ and $Bu^* = u^*$, i.e., $Au^* = u^*$. Moreover, $\forall x_0 \in [\theta, A\theta]$, $x_n = Bx_{n-1}$, we get $u_n \leq x_n \leq v_n,\ x_n \to u^*\ (n \to \infty)$. The assertion (i) is valid.

(ii) Since A is order concave, in virtue of Theorem 23.1, Theorem 23.3 in [8] and Proposition 3.2 in [7], we have

$$A'_+(y) \leq A'_+(x), \quad \forall \theta \leq x \leq y \leq A\theta,$$

i.e.,

$$A'_+(y)h \leq A'_+(x)h, \quad \forall h \in P,$$

and

$$Ay - Ax \geq A'_+(y)(y - x) \geq A'_+(A\theta)(y - x) \geq -N(y - x), \tag{2.301}$$

i.e.,

$$(Ay + Ny) - (Ax + Nx) \geq \theta.$$

Similar to the proof of (i), we can prove the assertion (ii). $\square$

Example 2.5.2 Consider the following integral equation:

$$x(t) = Ax(t) = \int_0^1 k(t,s)\frac{1}{(1+x(s))^p}\,ds \quad (0 < p < 1). \tag{2.302}$$

Proposition 2.5.3 *If* $k : [0,1] \times [0,1] \to \mathbb{R}$ *is continuous and* $k(t,s) > 0$, *then* (2.302) *has a unique positive solution* $x^*(t)$, $\forall x_0(t) \geq 0$, $x_n = Ax_{n-1}$, *we have* $x_n \to x^*$ $(n \to \infty)$.

Proof Let $E = C[0,1]$, $P = \{x | x(t) \geq 0, x \in E\}$ is a normal and solid cone in E. We know that $A : P \to P$ is completely continuous, $A^2\theta > \theta$. If $0 \leq x_1(t) < x_2(t)$, then

$$Ax_1(t) - Ax_2(t) = \int_0^1 k(t,s)\left(\frac{1}{(1+x_1(s))^p} - \frac{1}{(1+x_2(s))^p}\right)ds. \tag{2.303}$$

Since $k(t,s) > 0$, and $\frac{1}{(1+x_1(s))^p} - \frac{1}{(1+x_2(s))^p} > \theta$, there exists a constant $\delta > 0$ such that $Ax_1(t) - Ax_2(t) \geq \delta > 0$, so A is strongly decreasing. Moreover, $\forall x > \theta$, $\forall \tau \in (0,1)$, we have

$$\begin{aligned} A(\tau x) &= \int_0^1 k(t,s)\frac{1}{(1+\tau x(s))^p}\,ds = \int_0^1 k(t,s)\frac{1}{\tau^p(\tau^{-1}+x(s))^p}\,ds \\ &\leq \frac{1}{\tau^p}\int_0^1 k(t,s)\frac{1}{(1+x(s))^p}\,ds < \tau^{-1}Ax. \end{aligned} \tag{2.304}$$

By Theorem 2.5.4, we get A has a unique fixed point $x^* \in P$, and (2.302) has a unique positive solution $x^*(t)$. Moreover, $\forall x_0(t) \in P$, let $x_n(t) = Ax_{n-1}(t)$, then $x_n(t) \to x^*(t)$ uniformly on $[0,1]$. $\square$

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